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The Irreducibility of the Monodromy Representation Associated with the Dotsenko–Fateev Equation

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Abstract: We study the monodromy representation on the solution space of the Fuchsian differential equation of order 3 derived by Dotsenko and Fateev. In particular, we give a condition for the representation being irreducible, construct the subsystems in reducible cases, and obtain the monodromy-invariant Hermitian form on the solution space.

1. Introduction

In [5], as a fundamental equation in conformal field theory, especially in minimal models, Dotsenko and Fateev considered the following Fuchsian differential equation of order 3 :

$$
\begin{aligned}
&z^2(z-1)^2\frac{d^3I}{dz^3}+(K_1z+K_2(z-1))z(z-1)\frac{d^2I}{dz^2}\\
&\quad+(L_1z^2+L_2(z-1)^2+L_3z(z-1))\frac{dI}{dz}+(M_1z+M_2(z-1))I=0, \qquad (1.1)
\end{aligned}
$$

where

$$
\begin{aligned}
K_1&=-g-3b-3c, \quad K_2=-g-3a-3c,\\
L_1&=(b+c)(2b+2c+g+1), \quad L_2=(a+c)(2a+2c+g+1),\\
L_3&=(b+c)(2a+2c+g+1)+(a+c)(2b+2c+g+1)\\
&\quad+(c-1)(a+b+c)+(3c+g)(a+b+c+g+1),\\
M_1&=-c(2b+2c+g+1)(2a+2b+2c+g+2),\\
M_2&=-c(2a+2c+g+1)(2a+2b+2c+g+2).
\end{aligned}
$$

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We call this the Dotsenko–Fateev equation. It has three singular points at 0, 1, and ∞. The indicial equations have the roots

$$0, \quad a+c+1, \quad 2(a+c+1)+g \quad \text{at } z=0,$$

$$0, \quad b+c+1, \quad 2(b+c+1)+g \quad \text{at } z=1,$$

and

$$-2c, \quad -2c-(a+b+1)-g, \quad -2(a+b+c+1)-g \quad \text{at } z=\infty.$$

Dotsenko–Fateev also studied a monodromy representation on the solution space and determined a monodromy-invariant Hermitian form on it. Mimachi–Yoshida [19] interpreted the coefficients of the Hermitian form as the intersection numbers of the twisted cycles which produce a fundamental set of solutions.

In this paper, to give a fundamental set of solutions of (1.1), we consider the set of functions which is different from that in [5]. The conditions on the parameters a, b, c and g in the present case might be weakest for functions being the solution of (1.1) and being linearly independent. To study the sabtlety of the condition for the (ir)reducibility of the monodromy representation, it is very important to use such a set of functions.

The contents are the following: (1) we clarify the analytic property of these functions as the functions of the parameters a, b, c and g, (2) realize the monodromy representation by using these functions, (3) give a condition in terms of the parameters a, b, c and g for the representaion being irreducible, (4) construct the subsystems in reducible cases, and (5) obtain the invariant Hermitian form by calculating the intersection numbers of the twisted cycles which give the fundamental set of solutions.

The method to obtain the monodromy representation and its irreducibility condition is an extension of our previous works on the monodromy representation for several hypergeometric functions [13,14,16–18]. Among the conditions for the irreducibility (in Theorem 11) we encounter the conditions: $a+c+\frac{g+1}{2}, b+c+\frac{g+1}{2} \in \mathbb{C}\backslash\mathbb{Z}$, which are not related with the *resonance conditions* on parameters. It is the first time for such conditions to appear in the study of the (ir)reducibility of the monodromy representations for hypergeometric type functions. To know the geometric meaning is the future problem.

For related works to the Dotsenko–Fateev equation, we refer the readers to [6] and [8]. In [6], relationship between the Dotsenko–Fateev equation and a special case of a Fuchsian system of rank 8 in 3 variables with 4 parameters is studied from the viewpoint of the *addition* and the *middle convolution*, which are introduced by Katz [4,9]. In [8], the Dotsenko–Fateev equation is studied from the viewpoint of the prolongability into the integrable system in two variables.

In this paper, the symbol

$$e(A)=\exp(\pi\sqrt{-1}A)$$

is frequently used.

2. A Fundamental Set of Solutions

2.1. Integral representation of the solutions. For a point z of $\mathbb{C}\backslash\{0,1\}$, let $u(t)=u(z;t)$ be a multivalued function

$$u(t)=u(a,b,c,g;z;t)=(t_2-t_1)^g\prod_{i=1}^{2}t_i^a(1-t_i)^b(z-t_i)^c$$

on

$$T_z = \{t = (t_1,\ t_2) \in \mathbb{C}^2 \mid t_1 \neq t_2,\ t_i \neq 0,\ 1,\ z\ (i = 1, 2)\ \}.$$

Let $\mathcal{L}_z$ be the locally constant sheaf (the local system) defined by $u(t)$: the sheaf consisting of the local solutions of $dL = L\omega$ for $\omega = du(t)/u(t)$. Let $H_m(T_z, \mathcal{L}_z)$ be the m-th homology group with coefficients in $\mathcal{L}_z$, $H_m^{\mathrm{lf}}(T_z, \mathcal{L}_z)$ the m-th locally finite homology group with coefficients in $\mathcal{L}_z$. Elements of these twisted homology groups, called *twisted cycles* or *loaded cycles*, are represented by ∂-closed twisted or loaded (finite or locally finite) chains

$$C = \sum_{\Delta} a_\Delta \Delta \otimes v_\Delta, \quad (a_\Delta \in \mathbb{C}),$$

where each Δ is an m-simplex and v_Δ a section of $\mathcal{L}_z$ on Δ. The boundary operator ∂ is defined to be a $\mathbb{C}$-linear mapping satisfying $\partial(\Delta \otimes v) = \sum_{i=0}^{m}(-1)^i \Delta^i \otimes v|_{\Delta^i}$, where Δ is an m-simplex, Δ^i denotes the i-th face of Δ, and $v|_{\Delta^i}$ is the restriction of v on Δ^i.

If each factor $f_i(t)$ of $u(t)$ is defined over $\mathbb{R}$, and D is a simply connected domain of the real manifold $T_{\mathbb{R}}$ (the real locus of T), then it is convenient to load D with a section

$$u_D(t) = \prod_i (\epsilon_i\ f_i(t))^{\alpha_i}$$

of $\mathcal{L}$ on D, and to make a loaded cycle $D \otimes u_D(t)$, where $\epsilon_i = \pm$ is so determined that $\epsilon_i\ f_i(t)$ is positive on D, and the argument of $\epsilon_i\ f_i(t)$ is assigned to be zero. This choice of a section is said to be *standard*. In this paper, we usually adopt the standard loading; we omit the assignment of loading and denote just the topological cycles for simplicity. For example, in case $T = \mathbb{C}\backslash\{0, 1\}$ and $u(t) = t^\alpha(1-t)^\beta$, we denote by $\overrightarrow{(0, 1)}$ to express $\overrightarrow{(0, 1)} \otimes u(t)$, and $\overrightarrow{(1, \infty)}$ for $\overrightarrow{(1, \infty)} \otimes t^\alpha(t-1)^\beta$.

Let $H_2^{\mathrm{lf}}(T_z, \mathcal{L}_z)_-^{\mathfrak{S}_2}$ and $H_2(T_z, \mathcal{L}_z)_-^{\mathfrak{S}_2}$ stand for the anti-symmetric part of $H_2^{\mathrm{lf}}(T_z, \mathcal{L}_z)$ and $H_2(T_z, \mathcal{L}_z)$ with respect to the action of the symmetric group $\mathfrak{S}_2$ on the coordinate $t = (t_1, t_2)$ of T_z. Let $H^2(T_z, \mathcal{L}_z^\vee)_-^{\mathfrak{S}_2}$ be the anti-symmetric part of the twisted de Rham cohomology $H^2(T_z, \mathcal{L}_z^\vee)$, where $\mathcal{L}_z^\vee$ is the sheaf of the local solutions of $dL = -\omega L$, $\omega = u(t)^{-1}\,du(t)$.

It follows from [3] (see also [10]) that, for example, if $a+b+c+g \notin \mathbb{Z}$ and $2(a+b+c)+g \notin \mathbb{Z}$; $H_j(T_z, \mathcal{L}_z) = H_j^{\mathrm{lf}}(T_z, \mathcal{L}_z) = 0$ for $j \neq 2$, $\dim H_2(T_z, \mathcal{L}_z)_-^{\mathfrak{S}_2} = \dim H_2^{\mathrm{lf}}(T_z, \mathcal{L}_z)_-^{\mathfrak{S}_2} = 3$, and $\dim H^2(T_z, \mathcal{L}_z^\vee)_-^{\mathfrak{S}_2} = 3$. It guarantees the existence of the ordinary differential equation of order 3 (although we do not necessarily impose such a condition).

On the other hand, there exists the natural map

$$\iota:\ H_2(T_z, \mathcal{L}_z)_-^{\mathfrak{S}_2}\ \rightarrow\ H_2^{\mathrm{lf}}(T_z, \mathcal{L}_z)_-^{\mathfrak{S}_2}.$$

For a loaded cycle $D \in H_2^{\mathrm{lf}}(T_z, \mathcal{L}_z)_-^{\mathfrak{S}_2}$, if there exists a loaded cycle $\widetilde{D} \in H_2(T_z, \mathcal{L}_z)_-^{\mathfrak{S}_2}$ satisfying $\iota(\widetilde{D}) = D$, the cycle D is said to be *regularizable* and the cycle $\widetilde{D}$ is called a *regularization* of D. The cycle $\widetilde{D}$ is also written by $\mathrm{reg}\, D$.

It is known ([15,21]) that, when

$$a,\ 2a+g,\ b,\ 2b+g,\ c,\ 2c+g,\ g,\ a+b+c+g,\ 2(a+b+c)+g \in \mathbb{C}\backslash\mathbb{Z},$$

the natural map ι is an isomorphism.

If the exponent of an irreducible component of the divisor which is obtained by the minimal blow-up π along the non-normally crossing loci of $(\mathbb{P}^1(\mathbb{C}))^2 \setminus T_z$ is an integer, the irreducible component or the exponent itself is said to be *resonant*. The numbers

$$a,\ 2a+g,\ b,\ 2b+g,\ c,\ 2c+g,\ g,\ a+b+c+g,\ 2(a+b+c)+g$$

are the union of such exponents. Indeed, the exponent of each of the divisors

$$\pi^{-1}(t_1=t_2=0),\ \pi^{-1}(t_1=t_2=1),\ \pi^{-1}(t_1=t_2=z),\ \pi^{-1}(t_1=t_2=\infty)$$

is $2a+g,\ 2b+g,\ 2c+g,\ -2(a+b+c)-g$, respectively, while the exponent of each of the divisors

$$\pi^{-1}(t_i=0),\ \pi^{-1}(t_i=1),\ \pi^{-1}(t_i=z),\ \pi^{-1}(t_i=\infty) \quad (i=1,2)$$

is $a,\ b,\ c,\ -a-b-c-g$, respectively,

When the complex variable z is real and $0<z<1$, we assign the name $D_j,\ 1 \le j \le 3$ to each domain of the real manifold $(T_z)_{\mathbb{R}}$ as in Fig. 1:

$$D_1 = \{(t_1,t_2) \mid 0 < t_1 < t_2 < z\},$$

$$D_2 = \{(t_1,t_2) \mid 0 < t_1 < z < t_2 < 1\},$$

$$D_3 = \{(t_1,t_2) \mid z < t_1 < t_2 < 1\}.$$

Accordingly we consider the integrals

$$I_j(z) = \int_{D_j} u_{D_j}(z;t)\, dt_1 dt_2, \quad j=1,2,3,$$

where $u_{D_j}(z;t)$ is a standard loading of $u(z;t)$ on D_j:

$$I_1(z) = I_1(a,\ b,\ c,\ g\,;\ z) = \int_0^z dt_1 \int_{t_1}^z dt_2\ (t_2-t_1)^g \prod_{i=1,2} t_i^a(1-t_i)^b(z-t_i)^c,$$

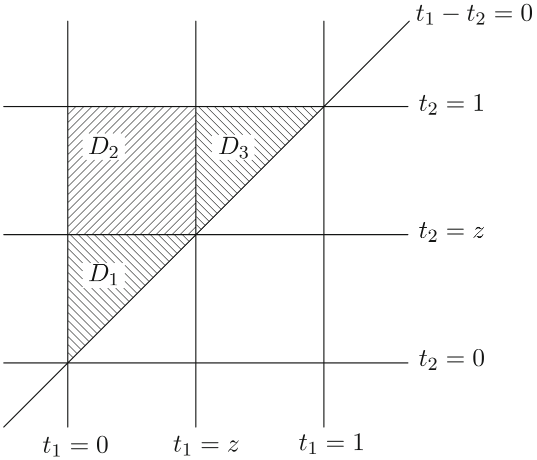


Fig. 1. Domains $D_1,\ D_2$ and $D3$

$$I_2(z) = I_2(a,\ b,\ c,\ g\ ;\ z) = \int_0^z dt_1 \int_z^1 dt_2\ (t_2 - t_1)^g \prod_{i=1,2} t_i^a (1-t_i)^b (z-t_1)^c (t_2 - z)^c,$$

$$I_3(z) = I_3(a,\ b,\ c,\ g\ ;\ z) = \int_z^1 dt_1 \int_{t_1}^1 dt_2\ (t_2 - t_1)^g \prod_{i=1,2} t_i^a (1-t_i)^b (t_i - z)^c,$$

where the argument of each factor of the integrand is fixed to be zero.

Remark. When we consider the functions like $I_j(z)$, it is enough to choose D_j's, which are the members of $H_2^{\mathrm{lf}}(T_z, \mathcal{L}_z)$ but are not the members of $H_2^{\mathrm{lf}}(T_z, \mathcal{L}_z)_-^{\mathfrak{S}_2}$. In Sect. 6, for the study of the invariant-Hermitian form, we need to choose their anti-symmetric counterpart.

2.2. Analytic continuations as functions of parameters. We start with the following elementary properties.

Lemma 1. *(1) The conditions for the existence of*

$$\iint_D t_1^{a_1} t_2^{a_2} (t_2 - t_1)^g\, dt_1 dt_2,$$

where

$$D := \{0 < t_1 < t_2 < R\} \quad \text{for a fixed positive number} \quad R,$$

are

$$\mathrm{Re}(a_1 + 1) > 0, \quad \mathrm{Re}(g + 1) > 0, \quad \text{and} \quad \mathrm{Re}(a_1 + a_2 + g + 2) > 0.$$

(2) The conditions for the existence of

$$\iint_D (-t_1)^{a_1} t_2^{a_2} (t_2 - t_1)^g dt_1 dt_2,$$

where

$$D := \{0 < t_2 - t_1 < R,\ t_1 < 0,\ 0 < t_2\} \quad \text{for a fixed positive number} \quad R,$$

are

$$\mathrm{Re}(a_1 + 1) > 0, \quad \mathrm{Re}(a_2 + 1) > 0, \quad \text{and} \quad \mathrm{Re}(a_1 + a_2 + g + 2) > 0.$$

Proof. (1) The change of integration variable t_1 to u_1 by $t_1 = t_2 u_1$ and $dt_1 = t_2 du_1$ leads to

$$\int_0^R dt_2 \int_0^{t_2} t_1^{a_1} t_2^{a_2} (t_2 - t_1)^g\, dt_1$$
$$= \int_0^R \int_0^1 u_1^{a_1} (1 - u_1)^g t_2^{a_1 + a_2 + g + 1}\, du_1 dt_2$$

$$= \int_0^1 u_1^{a_1}(1-u_1)^g\, du_1 \int_0^R t_2^{a_1+a_2+g+1}\, dt_2,$$

which converges when $\mathrm{Re}(a_1+1)>0,\ \mathrm{Re}(g+1)>0,$ and $\mathrm{Re}(a_1+a_2+g+2)>0.$
(2) The change of integration variables (t_1, t_2) to (u_1, u_2) by $t_1 = (u_1+u_2)/2,\ t_2 = (u_2-u_1)/2$ and $\partial(t_1,t_2)/\partial(u_1,u_2) = 1/2$ leads to

$$\int_D (-t_1)^{a_1} t_2^{a_2} (t_2-t_1)^g\, dt_1 dt_2 = \left(\frac{1}{2}\right)^{a_1+a_2+1} \int_{D'} (-u_1-u_2)^{a_1}(u_2-u_1)^{a_2}(-u_1)^g\, du_1 du_2,$$

where

$$D' = \{-R < u_1 < 0,\ u_1 < u_2 < -u_1\}.$$

Moreover, the change of variable $u_2 = u_1 v_2$ with $du_2 = u_1 dv_2$ leads to

$$\int_{D'} (-u_1-u_2)^{a_1}(u_2-u_1)^{a_2}(-u_1)^g\, du_1 du_2$$
$$= \int_{-R}^0 (-u_1)^{a_1+a_2+g+1}\, du_1 \int_{-1}^1 (1+v_2)^{a_1}(1-v_2)^{a_2}\, dv_2,$$

which converges when $\mathrm{Re}(a_1+a_2+g+2)>0,\ \mathrm{Re}(a_1+1)>0,$ and $\mathrm{Re}(a_2+1)>0.$
This completes the proof. □

Lemma 1 immediately implies the following: The conditions for the existence of $I_1(z)$ are

$$\mathrm{Re}(a+1),\ \mathrm{Re}(c+1),\ \mathrm{Re}(g+1),\ \mathrm{Re}(2a+g+2),\ \mathrm{Re}(2c+g+2) > 0,$$

the conditions for the existence of $I_2(z)$ are

$$\mathrm{Re}(a+1),\ \mathrm{Re}(b+1),\ \mathrm{Re}(c+1),\ \mathrm{Re}(2c+g+2) > 0,$$

and the conditions for the existence of $I_3(z)$ are

$$\mathrm{Re}(b+1),\ \mathrm{Re}(c+1),\ \mathrm{Re}(g+1),\ \mathrm{Re}(2b+g+2),\ \mathrm{Re}(2c+g+2) > 0.$$

Hence we have the following.

Lemma 2. *When all of*

$$\mathrm{Re}(a+1),\ \mathrm{Re}(a+1+\frac{g}{2}),\ \mathrm{Re}(b+1),\ \mathrm{Re}(b+1+\frac{g}{2}),\ \mathrm{Re}(c+1),\ \mathrm{Re}(c+1+\frac{g}{2}),\ \mathrm{Re}(g+1)$$

are positive, the functions $I_j(a,\ b,\ c,\ g\,;\ z),\ j = 1,\ 2,\ 3,$ *exist, and are holomorphic as functions of* $a,\ b,\ c$ *and* g.

Moreover, analytic continuation shows the following:

Proposition 3. *As functions of* $a,\ b,\ c$ *and* g, *the functions* $I_j(a,\ b,\ c,\ g;\ z),\ j = 1,\ 2,\ 3,$ *are meromorphic with the simple poles locating at*

$$a+1,\ a+1+\frac{g}{2},\ b+1,\ b+1+\frac{g}{2},\ c+1,\ c+1+\frac{g}{2},\ \frac{1}{2}(g+1) \in \mathbb{Z}_{\le 0}. \tag{2.1}$$

Proof. In this proof, we consider the function

$$\Phi = \Phi(\lambda_0,\ \lambda_1,\ \lambda_2,\ g) = (t_1 - t_2)^g \prod_{\substack{1 \le i \le 2 \\ 0 \le j \le 2}} (t_i - z_j)^{\lambda_j}$$

and set $z_{ij} = z_i - z_j,\ \lambda_{ij\ldots k} = \lambda_i + \lambda_j + \cdots + \lambda_k$ and $d\tau = dt_1 \wedge dt_2$ for brevity.

First, the equalities

$$\begin{aligned}
& d((t_2 - z_2)(t_1 - z_0)(t_1 - z_1)\Phi\, dt_2) \\
&= \left[(t_2 - z_2)\left\{(t_1 - z_0) + (t_1 - z_1) + (t_1 - z_0)(t_1 - z_1)\left(\sum_{j=0}^{2} \frac{\lambda_j}{t_1 - z_j} + \frac{g}{t_1 - t_2} \right)\right\} \right]\Phi\, d\tau \\
&= \left[(t_2 - z_2)\left\{(2 + \lambda_{012})(t_1 - z_2) + (1 + \lambda_0)z_{21} + (1 + \lambda_1)z_{20} + \lambda_2(z_{20} + z_{21}) \right.\right. \\
&\quad \left.\left. + \lambda_2 \frac{z_{20}z_{21}}{t_1 - z_2} + g\, \frac{(t_1 - z_2)^2 + (z_{20} + z_{21})(t_1 - z_2) + z_{20}z_{21}}{t_1 - t_2} \right\} \right]\Phi\, d\tau
\end{aligned}$$

and

$$\begin{aligned}
& (t_2 - z_2)\left\{(t_1 - z_2)^2 + (z_{20} + z_{21})(t_1 - z_2) + z_{20}z_{21}\right\} \\
&\quad - (t_1 - z_2)\left\{(t_2 - z_2)^2 + (z_{20} + z_{21})(t_2 - z_2) + z_{20}z_{21}\right\} \\
&= (t_1 - t_2)\{(t_1 - z_2)(t_2 - z_2) - z_{20}z_{21}\}
\end{aligned}$$

imply

$$\begin{aligned}
& d((t_2 - z_2)(t_1 - z_0)(t_1 - z_1)\Phi\, dt_2) - d((t_1 - z_2)(t_2 - z_0)(t_2 - z_1)\Phi\, dt_1) \\
&= \left[\left\{2(2 + \lambda_{012}) + g\right\}(t_1 - z_2)(t_2 - z_2) \right. \\
&\quad + \left\{ (1 + \lambda_0)z_{21} + (1 + \lambda_1)z_{20} + \lambda_2(z_{20} + z_{21}) \right\}\left\{(t_1 - z_2) + (t_2 - z_2)\right\} \\
&\quad \left. - g z_{20}z_{21} + \lambda_2 z_{20}z_{21}\left(\frac{t_2 - z_2}{t_1 - z_2} + \frac{t_1 - z_2}{t_2 - z_2} \right) \right]\Phi\, d\tau. \qquad (2.2)
\end{aligned}$$

Secondly, the equality

$$
\begin{aligned}
&d((t_1-z_0)(t_1-z_1)\Phi\,dt_2)\\
&=\Big[(2+\lambda_{012})(t_1-z_2)+(1+\lambda_0)z_{21}+(1+\lambda_1)z_{20}+\lambda_2(z_{20}+z_{21})+\lambda_2\frac{z_{20}z_{21}}{t_1-z_2}\\
&\quad+g\,\frac{(t_1-z_2)^2+(z_{20}+z_{21})(t_1-z_2)+z_{20}z_{21}}{t_1-t_2}\Big]\Phi\,d\tau
\end{aligned}
$$

implies

$$
\begin{aligned}
&d((t_1-z_0)(t_1-z_1)\Phi\,dt_2)-d((t_2-z_0)(t_2-z_1)\Phi\,dt_1)\\
&=\Big[(2+\lambda_{012}+g)\big\{(t_1-z_2)+(t_2-z_2)\big\}+2(1+\lambda_0)z_{21}+2(1+\lambda_1)z_{20}\\
&\quad+(2\lambda_2+g)(z_{20}+z_{21})+\lambda_2 z_{20}z_{21}\left(\frac{1}{t_1-z_2}+\frac{1}{t_2-z_2}\right)\Big]\Phi\,d\tau. \qquad (2.3)
\end{aligned}
$$

Thirdly, the equality

$$
\begin{aligned}
&d\left(\frac{(t_1-z_0)(t_1-z_1)}{t_2-z_2}\Phi\,dt_2\right)\\
&=\Big[(2+\lambda_{012})\frac{t_1-z_2}{t_2-z_2}+\frac{(1+\lambda_0)z_{21}+(1+\lambda_1)z_{20}+\lambda_2(z_{20}+z_{21})}{t_2-z_2}\\
&\quad+\lambda_2\frac{z_{20}z_{21}}{(t_1-z_2)(t_2-z_2)}+g\,\frac{(t_1-z_2)^2+(z_{20}+z_{21})(t_1-z_2)+z_{20}z_{21}}{(t_1-t_2)(t_2-z_2)}\Big]\Phi\,d\tau
\end{aligned}
$$

implies

$$
\begin{aligned}
&d\left(\frac{(t_1-z_0)(t_1-z_1)}{t_2-z_2}\Phi\,dt_2\right)-d\left(\frac{(t_2-z_0)(t_2-z_1)}{t_1-z_2}\Phi\,dt_1\right)\\
&=\Big[(2+\lambda_{012}+g)\left(\frac{t_1-z_2}{t_2-z_2}+\frac{t_2-z_2}{t_1-z_2}\right)\\
&\quad+\big\{(1+\lambda_0)z_{21}+(1+\lambda_1)z_{20}+(\lambda_2+g)(z_{20}+z_{21})\big\}\left(\frac{1}{t_1-z_2}+\frac{1}{t_2-z_2}\right)\\
&\quad+g+(2\lambda_2+g)\frac{z_{20}z_{21}}{(t_1-z_2)(t_2-z_2)}\Big]\Phi\,d\tau. \qquad (2.4)
\end{aligned}
$$

Combining equalities (2.2), (2.3) and (2.4), we have

$$
\begin{aligned}
&\lambda_2 z_{20}z_{21}\left\{d\Big(\frac{(t_1-z_0)(t_1-z_1)}{t_2-z_2}\Phi\,dt_2\Big)-d\Big(\frac{(t_2-z_0)(t_2-z_1)}{t_1-z_2}\Phi\,dt_1\Big)\right\}\\
&\quad-(2+\lambda_{012}+g)\big\{d((t_1-z_0)(t_1-z_1)\Phi\,dt_2)-d((t_2-z_0)(t_2-z_1)\Phi\,dt_1)\big\}\\
&\quad-\big\{(1+\lambda_0)z_{21}+(1+\lambda_1)z_{20}+(\lambda_2+g)(z_{20}+z_{21})\big\}
\end{aligned}
$$

$$\times \left\{ d((t_2 - z_2)(t_1 - z_0)(t_1 - z_1)\Phi\, dt_2) - d((t_1 - z_2)(t_2 - z_0)(t_2 - z_1)\Phi\, dt_1) \right\}$$

$$= \Big[-(2 + \lambda_{012} + g)(2(2 + \lambda_{012}) + g)(t_1 - z_2)(t_2 - z_2)$$

$$- (2 + \lambda_{012} + g)\big\{2(1 + \lambda_0)z_{21} + 2(1 + \lambda_1)z_{20} + (2\lambda_2 + g)(z_{20} + z_{21})\big\}$$

$$\times \big((t_1 - z_2) + (t_2 - z_2)\big) + g(2 + \lambda_{01} + 2\lambda_2 + g)z_{20}z_{21}$$

$$- \big\{(1 + \lambda_0)z_{21} + (1 + \lambda_1)z_{20} + (\lambda_2 + g)(z_{20} + z_{21})\big\}$$

$$\times \big\{2(1 + \lambda_0)z_{21} + 2(1 + \lambda_1)z_{20} + (2\lambda_2 + g)(z_{20} + z_{21})\big\}$$

$$+ \lambda_2(2\lambda_2 + g)\frac{z_{20}^2 z_{21}^2}{(t_1 - z_2)(t_2 - z_2)} \Big]\Phi\, d\tau,$$

which implies that there exists a polynomial $h_2 \in \mathbb{C}[t_1, t_2, \lambda_0, \lambda_1, \lambda_2, g, z_0, z_1, z_2, z_{20}^{-1}, z_{21}^{-1}]$ such that

$$\int_\Gamma \Phi(\lambda_0, \lambda_1, \lambda_2 - 1, g)\, d\tau = \frac{1}{\lambda_2(\lambda_2 + \frac{g}{2})} \int_\Gamma h_2\Phi(\lambda_0, \lambda_1, \lambda_2, g)\, d\tau \tag{2.5}$$

for any cycle Γ. The symmetry $(z_0, \lambda_0) \leftrightarrow (z_1, \lambda_1) \leftrightarrow (z_2, \lambda_2)$ leads to the existence of polynomials

$$h_0 \in \mathbb{C}[t_1, t_2, \lambda_0, \lambda_1, \lambda_2, g, z_0, z_1, z_2, z_{01}^{-1}, z_{02}^{-1}],$$

$$h_1 \in \mathbb{C}[t_1, t_2, \lambda_0, \lambda_1, \lambda_2, g, z_0, z_1, z_2, z_{10}^{-1}, z_{12}^{-1}]$$

such that

$$\int_\Gamma \Phi(\lambda_0 - 1, \lambda_1, \lambda_2, g)\, d\tau = \frac{1}{\lambda_0(\lambda_0 + \frac{g}{2})} \int_\Gamma h_0\Phi(\lambda_0, \lambda_1, \lambda_2, g)\, d\tau, \tag{2.6}$$

$$\int_\Gamma \Phi(\lambda_0, \lambda_1 - 1, \lambda_2, g)\, d\tau = \frac{1}{\lambda_1(\lambda_1 + \frac{g}{2})} \int_\Gamma h_1\Phi(\lambda_0, \lambda_1, \lambda_2, g)\, d\tau \tag{2.7}$$

for any cycle Γ.

On the other hand, the equality

$$d\left(\frac{1}{t_2 - t_1}\Phi\, dt_2\right) = \left[\left(\frac{1}{t_2 - t_1}\right)^2 + \frac{1}{t_2 - t_1}\left(\frac{g}{t_1 - t_2} + \sum_{j=0}^{2} \frac{\lambda_j}{t_1 - z_j}\right) \right]\Phi\, d\tau$$

implies

$$d\left(\frac{1}{t_2 - t_1}\Phi\, dt_2\right) - d\left(\frac{1}{t_1 - t_2}\Phi\, dt_1\right) = \left[2(1 - g)\left(\frac{1}{t_2 - t_1}\right)^2 + \sum_{j=0}^{2} \frac{\lambda_j}{(t_1 - z_j)(t_2 - z_j)} \right]\Phi\, d\tau,$$

which leads to

$$2(g-1)\int_\Gamma \Phi(\lambda_0,\,\lambda_1,\,\lambda_2,\,g-2)\,d\tau = \lambda_0\int_\Gamma \Phi(\lambda_0-1,\,\lambda_1,\,\lambda_2,\,g)\,d\tau$$
$$+\lambda_1\int_\Gamma \Phi(\lambda_0,\,\lambda_1-1,\,\lambda_2,\,g)\,d\tau+\lambda_2\int_\Gamma \Phi(\lambda_0,\,\lambda_1,\,\lambda_2-1,\,g)\,d\tau. \tag{2.8}$$

Substituting (2.5), (2.6) and (2.7) into (2.8), we have

$$2(g-1)\int_\Gamma \Phi(\lambda_0,\,\lambda_1,\,\lambda_2,\,g-2)\,d\tau = \frac{1}{\lambda_0+\frac{g}{2}}\int_\Gamma h_0\Phi(\lambda_0,\,\lambda_1,\,\lambda_2,\,g)\,d\tau$$
$$+\frac{1}{\lambda_1+\frac{g}{2}}\int_\Gamma h_1\Phi(\lambda_0,\,\lambda_1,\,\lambda_2,\,g)\,d\tau+\frac{1}{\lambda_2+\frac{g}{2}}\int_\Gamma h_2\Phi(\lambda_0,\,\lambda_1,\,\lambda_2,\,g)\,d\tau,$$

which implies that there exists a polynomial

$$h_3\in\mathbb{C}[t_1,\,t_2,\,\lambda_0,\,\lambda_1,\,\lambda_2,\,g,\,z_0,\,z_1,\,z_2,\,z_{01}^{-1},\,z_{02}^{-1},\,z_{12}^{-1}]$$

such that

$$\int_\Gamma \Phi(\lambda_0,\,\lambda_1,\,\lambda_2,\,g-2)\,d\tau = \frac{1}{(g-1)\prod_{j=0}^{2}(\lambda_j+\frac{g}{2})}\int_\Gamma h_3\Phi(\lambda_0,\,\lambda_1,\,\lambda_2,\,g)\,d\tau. \tag{2.9}$$

When we consider the case

$$\lambda_0=a,\ \lambda_1=b,\ \lambda_2=c,\ z_0=0,\ z_1=1,\ z_2=z,$$

equalities (2.5), (2.6), (2.7) and (2.9) imply that

$$I_j(a,\,b,\,c-1,\,g\,;\,z)=\frac{1}{c(c+\frac{g}{2})}\int_{D_j} h_2(a,\,b,\,c,\,g)u_{D_j}(a,\,b,\,c,\,g;\,z;\,t)\,d\tau,$$

$$I_j(a-1,\,b,\,c,\,g\,;\,z)=\frac{1}{a(a+\frac{g}{2})}\int_{D_j} h_0(a,\,b,\,c,\,g)u_{D_j}(a,\,b,\,c,\,g;\,z;\,t)\,d\tau,$$

$$I_j(a,\,b-1,\,c,\,g\,;\,z)=\frac{1}{b(b+\frac{g}{2})}\int_{D_j} h_1(a,\,b,\,c,\,g)u_{D_j}(a,\,b,\,c,\,g;\,z;\,t)\,d\tau$$

and

$$I_j(a,\,b,\,c,\,g-2;\,z)=\frac{1}{(g-1)(a+\frac{g}{2})(b+\frac{g}{2})(c+\frac{g}{2})}\int_{D_j} h_3(a,\,b,\,c,\,g)u_{D_j}(a,\,b,\,c,\,g)\,d\tau,$$

which are equivalent to

$$I_j(a,\,b,\,c,\,g\,;\,z)=\frac{1}{(c+1)(c+1+\frac{g}{2})}\int_{D_j} h_2(a,\,b,\,c+1,\,g)u_{D_j}(a,\,b,\,c+1,\,g;\,z;\,t)\,d\tau, \tag{2.10}$$

$$I_j(a,\, b,\, c,\, g\,;\, z) = \frac{1}{(a+1)(a+1+\frac{g}{2})} \int_{D_j} h_0(a+1,\, b,\, c,\, g) u_{D_j}(a+1,\, b,\, c,\, g;\, z;\, t)\, d\tau, \tag{2.11}$$

$$I_j(a,\, b,\, c,\, g\,;\, z) = \frac{1}{(b+1)(b+1+\frac{g}{2})} \int_{D_j} h_1(a,\, b+1,\, c,\, g) u_{D_j}(a,\, b+1,\, c,\, g;\, z;\, t)\, d\tau, \tag{2.12}$$

and

$$I_j(a,\, b,\, c,\, g\,;\, z) = \frac{1}{(g+1)(a+1+\frac{g}{2})(b+1+\frac{g}{2})(c+1+\frac{g}{2})} \times \int_{D_j} h_3(a,\, b,\, c,\, g+2) u_{D_j}(a,\, b,\, c,\, g+2;\, z;\, t)\, d\tau. \tag{2.13}$$

Therefore, (2.10), (2.11), (2.12) and (2.13), combined with Lemma 2, imply that each $I_j(a,\, b,\, c,\, g;\, z)$ is meromorphic as a function of $a,\, b,\, c$ and g over the domain

$$\mathrm{Re}(a+2),\ \mathrm{Re}(a+2+\frac{g}{2}),\ \mathrm{Re}(b+2),\ \mathrm{Re}(b+2+\frac{g}{2}),\ \mathrm{Re}(c+2),\ \mathrm{Re}(c+2+\frac{g}{2}),\ \mathrm{Re}(g+3) > 0,$$

with the simple poles locating at

$$a+1,\ a+1+\frac{g}{2},\ b+1,\ b+1+\frac{g}{2},\ c+1,\ c+1+\frac{g}{2},\ g+1 = 0.$$

Moreover, repeated argument shows that each function $I_j(a,\, b,\, c,\, g;\, z)$ is analytically continued to the meromorphic function with the simple poles locating at (2.1). This completes the proof of Proposition 3. □

Remark. Proposition 3 implies that

$$\widetilde{I}_j(a,\, b,\, c,\, g\,;\, z) := \frac{\Gamma(1+\frac{g}{2})}{\Gamma(a+1)\Gamma(a+1+\frac{g}{2})\Gamma(b+1)\Gamma(b+1+\frac{g}{2})\Gamma(c+1)\Gamma(c+1+\frac{g}{2})\Gamma(g+1)} \times I_j(a,\, b,\, c,\, g\,;\, z), \quad j = 1,\, 2,\, 3, \tag{2.14}$$

are holomorphic with respect to $a,\, b,\, c,$ and g. The functions $\widetilde{I}_j(a,\, b,\, c,\, g;\, z)$ will be used mainly in Sect. 5. (The factors $\Gamma(1+\frac{g}{2})/\Gamma(g+1)$ are chosen in stead of $1/\Gamma(\frac{g+1}{2})$ to have good chemisrty with the formula in Proposition 5.)

2.3. Differential equation. We show that the functions $I_j(z),\ j = 1, 2, 3$ satisfy the differential equation (1.1).

Theorem 4. *Suppose that*

$$a+1,\ a+1+\frac{g}{2},\ b+1,\ b+1+\frac{g}{2},\ c+1,\ c+1+\frac{g}{2},\ \frac{g+1}{2} \notin \mathbb{Z}_{\leq 0}.$$

Then each of $I_j(z),\ j = 1, 2, 3,$ satisfies the differential equation (1.1).

Proof. Until the last stage of the present proof we assume that $c \neq 0, 1$.

For the sake of brevity, we set $e_1,\ e_2,\ p_2,\ e_{(2,1)}$ to be

$$e_1 = \frac{1}{t_1 - z} + \frac{1}{t_2 - z}, \qquad e_2 = \frac{1}{(t_1 - z)(t_2 - z)},$$

$$p_2 = \left(\frac{1}{t_1 - z}\right)^2 + \left(\frac{1}{t_2 - z}\right)^2, \qquad e_{(2,1)} = \frac{1}{(t_1 - z)(t_2 - z)}\left(\frac{1}{t_1 - z} + \frac{1}{t_2 - z}\right),$$

and $\widetilde{c_0}, \widetilde{c_1}, \widetilde{c_2}$ to be

$$\widetilde{c_0} = a + b + c + 1, \quad \widetilde{c_1} = (a + b + 2c)z - a - c, \quad \widetilde{c_2} = (1 - c)z(1 - z).$$

Then we have

$$\partial_z u(z; t) = -ce_1 u(z; t),$$
$$\partial_z(e_1 u(z; t)) = \big\{(1 - c)p_2 - 2c\, e_2\big\} u(z; t),$$
$$\partial_z(e_2 u(z; t)) = (1 - c)e_{(2,1)} u(z; t),$$

hence

$$e_1 u(z; t) = \frac{1}{-c}\partial_z(u(z; t)), \tag{2.15}$$
$$(1 - c)p_2 u(z; t) = \frac{1}{-c}\partial_z^2(u(z; t)) + 2c\, e_2\, u(z; t), \tag{2.16}$$
$$(1 - c)e_{(2,1)} u = \partial_z(e_2 u(z; t)). \tag{2.17}$$

Combination of the equalities

$$\begin{aligned}
&\partial_{t_1}\left\{\frac{t_1(t_1 - 1)}{t_1 - z} u(z; t)\right\} \\
&= \left[\frac{\{(2t_1 - 1)(t_1 - z) - t_1(t_1 - 1)}{(t_1 - z)^2} + \frac{t_1(t_1 - 1)}{t_1 - z}\left(\frac{a}{t_1} + \frac{b}{t_1 - 1} + \frac{c}{t_1 - z} + \frac{g}{t_1 - t_2}\right)\right] u(z; t) \\
&= \left[\frac{\{(2 + a + b)t_1 - 1 - a\}(t_1 - z) + (c - 1)t_1(t_1 - 1)}{(t_1 - z)^2} + \frac{gt_1(t_1 - 1)}{(t_1 - z)(t_1 - t_2)}\right] u(z; t) \\
&= \left[(a + b + c + 1) + \frac{z(a + b + 2c) - a - c}{t_1 - z} + \frac{(c - 1)(z^2 - z)}{(t_1 - z)^2} + \frac{gt_1(t_1 - 1)}{(t_1 - z)(t_1 - t_2)}\right] u(z; t) \\
&= \left[\widetilde{c_0} + \frac{\widetilde{c_1}}{t_1 - z} + \frac{\widetilde{c_2}}{(t_1 - z)^2} + \frac{gt_1(t_1 - 1)}{(t_1 - z)(t_1 - t_2)}\right] u(z; t),
\end{aligned}$$

and

$$\partial_{t_2}\left\{\frac{t_2(t_2 - 1)}{t_2 - z} u(z; t)\right\} = \left[\widetilde{c_0} + \frac{\widetilde{c_1}}{t_2 - z} + \frac{\widetilde{c_2}}{(t_2 - z)^2} + \frac{gt_2(t_2 - 1)}{(t_2 - z)(t_2 - t_1)}\right] u(z; t)$$

with

$$\frac{t_1(t_1-1)}{(t_1-z)(t_1-t_2)}+\frac{t_2(t_2-1)}{(t_2-z)(t_2-t_1)}=1+\frac{z(1-z)}{(t_1-z)(t_2-z)}$$

leads to

$$\begin{aligned} &d_t\left\{\frac{t_1(t_1-1)}{t_1-z}u(z;t)\,dt_2-\frac{t_2(t_2-1)}{t_2-z}u(z;t)\,dt_1\right\} \\ &=\left[\,2\widetilde{c_0}+g+\widetilde{c_1}e_1+\widetilde{c_2}p_2+gz(1-z)e_2\,\right]u(z;t)\,d\tau. \end{aligned} \tag{2.18}$$

Substitution of (2.15) and (2.16) into (2.18) implies

$$\begin{aligned} &d_t\left\{\frac{t_1(t_1-1)}{t_1-z}u(z;t)\,dt_2-\frac{t_2(t_2-1)}{t_2-z}u(z;t)\,dt_1\right\} \\ &=\left[\,2\widetilde{c_0}+g+z(1-z)(2c+g)e_2\,\right]u(z;t)\,d\tau-\frac{\widetilde{c_1}}{c}\,\partial_z(u(z;t))\,d\tau \\ &+z(1-z)\frac{1}{-c}\,\partial_z^2(u(z;t))\,d\tau, \end{aligned}$$

which is equivalent to

$$\begin{aligned} e_2u(z;t)\,d\tau =&\frac{1}{z(1-z)(2c+g)}\left[\,d_t\left\{\frac{t_1(t_1-1)}{t_1-z}u(z;t)\,dt_2-\frac{t_2(t_2-1)}{t_2-z}u(z;t)\,dt_1\right\}\right. \\ &\left.-(2\widetilde{c_0}+g)u(z;t)\,d\tau+\frac{\widetilde{c_1}}{c}\,\partial_z(u(z;t))\,d\tau+z(1-z)\,\frac{1}{c}\,\partial_z^2(u(z;t))\,d\tau\,\right]. \end{aligned} \tag{2.19}$$

On the other hand, combination of the equalities

$$\begin{aligned} &\partial_{t_1}\left\{\frac{t_1(t_1-1)}{(t_1-z)(t_2-z)}u(z;t)\right\} \\ &=\left[\frac{\widetilde{c_0}}{t_2-z}+\frac{\widetilde{c_1}}{(t_1-z)(t_2-z)}+\frac{\widetilde{c_2}}{(t_1-z)^2(t_2-z)}+\frac{gt_1(t_1-1)}{(t_1-z)(t_2-z)(t_1-t_2)}\right]u(z;t) \end{aligned}$$

and

$$\partial_{t_2}\left\{\frac{t_2(t_2-1)}{(t_1-z)(t_2-z)}u(z;t)\right\}$$
$$=\left[\frac{\widetilde{c_0}}{t_1-z}+\frac{\widetilde{c_1}}{(t_1-z)(t_2-z)}+\frac{\widetilde{c_2}}{(t_1-z)(t_2-z)^2}+\frac{gt_2(t_2-1)}{(t_1-z)(t_2-z)(t_2-t_1)}\right]u(z;t)$$

with

$$\frac{1}{(t_1-z)(t_2-z)}\left\{\frac{t_1(t_1-1)}{(t_1-t_2)}+\frac{t_2(t_2-1)}{(t_2-t_1)}\right\}=e_1+(2z-1)e_{(2)}$$

leads to

$$d_t\left\{\frac{t_1(t_1-1)}{(t_1-z)(t_2-z)}u(z;t)\,dt_2-\frac{t_2(t_2-1)}{(t_1-z)(t_2-z)}u(z;t)\,dt_1\right\}$$
$$=\left[\,(\widetilde{c_0}+g)e_1+\{\,2\widetilde{c_1}+(2z-1)g\,\}e_2+\widetilde{c_2}\,e_{(2,1)}\,\right]u(z;t)\,d\tau. \tag{2.20}$$

Substitution of (2.15) and (2.17) into (2.20) implies

$$d_t\left\{\frac{t_1(t_1-1)}{(t_1-z)(t_2-z)}u(z;t)\,dt_2-\frac{t_2(t_2-1)}{(t_1-z)(t_2-z)}u(z;t)\,dt_1\right\}$$
$$=(\widetilde{c_0}+g)\frac{1}{-c}\,\partial_z(u(z;t))\,d\tau+\{\,2\widetilde{c_1}+(2z-1)g\,\}e_2u(z;t)\,d\tau+z(1-z)\partial_z(e_2u(z;t))\,d\tau. \tag{2.21}$$

Substitution of (2.19) into (2.21) implies

$$d_t\left\{\frac{t_1(t_1-1)}{(t_1-z)(t_2-z)}u(z;t)\,dt_2-\frac{t_2(t_2-1)}{(t_1-z)(t_2-z)}u(z;t)\,dt_1\right\}$$
$$=(\widetilde{c_0}+g)\frac{1}{-c}\partial_z(u(z;t))\,d\tau+\{\,2\widetilde{c_1}+(2z-1)g\,\}\frac{1}{z(1-z)(2c+g)}$$
$$\times\left[\,d_t\left\{\frac{t_1(t_1-1)}{t_1-z}u(z;t)\,dt_2-\frac{t_2(t_2-1)}{t_2-z}u(z;t)\,dt_1\right\}\right.$$
$$\left.-(2\widetilde{c_0}+g)u(z;t)\,d\tau+\frac{\widetilde{c_1}}{c}\,\partial_z(u(z;t))\,d\tau+z(1-z)\frac{1}{c}\,\partial_z^2(u(z;t))\,d\tau\,\right]$$
$$+z(1-z)\partial_z\left\{\frac{1}{z(1-z)(2c+g)}\left[\,d_t\left\{\frac{t_1(t_1-1)}{t_1-z}u(z;t)\,dt_2-\frac{t_2(t_2-1)}{t_2-z}u(z;t)\,dt_1\right\}\right.\right.$$
$$\left.\left.-(2\widetilde{c_0}+g)u(z;t)\,d\tau+\frac{\widetilde{c_1}}{c}\,\partial_z(u(z;t))\,d\tau+z(1-z)\frac{1}{c}\,\partial_z^2(u(z;t))\,d\tau\,\right]\right\}. \tag{2.22}$$

Here the third term of the right-hand side of (2.22) turns out to be

$$\frac{1}{2c+g}\left[\frac{2z-1}{z(1-z)}\left[d_t\left\{\frac{t_1(t_1-1)}{t_1-z}u(z;t)\,dt_2-\frac{t_2(t_2-1)}{t_2-z}u(z;t)\,dt_1\right\}\right.\right.$$
$$\left.-(2\widetilde{c_0}+g)u(z;t)\,d\tau+\frac{\widetilde{c_1}}{c}\partial_z(u(z;t))\,d\tau+z(1-z)\frac{1}{c}\,\partial_z^2(u(z;t))\,d\tau\,\right]$$

$$+\partial_z\left\{d_t\left\{\frac{t_1(t_1-1)}{t_1-z}u(z;t)\,dt_2-\frac{t_2(t_2-1)}{t_2-z}u(z;t)\,dt_1\right\}\right\}$$

$$-(2\widetilde{c_0}+g)\partial_z(u(z;t))\,d\tau+\frac{\partial_z(\widetilde{c_1})}{c}\partial_z(u(z;t))\,d\tau+\frac{\widetilde{c_1}}{c}\,\partial_z^2(u(z;t))\,d\tau$$

$$+\partial_z(z(1-z))\frac{1}{c}\,\partial_z^2(u(z;t))\,d\tau+z(1-z)\frac{1}{c}\,\partial_z^3(u(z;t))\,d\tau\ \Bigg].$$

Therefore, (2.22) is equivalent to

$$d_t\left\{\frac{t_1(t_1-1)}{(t_1-z)(t_2-z)}u(z;t)\,dt_2-\frac{t_2(t_2-1)}{(t_1-z)(t_2-z)}u(z;t)\,dt_1\right\}$$

$$=\frac{-1}{z(1-z)(2c+g)}(2\widetilde{c_0}+g)\big\{2\widetilde{c_1}+(2z-1)(g+1)\big\}u(z;t)\,d\tau$$

$$+\frac{1}{z(1-z)c(2c+g)}\Big[(\widetilde{c_0}+g)(g+2c)z(z-1)+\{2\widetilde{c_1}+(2z-1)g\}\widetilde{c_1}+(2z-1)\widetilde{c_1}$$

$$-z(z-1)\{(2\widetilde{c_0}+g)(-c)+\partial_z(\widetilde{c_1})\}\Big]\partial_z(u(z;t))\,d\tau$$

$$+\frac{1}{c(2c+g)}\{3\widetilde{c_1}+(2z-1)g\}\partial_z^2(u(z;t))\,d\tau+\frac{z(1-z)}{c(2c+g)}\partial_z^3(u(z;t))\,d\tau$$

$$=\frac{1}{z(1-z)c(2c+g)}$$

$$\times\Big\{(M_1z+M_2(z-1))u(z;t)\,d\tau+(L_1z^2+L_2(z-1)^2+L_3z(z-1))\partial_z(u(z;t))\,d\tau$$

$$+z(1-z)(-K_1z-K_2(z-1))\partial_z^2(u(z;t))\,d\tau+z^2(1-z)^2\partial_z^3(u(z;t))\,d\tau\Big\},\quad(2.23)$$

where

$$K_1=-g-3b-3c,\quad K_2=-g-3a-3c,$$

$$L_1=(b+c)(2b+2c+g+1),\quad L_2=(a+c)(2a+2c+g+1),$$

$$L_3=(b+c)(2a+2c+g+1)+(a+c)(2b+2c+g+1)$$
$$+(c-1)(a+b+c)+(3c+g)(a+b+c+g+1),$$

$$M_1=-c(2b+2c+g+1)(2a+2b+2c+g+2),$$

$$M_2=-c(2a+2c+g+1)(2a+2b+2c+g+2).$$

At this stage, we temporaly assume that

$$a,\ 2a+g,\ b,\ 2b+g,\ c,\ 2c+g,\ g\in\mathbb{C}\backslash\mathbb{Z}.$$

Then there exist reg $D_j \in H_2(T_z, \mathcal{L}_z)$ for $j = 1, 2, 3$, and equality (2.23) implies that the functions

$$\int_{\text{reg } D_j} u_{D_j}(z; t)\, dt_1 dt_2, \quad j = 1, 2, 3$$

satisfy the equation (1.1), since the support of each reg D_j is compact and the order of the integration and the differentiation can be changed.

Once it is established, analytic continuation implies the result. This completes the proof of Theorem 4. □

2.4. Wronskian formula. We need a Wronskian formula for $I_1(z), I_2(z), I_3(z)$, to clarify the condition for the set $\{I_1(z), I_2(z), I_3(z)\}$ being linearly independent. The formula is stated as follows.

Proposition 5. *Fix z to be $0 < z < 1$. Suppose that*

$$a+1,\ a+1+\frac{g}{2},\ b+1,\ b+1+\frac{g}{2},\ c+1,\ c+1+\frac{g}{2},\ \frac{g+1}{2} \notin \mathbb{Z}_{\leq 0}. \tag{2.24}$$

Then we have

$$\begin{aligned} W(z) &:= \det\left(\partial^{i-1} I_j(z)\right)_{1\leq i,\, j\leq 3} \\ &= -\frac{1}{2}\frac{\Gamma(1+a,\ 1+b,\ 1+c,\ 1+g)^2}{\Gamma(2+a+b+c+g,\ 1+\frac{g}{2})^2} \\ &\quad \times \frac{\Gamma(1+a+\frac{g}{2},\ 1+b+\frac{g}{2},\ 1+c+\frac{g}{2})}{\Gamma(2+a+b+c+\frac{g}{2})}\, z^{3a+3c+g}(1-z)^{3b+3c+g}, \end{aligned}$$

or

$$\begin{aligned} \widetilde{W}(z) &:= \det\left(\partial^{i-1} \widetilde{I}_j(z)\right)_{1\leq i,\, j\leq 3} \\ &= -\frac{1}{2}\frac{\Gamma(1+\frac{g}{2})}{\Gamma(1+a,\ 1+b,\ 1+c,\ 1+g,\ 2+a+b+c+\frac{g}{2})} \\ &\quad \times \frac{1}{\Gamma(1+a+\frac{g}{2},\ 1+b+\frac{g}{2},\ 1+c+\frac{g}{2},\ 2+a+b+c+g)^2}\, z^{3a+3c+g}(1-z)^{3b+3c+g}, \end{aligned}$$

where $\partial = d/dz$, $\Gamma(\lambda_1, \lambda_2, \ldots, \lambda_n) = \Gamma(\lambda_1)\Gamma(\lambda_2)\cdots\Gamma(\lambda_n)$ and $\widetilde{I}_j(z)$ are functions defined by (2.14).

Proof. We rewrite (1.1) as

$$\partial^3 I + p_1(z)\partial^2 I + \cdots = 0,$$

where

$$p_1(z) = \frac{-K_1}{z-1} + \frac{-K_2}{z}.$$

It is seen that the determinant $W(z) = \det\big(\partial^{i-1} I_j(z)\big)_{1\le i,j\le 3}$ satisfies the differential equation $\partial W(z) = -p_1(z)W(z)$. Hence $W(z)$ is a constant multiple of $z^{a+3c+g}(1-z)^{3b+3c+g}$. Therefore, what remains is to determine the multiplicative constant. For this purpose, we proceed as follows.

The change of integration variables $t_i = zu_i$ implies

$$\begin{aligned} I_1(z) &= \int_{D_1} (t_2-t_1)^g \prod_{i=1}^{2} t_i^a (1-t_i)^b (z-t_i)^c \, dt_1 dt_2 \\ &= z^{2(a+c+1)+g} \int_{0<u_1<u_2<1} (u_2-u_1)^g \prod_{i=1}^{2} u_i^a (1-u_i)^c (1-zu_i)^b \, du_1 du_2. \end{aligned}$$

The integrand converges to $(u_2-u_1)^g \prod_{i=1}^{2} u_i^a(1-u_i)^c$ uniformly on any compact subset of the integration domain $0 < u_1 < u_2 < 1$ as z reaches 0+. On the other hand, the conditions for the integral

$$\begin{aligned} &\int_{0<u_1<u_2<1} (u_2-u_1)^g \prod_{i=1}^{2} u_i^a (1-u_i)^c \, du_1 du_2 \\ &= \frac{1}{2} \frac{\Gamma(1+a,\ 1+a+\frac{g}{2},\ 1+c,\ 1+c+\frac{g}{2},\ 1+g)}{\Gamma(2+a+c+\frac{g}{2},\ 2+a+c+g,\ 1+\frac{g}{2})} \end{aligned} \tag{2.25}$$

converging are

$$a+1,\ a+1+\frac{g}{2},\ c+1,\ c+1+\frac{g}{2},\ \frac{g+1}{2} \notin \mathbb{Z}_{\le 0}.$$

The change of integration variables $t_1 = zu_1,\ t_2 = 1-(1-z)u_2$ implies

$$\begin{aligned} I_2(z) &= \int_{D_2} (t_2-t_1)^g ((z-t_1)(t_2-z))^c \prod_{i=1}^{2} t_i^a (1-t_i)^b \, dt_1 dt_2 \\ &= z^{1+a+c}(1-z)^{1+b+c} \int_{\substack{0<u_1<1 \\ 0<u_2<1}} (1-zu_1-(1-z)u_2)^g \\ &\quad \times u_1^a u_2^b (1-zu_1)^b (1-(1-z)u_2)^a \prod_{i=1}^{2} (1-u_i)^c \, du_1 du_2. \end{aligned}$$

The integrand converges to $u_1^a u_2^b (1-u_1)^c (1-u_2)^{a+c+g}$ uniformly on any compact subset of the integration domain $0 < u_1 < 1,\ 0 < u_2 < 1$ as z reaches 0+. On the other hand, the conditions for the integral

$$\int_{\substack{0<u_1<1\\0<u_2<1}} u_1^a u_2^b (1-u_1)^c (1-u_2)^{a+c+g}\, du_1 du_2 = B(a+1, c+1) B(b+1, a+c+g+1) \tag{2.26}$$

converging are

$$a+1,\ b+1,\ c+1,\ a+c+g+1 \notin \mathbb{Z}_{\leq 0}.$$

The change of integration variables $u_i = (t_i - z)/(1-z)$ implies

$$I_3(z) = \int_{D_3} (t_2 - t_1)^g \prod_{i=1}^{2} t_i^a (1-t_i)^b (t_i - z)^c \, dt_1 dt_2$$

$$= (1-z)^{2(b+c+1)+g} \int_{0<u_1<u_2<1} (u_2 - u_1)^g \prod_{i=1}^{2} (z + (1-z)u_i)^a (1-u_i)^b u_i^c \, du_1 du_2.$$

The integrand converges to $(u_2 - u_1)^g \prod_{i=1}^{2} u_i^{a+c} (1-u_i)^b$ uniformly on any compact subset of the integration domain $0 < u_1 < u_2 < 1$ as z reaches 0+. On the other hand, the conditions for the integral

$$\int_{0<u_1<u_2<1} (u_2 - u_1)^g \prod_{i=1}^{2} u_i^{a+c} (1-u_i)^b \, du_1 du_2$$

$$= \frac{1}{2} \frac{\Gamma(1+a+c,\ 1+a+c+\frac{g}{2},\ 1+b,\ 1+b+\frac{g}{2},\ 1+g)}{\Gamma(2+a+b+c+\frac{g}{2},\ 2+a+b+c+g,\ 1+\frac{g}{2})} \tag{2.27}$$

converging are

$$b+1,\ b+1+\frac{g}{2},\ a+c+1,\ a+c+1+\frac{g}{2},\ \frac{g+1}{2} \notin \mathbb{Z}_{\leq 0}.$$

Therefore, the functions

$$F_1(z) = z^{-\rho_1} I_1(z),\ \ F_2(z) = z^{-\rho_2} I_2(z),\ \ F_3(z) = I_3(z),$$

where

$$\rho_1 = 2(a+c+1) + g, \quad \rho_2 = a+c+1,$$

are holomorphic around the origin $z = 0$ under the conditions

$$a+c+1,\ a+c+1+\frac{g}{2},\ a+c+g+1 \notin \mathbb{Z}_{\leq 0} \tag{2.28}$$

and (2.24).

Consequently, we have

$$
\begin{aligned}
W(z) &= \det\Big(\partial_z^{i-1} I_j(z)\Big)_{1\le i,\, j\le 3} \\
&= \det\begin{pmatrix} I_1(z) & I_2(z) & I_3(z) \\ I_1'(z) & I_2'(z) & I_3'(z) \\ I_1''(z) & I_2''(z) & I_3''(z) \end{pmatrix} = \det\begin{pmatrix} z^{\rho_1}F_1 & z^{\rho_2}F_2 & F_3 \\ \partial_z(z^{\rho_1}F_1) & \partial_z(z^{\rho_2}F_2) & \partial_z(F_3) \\ \partial_z^2(z^{\rho_1}F_1) & \partial_z^2(z^{\rho_2}F_2) & \partial_z^2(F_3) \end{pmatrix} \\
&= \det\begin{pmatrix} z^{\rho_1}F_1 & z^{\rho_2}F_2 & F_3 \\ z^{\rho_1-1}(\rho_1F_1+zF_1') & z^{\rho_2-1}(\rho_2F_2+zF_2') & F_3' \\ z^{\rho_1-2}(\rho_1(\rho_1-1)F_1+2\rho_1zF_1'+z^2F_1'') & z^{\rho_2-2}(\rho_2(\rho_2-1)F_1+2\rho_2zF_2'+z^2F_2'') & F_3'' \end{pmatrix} \\
&= z^{\rho_1+\rho_2}\det\begin{pmatrix} F_1 & F_2 & F_3 \\ z^{-1}(\rho_1F_1+zF_1') & z^{-1}(\rho_2F_2+zF_2') & F_3' \\ z^{-2}(\rho_1(\rho_1-1)F_1+2\rho_1zF_1'+z^2F_1'') & z^{-2}(\rho_2(\rho_2-1)F_1+2\rho_2zF_2'+z^2F_2'') & F_3'' \end{pmatrix} \\
&= z^{\rho_1+\rho_2-1-2}\det\begin{pmatrix} F_1 & F_2 & F_3 \\ \rho_1F_1+zF_1' & \rho_2F_2+zF_2' & zF_3' \\ \rho_1(\rho_1-1)F_1+2\rho_1zF_1'+z^2F_1'' & \rho_2(\rho_2-1)F_1+2\rho_2zF_2'+z^2F_2'' & z^2F_3'' \end{pmatrix},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\lim_{z\to 0} z^{-\rho_1-\rho_2+1+2}W(z) &= \det\begin{pmatrix} F_1(0) & F_2(0) & F_3(0) \\ \rho_1F_1(0) & \rho_2F_2(0) & 0 \\ \rho_1(\rho_1-1)F_1(0) & \rho_2(\rho_2-1)F_2(0) & 0 \end{pmatrix} \\
&= F_1(0)F_2(0)F_3(0)\det\begin{pmatrix} \rho_1 & \rho_2 \\ \rho_1(\rho_1-1) & \rho_2(\rho_2-1) \end{pmatrix} \\
&= F_1(0)F_2(0)F_3(0)\rho_1\rho_2(\rho_2-\rho_1),
\end{aligned}
$$

where the right-most turns out to be

$$
\begin{aligned}
&\frac{1}{2}\frac{\Gamma(1+a,\ 1+a+\frac{g}{2},\ 1+c,\ 1+c+\frac{g}{2},\ 1+g)}{\Gamma(2+a+c+\frac{g}{2},\ 2+a+c+g,\ 1+\frac{g}{2})} \\
&\quad\times\frac{\Gamma(a+1,\ c+1,\ b+1,\ a+c+g+1)}{\Gamma(a+c+2,\ a+b+c+g+2)} \\
&\quad\times\frac{1}{2}\frac{\Gamma(1+a+c,\ 1+a+c+\frac{g}{2},\ 1+b,\ 1+b+\frac{g}{2},\ 1+g)}{\Gamma(2+a+b+c+\frac{g}{2},\ 2+a+b+c+g,\ 1+\frac{g}{2})}
\end{aligned}
$$

$$\times\{2(a+c+1)+g\}(a+c+1)(-a-c-1-g)$$

$$= -\frac{1}{2}\frac{\Gamma(1+a,\ 1+b,\ 1+c,\ 1+g)^2}{\Gamma(2+a+b+c+g,\ 1+\frac{g}{2})^2}$$

$$\times\frac{\Gamma(1+a+\frac{g}{2},\ 1+b+\frac{g}{2},\ 1+c+\frac{g}{2})}{\Gamma(2+a+b+c+\frac{g}{2})}$$

from (2.25), (2.26) and (2.27).

Finally, analytic continuation drops the conditions (2.28). □

Remark. The equalities of (2.25) and (2.27) come from the following Selberg integral formula (Theorem 8.1.1 of [1]):

$$\int_0^1\int_{t_1}^1 (t_2-t_1)^g \prod_{i=1,2} t_i^a(1-t_i)^b\,dt_1dt_2$$

$$= \frac{1}{2}\prod_{j=1,2}\frac{\Gamma(1+a+(j-1)\frac{g}{2},\ 1+b+(j-1)\frac{g}{2},\ 1+j\frac{g}{2})}{\Gamma(2+a+b+j\frac{g}{2},\ 1+\frac{g}{2})},$$

where $\mathrm{Re}(a+1)>0,\ \mathrm{Re}(b+1)>0$ and $\mathrm{Re}(\frac{g}{2})>-\min(\frac{1}{2},\ \mathrm{Re}(a+1),\ \mathrm{Re}(b+1))$.

3. Monodromy Representations

We fix a base point $z^0\in\mathbb{C}\backslash\{0,1\}$ to be real and $0<z^0<1$. Let V be $\sum_{1\le j\le 3}\mathbb{C}I_j(z)$. Let ρ be the monodromy representation

$$\rho:\ \pi_1(\mathbb{C}\backslash\{0,1\},\ z^0)\longrightarrow GL(V)$$

defined by

$$\gamma\longmapsto\rho(\gamma)f(z)$$

for $\gamma\in\pi_1(\mathbb{C}\backslash\{0,1\},\ z^0)$ and $f(z)\in V$, where $\rho(\gamma)f(z)$ means the consequence of the analytic continuation of $f(z)$ along the path γ. We call this representation ρ *the monodromy representation with respect to the basis* $\{I_1(z),\dots,I_3(z)\}$.

Let the symbols γ_0 and γ_1 designate the paths that express the moving of the coordinate z as in Fig. 2. It is known that these paths give a set of generators of the fundamental group $\pi_1(\mathbb{C}\backslash\{0,1\},z^0)$ with the base point z^0.

The path γ_i induces the action γ_i^* on the homology groups $H_2^{\mathrm{lf}}(T_{z^0},\mathcal{L}_{z^0})$ and $H_2(T_{z^0},\mathcal{L}_{z^0})$. The action γ_i^* on the homology groups induces the action $\rho(\gamma_i)$ on $V=\sum_{1\le j\le 3}\mathbb{C}I_j(z)$. For example, the actions

$$\gamma_0^*(D_1)=e(4a+4c+2g)\,D_1,\quad \gamma_1^*(D_3)=e(4b+4c+2g)\,D_3$$

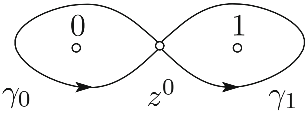


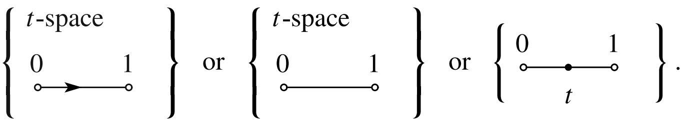
Fig. 2. Paths γ_0 and γ_1

on $H_2^{\mathrm{lf}}(T_{z^0}, \mathcal{L}_{z^0})$ imply

$$\rho(\gamma_0)(I_1(z)) = e(4a + 4c + 2g)\, I_1(z), \quad \rho(\gamma_1)(I_3(z)) = e(4b + 4c + 2g)\, I_3(z).$$

Before proceeding to derive the expression of $\rho(\gamma_1)$ on V, we explain the notations which will be used in our calculation.

When we consider the twisted chain $(0, 1) \otimes t^a(1-t)^b$ or $\{\, 0 < t < 1 \,\} \otimes t^a(1-t)^b$, we tacitly consider the integral $\int_0^1 t^a(1-t)^b\, dt$. To express the domain $\{\, 0 < t < 1 \,\}$ graphically, we write it as



Here the orientation of each domain of $\mathbb{R}\backslash\{0, 1\}$ is induced from the natural orientation of $\mathbb{C}\backslash\{0, 1\}$.

When we consider the twisted chain $\{\, 0 < t_2 < t_1 < 1 \,\} \otimes t_1^a t_2^c (t_1 - t_2)^d (1 - t_1)^b$, we tacitly consider the integral

$$\int_{0<t_2<t_1<1} t_1^a t_2^c (t_1 - t_2)^d (1 - t_1)^b\, dt_1 dt_2 = \int_0^1 dt_1 \int_0^{t_1} dt_2\, t_1^a t_2^c (t_1 - t_2)^d (1 - t_1)^b$$
$$= \int_0^1 t_1^a (1 - t_1)^b \left(\int_0^{t_1} t_2^c (t_1 - t_2)^d\, dt_2 \right) dt_1.$$

As a reflection of such relations, we denote

$$\{\, 0 < t_2 < t_1 < 1 \,\} \otimes t_1^a t_2^c (t_1 - t_2)^d (1 - t_1)^b$$

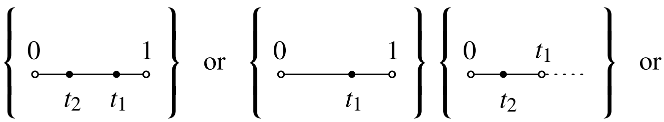
by

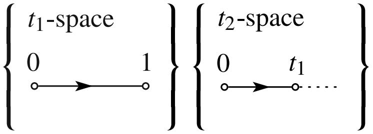
$$\Big[\, \{\, 0 < t_1 < 1 \,\} \times \{\, 0 < t_2 < t_1 \,\} \Big] \otimes t_1^a t_2^c (t_1 - t_2)^d (1 - t_1)^b$$

or

$$\Big[\{\, 0 < t_1 < 1 \,\} \otimes t_1^a (1 - t_1)^b \Big] \times \Big[\{ 0 < t_2 < t_1 \,\} \otimes t_2^c (t_1 - t_2)^d \Big].$$

Similarly, to express the domain $\{\, 0 < t_2 < t_1 < 1 \,\}$ graphically, we write it as



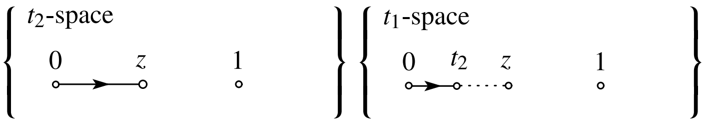


,

where the orientation of each domain of $\mathbb{R}^2 \setminus \{ t_1 = 0 \} \cup \{ t_2 = 0 \} \cup \{ t_1 = 1 \} \cup \{ t_1 = t_2 \}$ is induced from the natural orientation of $\mathbb{C}^2 \setminus \{ t_1 = 0 \} \cup \{ t_2 = 0 \} \cup \{ t_1 = 1 \} \cup \{ t_1 = t_2 \}$. We return to the derivation of $\rho(\gamma_1)$ on V.

First we consider the action of γ_1 on D_1. Since

$(0 < t_1 < t_2 < z) =$

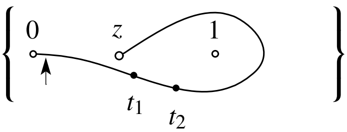


,

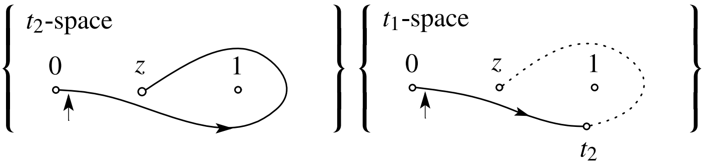
we have

$\gamma_1^*(D_1) = \gamma_1^*((0 < t_1 < t_2 < z))$

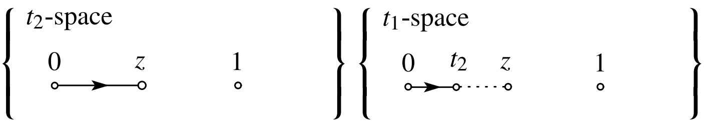
$=$



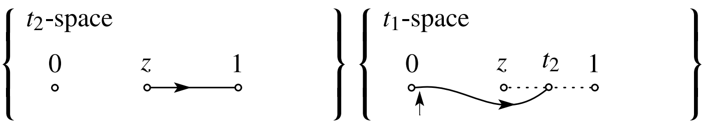
$=$



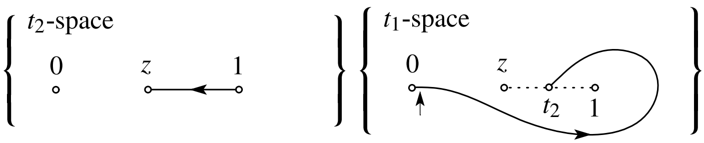
$=$



$+ e(c)$



$+ e(c + 2b)$



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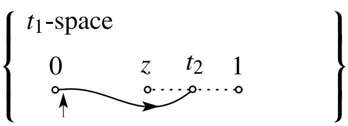
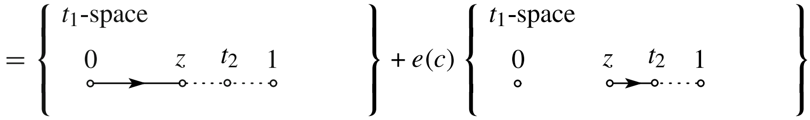
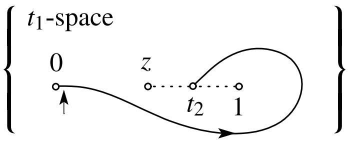
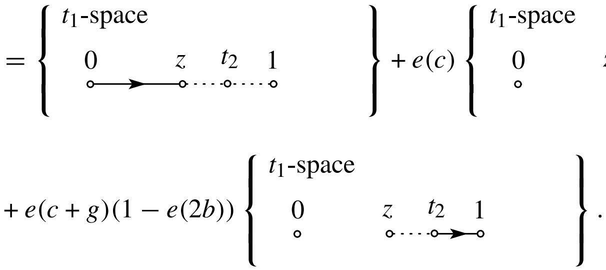




and

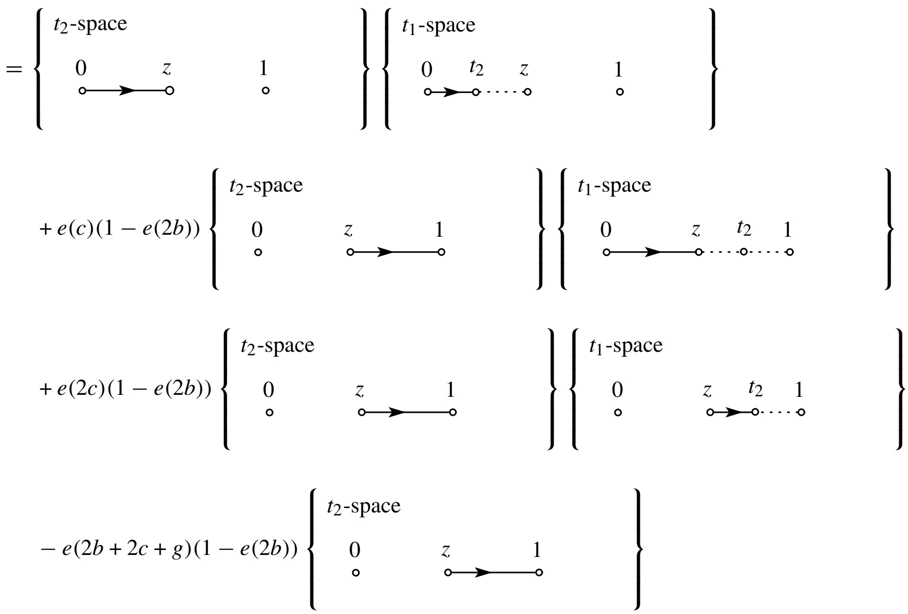


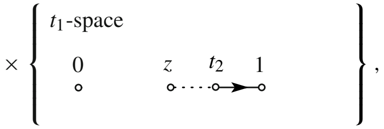


Hence,

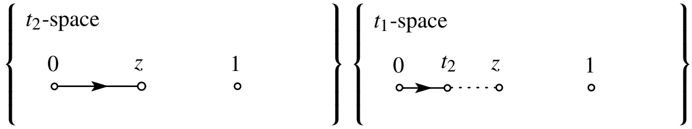
$\gamma_1^*(D_1)$

$$= \gamma_1^*((0 < t_1 < t_2 < z))$$

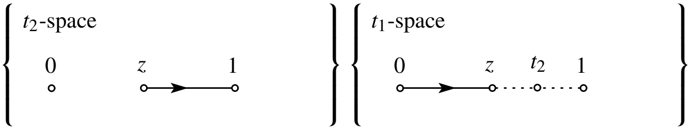




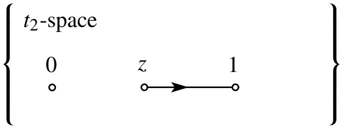
which is equal to

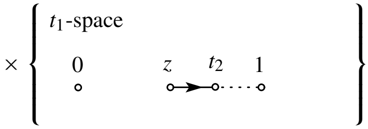


$+\, e(c)(1-e(2b))$



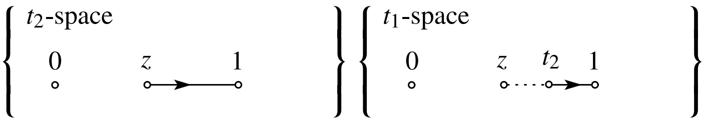
$+\, e(2c)(1-e(2b))(1-e(2b+g))$

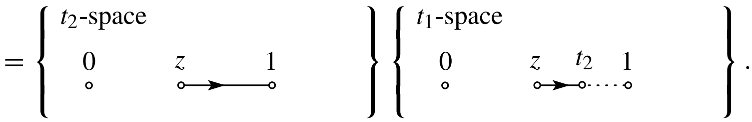




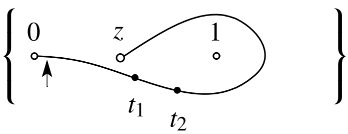
$$= D_1 + e(c)(1-e(2b))D_2 + e(2c)(1-e(2b))(1-e(2b+g))D_3.$$

Here we have used the symmetry of the integrals

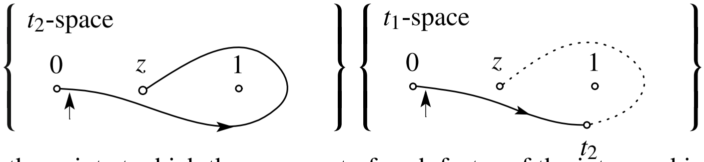




Here and in what follows, the vertical arrow in each picture like



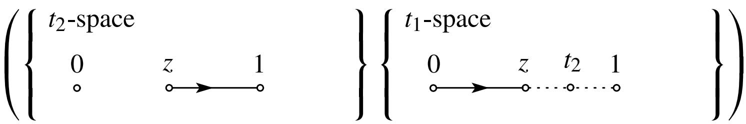
or

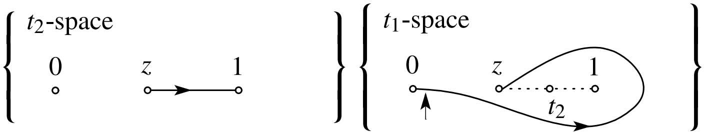


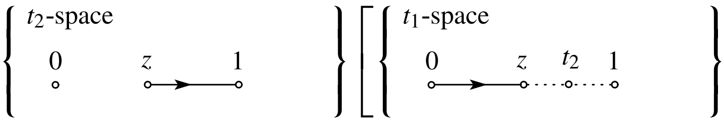
indicates the point at which the argument of each factor of the integrand is fixed to be zero, while we omit such an arrow when the chain in the picture is an interval of $(T_z)_{\mathbb{R}}$. We return to proceed our calculations.

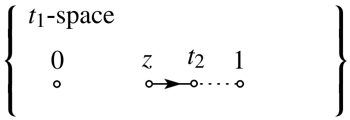
Secondly, we consider the action of γ_1 on D_2. We have

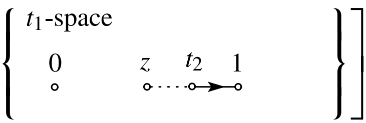
$$\gamma_1^*(D_2) = \gamma_1^*((0 < t_1 < z < t_2 < 1))$$

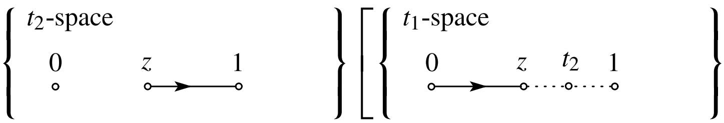
$= \gamma_1$


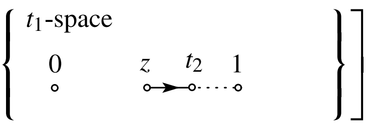
$= e(2b + 2c)$


$= e(2b + 2c)$


$+ e(c)(1 - e(2g + 2b))$


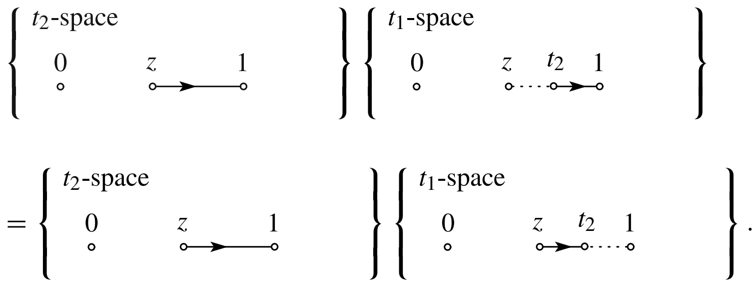
$+ e(c + g)(1 - e(2b))$


$= e(2b + 2c)$


$+ e(c)(1 + e(g))(1 - e(2b + g))$


$$= e(2b+2c)D_2 + e(2b+3c)(1+e(g))(1-e(2b+g))D_3.$$

Here we have used the symmetry of the integrals



Thirdly,

$$\gamma_1^*(D_3) = e(2(2b+2c+g))D_3$$

is easily derived.

Finally, the actions of γ_0 on $D_1,\ D_2,\ D_3$ are obtained from the formulas for γ_1 by changing $(a,\ b,\ z,\ D_1,\ D_3) \leftrightarrow (b,\ a,\ 1-z,\ D_3,\ D_1)$.

Consequently, we reach the following:

Proposition 6.

$$\rho(\gamma_0)(I_1, I_2, I_3) = (I_1, I_2, I_3)M_0,$$
$$\rho(\gamma_1)(I_1, I_2, I_3) = (I_1, I_2, I_3)M_1,$$

where

$$M_0 = \begin{pmatrix} e(4a+4c+2g) & e(2a+3c)(1+e(g))(1-e(2a+g)) & e(2c)(1-e(2a))(1-e(2a+g)) \\ 0 & e(2a+2c) & e(c)(1-e(2a)) \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ e(c)(1-e(2b)) & e(2b+2c) & 0 \\ e(2c)(1-e(2b))(1-e(2b+g)) & e(2b+3c)(1+e(g))(1-e(2b+g)) & e(4b+4c+2g) \end{pmatrix}.$$

4. A Sufficient Condition for the Irreducibility

In this section we consider the condition for the monodromy representation on V being irreducible.

For the representation

$$\rho(\gamma_0)(I_1, I_2, I_3) = (I_1, I_2, I_3)M_0,$$
$$\rho(\gamma_1)(I_1, I_2, I_3) = (I_1, I_2, I_3)M_1,$$

we use the name of each element $m_{ij}^{(0)}$ or $m_{ij}^{(1)}$ defined as

$$M_0 = (m_{ij}^{(0)})_{1\le i,j\le 3} \quad \text{and} \quad M_1 = (m_{ij}^{(1)})_{1\le i,j\le 3},$$

for our convenience.

Then we have the following.

Lemma 7. *Assume that*

$$(1-e(2c))(1-e(2c+g))(1-e(2(a+b+c)+g))(1-e(2(a+b+c+g)))$$
$$\times (1-e(2(a+c)+g+1))(1-e(2(b+c)+g+1)) \neq 0.$$

Then

$$c_1I_1 + c_2I_2 + c_3I_3 \in W \subset \sum_{1\le j\le 3} \mathbb{C}I_j$$

for some numbers $(c_1, c_2, c_3) \neq (0,0,0)$ *implies that*

$$I_1 \in W \qquad \textit{or} \qquad p_2I_2 + p_3I_3 \in W$$

for some numbers $(p_2, p_3) \neq (0,0)$.

Proof. Let v_0 express $c_1I_1 + c_2I_2 + c_3I_3$. Then we have

$$\begin{aligned}
&(\rho(\gamma_0) - m_{22}^{(0)})(\rho(\gamma_0) - m_{33}^{(0)})(v_0)\\
&\quad = I_1[c_1(m_{11}^{(0)} - m_{22}^{(0)})(m_{11}^{(0)} - m_{33}^{(0)}) + c_2(m_{11}^{(0)} - m_{33}^{(0)})m_{12}^{(0)}\\
&\quad + c_3\{(m_{11}^{(0)} - m_{22}^{(0)})m_{13}^{(0)} + m_{12}^{(0)}m_{23}^{(0)}\}]
\end{aligned}$$

and

$$\begin{aligned}
(\rho(\gamma_1) - m_{11}^{(1)})(v_0) &= I_2[c_1m_{21}^{(1)} + c_2(m_{22}^{(1)} - m_{11}^{(1)})]\\
&\quad + I_3[c_1m_{31}^{(1)} + c_2m_{32}^{(1)} + c_3(m_{33}^{(1)} - m_{11}^{(1)})].
\end{aligned}$$

If all of the coefficients of I_1, I_2, I_3 of the right-hand sides are zero, it holds that

$$B_1 \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$B_1 = \begin{pmatrix} (m_{11}^{(0)} - m_{22}^{(0)})(m_{11}^{(0)} - m_{33}^{(0)}) & (m_{11}^{(0)} - m_{33}^{(0)})m_{12}^{(0)} & (m_{11}^{(0)} - m_{22}^{(0)})m_{13}^{(0)} + m_{12}^{(0)}m_{23}^{(0)} \\ m_{21}^{(1)} & m_{22}^{(1)} - m_{11}^{(1)} & 0 \\ m_{31}^{(1)} & m_{32}^{(1)} & m_{33}^{(1)} - m_{11}^{(1)} \end{pmatrix}.$$

On the other hand, the determinant of B_1 is shown to be

$$e(2a+2c)(1-e(2c))(1-e(2c+g))(1-e(2(a+b+c)+g))(1-e(2(a+b+c+g)))$$
$$\times (1-e(2(a+c)+g+1))(1-e(2(b+c)+g+1)) \neq 0.$$

Thus we obtain the result. □

Lemma 8. *Assume that*

$$(1-e(2a+g))(1-e(2c+g))(1-e(g+1))(1-e(2a+2c+g+1))\\ \times(1-e(2(a+b+c+g)))\neq 0.$$

Then

$$c_2I_2+c_3I_3\in W\subset\sum_{1\le j\le 3}\mathbb{C}I_j$$

for some numbers $(c_2,c_3)\neq(0,0)$ *implies*

$$I_1\in W\qquad or\qquad I_3\in W.$$

Proof. Let v_0 express $c_2I_2+c_3I_3$. Then we have

$$(\rho(\gamma_0)-m_{22}^{(0)})(\rho(\gamma_0)-m_{33}^{(0)})(v_0)$$

$$=I_1\Big[c_2(m_{11}^{(0)}-m_{33}^{(0)})m_{12}^{(0)}+c_3\{(m_{11}^{(0)}-m_{22}^{(0)})m_{13}^{(0)}+m_{12}^{(0)}m_{23}^{(0)}\}\Big]$$

and

$$(\rho(\gamma_1)-m_{22}^{(1)})(v_0)=I_3[c_2m_{32}^{(1)}+c_3(m_{33}^{(1)}-m_{22}^{(1)})].$$

If both of the coefficients of I_1 and I_3 of the right-hand sides are zero, it holds that

$$B_2\begin{pmatrix}c_2\\c_3\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$$

where

$$B_2=\begin{pmatrix}(m_{11}^{(0)}-m_{33}^{(0)})m_{12}^{(0)} & (m_{11}^{(0)}-m_{22}^{(0)})m_{13}^{(0)}+m_{12}^{(0)}m_{23}^{(0)}\\ m_{32}^{(1)} & m_{33}^{(1)}-m_{22}^{(1)}\end{pmatrix}.$$

On the other hand, the determinant of B_2 is shown to be

$$e(2a+2b+5c)(1-e(g+1))(1-e(2a+g))(1-e(2c+g))\\ \times(1-e(2a+2c+g+1))(1-e(2(a+b+c+g))).$$

Thus we obtain the result. □

Lemma 9. (1) *Assume that*

$$(1-e(2a))(1-e(2a+g))(1-e(2a+2c+g+1))\neq 0.$$

Then $I_3\in W\subset\sum_{1\le j\le 3}\mathbb{C}I_j$ *implies* $I_1\in W$.
(2) *Assume that*

$$(1-e(2b))(1-e(2b+g))(1-e(2b+2c+g+1))\neq 0.$$

Then $I_1\in W\subset\sum_{1\le j\le 3}\mathbb{C}I_j$ *implies* $I_3\in W$.

Proof. (1) The equality

$$(\rho(\gamma_0) - m_{22}^{(0)})(\rho(\gamma_0) - m_{33}^{(0)})I_3 = (\rho(\gamma_0) - m_{22}^{(0)})(I_1 m_{13}^{(0)} + I_2 m_{23}^{(0)})$$

$$= I_1\left\{(m_{11}^{(0)} - m_{22}^{(0)})m_{13}^{(0)} + m_{12}^{(0)} m_{23}^{(0)}\right\}$$

$$= I_1\ e(2a + 4c + g)(1 - e(2a))(1 - e(2a + g))(1 - e(2a + 2c + g + 1))$$

implies the result.

(2) Similarly, the equality

$$(\rho(\gamma_1) - m_{22}^{(1)})(\rho(\gamma_1) - m_{11}^{(1)})I_1 = (\rho(\gamma_1) - m_{22}^{(1)})(I_2 m_{21}^{(1)} + I_3 m_{31}^{(1)})$$

$$= I_3\left\{m_{32}^{(1)} m_{21}^{(1)} + (m_{33}^{(1)} - m_{22}^{(1)})m_{31}^{(1)}\right\}$$

$$= I_3\ e(2b + 4c + g)(1 - e(2b))(1 - e(2b + g))(1 - e(2b + 2c + g + 1))$$

implies the result. □

Lemma 10. *Assume that* $1 - e(2a) \neq 0$ *or* $1 - e(2b) \neq 0$. *Then* $I_1, I_3 \in W \subset \sum_{1\leq j\leq 3} \mathbb{C} I_j$ *implies* $I_2 \in W$.

Proof. The equalities

$$(\rho(\gamma_0) - m_{33}^{(0)})I_3 = I_1 m_{13}^{(0)} + I_2 m_{23}^{(0)} \quad \text{with} \quad m_{23}^{(0)} = e(c)(1 - e(2a))$$

and

$$(\rho(\gamma_1) - m_{11}^{(1)})I_1 = I_2 m_{21}^{(1)} + I_3 m_{31}^{(1)} \quad \text{with} \quad m_{21}^{(1)} = e(c)(1 - e(2b))$$

imply the result. □

Consequently we obtain the following.

Theorem 11. *Assume that*

$$a,\ a + \frac{g}{2},\ b,\ b + \frac{g}{2},\ c,\ c + \frac{g}{2},\ a + b + c + \frac{g}{2},\ a + b + c + g,\ \frac{g+1}{2},$$
$$a + c + \frac{g+1}{2},\ b + c + \frac{g+1}{2} \in \mathbb{C}\backslash\mathbb{Z}.$$

Then $V = \sum_{1\leq j\leq 3} \mathbb{C} I_j$ *is irreducible.*

Remark. As is stated in [8], the Dotsenko–Fateev equation (1.1) has an accessory parameter: indeed, the constant term of the coefficient of I, i.e. $-M_2$, is the accessry parameter of (1.1), and is not uniquely determined by the datum of the local monodromy (or Riemann scheme).

5. Some Special Cases

In this section, we consider the special cases when the parameters a, b, c and g don't satisfy the assumption in Theorem 11. As a result, we obtain the sufficint condition that $V = \sum_{1\leq j\leq 3} \mathbb{C}\tilde{I}_j$ is reducible (Theorem 12). For such a purpose, in this section, we use the functions $\tilde{I}_j(a, b, c, g; z)$ defined by (2.14), in stead of $I_j(a, b, c, g; z)$.

5.1. The case $2a+2c+g+1=0$ *or* $2b+2c+g+1=0$. When $2b+2c+g+1=0$, the differential operator defining (1.1) becomes

$$\begin{aligned}
&z^2(z-1)^2\partial_z^3+\{K_1z+K_2(z-1)\}z(z-1)\partial_z^2\\
&\quad+\{L_1z^2+L_2(z-1)^2+L_3z(z-1)\}\partial_z+\{M_1z+M_2(z-1)\}\\
&=(1-z)\Big[z^2(1-z)\partial_z^3+\{K_2-(K_1+K_2)z\}z\partial_z^2 \quad +\{L_2-(L_2+L_3)z\}\partial_z-M_2\Big]\\
&=(1-z)\Big[z^2(1-z)\partial_z^3+\{1-3a+2b-c-(2-3a+b-2c)z\}z\partial_z^2\\
&\quad+\{(2a-2b)(a+c)-2(-a+a^2-ab+3ac-bc)z\}\partial_z+2c(1+2a)(a-b)\Big]\\
&=(1-z)\Big[z^2(1-z)\partial_z^3+\{\beta_1+\beta_2+1-(\alpha_1+\alpha_2+\alpha_3+3)z\}z\partial_z^2\\
&\quad+\{\beta_1\beta_2-(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3+\alpha_1+\alpha_2+\alpha_3+1)z\}\partial_z-\alpha_1\alpha_2\alpha_3\Big],
\end{aligned}\tag{5.1}$$

where

$$\alpha_1=-2c,\ \alpha_2=-1-2a,\ \alpha_3=b-a,\ \beta_1=-a-c,\ \beta_2=2b-2a.$$

The second factor of the right-most of (5.1) defines the differential equation ${}_3E_2(\alpha_1, \alpha_2, \alpha_3\,;\,\beta_1, \beta_2)$ which is satisfied by the generalized hypergeometric function ${}_3F_2(\alpha_1, \alpha_2, \alpha_3\,;\,\beta_1, \beta_2\,;\,z)$. It is known [13] that the roots of indicial equation for ${}_3E_2$ are

$$\begin{aligned}
&0,\ 1-\beta_1,\ 1-\beta_2 \quad \text{at} \quad z=0,\\
&0,\ 1,\ \beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3 \quad \text{at} \quad z=1,\\
&\alpha_1,\ \alpha_2,\ \alpha_3 \quad \text{at} \quad z=\infty,
\end{aligned}$$

and thus the exponents of the equation defined by the left-most of (5.1) are

$$\begin{aligned}
&0,\ 1+a+c,\ 1+2a-2b \quad \text{at} \quad z=0,\\
&0,\ 1,\ 1+b+c \quad \text{at} \quad z=1,\\
&-2c,\ -1-2a,\ b-a \quad \text{at} \quad z=\infty.
\end{aligned}$$

On the other hand, Beukers and Heckman [2] showed that the necessary and sufficient condition for ${}_3E_2$ being irreducible is $\beta_i-\alpha_j\notin\mathbb{Z}$ for all $i, j=1,2,3$ with $\beta_3=1$. Hence the necessary and sufficient condition for the equation defined by the left-most of (5.1) being irreducible is

$$2a,\ 2b,\ 2c,\ 2(a-b-c),\ a-b,\ a-c,\ b+c\notin\mathbb{Z}.$$

For instance, when $a=1/4, b=c=1/12$, the equation above is irreducible. This means that even if the assumption of Theorem 11 does not hold, the equation (1.1) is not necessarily reducible. Similarly, even if $2a+2c+g+1=0$, the differential equation (1.1) is not necessarily reducible.

On the other hand, when $a = -c - \frac{g+1}{2} + \mathbb{Z}$ and $b = -c - \frac{g+1}{2} + \mathbb{Z}$, the differential equation (1.1) is reducible. Indeed, the element

$$\begin{aligned} &e(c)(1+e(g))\widetilde{I}_1 + (1+e(g+2c))\widetilde{I}_2 \\ &+e(c)(1+e(g))\widetilde{I}_3 \end{aligned}$$

is invariant with respect to $\rho(\gamma_1)$ and $\rho(\gamma_2)$.

5.2. The case $a \in \mathbb{Z}$ *or* $b \in \mathbb{Z}$ *or* $a + \frac{g}{2} \in \mathbb{Z}$ *or* $b + \frac{g}{2} \in \mathbb{Z}$ *or* $\frac{g+1}{2} \in \mathbb{Z}$. It is easily seen from Proposition 6 that: (1) when $a \in \mathbb{Z}$, $\widetilde{I}_3$ is invariant. (2) when $b \in \mathbb{Z}$, $\widetilde{I}_1$ is invariant. (3) when $\frac{1+g}{2} \in \mathbb{Z}$ i.e. $g \in 1 + 2\mathbb{Z}$, $\widetilde{I}_2$ is invariant. (4) when $a + \frac{g}{2} \in \mathbb{Z}$ i.e. $2a + g \in 2\mathbb{Z}$, $\mathbb{C}\widetilde{I}_2 + \mathbb{C}\widetilde{I}_3$ is invariant. (5) when $b + \frac{g}{2} \in \mathbb{Z}$ i.e. $2b + g \in 2\mathbb{Z}$, $\mathbb{C}\widetilde{I}_1 + \mathbb{C}\widetilde{I}_2$ is invariant.
It implies that the monodromy representation is reducible.

5.3. The case $c \in \mathbb{Z}$. It is seen that the element

$$\widetilde{I}_1 + e(c)\widetilde{I}_2 + \widetilde{I}_3$$

is invariant with respect to $\rho(\gamma_1)$ and $\rho(\gamma_2)$. It implies that the monodromy representation is reducible.

5.4. The case $a + b + c + \frac{g}{2} \in \mathbb{Z}$. It is seen that the element

$$\begin{aligned} &(1 - e(2a))(1 - e(2b+2c))\widetilde{I}_1 + e(c)(1 - e(2a))(1 - e(2b))\widetilde{I}_2 \\ &+(1 - e(2a+2c))(1 - e(2b))\widetilde{I}_3 \end{aligned}$$

is invariant with respect to $\rho(\gamma_1)$ and $\rho(\gamma_2)$. It implies that the monodromy representation is reducible.

5.5. The case $c + \frac{g}{2} \in \mathbb{Z}$. When $g = -2c + 2\mathbb{Z}$, we have

$$\rho(\gamma_0)(\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3) = (\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3)M_0,$$

$$\rho(\gamma_1)(\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3) = (\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3)M_1,$$

where

$$M_0 = \begin{pmatrix} e(4a) & e(2a-c)(1+e(2c))(e(2c)-e(2a)) & (1-e(2a))(e(2c)-e(2a)) \\ 0 & e(2a+2c) & e(c)(1-e(2a)) \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ e(c)(1-e(2b)) & e(2b+2c) & 0 \\ (1-e(2b))(e(2c)-e(2b)) & e(2b-c)(1+e(2c))(e(c)-e(2b)) & e(4b) \end{pmatrix}.$$

If we denote by $V_\lambda^{(i)},\ i = 0, 1$, the eigenspace for $\rho(\gamma_i)$ with eigenvalue λ, we have

$$V_\lambda^{(i)} = \mathbb{C}v_\lambda^{(i)},$$

where

$$v_{e(4a)}^{(0)} = \widetilde{I}_1,$$

$$v_{e(2a+2c)}^{(0)} = (1+e(2c))\widetilde{I}_1 + e(c)\widetilde{I}_2,$$

$$v_1^{(0)} = (e(2c)-e(2a))\widetilde{I}_1 + e(c)(1-e(2a))\widetilde{I}_2 + (1-e(2a+2c))\widetilde{I}_3,$$

$$v_{e(4b)}^{(1)} = \widetilde{I}_3,$$

$$v_{e(2b+2c)}^{(1)} = e(c)\widetilde{I}_2 + (1+e(2c))\widetilde{I}_3,$$

$$v_1^{(1)} = (1-e(2b+2c))\widetilde{I}_1 + e(c)(1-e(2b))\widetilde{I}_2 + (e(2c)-e(2b))\widetilde{I}_3.$$

Using these elements, we obtain

$$\rho(\gamma_0)(v_{e(2b+2c)}^{(1)},\ v_1^{(0)}) = (v_{e(2b+2c)}^{(1)},\ v_1^{(0)}) \begin{pmatrix} e(2a+2c) & 0 \\ 1+e(2c) & 1 \end{pmatrix},$$

$$\rho(\gamma_1)(v_{e(2b+2c)}^{(1)},\ v_1^{(0)}) = (v_{e(2b+2c)}^{(1)},\ v_1^{(0)}) \begin{pmatrix} e(2b+2c) & -(1-e(2a+2b))(1-e(2c)) \\ 0 & 1 \end{pmatrix},$$

or

$$\rho(\gamma_0)(v_{e(2a+2c)}^{(0)},\ v_1^{(1)}) = (v_{e(2a+2c)}^{(0)},\ v_1^{(1)}) \begin{pmatrix} e(2a+2c) & -(1-e(2a+2b))(1-e(2c)) \\ 0 & 1 \end{pmatrix},$$

$$\rho(\gamma_1)(v_{e(2a+2c)}^{(0)},\ v_1^{(1)}) = (v_{e(2a+2c)}^{(0)},\ v_1^{(1)}) \begin{pmatrix} e(2b+2c) & 0 \\ 1+e(2c) & 1 \end{pmatrix}.$$

It implies that the monodromy representation is reducible.

Remark. Among the elements $v_\lambda^{(i)}$, we have

$$(1-e(2a+2c))v_{e(2b+2c)}^{(1)} = (e(2a)-e(2c))v_{e(2a+2c)}^{(0)} + (1+e(2c))v_1^{(0)},$$

$$(1-e(2a+2c))v_1^{(1)} = (1-e(2a+2b))(1-e(2c))v_{e(2a+2c)}^{(0)} + (e(2c)-e(2b))v_1^{(0)},$$

$$(1-e(2b+2c))v^{(0)}_{e(2a+2c)} = (e(2b)-e(2c))v^{(1)}_{e(2b+2c)} + (1+e(2c))v^{(1)}_1,$$

$$(1-e(2b+2c))v^{(0)}_1 = (1-e(2a+2b))(1-e(2c))v^{(1)}_{e(2b+2c)} + (e(2c)-e(2a))v^{(1)}_1.$$

5.6. The case $a+b+c+g \in \mathbb{Z}$. When $g=-a-b-c \in \mathbb{Z}$, we have

$$\rho(\gamma_0)(\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3) = (\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3)M_0,$$

$$\rho(\gamma_1)(\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3) = (\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3)M_1,$$

where M_0 and M_1 are given by

$$M_0 = \begin{pmatrix} e(2a-2b+2c) & e(a-2b+c)(1+e(a+b+c))(e(b+c)-e(a)) & e(c-b)(1-e(2a))(e(b+c)-e(a)) \\ 0 & e(2a+2c) & e(c)(1-e(2a)) \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ e(c)(1-e(2b)) & e(2b+2c) & 0 \\ e(c-a)(1-e(2b))(e(a+c)-e(b)) & e(b-2a+c)(1+e(a+b+c))(e(c+a)-e(b)) & e(2b-2a+2c) \end{pmatrix}.$$

If we denote by $V^{(i)}_\lambda$, $i=0,1$, the eigenspace for $\rho(\gamma_i)$ with eigenvalue λ, we have

$$V^{(i)}_\lambda = \mathbb{C}v^{(i)}_\lambda,$$

where

$$v^{(0)}_{e(2a-2b+2c)} = \widetilde{I}_1,$$

$$v^{(0)}_{e(2a+2c)} = (e(a)-e(b+c))(1+e(a+b+c))\widetilde{I}_1 + e(a+c)(1-e(2b))\widetilde{I}_2,$$

$$v^{(0)}_1 = -e(c)(1-e(2a))(e(a)-e(b+c))\widetilde{I}_1 + e(c)(1-e(2a))(e(b)-e(a+c))\widetilde{I}_2 + (e(b)-e(a+c))(1-e(2a+2c))\widetilde{I}_3,$$

$$v^{(1)}_{e(2b-2a+2c)} = \widetilde{I}_3,$$

$$v^{(1)}_{e(2b+2c)} = e(b+c)(1-e(2a))\widetilde{I}_2 + (e(b)-e(a+c))(1+e(a+b+c))\widetilde{I}_3,$$

$$v^{(1)}_1 = (e(a)-e(b+c))(1-e(2b+2c))\widetilde{I}_1 + e(c)(e(a)-e(b+c))(1-e(2b))\widetilde{I}_2 - e(c)(1-e(2b))(e(b)-e(a+c))\widetilde{I}_3.$$

Using these elements, we obtain

$$\rho(\gamma_0)(v^{(1)}_{e(2b+2c)}, v^{(0)}_1) = (v^{(1)}_{e(2b+2c)}, v^{(0)}_1)\begin{pmatrix} e(2a+2c) & 0 \\ 1+e(a+b+c) & 1 \end{pmatrix},$$

$$\rho(\gamma_1)(v^{(1)}_{e(2b+2c)}, v^{(0)}_1) = (v^{(1)}_{e(2b+2c)}, v^{(0)}_1)\begin{pmatrix} e(2b+2c) & -(1-e(a+b+c))(1-e(2c)) \\ 0 & 1 \end{pmatrix},$$

or

$$\rho(\gamma_0)(v^{(0)}_{e(2a+2c)}, v^{(1)}_1) = (v^{(0)}_{e(2a+2c)}, v^{(1)}_1)\begin{pmatrix} e(2a+2c) & -(1-e(a+b+c))(1-e(2c)) \\ 0 & 1 \end{pmatrix},$$

$$\rho(\gamma_1)(v^{(0)}_{e(2a+2c)}, v^{(1)}_1) = (v^{(0)}_{e(2a+2c)}, v^{(1)}_1)\begin{pmatrix} e(2b+2c) & 0 \\ 1+e(a+b+c) & 1 \end{pmatrix}.$$

It implies that the monodromy representation is reducible.

Remark. Among the elements $v^{(i)}_\lambda$, we have

$$(1-e(2a+2c))v^{(1)}_{e(2b+2c)} = e(c)(1-e(2a))v^{(0)}_{e(2a+2c)} + (1+e(a+b+c))v^{(0)}_1,$$

$$(1-e(2a+2c))v^{(1)}_1 = (1-e(2c))(1-e(a+b+c))v^{(0)}_{e(2a+2c)} - e(c)(1-e(2b))v^{(0)}_1,$$

$$(1-e(2b+2c))v^{(0)}_{e(2a+2c)} = e(c)(1-e(2b))v^{(1)}_{e(2b+2c)} + (1+e(a+b+c))v^{(1)}_1,$$

$$(1-e(2b+2c))v^{(0)}_1 = (1-e(a+b+c))(1-e(2c))v^{(1)}_{e(2b+2c)} - e(c)(1-e(2a))v^{(1)}_1.$$

5.7. Summary of the reducible cases. From Subsections 5.2–5.6, we obtain the following.

Theorem 12. *If one of*

$$a,\ a+\frac{g}{2},\ b,\ b+\frac{g}{2},\ c,\ c+\frac{g}{2},\ a+b+c+\frac{g}{2},\ a+b+c+g,\ \frac{g+1}{2}$$

is an integer, then $V = \sum_{1\le j\le 3}\mathbb{C}\widetilde{I}_j$ *is reducible.*

Remark. The same result is obtained in [7] as Theorem 12.11, where it is derived by using the shift operators for the shifts $a \to a\pm 1,\ b\to b\pm 1,\ c\to c\pm 1$ and $g \to g\pm 2$, which are given in A.6 of [7].

6. Intersection Numbers and the Invariant Hermitian Form

In this section we derive the monodromy-invariant Hermitian form $F(z, \overline{z})$ in terms of the functions $I_1(z),\ I_2(z),\ I_3(z)$ by using the theory of intersection numbers for twisted cycles [11,12], and thus for this purpose we restrict the exponents $a,\ b,\ c$ and g to be real numbers satisfying

$$a,\ a+\frac{g}{2},\ b,\ b+\frac{g}{2},\ c,\ c+\frac{g}{2},\ a+b+c+\frac{g}{2},\ a+b+c+g,\ \frac{g+1}{2}\notin\mathbb{Z}.$$

We use the symbols $s(A) = \sin(\pi A),\ \ c(A) = \cos(\pi A)$ and $d_A = e(2A) - 1 = \exp(2\pi\sqrt{-1}A) - 1$. We assign the names $D_j^{(i)}$, $1 \leq j \leq 3$, $i = 1, 2$, when z is real number and $0 < z < 1$, to the domains of the real manifold $(T_z)_{\mathbb{R}}$ as follows:

$$D_1^{(1)} = \{(t_1, t_2) \mid 0 < t_1 < t_2 < z\}, \qquad D_1^{(2)} = \{(t_1, t_2) \mid 0 < t_2 < t_1 < z\},$$

$$D_2^{(1)} = \{(t_1, t_2) \mid 0 < t_1 < z < t_2 < 1\}, \quad D_2^{(2)} = \{(t_1, t_2) \mid 0 < t_2 < z < t_1 < 1\},$$

$$D_3^{(1)} = \{(t_1, t_2) \mid z < t_1 < t_2 < 1\}, \qquad D_3^{(2)} = \{(t_1, t_2) \mid z < t_2 < t_1 < 1\}.$$

Remark. Each $D_j^{(1)}$ here is nothing but D_j defined in Subsection 2.1. In this Section, the name D_j will be used in different manner.

Remark. Each $D_j^{(i)}$ here is different from $D_j^{(i)}$ or $C_j^{(i)}$ in [19]. The domains $D_j^{(i)}$ in [19] correspond to the solutions around the point $z = 0$ with exponents $0,\ a + c + 1$ and $2(a + c + 1) + g$.

First, following [19] [20], we calculate the intersection numbers as follows:

$$D^{(1)} \bullet D_1^{(1)} = 1 + \frac{1}{d_a} + \frac{1}{d_c} + \frac{1}{d_g} + \frac{1}{d_a d_c} + \frac{d_{a+g}}{d_{2a+g} d_a d_g} + \frac{d_{c+g}}{d_{2c+g} d_c d_g},$$

$$D_1^{(1)} \bullet D_2^{(1)} = -\frac{e(c)}{d_c}\frac{d_{a+2c+g}}{d_{2c+g} d_a},$$

$$D_1^{(1)} \bullet D_3^{(1)} = -\frac{e(2c+g)}{d_{2c+g}}\frac{e(g)}{d_g},$$

$$D_1^{(1)} \bullet D_3^{(2)} = +\frac{e(2c+g)}{d_{2c+g}}\frac{d_{c+g}}{d_c d_g},$$

$$D_1^{(1)} \bullet D_2^{(2)} = -\frac{e(2c+g)}{d_{2c+g}}\frac{e(c)}{d_c},$$

$$D_1^{(1)} \bullet D_1^{(2)} = -\frac{e(g)}{d_g}\frac{d_{2a+2c+2g}}{d_{2a+g} d_{2c+g}},$$

$$D_2^{(1)} \bullet D_2^{(1)} = 1 + \frac{1}{d_a} + \frac{2}{d_c} + \frac{1}{d_b} + \frac{1}{d_{2c+g}} + \frac{2}{d_{2c+g} d_c} + \frac{1}{d_a}\left(\frac{1}{d_b} + \frac{1}{d_c}\right) + \frac{1}{d_b d_c},$$

$$D_2^{(1)} \bullet D_3^{(1)} = -\frac{e(c)}{d_c}\frac{d_{b+2c+g}}{d_b d_{2c+g}},$$

$$D_2^{(1)} \bullet D_3^{(2)} = -\frac{e(2c+g)}{d_{2c+g}}\frac{e(c)}{d_c},$$

$$D_2^{(1)} \bullet D_2^{(2)} = +\frac{e(2c+g)}{d_{2c+g}}\frac{d_{2c}}{d_c^2},$$

$$D_3^{(1)} \bullet D_3^{(1)} = 1 + \frac{1}{d_b} + \frac{1}{d_c} + \frac{1}{d_g} + \frac{1}{d_b d_c} + \frac{d_{b+g}}{d_{2b+g} d_b d_g} + \frac{d_{c+g}}{d_{2c+g} d_c d_g},$$

$$D_3^{(1)} \bullet D_3^{(2)} = -\frac{e(g)}{d_g}\frac{d_{2b+2c+2g}}{d_{2b+g} d_{2c+g}}$$

and

$$D_i^{(k)} \bullet D_j^{(l)} = D_j^{(l)} \bullet D_i^{(k)}, \qquad 1 \le i,\ j \le 3 \quad \text{and} \quad 1 \le k,\ l \le 2,$$
$$D_i^{(1)} \bullet D_j^{(2)} = D_i^{(2)} \bullet D_i^{(1)}, \qquad 1 \le i,\ j \le 3.$$

Secondly, we calculate the following:

$$D_1 \bullet D_1 = \frac{1}{2}(D_1^{(1)} \bullet D_1^{(1)} + D_1^{(1)} \bullet D_1^{(2)}) = \frac{1}{2}\frac{d_{a+c+\frac{g}{2}} d_{a+c+g}}{d_a d_c d_{a+\frac{g}{2}} d_{c+\frac{g}{2}} \left(1+e(g)\right)},$$

$$D_1 \bullet D_2 = \frac{1}{2}(D_1^{(1)} \bullet D_2^{(1)} + D_1^{(1)} \bullet D_2^{(2)}) = -\frac{1}{2}\frac{e(c) d_{a+c+\frac{g}{2}}}{d_a d_c d_{c+\frac{g}{2}}},$$

$$D_1 \bullet D_3 = \frac{1}{2}(D_1^{(1)} \bullet D_3^{(1)} + D_1^{(1)} \bullet D_3^{(2)}) = \frac{1}{2}\frac{e(2c+g)}{d_c d_{c+\frac{g}{2}} \left(1+e(g)\right)},$$

$$D_2 \bullet D_2 = \frac{1}{2}(D_2^{(1)} \bullet D_2^{(1)} + D_2^{(1)} \bullet D_2^{(2)}) = \frac{1}{2}\frac{(*)}{d_a d_b d_c d_{c+\frac{g}{2}}},$$

$$D_2 \bullet D_3 = \frac{1}{2}(D_2^{(1)} \bullet D_3^{(1)} + D_2^{(1)} \bullet D_3^{(2)}) = -\frac{1}{2}\frac{e(c) d_{b+c+\frac{g}{2}}}{d_b d_c d_{c+\frac{g}{2}}},$$

$$D_3 \bullet D_3 = \frac{1}{2}(D_3^{(1)} \bullet D_3^{(1)} + D_3^{(1)} \bullet D_3^{(2)}) = \frac{1}{2}\frac{d_{b+c+\frac{g}{2}} d_{b+c+g}}{d_b d_c d_{b+\frac{g}{2}} d_{c+\frac{g}{2}} \left(1+e(g)\right)},$$

and

$$D_j \bullet D_i = D_i \bullet D_j, \qquad 1 \le i < j \le 3,$$

where

$$(*) = 1 + e(2c) - e(2a+2c) - e(2b+2c) - e(2a+2c+g) - e(2b+2c+g) + e(2a+2b+2c+g) + e(2a+2b+4c+g).$$

When we consider the integrals $I_j(z)$ as the pairing between the cycle and the form $dt_1 \wedge dt_2$, it is not adequate to choose $D_j^{(1)}$ as a cycle, but adequate to choose $D_j = \frac{1}{2}(D_j^{(1)} + D_j^{(2)}) \in H_2^{\mathrm{lf}}(T_z, \mathcal{L}_z)_-^{\mathfrak{S}_2}$. Moreover, by using the results above, the determinant of $I_h = (D_i \bullet D_j)_{1 \le i, j \le 3}$ is given by

$$\det I_h = \frac{d_{a+b+c+\frac{g}{2}} d_{a+b+c+g}^2}{d_a^2 d_b^2 d_c^2 d_{a+\frac{g}{2}} d_{b+\frac{g}{2}} d_{c+\frac{g}{2}} (1+e(g))^2}.$$

The inverse I_h^{-1} of $I_h = (D_i \bullet D_j)_{1 \le i, j \le 3}$ is given by

$$I_h^{-1} = \frac{1}{d_{a+b+c+\frac{g}{2}} d_{a+b+c+g}} \begin{pmatrix} d_a d_{b+c} d_{a+\frac{g}{2}} d_{b+c+\frac{g}{2}} (1+e(g)) & e(c) d_a d_b d_{a+\frac{g}{2}} d_{b+c+\frac{g}{2}} (1+e(g)) & e(2c) d_a d_b d_{a+\frac{g}{2}} d_{b+\frac{g}{2}} (1+e(g)) \\ e(c) d_a d_b d_{a+\frac{g}{2}} d_{b+c+\frac{g}{2}} (1+e(g)) & (**) & e(c) d_a d_b d_{b+\frac{g}{2}} d_{a+c+\frac{g}{2}} (1+e(g)) \\ e(2c) d_a d_b d_{a+\frac{g}{2}} d_{b+\frac{g}{2}} (1+e(g)) & e(c) d_a d_b d_{b+\frac{g}{2}} d_{a+c+\frac{g}{2}} (1+e(g)) & d_b d_{a+c} d_{b+\frac{g}{2}} d_{a+c+\frac{g}{2}} (1+e(g)) \end{pmatrix},$$

where

$$\begin{aligned}(**) &= d_a d_b \Big(1 + e(2c+g) - e(2a+2c+g) - e(2b+2c+g) \\ &\quad - e(2a+2c+2g) - e(2b+2c+2g) + e(2a+2b+2c+2g) + e(2a+2b+4c+3g)\Big) \\ &= d_a d_b \left(d_{b+c+\frac{g}{2}} d_{a+c+g} + e(2c+g) d_a d_{b+\frac{g}{2}}\right) \\ &= d_a d_b \left(d_{a+c+\frac{g}{2}} d_{b+c+g} + e(2c+g) d_b d_{a+\frac{g}{2}}\right).\end{aligned}$$

Thus,

$$I_h^{-1} = \frac{\left(2\sqrt{-1}\right)^2}{s(a+b+c+\frac{g}{2})s(a+b+c+g)} \begin{pmatrix} s(a)s(b+c)s(a+\frac{g}{2})s(b+c+\frac{g}{2})2c(\frac{g}{2}) & s(a)s(b)s(a+\frac{g}{2})s(b+c+\frac{g}{2})2c(\frac{g}{2}) & s(a)s(b)s(a+\frac{g}{2})s(b+\frac{g}{2})2c(\frac{g}{2}) \\ s(a)s(b)s(a+\frac{g}{2})s(b+c+\frac{g}{2})2c(\frac{g}{2}) & (***) & s(a)s(b)s(b+\frac{g}{2})s(a+c+\frac{g}{2})2c(\frac{g}{2}) \\ s(a)s(b)s(a+\frac{g}{2})s(b+\frac{g}{2})2c(\frac{g}{2}) & s(a)s(b)s(b+\frac{g}{2})s(a+c+\frac{g}{2})2c(\frac{g}{2}) & s(b)s(a+c)s(b+\frac{g}{2})s(a+c+\frac{g}{2})2c(\frac{g}{2}) \end{pmatrix},$$

where

$$\begin{aligned}(***) &= s(a)s(b)\left(s(b+c+\frac{g}{2})s(a+c+g)+s(a)s(b+\frac{g}{2})\right)\\ &= s(a)s(b)\left(s(a+c+\frac{g}{2})s(b+c+g)+s(b)s(a+\frac{g}{2})\right).\end{aligned}$$

Therefore, we obtain the following invariant Hermitian form in terms of the functions $I_1(z),\ I_2(z),\ I_3(z)$ (see also [19] for the invariant Hermitian form in terms of another set of solutions of (1.1)).

Theorem 13. *A Hermitian form $F(z,\overline{z})$ which is invariant with respect to the action of the monodromy group is given by the following:*

$$F(z,\overline{z}) = \left(\overline{I_1(z)},\ \overline{I_2(z)},\ \overline{I_3(z)}\right) \begin{pmatrix} s(a)s(b+c)s(a+\frac{g}{2})s(b+c+\frac{g}{2})2c(\frac{g}{2}) & s(a)s(b)s(a+\frac{g}{2})s(b+c+\frac{g}{2})2c(\frac{g}{2}) & s(a)s(b)s(a+\frac{g}{2})s(b+\frac{g}{2})2c(\frac{g}{2}) \\ s(a)s(b)s(a+\frac{g}{2})s(b+c+\frac{g}{2})2c(\frac{g}{2}) & (***) & s(a)s(b)s(b+\frac{g}{2})s(a+c+\frac{g}{2})2c(\frac{g}{2}) \\ s(a)s(b)s(a+\frac{g}{2})s(b+\frac{g}{2})2c(\frac{g}{2}) & s(a)s(b)s(b+\frac{g}{2})s(a+c+\frac{g}{2})2c(\frac{g}{2}) & s(b)s(a+c)s(b+\frac{g}{2})s(a+c+\frac{g}{2})2c(\frac{g}{2}) \end{pmatrix} \begin{pmatrix} I_1(z) \\ I_2(z) \\ I_3(z) \end{pmatrix},$$

where

$$\begin{aligned}(***) &= s(a)s(b)\left\{s(b+c+\frac{g}{2})s(a+c+g)+s(a)s(b+\frac{g}{2})\right\}\\ &= s(a)s(b)\left\{s(a+c+\frac{g}{2})s(b+c+g)+s(b)s(a+\frac{g}{2})\right\}.\end{aligned}$$

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Conflict of interest The author has no competing interests to declare that are relevant to the content of this article.

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