Urakawa, H.
Osaka J. Math.
12 (1975), 117-137

ON THE HARDY CLASS OF HARMONIC SECTIONS AND VECTOR-VALUED POISSON INTEGRALS

HAJIME URAKAWA

(Received April 20, 1973)

1. Introduction

Let $D=\{z\in \boldsymbol{C};\ |z|<1\}$ be the unit disc in $\boldsymbol{C}$ and $B=\{e^{it};\ 0\leq t\leq 2\pi\}$ the boundary of D. For a complex-valued continuous function F on D and $0\leq r\leq 1$, we define a continuous function F_r on B by $F_r(u)=F(ru)$ for $u\in B$. We denote by $||F_r||_2$ the usual L^2-norm of F_r with respect to the normalized rotation invariant measure $\frac{1}{2\pi}dt$ on B. Let Δ be the Laplace-Beltrami operator on C^∞ functions on D with respect to the Poincaré metric on D. We denote by $H^2(D)$ the class of all C^∞ functions F on D such that $\Delta F=0$ and $\sup_{0\leq r\leq 1}||F_r||_2$ is finite. The Poisson kernel $P(z,u)$ on $D\times B$ for Δ is given by

$$P(re^{i\theta}, e^{it})=\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}, \qquad 0\leq r<1\,.$$

Then it is known (Zygmund [14]) that a function F on D belongs to $H^2(D)$ if and only if there exists a square integrable function f on B with respect to the measure $\frac{1}{2\pi}dt$ on B such that

$$F(z)=\frac{1}{2\pi}\int_0^{2\pi}f(e^{it})P(z, e^{it})dt$$

for $z\in D$. The integral is called the *Poisson integral* of f and $H^2(D)$ the *Hardy class* of harmonic functions on D. Our purpose is to extend these results to sections of a vector bundle on a symmetric space of non-compact type.

Now let G/K be a hermitian symmetric space of non-compact type. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G with respect to the Lie algebra $\mathfrak{k}$ of K. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$.

Now G/K can be holomorphically embedded (Harish-Chandra [2]) as a bounded domain $\mathscr{D}$ in a complex vector space $\mathfrak{p}^-$ and the Šilov boundary of $\mathscr{D}$ in $\mathfrak{p}^-$ is identified with the homogeneous space $G/B(E)$. Here the subgroup

$B(E)$ is a parabolic subgroup $M(E)AN$ where $M(E)$ is the centralizer in K of some element X^0 (cf. §3) in $\mathfrak{a}$, A is the analytic subgroup corresponding to $\mathfrak{a}$ and N is a nilpotent subgroup (cf. §2) of G. Let $\sum$ be the root system of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ with respect to the complexification $\mathfrak{h}^C$ of $\mathfrak{h}$. Fix an order on $\mathfrak{a}$ and choose an order (Satake [12]) of $\sum$ compatible with respect to the order on $\mathfrak{a}$. Let $2\rho_E$ be the restriction to $\mathfrak{a}$ of the sum of all positive roots α of $\sum$ with $\alpha(X^0)>0$.

Assume that G/K is holomorphically isomorphic with a tube domain. Let us consider a linear form $\lambda=z\rho_E$ on the complexification $\mathfrak{a}^C$ of $\mathfrak{a}$ where z is a complex number with the positive imaginary part. Let $\mathfrak{t}^C=Ad(u_1)^{-1}\mathfrak{h}^C$ be the Cartan subalgebra of the complexification $\mathfrak{k}^C$ of $\mathfrak{k}$, obtained from $\mathfrak{h}^C$ by the Cayley transform $Ad(u_1)$ (Moore [10]). Suppose that there exists an irreducible representation $(\tau_\Lambda, V_\Lambda)$ of K with the highest weight Λ on $\mathfrak{t}^C$ satisfying the condition

$$(C) \qquad {}^tAd(u_1^{-1})\Lambda = -(i\lambda+\rho_E) \qquad \text{on } \mathfrak{a}\,.$$

Let $\tau=\tau_\Lambda^*$ be the representation of K contragredient to τ_Λ. Let $L^2_{\tau,\lambda}(G/B(E))$ be the set of all measurable mapping ϕ of G into the dual space V_Λ^* of V_Λ satisfying

$$(1) \qquad \phi(\text{gman}) = e^{-(i\lambda+\rho_E)(\log a)}\tau(m^{-1})\phi(g)$$

for $m\in M(E)$, $a\in A$, $n\in N$, $g\in G$ where $\log a$ is the unique element in $\mathfrak{a}$ such that $\exp(\log a)=a$

$$(2) \qquad \int_K ||\phi(k)||^2 dk<\infty$$

where $||\cdot||$ is a $\tau(K)$-invariant norm on V_Λ^* and dk is the Haar measure of K, normalized by $\int_K dk=1$. G acts on $L^2_{\tau,\lambda}(G/B(E))$ by $U_{\tau,\lambda}(g)\phi(x)=\phi(g^{-1}x)$ for every g, $x\in G$.

Following K. Okamoto [11], we define the generalized Poisson integral $\mathcal{P}_{\tau,\lambda}$ as follows:

$$\mathcal{P}_{\tau,\lambda}\phi(g) = \int_K \tau(k)\phi(gk)dk(g\in G) \qquad \text{for } \phi\in L^2_{\tau,\lambda}(G/B(E))\,.$$

On the other hand, we define a norm $||\cdot||_2$, analogously in Knapp-Okamoto [6], for C^∞ sections of the vector bundle E_τ over G/K associated with the representation τ of K. We construct a representation $(U_\Lambda, \Gamma_2(\Lambda))$ of G on the completion of the space of all C^∞ sections f of E_τ satisfying the condition $||f||_2<\infty$ and certain boundary conditions (cf. §4). We define the *Hardy class* $H_2(\Lambda)$ as the space of all harmonic sections (cf. §5) in $\Gamma_2(\Lambda)$. Then we obtain the following results:

(i) The generalized Poisson integral $\mathcal{P}_{\tau,\lambda}$ maps $(U_{\tau,\lambda}, L^2_{\tau,\lambda}(G/B(E)))$ into $(U_\Lambda, \Gamma_2(\Lambda))$ G-equivariantly and strongly continuously (cf. Theorem 2 in §4).

(ii) The image of a certain G-submodule of $L^2_{\tau,\lambda}(G/B(E))$ under $\mathcal{P}_{\tau,\lambda}$ is contained in the Hardly class $H_2(\Lambda)$ (cf. Theorem 3 in §5).

The second result may be useful in proving the non-vanishingness of $H_2(\Lambda)$.

At the end of this paper we investigate the above condition (C) on the weight Λ for the unit disc D. If τ is the trivial representation, our Hardy class $H_2(\Lambda)$ is the usual Hardy class $H^2(D)$.

The author wishes to express his gratitude to Prof. K. Okamoto who suggested him to investigate this problem.

2. Asymptotic behavior of Poisson integrals

In this section we investigate the asymptotic behavior of Poisson integrals of symmetric space G/K of non-compact type. The results obtained in this section are natural generalizations of those obtained by Korányi [7], [8].

Let G be a non-compact semi-simple Lie group with finite center and let K be a maximal compact subgroup of G. Then the homogeneous space G/K is a symmetric space of non-compact type. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G with respect to the Lie algebra $\mathfrak{k}$ of K. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$; then we can find a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{a}$. Let $\sum$ be the set of all non-zero roots of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ with respect the complexification $\mathfrak{h}^C$ of $\mathfrak{h}$. The conjugation σ of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$ preserves $\mathfrak{h}$, and induces the permutation σ of $\sum$ defined by

$$\sigma(\alpha)(H) = \overline{\alpha(\sigma(H))}$$

for $\alpha\in\sum$, $H\in\mathfrak{h}$. We fix a σ-order $>$ of $\sum$, that is a linear order of $\sum$ such that $\sigma(\alpha)>0$ if $\alpha>0$ and if the restriction of α to $\mathfrak{a}$ does not vanish. Let $\sum_0$ be the set of all elements of $\sum$ which vanish on $\mathfrak{a}$. The restriction to $\mathfrak{a}$ of a root of $\sum-\sum_0$ is called a restricted root. The order $>$ on $\sum$ induces a linear order $>$ on the set of restricted roots. Let F be the fundamental system of restricted roots with respect to the order $>$.

Following Satake [12] and Moore [10], if E is a subset of F, let

$$\begin{aligned}\mathfrak{a}(E) &= \{H\in\mathfrak{a};\ \gamma(H)=0 \quad \text{for all } \gamma\in E\}\ ,\\ \textstyle\sum_0(E) &= \{\alpha\in\textstyle\sum;\ \pi(\alpha)=\sum n_\gamma\gamma(\gamma\in E,\ n_\gamma \text{ integers})\}\end{aligned}$$

where π is the restriction map of linear forms on $\mathfrak{h}$ to $\mathfrak{a}$. Let $\sum_+(E)$ (resp. $\sum_-(E)$) be the set of all $\alpha\in\sum-\sum_0(E)$ such that $\alpha>0$ (resp. $\alpha<0$). Then the subalgebras $\sum_{\alpha\in\sum_+(E)} CE_\alpha$, $\sum_{\alpha\in\sum_-(E)} CE_\alpha$ of $\mathfrak{g}^C$ are both invariant under σ where E_α's are root vectors for $\alpha\in\sum_\pm(E)$. Their intersections $\mathfrak{n}(E)$, $\bar{\mathfrak{n}}(E)$ with $\mathfrak{g}$ are the real forms of these subalgebras. The analytic subgroups of G corresponding to $\mathfrak{n}(E)$, $\bar{\mathfrak{n}}(E)$ will be denoted by $N(E)$, $\bar{N}(E)$. Let $\mathfrak{b}(E)$ be the normalizer of $\mathfrak{n}(E)$ in $\mathfrak{g}$ and $\mathfrak{m}(E)$ be the centralizer of $\mathfrak{a}(E)$ in $\mathfrak{k}$. Let $B(E)$ be the normalizer of $\mathfrak{n}(E)$ in G, and $M(E)$ the centralizer of $\mathfrak{a}(E)$ in K. Let $A(E)$ be the analytic subgroup of G corresponding to $\mathfrak{a}(E)$. If $E=\phi$, we write $\mathfrak{a}$, A, $\mathfrak{n}$, $\bar{\mathfrak{n}}$, N, $\bar{N}$, $\mathfrak{b}$, $\mathfrak{m}$, B and M instead of $\mathfrak{a}(E)$, $A(E)$, $\cdots$ respectively. We denote by $2\rho_E\in\mathfrak{a}^*$ the sum of all restrictions of roots in $\sum_+(E)$ with multiplicity counted where $\mathfrak{a}^*$ is the dual space of $\mathfrak{a}$. We also write ρ instead of ρ_E if $E=\phi$. We obtain the Iwasawa decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ and $G=KAN$. So for $g\in G$, it can be uniquely decomposed as $g=\kappa(g)\exp H(g)n(g)$ where $\kappa(g)\in K$, $H(g)\in\mathfrak{a}$ and $n(g)\in N$.

DEFINITION. For a complex number $z\in\boldsymbol{C}$, we put $\lambda=z\rho_E\in\mathfrak{a}_C^*$ where $\mathfrak{a}_C^*$ is the complexification of the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$. For a finite dimensional unitary representation τ of K on a complex vector space V_τ, we denote by $C_{\tau,\lambda}(G/B(E))$ the set of all continuous mappings ϕ of G into V_τ satisfying the following condition:

$$(1) \qquad \phi(gman)=e^{-(i\lambda+\rho_E)(\log a)}\tau(m^{-1})\phi(g)$$

for $m\in M(E)$, $a\in A$, $n\in N$ where $\log a$ denotes the unique element of $\mathfrak{a}$ such that $a=\exp(\log a)$. For a real number $p\geq 1$, we also denote by $L^p_{\tau,\lambda}(G/B(E))$ the set of all measurable mappings ϕ of G into V_τ satisfying (1) and

$$(2) \qquad \|\phi\|_p^p=\int_K\|\phi(k)\|^p dk<\infty$$

where $\|\cdot\|$ is a $\tau(K)$-invariant norm of V_τ and dk is the Haar measure of K, normalized by $\int_K dk=1$.

Following Okamoto [11], for every element ϕ of $C_{\tau,\lambda}(G/B(E))$ or $L^p_{\tau,\lambda}(G/B(E))$, we define a Poisson integral of ϕ by

$$(3) \qquad \mathcal{P}_{\tau,\lambda}\phi(g)=\int_K\tau(k)\phi(gk)dk\,.$$

Then $\mathcal{P}_{\tau,\lambda}\phi$ is a section of the vector bundle E_τ over G/K associated to the representation τ of K. Before investigating the asymptotic behavior of $\mathcal{P}_{\tau,\lambda}\phi$ we prepare the following Lemma.

Lemma 1. *Let G/K be a symmetric space of non-compace type. Then*

$$e^{2\rho_E(H(gm))}=e^{2\rho_E(H(g))}$$

for every $g\in G$, $m\in M(E)$.

Proof. For the proof, we notice (Korányi [8] Lemma 1.1) the fact that

$e^{2\rho_E(H(b))} = |\det(Ad(b))|$ for $b \in B(E)$ where $Ad(b)$ is the adjoint representation of $B(E)$ on $\mathfrak{b}(E)$. Then for $g = \kappa(g) \exp H(g) n(g)$ and $m \in M(E)$, using the decomposition $B(E) = M(E)AN$ (cf. Moore [10]), we may write

$$\exp H(g)n(g)m = m'a'n'$$

where $m' \in M(E)$, $a' \in A'$, $n' \in N$. Then $\exp H(gm) = a'$. Hence we obtain

$$\begin{aligned} e^{2\rho_E(H(gm))} &= |\det(Ad(H(gm)))| \\ &= |\det(Ad(m'a'n'))| \end{aligned}$$

since $M(E)$ is compact and N is nilpotent. Therefore we have

$$\begin{aligned} e^{2\rho_E(H(gm))} &= |\det(Ad(\exp(H(g))n(g)m))| \\ &= |\det(Ad(\exp H(g)))| \\ &= e^{2\rho_E(H(g))} \end{aligned} \qquad \text{Q.E.D.}$$

From Lemma 1, the right translation by $b \in B(E)$ of the measure $e^{-2\rho_E(H(g))}dg$ on G is equal to $e^{-2\rho_E(H(b))}e^{-2\rho_E(H(g))}dg$. Therefore the measure $e^{-2\rho_E(H(g))}dg$ on G induces (Bourbaki [1]) the measure $d\mu_E$ on $G/B(E)$ unique up to the constant factor such that

$$\int_{G/B(E)} \int_{B(E)} f(gb)db\, d\mu_E(gB(E)) = \int_G f(g)e^{-2\rho_E(H(g))}dg$$

for every continuous function f on G with compact support. Let $d\mu_E(guB(E))$ be the transform of the measure $d\mu_E$ under the mapping $G/B(E) \ni uB(E) \mapsto guB(E) \in G/B(E)$, then it follows (Korányi [8]) that

$$(4) \qquad d\mu_E(guB(E)) = e^{2\rho_E(H(gu) - H(u))} d\mu_E(uB(E)) .$$

Thus $d\mu_E$ is a K-invariant measure on $G/B(E)$. Let $d\mu_E$ be normalized by $\int_{G/B(E)} d\mu_E = 1$. And let us identify $K/M(E)$ and $G/B(E)$ under the mapping $K/M(E) \ni kM(E) \mapsto kB(E) \in G/B(E)$, then the measure $d\mu_E$ corresponds to the measure $dk_{M(E)}$ on $K/M(E)$ induced from the measure dk on K. And then the mapping $G/B(E) \ni uB(E) \mapsto guB(E) \in G/B(E)$ induces the transformation $K/M(E) \ni kM(E) \mapsto \kappa(gk)M(E) \in K/M(E)$. Put $h = \kappa(gk)$. Then we have, from the above equality (4), that $k = \kappa(g^{-1}h)$, $H(gk) = -H(g^{-1}h)$ and

$$(4') \qquad dh_{M(E)} = e^{2\rho_E(H(gk))} dk_{M(E)} .$$

In the case $E = \phi$, $M(E)$ is the centralizer M of $\mathfrak{a}$ in K and the equality (4′) is obtained in Harish-Chandra [3].

Corollary *G acts on $L^p_{\tau,\lambda}(G/B(E))$ by $U_{\tau,\lambda}(g)\phi(x) = \phi(g^{-1}x)$ for every g, $x \in G$. Then $U_{\tau,\lambda}(g)$ is a bounded operator on $L^p_{\tau,\lambda}(G/B(E))$ with respect to* *the norm* $||\cdot||_p$ *and* $C_{\tau,\lambda}(G/B(E))$ *is a G-invariant subspace of* $L^p_{\tau,\lambda}(G/B(E))$.

Indeed, we have

$$\int_K ||\phi(g^{-1}k)||^p dk \leq \sup_{k\in K} |e^{-(i\lambda+\rho_E)(H(g^{-1}k))}|^p \int_K ||\phi(\kappa(g^{-1}k))||^p dk$$
$$\leq \sup_{k\in K} |e^{-(i\lambda+\rho_E)(H(g^{-1}k))}|^p \sup_{k\in K} e^{-2\rho_E(H(gk))} \int_K ||\phi(k)||^p dk$$

since the mapping $k \mapsto ||\phi(\kappa(g^{-1}k))||$ is right $M(E)$-invariant, i.e. it is invariant under the right translation by elements of $M(E)$.

Proposition 1. *Let* $\mathfrak{a}^+(E)=\{H\in\mathfrak{a}(E);\ \alpha(H)>0$ *for all* $\alpha\in\sum_+(E)\}$. *For* $H\in\mathfrak{a}^+(E)$, *we put* $a_t=\exp tH$. *Then we have*

$$(5)\quad \lim_{t\to\infty} e^{(i\lambda+\rho_E)(\log a_t)}\mathcal{P}_{\tau,\lambda}\phi(ga_t) = \int_{\bar{N}(E)} e^{(i\lambda-\rho_E)(H(\bar{n}))}\tau(\kappa(\bar{n}))\phi(g)d\bar{n}$$

for all $g\in G$ *and* $\phi\in C_{\tau,\lambda}(G/B(E))$. *Here the measure* $d\bar{n}$ *is the Haar measure on* $\bar{N}(E)$, *normalized by* $\int_{\bar{N}(E)} e^{-2\rho_E(H(\bar{n}))}d\bar{n}=1$.

Proof. For every integrable right $M(E)$-invariant function f on K, it follows (cf. Korányi [8] Lemma 1.3) that

$$\int_K f(k)dk = \int_{\bar{N}(E)} f(\kappa(\bar{n}))e^{-2\rho_E(H(\bar{n}))}d\bar{n}\ .$$

For $\phi\in C_{\tau,\lambda}(G/B(E))$, the V_τ-valued function $\tau(k)\phi(gk)$ on K is right $M(E)$-invariant for fixed $g\in G$. Hence it follows that

$$\mathcal{P}_{\tau,\lambda}\phi(ga_t) = \int_{\bar{N}(E)} e^{-2\rho_E(H(\bar{n}))}\tau(\kappa(\bar{n}))\phi(ga_t\kappa(\bar{n}))d\bar{n}\ .$$

Since we have

$$a_t\kappa(\bar{n}) = \kappa(a_t\bar{n}a_t^{-1})\exp(H(a_t\bar{n}a_t^{-1})-H(\bar{n})+tH)n\ , \qquad n\in N$$

for $\bar{n}\in\bar{N}(E)$, it follows from (1) that

$$\mathcal{P}_{\tau,\lambda}\phi(ga_t) = \int_{\bar{N}(E)} e^{-(i\lambda+\rho_E)(H(a_t\bar{n}a_t^{-1})+H(n)+tH)}e^{-2\rho_E(H(\bar{n}))}\tau(\kappa(\bar{n}))\,\phi(g\kappa(a_t\bar{n}a_t^{-1}))d\bar{n}\ .$$

Then $a_t\bar{n}a_t^{-1}$ converges (cf. Korányi [8] Lemma 2.2) to the identity of G for every $H\in\mathfrak{a}^+(E)$ and $\bar{n}\in\bar{N}(E)$ as $t\to\infty$. Then we obtain the conclusion. Q.E.D.

If λ, $\mu\in\mathfrak{a}^*_C$, let $H_\lambda\in\mathfrak{a}^C$ be determined by $\lambda(H)=B(H_\lambda, H)$ for $H\in\mathfrak{a}$, where B is the Killing form of $\mathfrak{g}^C$. Put $\langle\lambda,\mu\rangle=B(H_\lambda, H_\mu)$. Then the integral of the right hand side of (5) converges if $\mathrm{Re}\langle i\lambda,\alpha\rangle<0$ for all $\alpha\in\sum_+(E)$.

From now on we shall assume $\lambda=z\rho_E$ satisfies the condition: $\mathrm{Re}\langle i\lambda, \alpha\rangle <0$ for all $\alpha\in\sum_+(E)$, that is, $y>0$ $(z=x+iy)$.

Lemma 2. *For $\phi\in L^1_{\tau,\lambda}(G/B(E))$, we obtain*

$$(6)\qquad \mathcal{P}_{\tau,\lambda}\phi(g)=\int_{K/M(E)} e^{(i\lambda-\rho_E)(H(g^{-1}k))}\tau(\kappa(g^{-1}k))\phi(k)dk_{M(E)}$$

where $dk_{M(E)}$ is the K-invariant measure on $K/M(E)$ induced from the Haar measure dk on K. Therefore $\mathcal{P}_{\tau,\lambda}$ may be ragarded as an integral operator with the kernel $K_{\tau,\lambda}(g, k)=e^{(i\lambda-\rho_E)(H(g^{-1}k))}\tau(\kappa(g^{-1}k))$.

Proof. For $\phi\in L^1_{\tau,\lambda}(G/B(E))$,

$$\mathcal{P}_{\tau,\lambda}\phi(g)=\int_{K/M(E)} e^{-(i\lambda+\rho_E)(H(gk))}\tau(k)\phi(\kappa(gk))dk_{M(E)}$$

from the condition (1). Put $h=\kappa(gk)$. Substituting (4′) into the right hand of the above equality, Lemma 2 is obtained. Q.E.D.

Corollary *For every $g\in G$, $k\in K$, let $||K_{\tau,\lambda}(g, k)||$ be the operator norm $||\cdot||$ of the transformation $K_{\tau,\lambda}(g, k)$ of V_τ with respect to the norm $||\cdot||$ in V_τ. Then it follows that*

(i) $||K_{\tau,\lambda}(g, k)|| = |e^{(i\lambda-\rho_E)(H(g^{-1}k))}|$

(ii) $||K_{\tau,\lambda}(g, km)|| = ||K_{\tau,\lambda}(g, k)||$ *for all $g\in G$, $k\in K$, $m\in M(E)$.*

Indeed, (i) is clear. Since we assume $\lambda=z\rho_E$ and z is a complex number, (ii) follows from Lemma 1.

Lemma 3. *For $H\in\mathfrak{a}^+(E)$, we put $a_t=\exp tH$. For every neighborhood V of $eM(E)$ in $K/M(E)$, we have*

$$\lim_{t\to\infty}|e^{(i\lambda+\rho_E)(\log a_t)}|\int_{K/M(E)-V}||K_{\tau,\lambda}(a_t, k)||dk_{M(E)}=0\,.$$

Proof. For every continuous function ϕ on $K/M(E)$, we have

$$(7)\qquad \int_{K/M(E)}||K_{\tau,\lambda}(a_t, k)||\,|\phi(kM(E))|\,dk_{M(E)}$$
$$=\int_{\bar{N}(E)}|e^{(i\lambda-\rho_E)(a_t^{-1}\kappa(\bar{n}))}|\,|\phi(\kappa(\bar{n})M)|\,e^{-2\rho_E(H(\bar{n}))}d\bar{n}\,.$$

Put $\bar{n}'=a_t^{-1}\bar{n}a_t$. Then it follows that $d\bar{n}=e^{-2\rho_E(H(\bar{n}'))}d\bar{n}'$ and $a_t^{-1}\kappa(\bar{n})=\kappa(\bar{n}')\exp(H(\bar{n}')-H(a_t\bar{n}'a_t^{-1})-tH)n$ for some $n\in N$. Then, substituting these into (7), we have

$$(7)=|e^{-(i\lambda+\rho_E)(\log a_t)}|\int_{\bar{N}(E)}|e^{-(i\lambda+\rho_E)(H(a_t\bar{n}a_t^{-1}))}e^{(i\lambda-\rho_E)(H(\bar{n}))}\phi(\kappa(a_t\bar{n}a_t^{-1})M(E))|\,d\bar{n}\,.$$

Therefore we obtain

$$(8)\quad \lim_{t\to\infty}|e^{(i\lambda+\rho_E)(\log a_t)}|\int_{K/M(E)}||K_{\tau,\lambda}(a_t, k)||\,|\phi(kM(E))|\,dk_{M(E)}$$
$$=|\phi(eM(E))|\int_{\bar{N}(E)}|e^{(i\lambda-\rho_E)(H(\bar{n}))}|\,d\bar{n}=|\phi(eM(E))|\,C_E(\lambda)<\infty$$

where $C_E(\lambda)=\int_{\bar{N}(E)}|e^{(i\lambda-\rho_E)(H(\bar{n}))}|\,d\bar{n}=\int_{\bar{N}(E)}e^{-(y+1)\rho_E(H(\bar{n}))}d\bar{n}<\infty$ because of $\lambda=z\rho_E$ $(z=x+iy,\ y>0)$. In particular, we have

$$(9)\quad \lim_{t\to\infty}|e^{(i\lambda+\rho_E)(\log a_t)}|\int_{K/M(E)}||K_{\tau,\lambda}(a_t, k)||dk_{M(E)}=C_E(\lambda)\,.$$

For every neighborhood V of $eM(E)$, there exists a continuous function ϕ on $K/M(E)$ such that $|\phi|\leq 1$, $\phi(eM(E))=1$ and $\sup_{kM(E)\notin V}|\phi(kM(E))|=m<1$. Then we have

$$(10)\quad \lim_{t\to\infty}|e^{(i\lambda+\rho_E)(\log a_t)}|\int_{K/M(E)}||K_{\tau,\lambda}(a_t, k)||\,|\phi(kM(E))|\,dk_{M(E)}=C_E(\lambda)\,.$$

On the other hand, we obtain

$$(11)\quad \int_{K/M(E)}||K_{\tau,\lambda}(a_t, k)||\,|\phi(kM(E))|\,dk_{M(E)}$$
$$\leqq\int_{K/M(E)}||K_{\tau,\lambda}(a_t, k)||dk_{M(E)}+(m-1)\int_{K/M(E)-V}||K_{\tau,\lambda}(a_t, k)||dk_{M(E)}\,.$$

Hence from $m-1<0$, (9), (10), (11), the proof is complete. Q.E.D.

Proposition 2. *Let $1<p<\infty$. For $H\in\mathfrak{a}^+(E)$, we put $a_t=\exp tH$. Then for every $\phi\in L^p_{\tau,\lambda}(G/B(E))$, we have*

$$\lim_{t\to\infty}\int_K\left\{\left\|e^{(i\lambda+\rho_E)(\log a_t)}\mathcal{P}_{\tau,\lambda}\phi(ka_t)-\int_{\bar{N}(E)}e^{(i\lambda-\rho_E)(H(\bar{n}))}\tau(\kappa(\bar{n}))\phi(k)d\bar{n}\right\|\right\}^p dk=0\,.$$

Proof. From Lemma 1, for every $\phi\in L^p_{\tau,\lambda}(G/B(E))$,

$$(12)\quad \mathcal{P}_{\tau,\lambda}\phi(ka_t)-\int_K K_{\tau,\lambda}(a_t, h)\phi(k)dh=\int_K K_{\tau,\lambda}(a_t, h)(\phi(kh)-\phi(k))dh\,.$$

For every function $g\in L^q(K/M(E))$ where $\frac{1}{p}+\frac{1}{q}=1$, we put $\tilde{g}(k)=g(kM(E))$. Then we have

$$\int_K\left\{\int_K||K_{\tau,\lambda}(a_t, h)||\,||\phi(kh)-\phi(k)||dh\right\}\tilde{g}(k)dk$$
$$\leq\int_K\left\{\left|\int_K||\phi_h(k)-\phi(k)||\tilde{g}(k)dk\right|\right\}||K_{\tau,\lambda}(a_t, h)||dh$$
$$\leq||\tilde{g}||_{L^p(K)}\int_K||\phi_h-\phi||_{L^p(K)}||K_{\tau,\lambda}(a_t, h)||dh$$

where $\phi_h(k)=\phi(kh)$ and $||\phi_h-\phi||_{L^p(K)}$ is the usual L^p-norm of the function $||\phi_h(k)-\phi(k)||$ on K with respect to the Haar measure dk on K. Hence together with (12), we obtain

$$(13)\qquad \left\{\int_K \left\| \mathcal{P}_{\tau,\lambda}\phi(ka_t)-\int_K K_{\tau,\lambda}(a_t, h)\phi(k)dh\right\|^p dk\right\}^{1/p} \leqq \int_K ||\phi_h-\phi||_{L^p(K)}||K_{\tau,\lambda}(a_t, h)||dh\,.$$

Here for every neighborhood V of $eM(E)$ in $K/M(E)$, the right hand side of

$$(13)\leq \sup_{hM(E)\in V} ||\phi_h-\phi||_{L^p(K)}\int_V ||K_{\tau,\lambda}(a_t, h)||dh_{M(E)} +2||\phi||_{L^p(K/M(E))}\int_{K/M(E)-V}||K_{\tau,\lambda}(a_t, h)||dh_{M(E)}\,.$$

Therefore by Lemma 3 and its proof, we get

$$(14)\qquad \lim_{t\to\infty}|e^{(i\lambda+\rho_E)(\log a_t)}|\left\{\int_K\left\{\left\|\mathcal{P}_{\tau,\lambda}\phi(ka_t)-\int_K K_{\tau,\lambda}(a_t, h)\phi(k)dh\right\|\right\}^p dk\right\}^{1/p}=0\,.$$

On the other hand, since $e^{(i\lambda+\rho_E)(\log a_t)}\int_K K_{\tau,\lambda}(a_t, h)\phi(k)dh$ is equal to $\int_{\overline{N}(E)} e^{(i\lambda-\rho_E)(H(\bar{n}))}e^{-(i\lambda+\rho_E)(H(a_t\bar{n}a_t^{-1}))}\tau(\kappa(\bar{n}))\phi(k)d\bar{n}$, it follows that

$$(15)\qquad \left\|e^{(i\lambda+\rho_E)(\log a_t)}\int_K K_{\tau,\lambda}(a_t, h)\phi(k)dh-\int_{\overline{N}(E)} e^{(i\lambda-\rho_E)(H(\bar{n}))}\tau(\kappa(\bar{n}))\phi(k)d\bar{n}\right\| \leq ||\phi(k)||\left\{\int_{\overline{N}(E)}|e^{-(i\lambda+\rho_E)(H(a_t\bar{n}a_t^{-1}))}-1|\,|e^{(i\lambda-\rho_E)(H(\bar{n}))}|d\bar{n}\,.\right.$$

From the fact that $C_E(\lambda)=\int_{\overline{N}(E)}|e^{(i\lambda-\rho_E)(H(\bar{n}))}|d\bar{n}<\infty$ and $a_t\bar{n}a_t^{-1}$ converges to the identity as $t\to\infty$, the right hand side converges to zero as $t\to\infty$. So, together with (14), Proposition 2 is proved. Q.E.D.

REMARK From Proposition 2, it follows that for every $\phi\in L^p_{\tau,\lambda}(G/B(E))$

$$\lim_{t\to\infty}\int_K ||e^{(i\lambda+\rho_E)(\log a_t)}\mathcal{P}_{\tau,\lambda}\phi(ka_t)||^p dk=\int_K\left\{\left\|\int_{\overline{N}(E)} e^{(i\lambda-\rho_E)(H(\bar{n}))}\tau(\kappa(\bar{n}))\phi(k)d\bar{n}\right\|\right\}^p dk$$

if $1<p<\infty$ and $\mathrm{Re}\langle i\lambda, \alpha\rangle<0$ for all $\alpha\in\sum_+(E)$. Now we denote the above limit by $(||\mathcal{P}_{\tau,\lambda}\phi||_{p,H})^p$. Then we have $||\mathcal{P}_{\tau,\lambda}\phi||_{p,H}\leq C_E(\lambda)||\phi||^p_{L^p(K/M(E))}$ where $||\phi||_{L^p(K/M(E))}$ is the usual L^p-norm of the function $||\phi(k)||$ on K.

3. Properties of hermitian symmetric spaces

From now on we shall assume G/K is an irreducible hermitian symmetric space of tube type; let G be a non-compact connected simple Lie group with a faithful matrix representation, K a maximal compact subgroup and we shall assume the homogeneous space G/K is an irreducible hermitian symmetric space holomorphically diffeomorphic with a tube domain. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ corresponding to $\mathfrak{k}$. For any subspace $\mathfrak{m}$ of $\mathfrak{g}$, we denote by $\mathfrak{m}^C$ the complexification of $\mathfrak{m}$. Since G has a faithful matrix representation, we can regard G as a subgroup of a connected Lie group G^C with Lie algebra $\mathfrak{g}^C$. Let K^C be the analytic subgroup of G^C with Lie algebra $\mathfrak{k}^C$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$, T the corresponding analytic subgroup of G, and let T^C be the analytic subgroup of G^C with Lie algebra $\mathfrak{t}^C$. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$ and of $\mathfrak{g}$. And T is also a Cartan subgroup of K and of G.

Let R be the set of all non-zero roots of $(\mathfrak{g}^C, \mathfrak{t}^C)$. For $\alpha\in R$, let $\mathfrak{g}_\alpha$ be the root space for α, then $\mathfrak{g}_\alpha\subset\mathfrak{k}^C$ or $\mathfrak{g}_\alpha\subset\mathfrak{p}^C$, and α is called a compact root or a non-compact root according to the respective cases. Let $R_\mathfrak{k}$ and $R_\mathfrak{n}$ be the set of compact and non-compact roots respectively.

We identify $\mathfrak{p}$ and $\mathfrak{p}^C$ with the tangent space $T_{eK}(G/K)$ of G/K at eK and its complefixication $T^C_{eK}(G/K)$, respectively, under the natural projection of G onto G/K. Let $\mathfrak{p}_-$ (resp. $\mathfrak{p}_+$) be the subspace of $\mathfrak{p}^C$ corresponding to the set of all holomorphic (anti-holomorphic) tangent vectors of $T^C_{eK}(G/K)$ respectively. Then $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are $ad(\mathfrak{k}^C)$-invariant abelian subalgebras of $\mathfrak{p}^C$ such that $\mathfrak{p}^C=\mathfrak{p}_++\mathfrak{p}_-$. Let P_+, P_- be the corresponding analytic subgroups of G^C. Moreover there exists a subset $P_\mathfrak{n}$ of R such that $\mathfrak{p}_+=\sum_{\alpha\in P_\mathfrak{n}}\mathfrak{g}_\alpha$. We can define a linear order $\succ$ on R such that the set P of all positive roots includes $P_\mathfrak{n}$. We put $P_\mathfrak{k}=P\cap R_\mathfrak{k}$.

Let τ be the conjugation of $\mathfrak{g}^C$ with respect to the compact real form $\mathfrak{g}_u=\mathfrak{k}+\sqrt{-1}\,\mathfrak{p}$ of $\mathfrak{g}^C$, and we choose root vectors $\{E_\alpha\}$ such that $\tau E_\alpha=-E_{-\alpha}$ for $\alpha\in R$. Let $\Delta=\{\gamma_1\succ\cdots\succ\gamma_m\}$ be the maximal set of strongly orthogonal non-compact positive roots of Harish-Chandra [2]. For $\alpha\in R$, let H_α be the unique element of $\sqrt{-1}\,\mathfrak{t}$ satisfying $B(H_\alpha, H)=\alpha(H)$ for all $H\in\mathfrak{t}^C$. For $\alpha\in\Delta$, we put $X^0_\alpha=E_\alpha+E_{-\alpha}$, $Y^0_\alpha=(-\sqrt{-1})(E_\alpha-E_{-\alpha})$ and $H'_\alpha=\dfrac{2}{\langle\alpha, \alpha\rangle}H_\alpha$. Moreover we put $X^0=\sum_{\alpha\in\Delta}X^0_\alpha$ and $Z^0=-\dfrac{\sqrt{-1}}{2}\sum_{\alpha\in\Delta}H'_\alpha$. Let $\mathfrak{t}^-=\sqrt{-1}\sum_{\alpha\in\Delta}\boldsymbol{R}H'_\alpha$ be the subalgebra of $\mathfrak{t}$ spanned by $\sqrt{-1}\,H'_\alpha$, $\alpha\in\Delta$ over the real number field $\boldsymbol{R}$. Let $\mathfrak{t}^+$ be the orthocomplement of $\mathfrak{t}^-$ in $\mathfrak{t}$ with respect to the Killing form B, and let T^-, T^+ be the analytic subgroups of T corresponding to $\mathfrak{t}^-$, $\mathfrak{t}^+$ respectively. We have the decomposition $\mathfrak{t}^C=(\mathfrak{t}^+)^C+(\mathfrak{t}^-)^C$, and corresponding to this, we can decompose each element μ of the complexification $\mathfrak{t}^*_C$ of the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$, as

$$\mu=\mu_++\mu_- \tag{16}$$

where μ_+ (resp. μ_-) is the same as the restriction of μ on $(\mathfrak{t}^+)^C$ (resp. $(\mathfrak{t}^-)^C$) and vanishes identically on $(\mathfrak{t}^-)^C$ (resp. $(\mathfrak{t}^+)^C$). The vectors X^0_α, $\alpha\in\Delta$ span a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ and $\mathfrak{h}=\mathfrak{t}^+ + \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let A, H be the analytic subgroups of G corresponding to $\mathfrak{a}$, $\mathfrak{h}$ respectively.

Now we define, analogously in Knapp-Okamoto [6]

$$u_t = \exp\left(\frac{\pi t}{4}\sum_{\alpha\in\Delta}(-\sqrt{-1})Y^0_\alpha\right)\in G^C \qquad \text{for} \quad 0\le t\le 1 .$$

We have the following lemma:

Lemma 4. *Let G/K be an irreducible hermitian symmetric space, not necessarily of tube type. Then we have the following decomposition of u_t:*

$$(17) \qquad u_t = \zeta_t k_t z_t \qquad \text{for} \quad 0\le t\le 1$$

where $\zeta_t=\exp\left(\tan\frac{\pi t}{4}\sum_{\alpha\in\Delta}E_{-\alpha}\right)\in P_-$, $k_t=\exp\left(\log\left(\cos\frac{\pi t}{4}\right)\sum_{\alpha\in\Delta}H'_\alpha\right)\in T^C$ *and* $z_t=\exp\left(-\tan\frac{\pi t}{4}\sum_{\alpha\in\Delta}E_\alpha\right)\in P_+$. *Moreover for* $0<t<1$,

$$(18) \qquad \zeta_t = a_s h_r \eta_t$$

where $a_s=\exp(sX^0)\in A\left(\tanh(s)=\tan\frac{\pi t}{4}\right)$, $h_r=\exp(r\sum_{\alpha\in\Delta}H'_\alpha)\in T^C\left(e^r=\frac{1}{\cosh(s)}\right)$ *and* $\eta_t=\exp(-\tanh(s)e^{-2r}\sum_{\alpha\in\Delta}E_\alpha)\in P_+$.

The proof follows from a straightforward calculation in $SL(2, \boldsymbol{C})$, analogously in Knapp-Okamoto [6].

Now it is well-known that

$$(19) \qquad Ad(u_1) = id \text{ on } \mathfrak{t}^+ \text{ and } Ad(u_1)(H'_\alpha) = X^0_\alpha , \qquad \alpha\in\Delta .$$

Hence we obtain $Ad(u_1)(\mathfrak{t}^C)=\mathfrak{h}^C$. $Ad(u_1)$ is called a *Cayley transform* (cf. Moore [10]). Let $\sum$ be the set of all non-zero roots of $(\mathfrak{g}^C, \mathfrak{h}^C)$. For $\lambda\in R$, we put ${}^tAd(u_1^{-1})\lambda(X)=\lambda(Ad(u_1^{-1})X)$, $X\in\mathfrak{h}$. Then ${}^tAd(u_1^{-1})\lambda$ belongs to $\sum$ if $\lambda\in R$; ${}^tAd(u_1^{-1})$ sends R onto $\sum$. We can define a linear order $>$ on $\sum$ such that the set of all positive roots in $\sum$ coincides with ${}^tAd(u_1^{-1})P$.

Let Π be a fundamental system of R with respect to the order $\succ$. Then under the assumption of *tube type*, it follows (Moore [10]) that

$$\pi(\Pi)-\{0\} = \left\{\frac{1}{2}(\gamma_2-\gamma_1), \cdots, \frac{1}{2}(\gamma_m-\gamma_{m-1}), \gamma_1\right\}$$

where for a linear form λ on $\mathfrak{t}^C$, $\pi(\lambda)$ means the restriction of λ to $(\mathfrak{t}^-)^C$. Therefore it follows immediately that the above linear order $>$ on $\sum$ is a σ-order. Then, as in §2, we can consider $\sum_0$, $\sum_\pm$ and F. The σ-invariantness of $\sum_+$ implies the following equality:

$$\sum_{\alpha' \in \Sigma_+} H_{\alpha'} = 2H_\rho, \tag{20}$$

where H_ρ is an element of $\mathfrak{a}$ defined by $\rho(H)=B(H, H_\rho)$ for all $H \in \mathfrak{a}$ and $H_{\alpha'}$ is an element of $\mathfrak{h}^C$ defined by $\alpha'(H)=B(H, H_{\alpha'})$ for all $H \in \mathfrak{h}$.

4. Construction of Hardly class (I)

We shall always assume that G/K is an irreducible hermitian symmetric space of tube type. We take $\{\alpha \in F: \alpha(X^0)=0\}$ as the subset E of F in §2. Then, under the notation in §2, $\mathfrak{a}(E)$ is spanned by X^0, and $M(E)$ is the centralizer of X^0 in K. Let 2δ be the sum of all roots in P. Then we obtain

$$\begin{aligned} \rho &= {}^t Ad(u_1^{-1})\delta \qquad \text{on } \mathfrak{a}, \\ \rho_E(X^0) &= \rho(X^0) = \delta(\sum_{\alpha \in \Delta} H'_\alpha). \end{aligned} \tag{21}$$

Let Λ be an integral linear form on $\mathfrak{t}^C$, dominant with respect to $\mathfrak{k}$, that is, Λ satisfies

(i) $\quad \Lambda(H) \in 2\pi\sqrt{-1}\, \boldsymbol{Z} \qquad$ for every $H \in \mathfrak{t}$, $\exp(H) = e$

(ii) $\quad \langle \Lambda, \alpha \rangle \geqq 0 \qquad$ for every $\alpha \in P_{\mathfrak{k}}$.

Let τ_Λ be the irreducible unitary representation of K with the highest weight Λ on the complex vector space V_Λ. Then τ_Λ is uniquely extended to a holomorphic representation of K^C. Since P_+ is a normal subgroup in the subgroup $K^C P_+$ of G^C, we can extend τ_Λ uniquely to a holomorphic representation of $K^C P_+$ which is trivial on P_+. We denote by the same notation τ_Λ this extended representation. Let $\tau=\tau_\Lambda^*$ be the representation contragredient to τ_Λ on the dual space V_Λ^* of V_Λ. Let $\tilde{E}_\Lambda$ be the vector bundle over $G^C/K^C P_+$ associated to the representation τ of $K^C P_+$. We notice that $G \cap K^C P_+ = K$. Then, as is well-known, G/K can be identified with the open G-orbit of the origin in $G^C/K^C P_+$. We denote by E_Λ the restriction of $\tilde{E}_\Lambda$ to the open submanifold G/K of $G^C/K^C P_+$.

DEFINITION. Let $\Gamma(\Lambda)$ be the set of all C^∞ mappings f of $GK^C P_+$ into V_Λ^* satisfying

$$f(gb) = \tau(b^{-1})f(g), \quad g \in GK^C P_+, \quad b \in K^C P_+ \tag{22}$$

$$\|f\|_2^2 = \lim_{t \uparrow 1} \int_K \|f(ku_t)\|^2 dk < \infty \tag{23}$$

where $\|\cdot\|$ is the operator norm in V_Λ^* with respect to $\tau_\Lambda(K)$ invariant norm $\|\cdot\|$ in V_Λ. From Lemma 4, $f(ku_t)$ is well-defined. We remark that the space $\Gamma(\Lambda)$ can be regarded as a space of C^∞ sections of E_Λ. For an element $\phi \in L^2_{\tau,\lambda}(G/B(E))$, a Poisson integral $\mathcal{P}_{\tau,\lambda}\phi$ of ϕ can be considered as a C^∞ section of E_Λ since $\mathcal{P}_{\tau,\lambda}$ is an integral operator with the kernel $K_{\tau,\lambda}$. Moreover from the results in §2, we have the following theorem.

Theorem 1. *Let G/K be an irreducible hermitian symmetric space of tube type. Suppose that $\lambda = z\rho_E \in \mathfrak{a}_C^*$, $z = x+iy \in \boldsymbol{C}$, $y>0$ satisfies the following condition*:

$$\text{(C)} \qquad {}^tAd(u_1^{-1})\Lambda = -(i\lambda+\rho_E) \text{ on } \mathfrak{a}\,.$$

Then we have

$$\mathcal{P}_{\tau,\lambda} L^2_{\tau,\lambda}(G/B(E)) \subset \Gamma(\Lambda)\,.$$

Before proving the Theorem, we prepare the following Lemma.

Lemma 5. *Let G/K be an irreducible hermitian symmetric space of tube type. Under the above notation, for $a=\exp X$, $X \in Cl(\mathfrak{a}^+)$, we have*

$$(24) \qquad \|\tau(u_1^{-1}au_1)^{-1}v\| \leqq e^{\Lambda(Ad(u_1^{-1})X)}\|v\| \qquad \text{for all } \quad v \in V_\Lambda^*$$

where $\mathfrak{a}^+ = \{H \in \mathfrak{a};\ \alpha'(H)>0$ for all $\alpha' \in \sum_+\}$ and $Cl(\mathfrak{a}^+)$ is the closure of $\mathfrak{a}^+$in $\mathfrak{a}$. In particular, for $a_t = \exp tX^0$, we have

$$(25) \qquad \tau(u_1^{-1}a_tu_1)^{-1}v = e^{t\Lambda(\sum_{\alpha\in\Delta} H_\alpha')}v \qquad \text{for all } \quad v \in V_\Lambda^*$$

where $u_1^{-1}a_tu_1 = \exp(t\sum_{\alpha\in\Delta} H_\alpha')$.

Proof. From C. Moore [10],

$$Cl(\mathfrak{a}^+) = \{\sum_{i=1}^m a_i X_{\gamma_i}^0;\ 0 \leq a_1 \leq \cdots \leq a_m\}\,.$$

Let $a=\exp(\sum_{i=1}^m a_i X_{\gamma_i}^0)$, $0 \leq a_1 \leq \cdots \leq a_m$. Then by means of (19), we have $u_1^{-1}au_1 = \exp(\sum_{i=1}^m a_i H_{\gamma_i})$. On the other hand, all the weights of τ_Λ are of the form $\Lambda - \sum_{i=1}^p m_i\alpha_i$ when $D=\{\alpha_i\}_{i=1}^p$ is the set of all simple roots in $R_{\mathfrak{k}}$ with respect to the order $\mathfrak{S}$ in R and $m_i \geq 0$ are integers. Let $V_{\Lambda-\sum m_i\alpha_i}$ be the weight space for $\Lambda-\sum m_i\alpha_i$, and let $V^*_{\Lambda-\sum m_i\alpha_i}$ be the dual space of $V_{\Lambda-\sum m_i\alpha_i}$ which is identified with the subspace of all elements in V_Λ^* vanishing on the orthocomplement of $V_{\Lambda-\sum m_i\alpha_i}$ in V_Λ. Let $\{v^j_{m_1\cdots m_p};\ j=1, \cdots, \dim V_{\Lambda-\sum m_i\alpha_i}\}$ be an orthonomal base in $V_{\Lambda-\sum m_i\alpha_i}$, and let $\omega^j_{m_1\cdots m_p}$ be its dual base in $V^*_{\Lambda-\sum m_i\alpha_i}$. For $v \in V_\Lambda^*$, we put $v=\sum a^j_{m_1\cdots m_p}\omega^j_{m_1\cdots m_p}$, $a^j_{m_1\cdots m_p} \in \boldsymbol{C}$. Then we have

$$\begin{aligned} \tau(u_1^{-1}au_1)^{-1}v &= \sum a^j_{m_1\cdots m_p}\tau(u_1^{-1}au_1)^{-1}\omega^j_{m_1\cdots m_p} \\ &= \sum a^j_{m_1\cdots m_p} e^{(\Lambda-\sum m_i\alpha_i)(\sum_{k=1} a_k H_{\gamma_k})}\omega^j_{m_1\cdots m_p}\,. \end{aligned}$$

From C. Moore [10], the non-zero vectors in $\pi(D)$ are of the form

$$\frac{1}{2}(\gamma_{j+1}-\gamma_j), \qquad j=1, \cdots, m-1$$

if G/K is of tube type. Then we have

$$e^{(\Lambda-\sum m_i\alpha_i)(\sum_{k=1}^m a_k H'_{\gamma_k})} = e^{\Lambda(\sum_{k=1}^m a_k H'_{\gamma_k})} e^{-\sum_{k=1}^{m-1}\frac{d}{2}(a_{k+1}-a_k)n_k}$$
$$\leqq e^{\Lambda(\sum_{k=1}^m a_k H'_{\gamma_k})}$$

for some non-negative integers n_k $(k=1, \cdots, m-1)$ and the equality holds if $a_1=\cdots=a_m$. It follows that

$$||\tau(u_1^{-1}au_1)^{-1}v||^2 = \sum |a^j_{m_1\cdots m_p}|^2 ||\tau(u_1^{-1}au_1)^{-1}\omega^j_{m_1\cdots m_p}||^2$$
$$\leqq \left\{e^{\Lambda(\sum_{i=1}^m a_i H'_{\gamma_i})}\right\}^2 \sum |a^j_{m_1\cdots m_p}|^2$$
$$= \left\{e^{\Lambda(\sum_{i=1}^m a_i H'_{\gamma_i})}\right\}^2 ||v||^2 .$$

In partiqular, if $a=\exp tX^0$,

$$\tau(u_1^{-1}au_1)^{-1}v = e^{t\Lambda(\sum_{k=1}^m H'_{\gamma_k})}v . \qquad \text{Q.E.D.}$$

Proof of Theorem 1. For $\phi\in L^2_{\tau,\lambda}(G/B(E))$, from Lemma 4,

$$(26) \qquad \mathcal{P}_{\tau,\lambda}\phi(gu_t) = \tau(\exp(r\sum_{\alpha\in\Delta}H'_\alpha))^{-1}\tau(k_t)^{-1}\mathcal{P}_{\tau,\lambda}\phi(ga_s)$$
$$= e^{(r+\log(\cos\frac{\pi}{4}t))\Lambda(\sum_{\alpha\in\Delta}H'_\alpha)}\mathcal{P}_{\tau,\lambda}\phi(ga_s)$$
$$= e^{-(r+\log(\cos\frac{\pi}{4}t))(i\lambda+\rho_E)(X^0)}\mathcal{P}_{\tau,\lambda}\phi(ga_s)$$

under the notations in Lemma 4. We put $C=\lim_{t\to1} e^{\log(\cos\frac{\pi}{4}t)(i\lambda+\rho_E)(X^0)}<\infty$. We notice that $e^{-r(i\lambda+\rho_E)(X^0)}=(\cosh(s))^{(i\lambda+\rho_E)(sX^0)}\sim\frac{1}{2}e^{(i\lambda+\rho_E)(sX^0)}$ as $s\to\infty$. Hence we obtain

$$\lim_{t\uparrow1}\int_K ||\mathcal{P}_{\tau,\lambda}\phi(ku_t)||^2 dk = \frac{C}{2}\lim_{s\to\infty}|e^{(i\lambda+\rho_E)(sX^0)}|\int_K ||\mathcal{P}_{\tau,\lambda}\phi(ka_s)||dk$$
$$\leq \frac{C}{2}C_E(\lambda)||\phi||_{L^2(K/M(E))}<\infty .$$

from Remark of Proposition 2. Q.E.D.

Moreover, by considering the subspace $\Gamma_0(\Lambda)$ of $\Gamma(\Lambda)$ consisting of all elements of $\Gamma(\Lambda)$ which satisfies the following boundary conditions (iii), (iv), we construct a representation of G.

DEFINITION. Let $\Gamma_0(\Lambda)$ be the set of all $f\in\Gamma(\Lambda)$ satisfying

(iii) for every $g\in G$, there exists a limit $\lim_{t\uparrow 1} f(gu_t)$, say $f(gu_1)$, and the boundary value $f(gu_1)$ satisfies

$$f(gmanu_t)=e^{t\,Ad(u_1^{-1})\Lambda(\log a)}\tau(m^{-1})f(gu_1) \tag{27}$$

for $g\in G$, $m\in M$, $a\in A$ and $n\in N$ where M is the centralizer of $\mathfrak{a}$ in K.

(iv) $G\ni g\mapsto ||f(gu_1)||$ is continuous.

Then we can apply Theorem of bounded convergence to the sequence of functions $k\mapsto||f(ku_t)||$ $(0\leq t\leq 1)$ by means of the conditions (iii), (iv), and then it follows that

$$||f||_2^2=\lim_{t\uparrow 1}\int_K ||f(ku_t)||dk=\int_K ||f(ku_1)||dk \qquad \text{for } f\in\Gamma_0(\Lambda). \tag{28}$$

Let us define the action $U_\Lambda(g)$ of G on $\Gamma_0(\Lambda)$ by $U_\Lambda(g)f(x)=f(g^{-1}x)$. Let us consider the factor space of $\Gamma_0(\Lambda)$ by the subspace $\{f\in\Gamma_0(\Lambda);\ ||f||_2=0\}$, and let $\Gamma_2(\Lambda)$ be its completion with respect to the norm induced from the norm $||\cdot||_2$. Then we have the following Proposition.

Proposition 3. *Let us preserve the assumption in Theorem* 1. *Then* $\Gamma_0(\Lambda)$ *is stable under* $U_\Lambda(g)$ *and* $U_\Lambda(g)$ *acts by a bounded operator on it with respect to the norm* $||\cdot||_2$. *Moreover* $U_\Lambda(g)$ *acts on* $\Gamma_2(\Lambda)$ *by a bounded representation of* G.

Proof. For $g\in G$,

$$\int_K ||f(g^{-1}ku_1)||^2 dk \leqq \sup_{k\in K} |e^{t\,Ad(u_1^{-1})\Lambda(H(g^{-1}k))}|^2 \int_K ||f(\kappa(g^{-1}k)u_1||^2 dk .$$

The function $k\rightarrow||f(\kappa(g^{-1}k)u_1)||^2$ is a right M-invariant because of $\kappa(g^{-1}km)M=\kappa(g^{-1}k)M$ in K/M and the condition (27). Put $h=\kappa(g^{-1}k)$. Then it follows from (4′) that

$$k=\kappa(gh)\,,\ H(g^{-1}k)=-H(gh) \quad\text{and}\quad dk_M=e^{-2\rho(H(gh))}dh_M.$$

Therefore $\int_K ||f(\kappa(g^{-1}k)u_1)||^2 dk\leqq \sup_{h\in K} e^{-2\rho(H(gh))}\int_K ||f(hu_1)||^2 dh$. Hence $\Gamma_0(\Lambda)$ is stable under $U_\Lambda(g)$ and $U_\Lambda(g)$ acts by a bounded operator on it with respect to the norm $||\ ||_2$.

For the proof of the last statement, let $L^2_\lambda(G/MAN)$ be the set of all measurable mappings ϕ of G into $\boldsymbol{C}$ satisfying $\phi(\text{gman})=e^{(-\nu+1)\rho_E(\log a)}\phi(g)$ and $||\phi||_2^2=\int_K|\phi(k)|^2 dk$ is finite. Then G acts on $L^2_\lambda(G/MAN)$ by $U_\lambda(g)\phi(x)=\phi(g^{-1}x)$. Then $U_\lambda(g)$ is a bounded operator on $L^2_\lambda(G/MAN)$ with respect to the above norm $\|\ \|_2$. Now we define the linear map $\mathcal{L}$ of $\Gamma_0(\Lambda)$ into $L^2_\lambda(G/MAN)$ by $(\mathcal{L}f)(g)=f(gu_1)$ for $f\in\Gamma_0(\Lambda)$. Then $\mathcal{L}$ is a G-equivariant isometry of $\Gamma_0(\Lambda)$ into $L^2_\lambda(G/MAN)$, that is, $\mathcal{L}U_\Lambda(g)=U_\lambda(g)\mathcal{L}$ and $\|\mathcal{L}f=f\|_2$, i.e. $\int_K \|f(ku_2)\|dk$ $=\lim_{t\to 1}\int_K \|f(ku_t)\|^2dk$, for $f\in\Gamma_0(\Lambda)$ because of (28). Therefore $U_\Lambda(g)$ can be extended to a bounded operator on $\Gamma_2(\Lambda)$. Q.E.D.

Summing up the above results, we have the following theorem as a Corallary of Theorem 1.

Theorem 2. *Let G/K be an irreducible hermitian symmetric space of tube type. Suppose that $\lambda=z\rho_E$, $z=x+iy$, $y>0$ and Λ satisfy the condition (C). Then $\mathcal{P}_{\tau,\lambda}$ is a G-equivariant bounded operator from $L^2_{\tau,\lambda}(G/B(E))$ into $\Gamma_2(\Lambda)$, that is,*

$$(29)\qquad U_\Lambda(g)\circ\mathcal{P}_{\tau,\lambda}=\mathcal{P}_{\tau,\lambda}\circ U_{\tau,\lambda}(g)\qquad \text{on}\quad L^2_{\tau,\lambda}(G/B(E))\,.$$

Proof. The boundedness of $\mathcal{P}_{\tau,\lambda}$ has been proved in Theorem 1 and, by the definition of Poisson integrals, we have the G-equivariantness (29) of $\mathcal{P}_{\tau,\lambda}$. Since $C_{\tau,\lambda}(G/B(E))$ is dense in $L^2(G/B(E))$, it suffices to prove that $\mathcal{P}_{\tau,\lambda}C_{\tau,\lambda}(G/B(E))\subset\Gamma_0(\Lambda)$.

For $\phi\in C_{\tau,\lambda}(G/B(E))$, we have

$$(26)\qquad \mathcal{P}_{\tau,\lambda}\phi(gu_t)=e^{-(r+\log(\cos\frac{\pi}{4}t))(i\lambda+\rho_E)(X^0)}\mathcal{P}_{\tau,\lambda}\phi(ga_s)\,.$$

Then, from Proposition 1, we obtain

$$\begin{aligned}\lim_{t\to1}\mathcal{P}_{\tau,\lambda}\phi(gu_t)&=\frac{C}{2}\lim_{s\to\infty}e^{(i\lambda+\rho_E)(sX^0)}\mathcal{P}_{\tau,\lambda}\phi(ga_s)\\&=\frac{C}{2}\int_{\bar N(E)}e^{(i\lambda-\rho_E)(H(\bar n))}\tau(\kappa(\bar n))\phi(g)d\bar n\,,\end{aligned}$$

that is, $\mathcal{P}_{\tau,\lambda}\phi(gu_1)=\frac{C}{2}\int_{\bar N(E)}e^{(i\lambda-\rho_E)(H(\bar n))}\tau(\kappa(\bar n))\phi(g)d\bar n$. From the condition (1), we have, for $m\in M$, $a\in A$, $n\in N$,

$$\mathcal{P}_{\tau,\lambda}\phi(gmanu_1)=\frac{C}{2}e^{-(i\lambda+\rho_E)(\log a)}\int_{\bar N(E)}e^{(i\lambda-\rho_E)(H(n))}\tau(\kappa(\bar n)m^{-1})\phi(g)d\bar n\,.$$

Here $\kappa(\bar n)m^{-1}=m^{-1}\kappa(m\bar nm^{-1})$ for $m\in M$. We put $\bar n'=m\bar nm^{-1}$, then $H(\bar n')=H(\bar n)$ and $d\bar n'=d\bar n$. Therefore we have $\mathcal{P}_{\tau,\lambda}\phi(gmanu_1)=\frac{C}{2}e^{-(i\lambda+\rho_E)(\log a)}\tau(m^{-1})\mathcal{P}_{\tau,\lambda}\times\phi(gu_1)$. It follows from the assumption (C) that the condition (27) is satisfied. Q.E.D.

5. Construction of Hardy class (II)

We preserve the notation and the assumption in §4. Let $C^\infty(G, V_\Lambda^*)$ be the set of all C^∞ mappings of G into V_Λ^*. Let ν be the left regular representation of G on $C^\infty(G, V_\Lambda^*)$. We define a representation ν of $\mathfrak{g}^C$ on $C^\infty(G, V_\Lambda^*)$ by

$$\nu(X)f(g) = \left[\frac{d}{dt} f(\exp(-tx)g)\right]_{t=0}$$

for $g \in G, f \in C^\infty(G, V_\Lambda^*)$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}^C$. Then ν defines a representation ν of $U(\mathfrak{g})$ on $C^\infty(G, V_\Lambda^*)$. Let $\nu(C)$ be the Casimir operator of ν with respect to the Killing form B on $C^\infty(G, V_\Lambda^*)$.

We put $C^\infty_{\tau,\lambda}(G/B(E)) = C_{\tau,\lambda}(G/B(E)) \cap C^\infty(G, V_\Lambda^*)$. Then the representations $(\Gamma_0(\Lambda), U_\Lambda)$ and $(C^\infty_{\tau,\lambda}(G/B(E)), U_{\tau,\lambda})$ are subrepresentations of the left regular representation of $(C^\infty(G, V_\Lambda^*), \nu)$ of G.

DEFINITION. Let $H_0(\Lambda)$ be the set of elements f in $\Gamma_0(\Lambda)$ satisfying

$$(\nu(C) - \langle \Lambda + 2\delta, \Lambda \rangle) f = 0 . \tag{30}$$

Let us consider the factor space of $H_0(\Lambda)$ by the subspace $\{f \in H_0(\Lambda);\ \|f\|_2 = 0\}$ and let $H_2(\Lambda)$ be its completion with respect to the norm $\|\cdot\|_2$. Then, for $g \in G$, $U_\Lambda(g)$ acts on $H_2(\Lambda)$ as a bounded operator with respect to this norm. $H_2(\Lambda)$ is called the *Hardy class* of the vector bundle E_Λ over G/K.

Now we can write Λ and δ as $\Lambda = \Lambda_+ + \Lambda_-$, $\delta = \delta_+ + \delta_-$ according to (16). Let M_0 be the connected component of the centralizer M of $\mathfrak{a}$ in K. Then $\mathfrak{t}^+$ is a Cartan subalgebra of the Lie algebra of M, M_0 and Λ_+ satisfies the following conditions:

(i) $\Lambda_+(H) = \Lambda(H) \in 2\pi\sqrt{-1}\,\boldsymbol{Z}$ for all $H \in \mathfrak{t}_+ \subset \mathfrak{t}$, $\exp H = e$

(ii) $\langle \Lambda_+, \alpha \rangle \geq 0$ for all $\alpha \in P_{\mathfrak{t}}$ such that $\pi(\alpha) = 0$.

Hence there exists an irreducible unitary representation π_{Λ_+} of M_0 with the highest weight Λ_+ on a representation space V_{Λ_+}. We define the projection operator e_{Λ_+} of $C^\infty_{\tau,\lambda}(G/B(E))$ as follows:

$$e_{\Lambda_+}\phi(g) = d_{\Lambda_+} \int_{M_0} \bar{\theta}_{\Lambda_+}(m)\phi(gm)dm \qquad \text{for} \quad \phi \in C^\infty_{\tau,\lambda}(G/B(E))$$

where $d_{\Lambda_+} = \dim V_{\Lambda_+}$, θ_{Λ_+} the character of τ_{Λ_+} and $\bar{\theta}_{\Lambda_+}(m)$ is the complex conjugate of $\theta_{\Lambda_+}(m)$.

Then $e_{\Lambda_+} C^\infty_{\tau,\lambda}(G/B(E))$ is a G-invariant subspace of $C^\infty_{\tau,\lambda}(G/B(E))$. Moreover we have the following theorem.

Theorem 3. *Under the assumption of theorem 2, we have*

$$\mathcal{P}_{\tau,\lambda}\, e_{\Lambda_+} C^\infty_{\tau,\lambda}(G/B(E)) \subset H_2(\Lambda) .$$

Proof. We will prove that $\nu(C)\mathcal{P}_{\tau,\lambda}e_{\Lambda_+}\phi=\langle\Lambda+2\delta,\Lambda\rangle\mathcal{P}_{\tau,\lambda}e_{\Lambda_+}\phi$ for $\phi\in C_{\tau,\lambda}(G/B(E))$. Since $U_\Lambda(g)\circ\mathcal{P}_{\tau,\lambda}=\mathcal{P}_{\tau,\lambda}U_{\tau,\lambda}(g)$, it suffices to prove that

$$\nu(C)e_{\Lambda_+}\phi=\langle\Lambda+2\delta,\Lambda\rangle e_{\Lambda_+}\phi \qquad \text{for} \quad \phi\in C^\infty_{\tau,\lambda}(G/B(E)).$$

Now let $\tilde{\nu}$ be the right regular representation of G on $C^\infty(G, V^*_\Lambda)$. We define a representation $\tilde{\nu}$ of $\mathfrak{g}^C$ on $C^\infty(G, V^*_\Lambda)$ by

$$\tilde{\nu}(X)f(g)=\left[\frac{d}{dt}f(g\exp tX)\right]_{t=0}$$

for $g\in G$, $X\in\mathfrak{g}$ and $f\in C^\infty(G, V^*_\Lambda)$. $\tilde{\nu}$ defines a representation $\tilde{\nu}$ of $U(\mathfrak{g})$ on $C^\infty(G, V^*_\Lambda)$. Then it follows (Harish-Chandra [4]) that

$$\nu(C)\phi=\tilde{\nu}(C)\phi \qquad \text{for every} \quad \phi\in C^\infty(G, V^*_\Lambda).$$

So we will show that $\tilde{\nu}(C)e_{\Lambda_+}\phi=\langle\Lambda+2\delta,\Lambda\rangle e_{\Lambda_+}\phi$ for $\phi\in C^\infty_{\tau,\lambda}(G/B(E))$.

Following Harish-Chandra [3], let $\{X_{\alpha'}\}$ be the root vector for $\alpha'\in\sum$ such that $\tau X_{\alpha'}=-X_{-\alpha'}$ and $B(X_{\alpha'}, X_{-\alpha'})=1$, and let $H_{\alpha'}$ be an element of $\mathfrak{t}^C$ such that $B(H, H_{\alpha'})=\alpha'(H)$, for $H\in\mathfrak{h}$. Then $[X_{\alpha'}, X_{-\alpha'}]=H_{\alpha'}$. Let $\{H_i\}_{i=1}^l$ be a base of $\mathfrak{h}^C$ such that $H_1,\cdots,H_m$ is an orthonormal base of $\mathfrak{a}$ with respect to the Killing form B of $\mathfrak{g}^C$ and $H_{m+1},\cdots,H_l$ is that of $\sqrt{-1}\mathfrak{t}^+$ with respect to B. Then $\{H_1,\cdots,H_l, X_{\alpha'}, X_{-\alpha'}; \alpha'\in\sum, \alpha'>0\}$ is a base of $\mathfrak{g}^C$. Then we have

$$\begin{aligned}\tilde{\nu}(C)&=\sum_{i=1}^{l}\tilde{\nu}(H_i)^2+\sum_{\substack{\alpha'\in\Sigma\\ \alpha'>0}}(\tilde{\nu}(X_{\alpha'})\tilde{\nu}(X_{-\alpha'})+\tilde{\nu}(X_{-\alpha'})\tilde{\nu}(X_{\alpha'}))\\ &=D_1+D_2+D_2\end{aligned}$$

where
$$D_1=\sum_{i=m+1}^{l}\tilde{\nu}(H_i)^2+\sum_{\substack{\alpha'\in\Sigma_0\\ \alpha'>0}}(\tilde{\nu}(X_{\alpha'})\tilde{\nu}(X_{-\alpha'})+\tilde{\nu}(X_{-\alpha'})\tilde{\nu}(X_{\alpha'}))$$

$$D_2=\sum_{i=1}^{m}\tilde{\nu}(H_i)^2+\sum_{\alpha'\in\Sigma_+}\tilde{\nu}(H_{\alpha'})^2$$

and
$$D_3=2\sum_{\alpha'\in\Sigma_+}\tilde{\nu}(X_{-\alpha'})\tilde{\nu}(X_{\alpha'}).$$

Since $e_{\Lambda_+}\phi$ belongs to $C^\infty_{\tau,\lambda}(G/B(E))$, we have

$$(31) \qquad D_3e_{\Lambda_+}\phi=0$$

because of $e_{\Lambda_+}\phi(gn)=e_{\Lambda_+}\phi(g)$, $n\in N$.
We note (20) $\sum_{\alpha'\in\Sigma_+}H_{\alpha'}=2H_\rho\in\mathfrak{a}$. Then since we have

$$\phi(g\exp H)=e^{-(i\lambda+\rho_B)(H(g)+H)}\phi(\kappa(g))$$

for every $\phi\in C^\infty_{\tau,\lambda}(G/B(E))$, $H\in\mathfrak{a}$, it follows that

(32) $\quad D_2 e_{\Lambda_+}\phi = (\langle i\lambda+\rho_E, i\lambda+\rho_E\rangle - \langle i\lambda+\rho_E, 2\rho\rangle) e_{\Lambda_+}\phi\,.$

On the other hand, let $\tau_{\Lambda_+}(C_M)$ be the Casimir operator of the representation τ_{Λ_+} of M_0 with respect to the form B. Then we have

$$\tau_{\Lambda_+}(C_M) = \langle \Lambda_+ + 2\delta_+, \Lambda_+\rangle \boldsymbol{I}$$

where $\boldsymbol{I}$ is the identity operator on V_{Λ_+}. And we have (cf. Harish-Chandra [4])

$$\begin{aligned} D_1 \xi_{\Lambda_+}(m) &= \sum_i (v_i, \tau_{\Lambda_+}(m)\tau_{\Lambda_+}(C_M)v_i) \\ &= \langle \Lambda_+ + 2\delta_+, \Lambda_+\rangle \xi_{\Lambda_+}(m) \end{aligned}$$

where $\{v_i\}$ is an orthonormal basis of V_{Λ_+} with respect to the inner product $(\ ,\ )$ on V_{Λ_+}. Then we have (cf. Harish-Chandra [4])

$$(33) \qquad \begin{aligned} D_1 e_{\Lambda_+}\phi(g) &= d_{\Lambda_+}\int_{M_0} \overline{\xi}_{\Lambda_+}(m)(D_1\phi)(gm)dm \\ &= d_{\Lambda_+}\int_{M_0} D_1\overline{\xi}_{\Lambda_+}(m)\phi(gm)dm \\ &= \langle \Lambda_+ + 2\delta_+, \Lambda_+\rangle e_{\Lambda_+}\phi(g)\,. \end{aligned}$$

Hence together with (31), (32), (33), we have

(34) $\quad \mathfrak{D}(C) e_{\Lambda_+}\phi = \{\langle i\lambda+\rho_E, i\lambda+\rho_E\rangle - \langle i\lambda+\rho_E, 2\rho\rangle + \langle \Lambda_+ + 2\rho_+, \Lambda_+\rangle\} e_{\Lambda_+}\phi.$

Since we have ${}^tAd(u_1^{-1})\Lambda_- = -(i\lambda+\rho_E)$ and (21) $\rho = {}^tAd(u_1^{-1})\delta$ on $\mathfrak{a}$, it follows that

$$\begin{aligned} (34) &= \{\langle \Lambda_-, \Lambda_-\rangle + \langle \Lambda_-, 2\delta\rangle + \langle \Lambda_+ + 2\delta_+, \Lambda_+\rangle\} e_{\Lambda_+}\phi \\ &= \langle \Lambda + 2\delta, \Lambda\rangle e_{\Lambda_+}\phi\,. \end{aligned}$$

Q.E.D.

EXAMPLE. Let $G=SU(1,1)$, $K=T=S(U(1)\times U(1))=\left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix} : \theta\in\boldsymbol{R}\right\}$, and so G/K is the unit disc D. Then $G^C=SL(2,\boldsymbol{C})$, $K^C=T^C=\left\{\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1}\end{pmatrix} : \gamma\in\boldsymbol{C}-(0)\right\}$. Then $\mathfrak{g}=\mathfrak{su}(1,1)$, $\mathfrak{k}=\mathfrak{t}=\left\{\begin{pmatrix} i\theta & 0 \\ 0 & -i\theta\end{pmatrix} : \theta\in\boldsymbol{R}\right\}$, $\mathfrak{g}^C=\mathfrak{sl}(2,\boldsymbol{C})$, $\mathfrak{k}^C=\mathfrak{t}^C=\left\{\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha\end{pmatrix} ; \alpha\in\boldsymbol{C}\right\}$ and the set R of roots of $(\mathfrak{g}^C, \mathfrak{t}^C)$ is given by

$$R = \{\pm\gamma\}\,, \quad \text{where} \quad \gamma\colon \mathfrak{t}^C \ni \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha\end{pmatrix} \mapsto -2\alpha\,.$$

A linear order $\succ$ on R is defined as $\gamma \succ 0$. Let $E_\gamma=\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}$, $E_{-\gamma}=\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$. We have

$$X_\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad Y_\gamma^0=-\sqrt{-1}\begin{pmatrix}0&-1\\1&0\end{pmatrix},$$

$$u_t=\exp\left(-\frac{\pi t}{4}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right)=\begin{pmatrix}\cos\frac{\pi t}{4}&\sin\frac{\pi t}{4}\\-\sin\frac{\pi t}{4}&\cos\frac{\pi t}{4}\end{pmatrix},$$

$$\mathfrak{t}^-=\mathfrak{t}\,,\ \mathfrak{t}^+=(0)\,,\quad M=M(E)=\left\{\begin{pmatrix}\varepsilon&0\\0&\varepsilon\end{pmatrix};\ \varepsilon=\pm 1\right\},$$

$$\delta:\mathfrak{t}\ni\begin{pmatrix}i\theta&0\\0&-i\theta\end{pmatrix}\mapsto -i\theta\quad\text{and}\quad \rho:\mathfrak{a}\ni\begin{pmatrix}0&t\\t&0\end{pmatrix}\mapsto t\,.$$

Let $\Lambda=-n\delta$, $n\in\boldsymbol{Z}$. Then we obtain a holomorphic representation $\tau=\tau_\Lambda^*$ of K^CP_+ given by

$$K^CP_+\ni\begin{pmatrix}\gamma&0\\\alpha&\gamma^{-1}\end{pmatrix}\mapsto\gamma^{-n}\in\boldsymbol{C}-(0)\,.$$

Now our conditions "$\mathrm{Re}\langle i\lambda,\alpha\rangle<0$, $\alpha=2\rho$ and ${}^tAd(u_1^{-1})\Lambda=-(i\lambda+\rho)$ on $\mathfrak{a}$" coincide with (cf. Okamoto [11])

$$i\lambda=(n-1)\rho\,,\qquad n<1\,,\ n\in\boldsymbol{Z}\,.$$

If $n=0$ i.e., $\Lambda=0$, then $i\lambda=-\rho$ and τ_Λ is the trivial representation of K. Then our Hardy class $H_2(\Lambda)$ is the usual Hardy class $H^2(D)$ given in the introduction.

NAGOYA UNIVERSITY

References

[1] N. Bourbaki: "Éléments de Mathématique: Intégration," Livre VI, Hermann, Paris, 1963.

[2] Harish-Chandra: *Representations of semi-simple Lie groups*: VI, Amer. J. Math. **78** (1956), 564–628.

[3] ————: *Spherical functions on a semi-simple Lie groups*: I, II, Amer. J. Math. **80** (1958), 241–310; 553–613.

[4] ————: *Discrete series for semisimple Lie groups* II, Acta Math. **116** (1966), 1–111.

[5] S. Helgason: *A duality for symmetric spaces with applications to group representattions*, Advances in Math. **5** (1970).

[6] A.W. Knapp and K. Okamoto: *Limits of holomorphic discrete series*, J. Functional Analysis **9** (1972), 375–409.

[7] A. Korányi: *The Poisson integrals for generalized half-planes and bounded symmetric domains*, Ann. of Math. **82** (1965), 332–350.

[8] ———————: *Boundary behavior of Poisson integrals on symmetric spaces*, Trans. Amer. Math. Soc. **140** (1969), 393–409.

[9] ——————— and J.A. Wolf: *Realization of hermitian symmetric spaces as generalized half-planes*, Ann. of Math. (2) **81** (1965), 265–288.

[10] C.C. Moore: *Compactifications of symmetric spaces*: II, The Cartan domains, Amer. J. Math. **86** (1964), 358–378.

[11] K. Okamoto: *Harmonic analysis on homogeneous vector bundles*, Springer, Lecture Note in Math. 266 (1971), 255–271.

[12] I. Satake: *On representations and compactifications of symmetric Riemannian spaces*, Ann. of Math. **71** (1960),

[13] G. Warner: Harmonic Analysis on Semi-simple Lie Groups, I, Springer, 1972.

[14] A. Zygmund: Trigonometric Series, 2nd. ed. Cambridge Univ. Press, New York, 1959.