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EQUIVARIANT DESUSPENSION OF G-MAPS

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1. Introduction

In this paper we will give sufficient conditions for a G-map to desuspend equivariantly. Throughout this paper G always denotes a compact Lie group.

For a G-space M let M^Σ be the unreduced suspension defined to be the quotient space of $M\times[0,1]$ in which $M\times\{0\}$ is collapsed to one point (called the south pole) and $M\times\{1\}$ is collapsed to another point (called the north pole). Giving the trivial G-action on $[0,1]$, a G-action on M^Σ is naturally induced. The unreduced suspension $f^\Sigma: M^\Sigma\to N^\Sigma$ of a G-map $f: M\to N$ is also a G-map.

If H is a closed subgroup of G, then (H) and $N(H)$ denote the conjugacy class and the normalizer of H in G, respectively. For a point x of a G-space M, G_x denotes the isotropy subgroup of G at x. The conjugacy class of an isotropy subgroup is called an isotropy type on M. Define $\mathcal{J}(M)$ to be the set of all isotropy types on M. Define

$$M^H=\{x\in M\,|\,H\subset G_x\}\ .$$

If M is a smooth G-manifold, then M^H is an $N(H)$-invariant submanifold of M, which possibly has various dimensional components. Define dim M^H to be the maximum of those dimensions.

The main result of this paper is:

Theorem. *Let M be a compact, smooth G-manifold, and N a G-space. Let $f: M^\Sigma\to N^\Sigma$ be a G-map such that $f(z_\varepsilon)=z'_\varepsilon$ for $\varepsilon=0,1$, where z_0 and z_1 are the south pole and the north pole of M^Σ respectively, and z'_0 and z'_1 are those of N^Σ. Suppose that for all $(H)\in\mathcal{J}(M)$ there are non-negative integers n_H satisfying the following conditions:*

(i) $\dim M^H-\dim N(H)/H\leq n_H+1$,

(ii) *N^H is n_H-connected, and*

(iii) *if $n_H=0$, $\pi_1(N^H)$ is abelian.*

Then f is G-homotopic to h^Σ relative to $\{z_0, z_1\}$ for some G-map $h: M\to N$.

$S(V)$ denotes the unit sphere in an orthogonal representation V of G. $\boldsymbol{R}$ denotes the trivial one-dimensional representation of G. Then $S(V\oplus\boldsymbol{R})$ may be equivariantly identified with $S(V)^{\Sigma}$. So we obtain:

Corollary. *Let U and V be orthogonal representations of G. Let f: $S(U\oplus \boldsymbol{R})\to S(V\oplus \boldsymbol{R})$ be a G-map such that $f(z_\varepsilon)=z'_\varepsilon$ for $\varepsilon=0,1$. Suppose that*

$$2\leq \dim V^H, \textit{ and } \dim U^H-\dim N(H)/H\leq \dim V^H$$

for any $(H)\in\mathcal{J}(S(U))$. Then f is G-homotopic to h^{Σ} relative to $\{z_0, z_1\}$ for some G-map h: $S(U)\to S(V)$.

REMARKS. Let M and N be as in the Theorem.

(1) Assume $N^G\neq\phi$. Then any G-map f: $M^{\Sigma}\to N^{\Sigma}$ is G-homotopic to a G-map f': $M^{\Sigma}\to N^{\Sigma}$ such that $f'(z_\varepsilon)=z'_\varepsilon$ for $\varepsilon=0,1$. Thus f is G-homotopic to h^{Σ} for some G-map h: $M\to N$.

(2) Consider the case in which the degree of a map from M to N is defined. Then the Theorem shows that the existence of a G-map f: $M^{\Sigma}\to N^{\Sigma}$ with $f(z_\varepsilon)=z'_\varepsilon$ implies the existence of a G-map h: $M\to N$ with $\deg h=\deg f$. This seems to be useful for the existence problem of G-maps with given degree.

2. Cochain groups based on a bundle of coefficients

Throughout this section M is a compact, smooth, free G-manifold, and N is a path connected, m-simple G-space, where $m=\dim M/G\geq 1$. Define

$$\begin{aligned}
&M^\sigma = M\times[0,1]\,,\\
&\tilde{M} = M/G\,,\\
&\tilde{M}^\sigma = M^\sigma/G = M/G\times[0,1]\,,\\
&E(M,N) = M\times_G N\,,\\
&E(M^\sigma,N^{\Sigma}) = M^\sigma\times_G N^{\Sigma} = (M\times_G N^{\Sigma})\times[0,1]\,.
\end{aligned}$$

Then we obtain the two fibre bundles

$$\begin{aligned}
&E(M,N)\to\tilde{M} \text{ with fibre } N, \text{ and}\\
&E(M^\sigma,N^{\Sigma})\to\tilde{M}^\sigma \text{ with fibre } N^{\Sigma}\,.
\end{aligned}$$

There is a bijective correspondence between the set of cross sections s: $\tilde{M}\to E(M, N)$ and the set of G-maps f: $M\to N$. The bijective correspondence is given by the equation

$$s([x]) = [x, f(x)]\in M\times_G N$$

for any $[x]\in\tilde{M}$. Similarly there is also a bijective correspondence between the set of cross sections $\tilde{M}^\sigma\to E(M^\sigma, N^{\Sigma})$ and the set of G-maps $M^\sigma\to N^{\Sigma}$. These correspondences will be used repeatedly in this paper.

Since N is m-simple, we obtain the bundle of coefficients associated with the bundle $E(M,N)$ by the m-th homotopy group, which is denoted by $\mathscr{B}(\pi_m)$. (See Steenrod [2; 30.2].) Since N is path connected, N^{Σ} is simply connected, and hence $(m+1)$-simple. So we also obtain the bundle of coefficients associated with the bundle $E(M^{\sigma},N^{\Sigma})$ by the $(m+1)$-th homotopy group, which is denoted by $\bar{\mathscr{B}}(\pi_{m+1})$.

Since $\tilde{M}$ is a smooth manifold, $\tilde{M}$ is triangulable. So $\tilde{M}$ admits a cell structure in the sense of Steenrod [2; 19.1]. We fix one of cell structures on $\tilde{M}$, and give a cell structure on $\tilde{M}^{\sigma}=\tilde{M}\times[0,1]$ as in [2; 19.1]. Then we obtain the cochain groups $C^k(\tilde{M}; \mathscr{B}(\pi_m))$ and $C^k(\tilde{M}^{\sigma}; \bar{\mathscr{B}}(\pi_{m+1}))$, where the former is the group of k-cochains of $\tilde{M}$ with coefficients in $\mathscr{B}(\pi_m)$, and the latter is the group of k-cochains of $\tilde{M}^{\sigma}$ with coefficients in $\bar{\mathscr{B}}(\pi_{m+1})$. (See [2; 31.2].)

Let s, t: $\tilde{M}\to E(M,N)$ be two cross sections, and let

$$K: \tilde{M}^{m-1}\times[0, 1]\to E(M, N)\,|\,\tilde{M}^{m-1}$$

be a homotopy of cross section such that

$$K_0=s\,|\,\tilde{M}^{m-1}, \text{ and } K_1=t\,|\,\tilde{M}^{m-1},$$

where $\tilde{M}^{m-1}$ is the $(m-1)$-skeleton of $\tilde{M}$, and K_i is the i-level of K. Then we may define the deformation m-cochain $d(s,K,t)\in C^m(\tilde{M}; \mathscr{B}(\pi_m))$. (See [2; 33.4].) If s coincides with t on $\tilde{M}^{m-1}$ and K is the constant homotopy, we abbreviate $d(s,K,t)$ by $d(s,t)$.

Let f: $M\to N$ be the G-map corresponding to s, and let $\bar{s}$: $\tilde{M}^{\sigma}\to E(M^{\sigma},N^{\Sigma})$ be the cross section corresponding to the G-map

$$p\circ f^{\sigma}: M^{\sigma}\to N^{\sigma}\to N^{\Sigma},$$

where $f^{\sigma}=f\times id$: $M^{\sigma}\to N^{\sigma}$ and p: $N^{\sigma}\to N^{\Sigma}$ is the projection. Then $\bar{s}$ satisfies

$$\bar{s}([x, r])=[(x, r), p(f(x), r)]\in M^{\sigma}\times_G N^{\Sigma}$$

for $[x,r]\in\tilde{M}^{\sigma}$ ($x\in M, r\in[0,1]$). Similarly we may define the cross section $\bar{t}$: $\tilde{M}^{\sigma}\to E(M^{\sigma}, N^{\Sigma})$.

Define $L=\pi^{-1}(\tilde{M}^{m-1})$, where π: $M\to\tilde{M}$ is the projection. Let F: $L\times[0,1]\to N$ be the G-homotopy corresponding to K. Consider the G-invariant subspace $L^{\sigma}\cup M\times\{0,1\}$ of M^{σ}, and define a G-homotopy

$$F': (L^{\sigma}\cup M\times\{0, 1\})\times[0, 1]\to N^{\Sigma}$$

by

$$F'\,|\,L^{\sigma}\times[0, 1]=p\circ F^{\sigma}, \text{ and}$$
$$F'(M\times\{\varepsilon\}\times[0, 1])=z'_{\varepsilon} \text{ for } \varepsilon=0, 1\,.$$

Note

$$\pi^{\sigma}(L^{\sigma}\cup M\times\{0,1\})=(\tilde{M}^{m-1})^{\sigma}\cup\tilde{M}\times\{0,1\}=(\tilde{M}^{\sigma})^{m}.$$

Let

$$\bar{K}\colon(\tilde{M}^{\sigma})^{m}\times[0,1]\to E(M^{\sigma},N^{\Sigma})\,|\,(\tilde{M}^{\sigma})^{m}$$

be the homotopy corresponding to F'. Then

$$\bar{K}_0=\bar{s}\,|\,(\tilde{M}^{\sigma})^{m},\ \text{ and }\ \bar{K}_1=\bar{t}\,|\,(\tilde{M}^{\sigma})^{m}.$$

So we may define the deformation $(m+1)$-cochain

$$d(\bar{s},\bar{K},\bar{t})\in C^{m+1}(\tilde{M}^{\sigma};\bar{\mathscr{B}}(\pi_{m+1})).$$

Then

Lemma 1. *There is a homomorphism*

$$\Phi\colon C^{m}(\tilde{M};\mathscr{B}(\pi_m))\to C^{m+1}(\tilde{M}^{\sigma};\bar{\mathscr{B}}(\pi_{m+1}))$$

such that $\Phi(d(s,K,t))=d(\bar{s},\bar{K},\bar{t})$. *Moreover, if N is n-connected and $m\leq 2n$, then Φ is an isomorphism, and if N is n-connected and $m=2n+1$, then Φ is an epimorphism.*

Proof. The suspension homomorphism $\pi_m(N)\to\pi_{m+1}(N^{\Sigma})$ is an isomorphism if $m\leq 2n$, and is an epimorphism if $m=2n+1$. There is a bijective correspondence between the m-cells of $\tilde{M}$ and the $(m+1)$-cells of $\tilde{M}^{\sigma}$. This lemma follows from the above two facts. Q.E.D.

3. Homotopy extension lemma (Free case)

In this section we prove the following lemma:

Lemma 2. *Let M be a compact, smooth, free G-manifold (with or without boundary), and N a G-space. Let $f\colon M^{\Sigma}\to N^{\Sigma}$ be a G-map such that $f(z_{\varepsilon})=z'_{\varepsilon}$ for $\varepsilon=0,1$. If $\partial M\neq\phi$, let $K\colon(\partial M)^{\Sigma}\times[0,1]\to N^{\Sigma}$ be a G-homotopy such that*

(i) $K(\{z_{\varepsilon}\}\times[0,1])=z'_{\varepsilon}$ *for* $\varepsilon=0,1$,

(ii) $K_0=f\,|\,(\partial M)^{\Sigma}$, and

(iii) $K_1=g^{\Sigma}$ *for some G-map* $g\colon\partial M\to N$.

Suppose that there is a non-negative integer n satisfying the following conditions:

(i) $\dim M-\dim G\leq n+1$,

(ii) *N is n-connected, and*

(iii) *if $n=0$, $\pi_1(N)$ is abelian.*

Then there is a G-homotopy $L\colon M^{\Sigma}\times[0,1]\to N^{\Sigma}$ such that

(i) $L(\{z_{\varepsilon}\}\times[0,1])=z'_{\varepsilon}$ *for* $\varepsilon=0,1$,

(ii) *L is an extension of K,*

(iii) $L_0=f$, *and*

(iv) $L_1=h^\Sigma$ *for some G-map* $h: M\to N$.

Proof. Define

$$f' = f\circ p: M^\sigma \to M^\Sigma \to N^\Sigma, \text{ and}$$
$$K' = K\circ(p\times id): (\partial M)^\sigma\times[0,1] \to (\partial M)^\Sigma \times [0,1] \to N^\Sigma .$$

Let A be the G-invariant subspace $(\partial M)^\sigma\cup M\times\{0,1\}$ of M^σ. Define a G-homotopy $K'': A\times[0,1]\to N^\Sigma$ by

$$K''|(\partial M)^\sigma\times[0,1] = K', \text{ and}$$
$$K''(M\times\{\varepsilon\}\times[0,1]) = z'_\varepsilon \text{ for } \varepsilon=0,1 .$$

Let $s: \tilde{M}^\sigma\to E(M^\sigma,N^\Sigma)$ be the cross section corresponding to f', and let

$$P: \tilde{A}\times[0,1] \to E(A,N^\Sigma) = E(M^\sigma,N^\Sigma)|\tilde{A}$$

be the homotopy corresponding to K''. Then $P_0=s|\tilde{A}$, and $P_1|(\partial\tilde{M})^\sigma=\bar{t}$, where $\bar{t}$ is defined from $t: \partial\tilde{M}\to E(\partial M,N)$ as in section 2 and t is the cross section corresponding to g. t extends to a cross section $u: \tilde{M}\to E(M,N)$, since $\dim\tilde{M}\leq n+1$ and the fibre N of $E(M,N)$ is n-connected. Note that the $(n+1)$-skeleton $(\tilde{M}^\sigma)^{n+1}$ of $\tilde{M}^\sigma$ contains $\tilde{A}$. Since the fibre N^Σ of $E(M^\sigma,N^\Sigma)$ is $(n+1)$-connected, P extends to a homotopy of cross section

$$Q: (\tilde{M}^\sigma)^{n+1}\times[0,1] \to E(M^\sigma,N^\Sigma)|(\tilde{M}^\sigma)^{n+1},$$

such that $Q_0=s|(\tilde{M}^\sigma)^{n+1}$ and $Q_1=\bar{u}|(\tilde{M}^\sigma)^{n+1}$.

If $\dim\tilde{M}^\sigma\leq n+1$, then $\tilde{M}^\sigma=(\tilde{M}^\sigma)^{n+1}$, and Q corresponds to a G-homotopy $R: M^\sigma\times[0,1]\to N^\Sigma$ which satisfies

$$R(M\times\{\varepsilon\}\times[0,1]) = K''(M\times\{\varepsilon\}\times[0,1]) = z'_\varepsilon$$

for $\varepsilon=0,1$. Thus R induces the desired G-homotopy $L: M^\Sigma\times[0,1]\to N^\Sigma$.

Since $\dim\tilde{M}^\sigma\leq n+2$ by the assumption, it only remains to show the case $\dim\tilde{M}^\sigma=n+2$. Let $m=\dim\tilde{M}$, then $m=n+1$. In this case M and N satisfy the conditions in section 2. So we can apply Lemma 1. Let

$$d = d(s,Q,\bar{u})\in C^{m+1}(\tilde{M}^\sigma;\bar{\mathcal{B}}(\pi_{m+1})) .$$

Since Φ is epic, there is $d'\in C^m(\tilde{M};\mathcal{B}(\pi_m))$ with $\Phi(d')=d$. From [2; 33.9] there is a cross section $v: \tilde{M}\to E(M,N)$ such that u coincides with v on $\tilde{M}^{m-1}$ and $d(u,v)=-d'$. $\bar{u}$ coincides with $\bar{v}$ on $(\tilde{M}^\sigma)^m$. So

$$d(\bar{u},\bar{v})\in C^{m+1}(\tilde{M}^\sigma;\bar{\mathcal{B}}(\pi_{m+1}))$$

is defined. By Lemma 1,

$$d(\bar{u},\bar{v}) = \Phi(d(u,v)) = -d .$$

Define a homotopy

$$R\colon (\tilde{M}^\sigma)^m \times [0, 1] \to E(M^\sigma, N^\Sigma) \,|\, (\tilde{M}^\sigma)^m$$

by

$$R_i = Q_{2i} \quad \text{for} \quad 0 \le i \le 1/2, \ \text{and}$$
$$R_i = \bar{u} \,|\, (\tilde{M}^\sigma)^m = \bar{v} \,|\, (\tilde{M}^\sigma)^m \quad \text{for} \quad 1/2 \le i \le 1\,.$$

By [2; 33.7],

$$\begin{aligned} d(s, R, \bar{v}) &= d(s, Q, \bar{u}) + d(\bar{u}, \bar{v}) \\ &= d - d \\ &= 0\,. \end{aligned}$$

$d(s,Q,\bar{v}) = d(s,R,\bar{v})$ follows from the definition of deformation cochain. Hence $d(s,Q,\bar{v})=0$. By [2;33.8] Q extends to a homotopy of cross section,

$$S\colon \tilde{M}^\sigma \times [0, 1] \to E(M^\sigma, N^\Sigma)$$

such that $S_0 = s$ and $S_1 = \bar{v}$. S corresponds to a G-homotopy $T\colon M^\sigma \times [0,1] \to N^\Sigma$ which satisfies

$$T(M \times \{\varepsilon\} \times [0, 1]) = K''(M \times \{\varepsilon\} \times [0, 1]) = z'_\varepsilon$$

for $\varepsilon = 0, 1$. Thus T induces the desired G-homotopy $L\colon M^\Sigma \times [0,1] \to N^\Sigma$. Q.E.D.

4. Homotopy extension lemma (General case)

In this section we generalize Lemma 2 to a general smooth G-action on M as follows:

Lemma 3. *Let M be a compact, smooth G-manifold (with or without boundary), and N a G-space. Let $f\colon M^\Sigma \to N^\Sigma$ be a G-map such that $f(z_\varepsilon) = z'_\varepsilon$ for $\varepsilon = 0, 1$. If $\partial M \neq \phi$, let $K\colon (\partial M)^\Sigma \times [0,1] \to N^\Sigma$ be a G-homotopy such that*

(i) *$K(\{z_\varepsilon\} \times [0,1]) = z'_\varepsilon$ for $\varepsilon = 0, 1$,*

(ii) *$K_0 = f \,|\, (\partial M)^\Sigma$, and*

(iii) *$K_1 = g^\Sigma$ for some G-map $g\colon \partial M \to N$.*

Suppose that for all $(H) \in \mathcal{J}(M)$ there are non-negative integers n_H satisfying the following conditions:

(i) *$\dim M^H - \dim N(H)/H \le n_H + 1$,*

(ii) *N^H is n_H-connected, and*

(iii) *if $n_H = 0$, $\pi_1(N^H)$ is abelian.*

Then there is a G-homotopy $L\colon M^\Sigma \times [0,1] \to N^\Sigma$ such that

(i) *$L(\{z_\varepsilon\} \times [0,1]) = z'_\varepsilon$ for $\varepsilon = 0, 1$,*

(ii) *L is an extension of K,*

(iii) *$L_0 = f$, and*

(iv) $L_1 = h^{\Sigma}$ *for some G-map* $h: M \to N$.

Proof. We proceed by induction on $\#\mathcal{I}(M)$, the number of isotropy types on M.

First assume $\#\mathcal{I}(M)=1$. Let (H) be the isotropy type on M, then M^H is a compact, smooth, free $N(H)/H$-manifold. Since M^H and N^H are nonempty, it follows $(M^{\Sigma})^H = (M^H)^{\Sigma}$ and $(N^{\Sigma})^H = (N^H)^{\Sigma}$. So f induces an $N(H)/H$-map

$$f^H = f \mid (M^H)^{\Sigma}: (M^H)^{\Sigma} \to (N^H)^{\Sigma}.$$

Similarly K induces an $N(H)/H$-homotopy

$$K^H = K \mid (\partial M^H)^{\Sigma} \times [0,1]: (\partial M^H)^{\Sigma} \times [0,1] \to (N^H)^{\Sigma}.$$

Applying Lemma 2 to f^H and K^H, we obtain an $N(H)/H$-homotopy $P: (M^H)^{\Sigma} \times [0,1] \to (N^H)^{\Sigma}$ such that

(i) $P(\{z_\varepsilon\} \times [0,1]) = z'_\varepsilon$ for $\varepsilon = 0, 1$,

(ii) P is an extension of K^H,

(iii) $P_0 = f^H$, and

(iv) $P_1 = u^{\Sigma}$ for some $N(H)/H$-map $u: M^H \to N^H$.

Since $M = G(M^H)$, we may extend P to a G-homotopy $L: M^{\Sigma} \times [0,1] \to N^{\Sigma}$, and this is the desired G-homotopy.

Now assume that Lemma 3 is true for the case in which the number of isotropy types is equal to or less than a, and assume $\#\mathcal{I}(M) = a+1$. Let (H) be a maximal isotropy type on M. Then

$$M_{(H)} = \{x \in M \mid (G_x) = (H)\}$$

is a compact, smooth, G-invariant submanifold of M with $\partial M_{(H)} = M_{(H)} \cap \partial M$. By Rubinsztein [1; Lemma 1.1] there are compact, smooth, G-invariant submanifolds A, B of M such that

(1) $M = A \cup B$,

(2) $\partial A = A \cap B$, $\partial B = \partial A \cup \partial M$, $\partial A \cap \partial M = \phi$,

(3) $B \supset M_{(H)} \cup \partial M$, and

(4) B is a mapping cylinder of some G-map $\partial A \to M_{(H)} \cup \partial M$.

Since $\#\mathcal{I}(M_{(H)}) = 1$, there is a G-homotopy $E: (M_{(H)})^{\Sigma} \times [0,1] \to N^{\Sigma}$ such that

(i) $E(\{z_\varepsilon\} \times [0,1]) = z'_\varepsilon$ for $\varepsilon = 0, 1$,

(ii) E coincides with K on $(\partial M_{(H)})^{\Sigma} \times [0,1]$,

(iii) $E_0 = f \mid (M_{(H)})^{\Sigma}$, and

(iv) $E_1 = k^{\Sigma}$ for some G-map $k: M_{(H)} \to N$.

K and E give a G-homotopy on $(M_{(H)} \cup \partial M)^{\Sigma}$, and by (3), (4) this G-homotopy extends to a G-homotopy $F: B^{\Sigma} \times [0,1] \to N^{\Sigma}$ such that

(i) $F(\{z_\varepsilon\} \times [0,1]) = z'_\varepsilon$ for $\varepsilon = 0, 1$,

(ii) F is an extension of K,

(iii) $F_0=f|B$, and

(iv) $F_1=v^{\Sigma}$ for some G-map $v: B\to N$.

Since $\sharp\mathcal{J}(A)=a$, there is a G-homotopy $J: A^{\Sigma}\times[0,1]\to N^{\Sigma}$ such that

(i) J coincides with F on $(\partial A)^{\Sigma}\times[0,1]$,

(ii) $J_0=f|A$, and

(iii) $J_1=w^{\Sigma}$ for some G-map $w: A\to N$.

F and J give the desired G-homotopy on M^{Σ}. Q.E.D.

5. Proof of the Theorem

Let $f: M^{\Sigma}\to N^{\Sigma}$ be the G-map in the Theorem. Applying Lemma 3 to the G-map $f|(\partial M)^{\Sigma}: (\partial M)^{\Sigma}\to N^{\Sigma}$, we obtain a G-homotopy $K: (\partial M)^{\Sigma}\times[0,1]\to N^{\Sigma}$ such that

(i) $K(\{z_\varepsilon\}\times[0,1])=z'_\varepsilon$ for $\varepsilon=0,1$,

(ii) $K_0=f|(\partial M)^{\Sigma}$, and

(iii) $K_1=g^{\Sigma}$ for some G-map $g: \partial M\to N$.

Again applying Lemma 3 to f and K, we obtain a G-homotopy $L: M^{\Sigma}\times[0,1]\to N^{\Sigma}$ such that

(i) $L(\{z_\varepsilon\}\times[0,1])=z'_\varepsilon$ for $\varepsilon=0,1$,

(ii) $L_0=f$, and

(iii) $L_1=h^{\Sigma}$ for some G-map $h: M\to N$.

This shows that f is G-homotopic to h^{Σ} relative to $\{z_0, z_1\}$.

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