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# DOUBLY TRANSITIVE PERMUTATION REPRESENTATIONS OF THE FINITE PROJECTIVE SPECIAL LINEAR GROUPS $PSL(n, q)$

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## 1. Introduction

In this note we will determine all doubly transitive permutation representations of the projective special linear groups $PSL(n, q)$ over the finite field $\boldsymbol{F}_q$. Our main result (Theorem 1) asserts that these are all well known ones, namely

**Theorem 1.** *If the group $G=PSL(n, q)$ is reperesented as a faithful doubly transitive permutation group on a set $\Omega$, $|\Omega|=m$, then $(G, \Omega)$ is isomorphic with one of the members in the following list:*

I) *$G$ acts on the set $\Omega$ of points of the $(n-1)$-dimensional projective space over $\boldsymbol{F}_q$: $\mathcal{P}(n-1, q)$, $m=(q^n-1)/(q-1)$, via the natural action.*

II) *$G$ acts on the set $\Omega$ of hyperplanes of $\mathcal{P}(n-1, q)$ via the natural action, $m=(q^n-1)/(q-1)$.*

III) $G=PSL(2, 5)$ $(\cong A_5)$, $m=5$.

IV) $G=PSL(2, 7)$ $(\cong PSL(3, 2))$, $m=7$.

V) $G=PSL(2, 9)$ $(\cong A_6)$, $m=6$.

VI) $G=PSL(2, 11)$, $m=11$.

VII) $G=PSL(3, 2)$ $(\cong PSL(2, 7))$, $m=8$.

VIII) $G=PSL(4, 2)$ $(\cong A_8)$, $m=8$.

For $n=2$, Theorem 1 has been given by E. Galois, L. E. Dickson and others (*cf.* B. Huppert [4]). Furthermore, for $n=3$, or also for particular pairs of $(n, q)$ provided $n$, $q$ are small the result above might have been proved by making use of the classifications of the maximal subgroups due to H.H. Mitchell [7], R.E. Hartley [3] and others.

Recently N. Ito [5] classified all premutation representations of the group $PSL(n, q)$ whose degrees are prime numbers. On the other hand, T. Tsuzuku [10] has shown that, if a finite simple group of Lie type has a primitive permutation representation whose degree is relatively prime to the characteristic of the basic field, then the stabilizer of a point must be a maximal parablilc subgroup. (This was also obtained independently by J. Tits). Especially Tsuzuku has shown that, if $PSL(n, q)$ is repersented as a doubly transitive perumtation group whose degree is relatively prime to $q$, then this permutation group must be either the case (I) or (II) in Theorem 1.

Nevertheless, it seems to the author that Theorem 1 has not yet been given in such a general form as was stated above as Theorem 1.

The outline of the proof of Theorem 1 is as follows: to begin with, it is shown that if $n \geqslant 4$ and $q^{n-2} \nmid m$, then the case (I) or (II) must hold. The proof depends heavily on a theorem of F.C. Piper [8 and 9] which characterizes the group $PSL(n, q)$ from a geometric view point.

Next we show that $m-1$ is bounded by a fixed value depending only on $q$ and $n$, say $(q^n-1)(q^{n-1}-1)/(q-1)$. Then we determine irreducible characters $\varphi$ of $G=PSL(n, q)$ which satisfy the conditions

1) $\varphi(1) \leqslant (q^n-1)(q^{n-1}-1)/(q-1)$,
2) $q^{n-2} | (\varphi(1)+1)$,

$\varphi(1)$ being the degree of the character $\varphi$. There, we are deeply indebted to the well-known construction of irreducible characters of the group $GL(n, q)$ by J.A. Green [2].

Suppose now that $n \geqslant 4$ and $q^{n-2} | m$. Since $G$ is doubly transitive, $G$ must have an irreducible character $\varphi$ satisfying the above conditions (1) and (2). However we can easily show that, there exists no such irreducible character $\varphi$ for $n \geqslant 5$, and so there exists no such doubly transitive permutation representation of $G$. Finally we will make some further observations for $m \leqslant 4$, and complete the proof of Theorem 1.

Our method is rather unrefined, because of its heavy dependence on other papers (especially on [2], [8] and [9]). Thus it is far from self-containedness. Therefore it is desirable to give a simple proof of Theorem 1 without using the character theory of $GL(n, q)$.

We use the following notation: let $G$ be a permutation group on a set $\Omega$, and let, $\Delta \subset \Omega$ then $G_\Delta$ (resp. $G_{(\Delta)}$) denotes the pointwise (resp.setwise) stabilizer of $\Delta$. Moreover let $\Delta$ be invariant by $G$, then $G^\Delta$ denotes the constituent of $G$ on $\Delta$. Moreover let us set $G^{(\Delta)}=(G_{(\Delta)})^\Delta$.

## 2. A review of a theorem of Piper. Proof of Theorem 1 for the case $n \geqslant 4$ and $q^{n-2} \nmid m$

A projective space is defined as a system of points and lines (i.e., subsets of points) connected by axioms of incidence in the usual way (see, for example, O. Veblen and J.W. Young [11]).

We dentote by $\mathscr{P}(d, q)$ the $d$-dimensional projective space defined over a finite field $\boldsymbol{F}_q$ with $q$ elements, and denote by $\boldsymbol{P}$ (resp. $\boldsymbol{L}$) the set of points (resp. lines) in $\mathscr{P}(d, q)$.

A system $S$ of points $\boldsymbol{P}'$ and lines $\boldsymbol{L}'$ is said to be a subspace of $\mathcal{P}(d, q)$, if $\boldsymbol{P}' \subset \boldsymbol{P}$ and any line $l' \in \boldsymbol{L}'$ is contained in some line $l \in \boldsymbol{L}$, and if $\boldsymbol{P}'$ and $\boldsymbol{L}'$ themselves form a projective space. A subspace $S$ is said to be complete, if $l \in \boldsymbol{L}'$ implies $l \in \boldsymbol{L}$. Note that every complete subspace is a subspace of $\mathcal{P}(d, q)$ naturally induced from a linear subspace of the $(d+1)$-dimensional vector space over $\boldsymbol{F}_q$ defining $\mathcal{P}(d, q)$, and vice versa.

A collineation of $\mathcal{P}(d, q)$ is a permutation of the points which transforms every three collinear points onto three collinear points, and this is equivalent to say that a collineation is a permutation of the complete subspaces preserving their dimension and incidence.

A collineation $\sigma$ of $\mathcal{P}(d, q)$ is said to be an elation, if it fixes every point on a fixed hyperplane (called an axis of $\sigma$) and every hyperplane through a fixed point (called center of $\sigma$) lying on the hyperplane and fixes no other points or hpyerplanes. Let $\pi$ be a collineation group of $\mathcal{P}(d, q)$, and let there exist two elations in $\pi$ which have same axis and distinct centers, then the line joining the two centers is called an axis line for $\pi$.

In [8, 9] F.C. Piper proved the following theorem.

**Theorem of Piper.** *Let $\pi$ be a collineation group of $\mathcal{P}(d, q)$ such that* (i) *$\pi$ fixes no subspace of $\mathcal{P}(d, q)$,* (ii) *some hyperplane is the axis of elations in $\pi$ for more than one centers. Then either $\pi$ contains the little projective group $PSL(d+1, q)$, or $(d, q)=(2, 4)$ and $\pi \cong A_6$ or $S_6$.*

We will prove the following lemma which is a slight extension of Theorem of Piper.

**Lemma 1.** *Let a proper subgroup $\pi$ of $PSL(d+1, q)$ $(d \geqslant 3)$, regarded as a collineation group of $\mathcal{P}(d, q)$, fix no complete subspace of $\mathcal{P}(d, q)$ $(d \geqslant 3)$, and let some axis has more than one center, then $\pi$ fixes the subspace $S$ consisting of all the elation centers and the axis lines for $\pi$. Moreover, $S$ is a desarguesian projective space of dimension $d$ defined over $F_{q'}$ with $(q')^j = q$ for some $j \geqslant 2$.*

Proof. By examining the proof of the theorem of Piper in [8 and 9], we can easily see that $\pi$ fixes the subspace $S$ consisting of all the elation centers and the axis lines for $\pi$. Therefore we have only to prove the latter assertion that $S \cong \mathcal{P}(d, q')$ with $(q')^j = q$ for some $j \geqslant 2$. Since $\pi$ fixes no complete subspace, the complete subspace generated by $S$ in $\mathcal{P}(d, q)$ is $\mathcal{P}(d, q)$ itself. So we have $\dim S \geqslant d$, because there exist $d+1$ points of $S$ which are in general position in $\mathcal{P}(d, q)$ and these $d+1$ points are of course in general position in $S$. Thus $S$ is desarugesian, since $\dim S \geqslant d \geqslant 3$. Next we will show that $\dim S \leqslant d$. Let $H^{(1)}$ be an axis for $\pi$. Then $S \cap H^{(1)}$ is clearly a subspace of $S$, and moreover is a complete subspace, since every line in $S$ meets the complete subspace according to Lemma 3 in [8] and Remark 4 in [9]. (Note that the conclusion of Lemma 3 in [8] and Remark 4 in [9] are both valid under the assumption of our Lemma 1.) Thus we have $\dim S \leqslant \dim(S \cap H^{(1)})+1$. Now there exists an axis $H^{(2)}$ for $\pi$ such that $H^{(1)} \supsetneqq H^{(1)} \cap H^{(2)}$, according to an extension of Lemma 5 in [8]. (Note that the conclusion of Lemma 5 in [8] is valid for $\pi$ under the assumption of this lemma. Especially this is valid even if $q$ is even.) Thus $S \cap H^{(1)} \cap H^{(2)}$ is a complete subspace of $S \cap H^{(1)}$, and we have $\dim(S \cap H^{(1)}) \leqslant \dim(S \cap H^{(1)} \cap H^{(2)})+1$ by Lemma 3 in [8] and Remark 4 in [9], since every line in $S \cap H^{(1)}$ meets the complete subspace $S \cap H^{(1)} \cap H^{(2)}$. Thus, there exists inductively for $i=3, 4, \cdots, d-1$ an axis $H^{(i)}$ for $\pi$ such that $S \cap H^{(1)} \cap \cdots \cap H^{(i)}$ is a complete subspace of $S \cap H^{(1)} \cap \cdots \cap H^{(i-1)}$ by Lemma 5 in [8], and we have

$$\dim(S \cap H^{(1)} \cap \cdots \cap H^{(i-1)}) \leqslant \dim(S \cap H^{(1)} \cap \cdots \cap H^{(i)})+1$$

by Lemma 3 in [8] and Remark 4 in [9]. Clearly $\dim(S \cap H^{(1)} \cap \cdots \cap H^{(d-1)}) \leqslant 1$. Hence, we have $\dim S \leqslant d$, and so we have $\dim S = d$. Let $S \simeq \mathcal{P}(d, q')$. We have obviously from the existence of an elation, $q' \mid q$ $(q' \lneqq q)$. Now we can assume that $q'$ is not a prime. Let $l \in L$ be an axis line. Then $PSL(d+1, q)^{(l)}$ [1] is a subgroup of $PGL(2, q)$, the group of projective collineations of the projective line $l$, and so $\pi^{(l)}$ is a subgroup of $PGL(2, q)$. While $\pi^{(l \cap S)}$ is a subgroup of $PGL(2, q')$. By Result 1 in [8] together with Lemma 5 in [8] $\pi^{(l \cap S)}$ is transitive on $S \cap l$, and the classification of subgroups of $PGL(2, q')$ shows that either $\pi^{(l \cap S)} \supseteqq PSL(2, q')$ or $q=$even and $\pi^{(l \cap S)}$ is the dihedral group of order $2(q'+1)$ [2]. Since $|\pi^{(l \cap S)}|$ must divide $|PGL(2, q)|$, we have $(q')^j = q$ for some $j$, owing to the classification of subgroups of $PGL(2, q)$. Hence we completed the proof of Lemma 1.

**Lemma 2.** *Let $H$ be a subgroup of index $m$ of $G=PSL(n, q)$ with $n \geqslant 4$, and let $q^{n-2} \nmid m$. Then $H$ fixes some complete subspace of $\mathcal{P}(n-1, q)$.*

(This is a generalization of the result concerning $PSL(n, q)$ in [11]. The result of this lemma may have an independent interest.)

Proof. Let $x=\begin{pmatrix} 1 & a_2 \cdots a_n \\ & 1 \quad 0 \\ & \ddots \\ & 0 \quad 1 \end{pmatrix} \in GL(n, q)$ with some $a_i \neq 0$, then the collineation $\bar{x}$ of $\mathcal{P}(n-1, q)$ is an elation with the axis $H_{n-1}=\overline{\{(x_1, \cdots, x_n)}; x_i \in F_q, x_1=0\}$, and the center $\overline{(0, a_2, \cdots, a_n)}$. And the Sylow's theorem shows that $H$ contains two elations with the same axis and distinct centers. (Note that a Sylow $p$-subgroup of some conjugate of $H$ is contained in the group of upper triangular unipotent matrices (i.e., a Sylow $p$-subgroup of $G$) and the index of the Sylow

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1) See the notation at the end of Section 1.

2) Cf. D.G. Higman and J.E. McLaughlin, Rank 3 subgroups of finite symplectic and unitary groups, Lemma 1, page 179.

$p$-subgroup of the conjugate subgroup of $H$ in the upper triangular unipotent matrices is not divisible by $q^{n-2}$, and that the Sylow $p$-subgroup of the conjugate subgroup of $H$ (hence the conjugate subgroup of $H$) contains two such elations with the axis $H_{n-1}$.) Let us assume that $H$ fixes no complete subspace of $\mathcal{P}(d, q)$. Then, by Lemma 1, $H$ fixes the subspace $S$, and we have $|H|=|H_S|\cdot|H^S|$. But $H_S$ is not divisible by $p$, because the set of the fixed points by an element of order $p$ of $PSL(n, q)$ is contained in some hyperplane and $S$ is not contained in any hyperplane. While, since every element of $PSL(n, q)$ which fixes the subspace $S$ induces a collineation of $S$, (because, since $S$ is a subspace, any three collinear points in $S$ is transformed onto three collinear points) $H^S$ is regared as a subgroup of the full collineation group $P\Gamma L(n, q')$ of $S$. But clearly $|P\Gamma L(n, q')|$ is not divisible by $q'\cdot q'^{(n/2)(n-1)}$. Therefore index $m$ is divisible by $q^{n(n-1)/2}/(q'\cdot q'^{(n/2)(n-1)})\geqslant q^{n-2}$, but this is a contradication and the lemma is proved.

*Proof of Theorem* 1 *for the case* $n\geqslant 4$ *and* $q^{n-2}\nmid m$. Let $n\geqslant 4$ and $q^{n-2}\nmid m$. Then by Lemma 2, the stabilizer $H$ of a point of $\Omega$, must fix some complete subspace of $\mathcal{P}(n-1, q)$. Since $H$ is maximal in $G$, $H$ is the subgroup consisting of all elements of $G$ which fix an $r$-dimentional complete subspace of $\mathcal{P}(n-1, q)$, and it is well known that the number of orbits of $H$ on $\Omega$ (i.e., the rank of the permutation group $(G, \Omega)$) is equal to $\min\{2+r, n+1-r\}$. Especially this is equal to 2 if and only if $r=0$ or $r=n-1$, hence the assertion is proved.

## 3. A bound of the degree $m$

**Lemma 3.** *Let a finite group $G$ be doubly transitive on a set $\Omega$, $|\Omega|=m$, then for each non-identity element of $G$, there exist at least $m-1$ elements of $G$ which are conjugate to the element.*

(This in the Lemma 1 in Ed. Maillet [6], However we repeat the proof for completeness.)

Proof. Let a non-identity element $x$ of $G$ be expressed as a cyclic permutation on the set $\Omega$ as follows:

$$x=(a, b, \cdots)\cdots, \qquad a, b\in\Omega,$$

where the cycle containing $a$ is of length greater than 1. Since $G$ is doubly transitive on $\Omega$, $G_a$, the stabilizer of a point $a\in\Omega$, is transitive on the set $\Omega-\{a\}$, hence for every $b_i\in\Omega-\{a\}$ $(i=s, \cdots, m-1)$ there exists an element $y_i\in G_a$ such that $b^{y_i}=b_i$. But $y_i^{-1}xy_i$ $(i=1, \cdots, m-1)$ are all distinct from each other, and the assertion is proved.

**Lemma 4.** *Under the assumption of Theorem* 1, *we have* $m-1\leqslant(q^n-1)\cdot(q^{n-1}-1)/(q-1)$.

Proof. The number of elements of $PSL(n, q)$ which are conjugate to a fixed elation is $\leq (q^n-1)(q^{n-1}-1)/(q-1)$, hence we have the sasertion by Lemma 3.

## 4. Characters of the group $GL(n, q)$. Proof of Theorem 1 for the case $n \geqslant 5$ and $q^{n-2} | m$

Let $G=PSL(n, q)$ be doubly transitive on a set $\Omega$, $|\Omega|=m$, and let us assume that $n \geqslant 4$ and $q^{n-2}|m$. Then $G$ has the irreducible character $\varphi_1$ such that $\varphi_1(x)=I(x)-1$ $(x \in G)$ where $I$ denotes the permutation character of $(G, \Omega)$.

Now we will determine which irreducible character $\varphi$ of $G$ satisfy the following two conditions (1) and (2).

1) $\varphi(1) \leqslant (q^n-1)(q^{n-1}-1)/(q-1)$,
2) $q^{n-2} | (\varphi(1)+1)$.

Clearly, from our assumption and Lemma 4, the irreducible character $\varphi_1$ must satisfy the conditions (1) and (2).

As is obvious from the theorem of Clifford, for any irreducible character $\varphi$ of $G=PSL(n, q)$, there is associated some irreducible character $\chi$ of $GL(n, q)$ such that

$$\varphi(1) = \frac{1}{\alpha}\chi(1)\,,$$

where $\alpha | (n, q-1)$.

(Note that $PGL(n, q)$ is a factor group of $GL(n, q)$ and that $PSL(n, q)$ is a normal subgroup of $PGL(n, q)$ such that the factor group $PGL(n, q)/PSL(n, q)$ is a cyclic group of order $(n, q-1)$).

As the first step of the determination of irreducible characters of $G$ satisfying the conditions (1) and (2), we will determine which irreducible character $\chi$ of $GL(n, q)$ with $n \geqslant 4$ satisfy the following two conditions,

1') $\chi(1) \leqslant (q^n-1)(q^{n-1}-1)$,
2) $\chi(1)$ is prime to $q$.

Clearly, if $\chi$ is an irreducible character of $GL(n, q)$ associated to an irreducible character of $G$ satisfying the conditions (1) and (2), then $\chi$ satisfies the conditions (1') and (2').

Owing to J.A. Green [2], we have the following lemma.

**Lemma 5.** *Let $\chi$ be an irredubcile character of $GL(n, q)$ whose degree $\chi(1)$ is prime to $q$, then there exists a partition of $n$, $n_1+n_2+\cdots+n_r=n$, positive integers $s_i$ and $v_i$ such that $s_i v_i=n_i$ $(i=1, \cdots, r)$ and $s_i$-simplexes $k^{(i)}(i=1, \cdots, r)$, and we have*

$$\chi = I_{s_1}^{k^{(1)}}[v_1]\circ\cdots\circ I_{s_r}^{k^{(r)}}[v_r]\,.^{3)}$$

3) In this notation we understand that if the right hand side is a negative character then $\chi$ is $(-1)$ multiple of the negative character.

*Moreover,*

$$\chi(1)=\frac{\psi_n(q)}{\psi_{n_1}(q)\cdots\psi_{n_r}(q)}\frac{\psi_{n_1}(q)}{\psi_{v_1}(q^{s_1})}\cdots\frac{\psi_{n_r}(q)}{\psi_{v_r}(q^{s_r})},$$

*where* $\psi_l(q)=(q^l-1)(q^{l-1}-1)\cdots(q-1)$.

(For the notation and the proof of the lemma, see [2], especially Lemma 2.7, Lemma 7.4 and Theorem 13 in [2].)

Using Lemma 5, we can get the following lemma. Since the proof is straightforward and easy, we omit it.

**Lemma 6.** *If an irredicible character* $\chi$ *of* $GL(n, q)$ *with* $n\geqslant 4$ *satisfies the conditions* (1′) *and* (2′), *then one of the following cases occurs.*

(Here we may assume that $n_1\leqslant n_2\leqslant\cdots\leqslant n_n$, and that $s_i\leqslant s_j$, if $n_i=n_j$ and $i\leqslant j$. Here we omit the parameter $k^{(i)}$ of $I_{s_i}^{k^{(i)}}[v_i]$. The $s_i$-simplexes $k^{(i)}$ must be suitably chosen. Especially, if $q=2$, then the cases 1°) 2°), 4°), 13°) and 16°) do not occur, because there exists only one 1-simplex if $q=2$, see [2].)

1°) $\chi=I_1[1]\circ I_1[n-1]$, $\chi(1)=(q^{n-1}+\cdots+q+1)$.
2°) $\chi=I_1[2]\circ I_1[n-2]$, $\chi(1)=(q^{n-1}+\cdots+q+1)(q^{n-2}+\cdots+q+1)/(q+1)$.
3°) $\chi=I_2[1]\circ I_1[n-2]$. $\chi(1)=(q^{n-1}+\cdots+q+1)(q^{n-2}+\cdots+q+1)(q-1)/(q+1)$.
4°) $\chi=I_1[1]\circ I_1[1]\circ I_1[n-2]$, $\chi(1)=(q^{n-1}+\cdots+q+1)(q^{n-2}+\cdots+q+1)$.
5°) $n=4$, $\chi=I_4[1]$, $\chi(1)=(q^3-1)(q^2-1)(q-1)$
6°) $n=4$, $\chi=I_2[2]$, $\chi(1)=(q^3-1)(q-1)$
7°) $n=4$, $\chi=I_1[1]\circ I_3[1]$, $\chi(1)=(q^3+q^2+q^2+1)(q^2-1)(q-1)$.
8°) $n=4$, $\chi=I_2[1]\circ I_2[1]$, $\chi(1)=(q^3+q^2+q+1)(q^2+q+1)(q-1)^2/(q+1)$.
9°) $n=4$, $\chi=I_1[2]\circ I_2[1]$, $\chi(1)=(q^3+q^2+q+1)(q^2+q+1)(q-1)/(q+1)$
10°) $n=5$, and $q=2$, $\chi=I_5[1]$, $\chi(1)=(q^4-1)(q^3-1)(q^2-1)/(q-1)$.
11°) $n=5$, $\chi=I_1[1]\circ I_2[2]$, $\chi(1)=(q^4+q^3+q^2+q+1)(q^3-1)(q-1)$.
12°) $n=5$, $\chi=I_1[2]\circ I_3[1]$, $\chi(1)=(q^4+q^3+q^2+q+1)(q^3+q^2+q+1)(q-1)^2$.
13°) $n=5$, $\chi=I_1[1]\circ I_1[2]\circ I_1[2]$,
$\chi(1)=(q^4+q^3+q^2+q+1)(q^3+q^2+q+1)(q^2+q+1)/(q+1)$.
14°) $n=6$, $\chi=I_2[3]$, $\chi(1)=(q^5-1)(q^3-1)(q-1)$.
15°) $n=6$, $\chi=I_3[2]$, $\chi(1)=(q^5-1)(q^4-1)(q^2-1)(q-1)$.
16°) $n=6$, $\chi=I_1[3]\circ I_1[3]$,
$\chi(1)=(q^5+\cdots+q+1)(q^4+\cdots+q+1)(q^3+q^2+q+1)/(q^2+q+1)(q+1)$.
17°) $n=8$ and $q=2$, $\chi=I_2[4]$, $\chi(1)=(q^7-1)(q^5-1)(q^3-1)(q-1)$.

Using Lemma 6 together with the following easily verified Remark, we have the next Lemma 7.

REMARK. Let $f(x)$ be a polynomial with integral coefficients such that $f(0)=1$ (resp. $f(0)=-1$). If $\frac{1}{\alpha}f(q)+1$, where $\alpha\,|\,(q-1)$, is an integer and is divisible by $q$, then $\alpha=q-1$ (resp. $\alpha=1$).

**Lemma 7.** *If $\varphi$ is an irreducible character of $G=PSL(n, q)$ satisfying the conditions* (1) *and* (2), *then one of the following cases occurs.*
i) $n=4$, $\varphi(1)=(q^2+1)(q^2+q+1)(q-1)$, *the associated character* $\chi$ *of* $\varphi$ *is* $I_1[2]\circ I_2[1]$ *and* $\alpha=1$
ii) $n=4$, $\varphi(1)=q^3-1$, *the associated character* $\chi$ *of* $\varphi$ *is* $I_2[2]$ *and* $\alpha=q-1$.
iii) $n=4$, $\varphi(1)=(q^2+1)(q^2+q+1)(q-1)$, *the associated character* $\chi$ *of* $\varphi$ *is* $I_2[1]\circ I_2[1]$ *and* $\alpha=q-1$.

Proof. Let $\chi$ be an irreducible character of $GL(n, q)$ associated to $\varphi$. Then, $\chi$ is one of the characters (1°)~(17°) in Lemma 6. Let us assume that for $\chi$ the case (1°) or (2°) holds. Then $\alpha=q-1$ by the above Remark, and $q^{n-2}\not\mid\left(\frac{1}{\alpha}\chi(1)+1\right)$, since $n\geqslant 4$ and $q\neq 2$. But this contradicts the assumption that $\varphi$ satisfies the condition (2). Let us assume that for $\chi$ the case (3°) of Lemma 6 holds. Then $\alpha=1$, and $q^{n-2}\left|\left(\frac{1}{\alpha}\chi(1)+1\right)\right.$ if and and only if $n=4$, hence the case (i) holds. By the similar argument we can easily show that only the cases (ii) and (iii) hold, if one of the cases (4°)~(17°) of Lemma 6 holds for $\chi$.

*Proof of Theorem* 1 *for the case* $n\geqslant 5$ *and* $q^{n-2}|m$. This case does not occur, because by Lemma 7, there exists no irreducible character $\varphi$ of $G$ satisfying the conditions (1) and (2).

## 5. Proof of Theorem 1 for the case $n\leqslant 4$

*The case* $n=4$. Let $n=4$, then we may assume that $q^2|m$. By Lemma 7, we have either $m=q^2(q^3+q-1)$ ($q$ being arbitrary) or $m=q^3$ ($q=2$, 3 or 5). The first case is impossible, since it is easily verified that $m$ does not divide the order of $G$.

(1) Let $q=2$ ($m=8$). Then this case does occur because $PSL(4, 2)\cong A_8$ and doubly transitive on 8 points. Therefore the case (VIII) in Theorem 1 holds. Uniqueness of doubly transitive permutation representation of $PSL(4, 2)$ on 8 points is clear.

(2) Let $q=3$ ($m=27$). Let $H$ be the stabilizer of a point in $\Omega$. Obviously, $G$ contains a subgroup $K$ which is isomorphic to $PSp(4, 3)$. But according to L.E. Dickson [1], $PSp(4, 3)$ is represented as a permutation group on 27 points, and is not represented on less than 27 points if the action is nontrivial. Moreover we have that the minimal degree (=class) of $PSp(4, 3)$ on the 27 points is 12. But the result of W.A. Manning (cf. [12], page 43) on permutation groups of small minimal degrees shows that $PSL(4, 3)$ is not represented as a doubly transitive permutation group on 27 points. Hence, this case does not occur.

(3) Let $q=5$ ($m=125$). Let $H$ be the stabilizer of a point in $\Omega$. Obviously, $G$ contains a subgroup $K$ which is isomorphic to $PSp(4, 5)$. We have $|K: K\cap H|=|KH|/|H|\leqslant|H: G|=125$. But according to L.E. Dickson [1],

$PSp(4, 5)$ contains no proper subgroup whose index is not greater than 125 (this also due to C. Jordan). Hence $H \supseteqq K$, but this is a contracdition, since $125 \nmid |G: K|$.

*The case* $n=3$. Let $n=3$. If $m$ is prime to $q$, then owing to the Theorem of F.C. Piper [8 and 9], the case (I) or (II) of Theorem 1 hold. Let us assume that $m$ is not prime to $q$. If $q=2$, $G=PSL(3, 2) \cong PSL(2, 7)$ and it has a doubly transitive permutation representation on 8 points, and has no other doubly transitive permutation representation of even degree. Uniqueness of doubly transitive representations on 8 points is clear. Let $q \neq 2$, then the same methods as previous section shows that the degree of $\varphi_1$ must be either 7 (for $q=4$), 28 (for $q=4$), $q^3-1$ ($q$ being arbitrary), 15 (for $q=4$). But according to H.H. Mitchell [7] and R.W. Hartley [3], there exist no subgroups of index 8 (for $q=4$), 29 (for $q=4$), 16 (for $q=4$) and $q^3$ (for arbitrarily $q$). This is a contradiction.

The assertion of Theorem 1 for $n=2$ is well known, and we omit the proof.

Thus, we completed the proof of Theorem 1.

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