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CHARACTER CORRESPONDENCES IN p-SOLVABLE GROUPS

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Introduction

Let G and A be finite groups and suppose that A acts on G by automorphisms. We write $Irr(G)$ to denote the set of all irreducible characters of G over the complex number field. Then A induces permutation action on $Irr(G)$. For $\chi \in Irr(G)$ and $a \in A$, the character χ^a is defined by $\chi^a(g^a)=\chi(g)$ for $g \in G$. The set of all A-invariant characters in $Irr(G)$ is denoted by $Irr_A(G)$.

Assume further that $(|G|, |A|)=1$. G. Glauberman [2] first showed that if A is solvable then there is a bijection

$$\pi(G, A)\colon Irr_A(G) \to Irr(C_G(A))$$

which is uniquely defined by the action of A on G.
When A is not solvable, the Odd-Order Theorem of Feit and Thompson implies that $|A|$ is even and hence $|G|$ is odd. E.C. Dade and I.M. Isaacs [3] developed the correspondence when $|G|$ is odd, and T.R. Wolf [7] showed the correspondences of Glauberman and Isaacs are equal when both are defined.

For a fixed prime p, $IBr(G)$ denotes the set of all irreducible p-modular characters of G, chosen with respect to some fixed pullback of the p-modular roots of unity to the complex numbers. Then A also induces permutation action on $IBr(G)$ by the same manner as on $Irr(G)$. Now the question arises whether there is a bijection from $IBr_A(G)$ onto $IBr(C_G(A))$ or not. The purpose of this paper is to show that it exists when G is p-solvable, namely, we shall prove the following.

Theorem. *Let A act on G such that $(|G|, |A|)=1$. Suppose that G is p-solvable. Then there exists a bijection*

$$\tilde{\pi}(G, A)\colon IBr_A(G) \to IBr(C_G(A)).$$

And the following hold.

(i) *If $B \trianglelefteq A$, then $\tilde{\pi}(G, A)=\tilde{\pi}(G, B)\tilde{\pi}(C_G(B), A/B)$.*

(ii) *If A is a q-group for a prime q, then, for $\phi \in IBr_A(G)$, $(\phi)\tilde{\pi}(G, A)$ is the unique irreducible constituent of $\phi_{C_G(A)}$ with multiplicity prime to q.*

The proof of the above Theorem is divided into two parts. It is proved when A is solvable in Section 3 (Theorem 3.10). If A is nonsolvable, then $2 \mid |A|$ by the Odd-Order Theorem. Thus $|G|$ is odd and we may assume $p \neq 2$. In this case it is done in Section 4 (Theorem 4.3).

We follow the notation of [5].

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1. Preliminaries

In this section, we mention some properties of co-prime actions.

The first lemma, which can be proved via the Schur-Zassenhaus Theorem, is quite useful when looking at co-prime actions. It is due to Glauberman [1]. Also a proof can be found in Lemma 13.8 and Corollary 13.9 of [5].

Lemma 1.1. *Suppose that a group A acts on a group G with $(|G|, |A|)=1$. Let A and G both act on a set Ω and assume*

(i) $(x\cdot g)\cdot a=(x\cdot a)\cdot g^a$ *for all* $x\in\Omega$, $g\in G$ *and* $a\in A$.

(ii) *G is transitive on Ω.*

Then A fixes a point of Ω and $C_G(A)$ acts transitively on the set of fixed points of A.

The following lemma is easily seen by using the above.

Lemma 1.2. *Assume A acts on G, $N\trianglelefteq G$, N is A-invariant, $(|G:N|, |A|)=1$, and $\chi\in Irr_A(G)$. Then*

(i) *χ_N has an A-invariant irreducible constituent θ.*

(ii) *If $C_{G/N}(A)=1$, then the above θ is unique.*

(iii) *If $C_{G/N}(A)=G/N$, then every irreducible constituent of χ_N is A-invariant.*

The next result is in some sense dual to the above lemma.

Lemma 1.3. *Assume A acts on G, $N\trianglelefteq G$, N is A-invariant, $(|G:N|, |A|)=1$, and $\theta\in Irr_A(N)$. Then*

(i) *θ^G has an A-invariant irreducible constituent χ.*

(ii) *If $C_{G/N}(A)=1$, then the above χ is unique.*

(iii) *If $C_{G/N}(A)=G/N$ then every irreducible constituent of θ^G is A-invariant.*

Proof. This Lemma follows from Theorem 13.31 and Problems 13.10 and 13.13 of [5].

2. Preliminaries for character correspondence

In this section, we recall some properties of the character correspondence of Glauberman and Dade-Isaacs. Since we will be frequently looking at coprime actions, we make the following hypothesis.

HYPOTHESIS 2.1. Let A act on G such that $(|G|, |A|)=1$. Let $C=C_G(A)$ and let $\Gamma=GA$ be the semi-direct product of G and A.

The results of Glauberman, Isaacs and Wolf may be summarized as follows.

Theorem 2.2. *Assume Hypothesis* 2.1. *Then there is a uniquely defined map*

$$\pi(G, A): Irr_A(G) \to Irr(C)$$

and the following hold.

(i) $\pi(G, A)$ *is bijective.*

(ii) *If* $B \trianglelefteq A$, *then* $\pi(G, A)=\pi(G, B)\pi(C_G(B), A/B)$.

(iii) *If A is a q-group for a prime q and* $\chi\in Irr_A(G)$, *then* $(\chi)\pi(G, A)$ *is the unique* $\xi\in Irr(C)$ *such that* $q \nmid [\chi_C, \xi]$.

(iv) *If* $|G|$ *is odd and* $\chi\in Irr_A(G)$, *then there exists the unique* $\xi\in Irr_A([G, A]'C)$ *such that* $2\nmid [\chi_{[G,A]'C}, \xi]$. *Also* $(\chi)\pi(G, A)=(\xi)\pi([G, A]'C, A)$. *Moreover suppose α is an automorphism of Γ which leaves G and A invariant. Then C is α-invariant and we have*

$$(\chi^{\alpha})\pi(G, A) = \{(\chi)\pi(G, A)\}^{\alpha} \quad \textit{for all } \chi\in Irr_A(G).$$

Proof. See Corollary 5.2 of [7] for (i)~(iv). The last statement holds since $\pi(G, A)$ is ultimately determined uniquely by multiplicities. A similar argument can be found, for example, in the discussion preceding Corollary 13.19 of [5].

By saying that $\pi(G, A)$ is uniquely defined, we mean that $\pi(G, A)$ is determined by the action of A on G. If A is solvable, then (ii) and (iii) give an algorithm for computing $\pi(G, A)$. Suppose that $|G|$ is odd. If $[G, A]=1$, then $C=G$ and $\pi(G, A)$ is the identity map on $Irr(G)$. Assume that $[G, A]\neq 1$. The Odd-Order Theorem implies $[G, A]'<[G, A]$ and hence $[G, A]'C<G$. Thus (iv) provides an algorithm for computing $\pi(G, A)$ when $|G|$ is odd.

In the next lemma, we mention some useful properties of $\pi(G, A)$, which relate $\pi(G, A)$ and $\pi(N, A)$ for an A-invariant normal subgroup N of G.

Lemma 2.3. *Assume Hypothesis* 2.1 *and that N is an A-invariant normal subgroup of G. Let* $\chi\in Irr_A(G)$, $\theta\in Irr_A(N)$, $T=I_G(\theta)$, $\xi=(\chi)\pi(G, A)$, *and* $\phi=$

$(\theta)\pi(N, A)$, where $I_G(\theta)$ denotes the inertia group of θ in G. Then
(i) *$[\chi_N, \theta]\neq 0$ if and only if $[\xi_{N\cap C}, \phi]\neq 0$.*
(ii) *$T\cap C=I_C(\phi)$ and $(\psi^G)\pi(G, A)=((\psi)\pi(T, A))^C$ for $\psi\in Irr_A(T\mid\theta)$.*

Proof. See Lemma 2.5 of [8].

Assume Hypothesis 2.1. For $\chi\in Irr_A(G)$ there exists the unique extension $\chi^*\in Irr(\Gamma)$ of χ such that $A\leq \ker(\det\chi^*)$. (See Lemma 13.3 of [5].) χ^* is called the canonical extension of χ.

Lemma 2.4. *Assume Hypothesis 2.1 and that A is cyclic. Let $\chi\in Irr_A(G)$ and $\xi=(\chi)\pi(G, A)$. Let χ^* be the canonical extension of χ to Γ. Then there exists $\varepsilon=\pm 1$ such that*

$$\chi^*(ca)=\varepsilon\xi(c) \quad \text{for all } c\in C \text{ and all generators } a \text{ of } A.$$

Proof. See Theorem 13.6 of [5].

3. Correspondence of Brauer characters

Let p be a fixed prime. In this section, we construct, under Hypothesis 2.1, a bijection from $IBr_A(G)$ onto $IBr(C)$ when G is p-solvable. We begin with two useful results of Isaacs [4], [6]. For a character χ of G let $\hat{\chi}$ denote the restriction of χ to the p-regular elements of G.

Lemma 3.1. *Let $N\trianglelefteq G$ with $p\nmid |G: N|$. Let $\theta\in Irr(N)$ and assume*
(i) *$\hat{\theta}\in IBr(N)$ and*
(ii) *$\theta^g=\theta$ for those $g\in G$ with $\hat{\theta}^g=\hat{\theta}$.*
Then $\wedge$ defines a bijection from $Irr(G\mid\theta)$ onto $IBr(G\mid\hat{\theta})$.

Proof. See Lemma 2.6 of [6].

Lemma 3.2. *Let $N\trianglelefteq G$ with G/N a p-group. Let $\phi\in IBr(N)$. Then $IBr(G\mid\phi)$ consists of a single element ψ. Moreover if $I_G(\phi)=\{g\in G\mid \phi^g=\phi\}=G$, then $\psi_N=\phi$.*

Proof. See Lemma 4.4 of [6].

To construct the bijection, we need a definition. If $\Omega\subset Irr(G)$ and $H\leq G$, we write $\Omega(H)$ to denote $\{\theta\in Irr(H)\mid[\chi_H, \theta]\neq 0$ for some $\chi\in\Omega\}$. Note that if $K\leq H\leq G$ and $\psi\in\Omega(H)$, then every irreducible constituent of ψ_K lies in $\Omega(K)$.

DEFINITION 3.3. Assume A acts on G. Let Ω be a subset of $Irr(G)$ and let $G=G_0\trianglerighteq G_1\trianglerighteq\cdots\trianglerighteq G_n=\{1\}$ be a normal series of G. We say that Ω has *the lifting property with respect to A and $\{G_i\}_{i=0}^n$* if the following is satisfied.

(i) Ω is A-invariant.

(ii) $\wedge$ defines the bijection from $\Omega(G_i)$ onto $IBr(G_i)$ for each i, $0 \leq i \leq n$.
Furthermore, we simply say that Ω has *the lifting property with respect to* A if Ω has the lifting property with respect to A and every normal series of G.

It is easily seen that an A-invariant subset Ω of $Irr(G)$ has the lifting property with respect to A if and only if $\wedge$ defines the bijection from $\Omega(N)$ onto $IBr(N)$ for any subnormal subgroup N of G.

REMARK. For each p-solvable group G, Isaacs [4], [6] constructed a characteristic subset $\mathcal{Y}(G)$ of $Irr(G)$ such that

(i) $\wedge$ defines a bijection from $\mathcal{Y}(G)$ onto $IBr(G)$, and

(ii) if $N \trianglelefteq \trianglelefteq G$ and $\chi \in \mathcal{Y}(G)$, then every irreducible constituent of χ_N lies in $\mathcal{Y}(N)$.

Moreover the above properties (i) and (ii) of $\mathcal{Y}(G)$ imply that for each subnormal subgroup N of G and $\theta \in \mathcal{Y}(N)$, θ^G has an irreducible constituent which lies in $\mathcal{Y}(G)$. This can be shown using the same argument as in Lemma 3.4. So $\mathcal{Y}(G)$ has the lifting property with respect to any A.

Lemma 3.4. *Assume A acts on G. Let $G=G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n=\{1\}$ be a normal series of G. Let $\Omega \subset Irr(G)$ have the lifting property with respect to A and $\{G_i\}_{i=0}^{n}$.*

(i) *If $p \nmid |G_k : G_l|$ for $0 \leq k \leq l \leq n$, then $Irr(G_k|\theta) \subset \Omega(G_k)$ for all $\theta \in \Omega(G_l)$.*

(ii) *If $|G_k : G_l|$ is a power of p for $0 \leq k \leq l \leq n$, then $Irr(G_k|\theta) \cap \Omega(G_k)$ consists of a single element for every $\theta \in \Omega(G_l)$.*

To prove Lemma 3.4 we need another lemma.

Lemma 3.5. *Under the hypothesis of Lemma 3.4 let $\theta \in \Omega(G_l)$ and $\psi \in \Omega(G_k)$. Suppose $\hat{\psi} \in IBr(G_k|\hat{\theta})$. Then $\psi \in Irr(G_k|\theta)$.*

Proof. We have $\psi_{G_l}=\sum_{i=1}^{t}\eta_i$, where the η_i are irreducible. We also have $\hat{\psi}_{G_l}=\sum_{i=1}^{t}\hat{\eta}_i$. Since $\eta_i \in \Omega(G_l)$, $\hat{\eta}_i \in IBr(G_l)$ by the lifting property of Ω. Since $\hat{\psi} \in IBr(G_k|\hat{\theta})$, it follows that $\hat{\theta}=\hat{\eta}_i$ for some i. Since $\wedge$ is the bijection from $\Omega(G_l)$ onto $IBr(G_l)$, we have $\theta=\eta_i$ and the result follows.

Proof of Lemma 3.4. It suffices to prove only when $l=k+1$. Thus we may assume that G_l is normal in G_k.

(i) Let $\theta \in \Omega(G_l)$ and $\chi \in Irr(G_k|\theta)$. For $g \in G_k$ $\theta^g \in \Omega(G_l)$ by the definition of $\Omega(G_l)$ and it follows by the lifting property of Ω that $\theta^g=\theta$ for those $g \in G_k$ with $\hat{\theta}^g=\hat{\theta}$. From Lemma 3.1, $\hat{\chi}$ belongs to $IBr(G_k|\hat{\theta})$, and then we can find $\psi \in \Omega(G_k)$ such that $\hat{\psi}=\hat{\chi}$. Since $\hat{\psi} \in IBr(G_k|\hat{\theta})$, Lemma 3.4 yields that $\psi\in Irr(G_k|\theta)$. Thus by Lemma 3.1 again, we have $\chi=\psi\in\Omega(G_k)$.

(ii) Let $\theta\in\Omega(G_l)$. Let $\phi\in IBr(G_k|\hat{\theta})$. Then by the lifting property of Ω, there exists $\psi\in\Omega(G_k)$ such that $\hat{\psi}=\phi$. Since $\hat{\psi}\in IBr(G_k|\hat{\theta})$, we have $\psi\in Irr(G_k|\theta)$ by Lemma 3.5. Thus $\psi\in Irr(G_k|\theta)\cap\Omega(G_k)$. It follows from Lemma 3.2 and the lifting property of Ω that such a ψ is unique. Now the proof is complete.

In the next proposition, we see that $\pi(G, A)$ preserves the lifting property with respect to A and any given A-composition series of G.

Proposition 3.6. *Assume Hypothesis* 2.1 *and that G is p-solvable. Let $G=G_0\trianglerighteq G_1\trianglerighteq\cdots\trianglerighteq G_n=\{1\}$ be an A-composition series of G and let $\Omega\subset Irr(G)$ have the lifting property with respect to A and $\{G_i\}_{i=0}^n$. Then the image of $\Omega_A=\Omega\cap Irr_A(G)$ by $\pi(G, A)$ has the lifting property with respect to $\{1\}$ (the trivial automorphism of C) and $\{G_i\cap C\}_{i=0}^n$.*

Proof. Use induction on $|G|$.

Let Λ be the image of Ω_A by $\pi(G, A)$. Let $\eta\in\Omega(G_i)_A=\Omega(G_i)\cap Irr_A(G_i)$ for $i\geq 1$. If G_{i-1}/G_i is a p'-group, then it is clear from Lemma 1.3 and Lemma 3.4 (i) that $\Omega(G_{i-1})_A\cap Irr(G_{i-1}|\eta)$ is nonempty. If G_{i-1}/G_i is a p-group, then by Lemma 3.4 (ii) $\Omega(G_{i-1})\cap Irr(G_{i-1}|\eta)$ has exactly one element and since both $\Omega(G_{i-1})$ and $Irr(G_{i-1}|\eta)$ are A-invariant, it must be A-invariant. So $\Omega(G_{i-1})_A\cap Irr(G_{i-1}|\eta)$ is nonempty in any case.

Suppose $\psi\in\Omega(G_{i-1})_A\cap Irr(G_{i-1}|\eta)$. Applying Lemma 2.3 (i) repeatedly we can find $\chi\in\Omega_A\cap Irr(G|\eta)$ such that $(\chi)\pi(G, A)\in Irr(C|(\eta)\pi(G_i, A))$. Since $(\chi)\pi(G, A)\in\Lambda$, it follows that $(\eta)\pi(G_i, A)\in\Lambda(G_i\cap C)$. Conversely we suppose $\xi\in\Lambda(G_i\cap C)$ and set $\eta=(\xi)\pi^{-1}(G_i, A)$. By the definition of Λ and $\Lambda(G_i\cap C)$ there exists $\chi\in\Omega_A$ such that $(\chi)\pi(G, A)\in Irr(C|\xi)$. From Lemma 2.3 (i), we have $\chi\in Irr(G|\eta)$, so $\eta\in\Omega(G_i)_A$. Thus we can conclude that the image of $\Omega(G_i)_A$ by $\pi(G_i, A)$ is precisely $\Lambda(G_i\cap C)$ for each i, $0\leq i\leq n$. Since $\Omega(G_i)$ has the lifting property with respect to A and $\{G_i\}_{i=1}^n$ and $\{G_i\}_{i=1}^n$ is an A-composition series of G_1, it follows from the inductive hypothesis that for each i, $1\leq i\leq n$,

$$\wedge\colon \Lambda(G_i\cap C)\rightarrow IBr(G_i\cap C)$$

is a bijection. Therefore the proof will be complete if we show that $\wedge$ gives a bijection from Λ onto $IBr(C)$.

If $C\leq G_1$, then $G_1\cap C=C=G_0\cap C$ and $\Lambda(G_1\cap C)=\Lambda$. Thus we have nothing to prove.

Now assume $G_1\ngeq C$. Let $\theta_1, \cdots, \theta_k$ be representatives of C-orbits of $\Omega(G_1)_A$. For each i, $1\leq i\leq k$, set $\phi_i=(\theta_i)\pi(G_1, A)$. If $g\in C$, then θ_i^g is also A-invariant and from Theorem 2.2 we have

$$(\theta_i^g)\pi(G_1, A) = (\theta_i)\pi(G_1, A)^g = \phi_i^g .$$

Thus $\phi_1, \cdots, \phi_k$ are representatives of C-orbits of $\Lambda(G_1 \cap C)=(\Omega(G_1)_A)\pi(G_1, A)$. Furthermore $\theta_i^g \in \Omega(G_1)$ for $g \in G$ and i, $1 \leq i \leq k$, by the definition of $\Omega(G_1)$. Since $\wedge$ gives the bijection from $\Omega(G_1)$ onto $IBr(G_1)$, it follows that $\theta_i^g=\theta_i$ for those $g \in G$ with $\hat{\theta}_i^g=\hat{\theta}_i$. Thus we have $I_G(\theta_i)=I_G(\hat{\theta}_i)$ for i, $1 \leq i \leq k$. Also we obtain $I_C(\phi_i)=I_C(\hat{\phi}_i)$ for each i, $1 \leq i \leq k$.

We distinguish two cases.

Case 1. G/G_1 is a p-group.

Since G/G_1 is abelian, we have $G_1C=G$. Then $Irr(G|\theta_i) \cap Irr(G|\theta_j)$ is empty for $i \neq j$. And by Lemma 3.4 (ii), $Irr(G|\theta_i) \cap \Omega$ has exactly one element which is of course A-invariant. So we have $\Omega_A=\bigcup_{i=1}^{k} Irr(G|\theta_i) \cap \Omega$ and especially $|\Omega_A|=k$. Thus $|\Lambda|=k$.

Recall that $\phi_1, \cdots, \phi_k$ are representatives of C-orbits of $\Lambda(G_1 \cap C)$. By the lifting property of $\Lambda(G_1 \cap C)$, $\hat{\phi}_1, \cdots, \hat{\phi}_k$ are representatives of C-orbits of $IBr(G_1 \cap C)$ and thus by Lemma 3.2 we have $|IBr(C)|=k$. Therefore it suffices to prove that, for $\chi \in \Omega_A \cap Irr(G|\theta_i)$, $\widehat{(\chi)\pi(G, A)}$ is modularly irreducible. Let $\chi \in \Omega_A \cap Irr(G|\theta_i)$. Then there exists $\xi \in Irr_A(I_G(\theta_i)|\theta_i)$ such that $\xi^G=\chi$. (See Theorem 6.11 of [5].) Since $\hat{\xi}^G=\widehat{\xi^G}=\hat{\chi} \in IBr(G)$, $\hat{\xi}$ must be irreducible. Also $[\theta_i, \xi_{G_1}] \neq 0$ yields that $\hat{\theta}_i$ is an irreducible constituent of $\hat{\xi}_{G_1}$. By Lemma 2.3, $(\xi)\pi(I_G(\theta_i), A) \in Irr(I_C(\phi_i)|\phi_i)$ and $(\chi)\pi(G, A)=((\xi)\pi(I_G(\theta_i), A))^C$. And by Lemma 3.2, $\hat{\xi}$ is the extension of $\hat{\theta}_i$. Since $I_G(\theta_i)/G_1$ is abelian, $|Irr(I_G(\theta_i)|\theta_i)| =|I_G(\theta_i)\colon G_1|$. (See Corollary 6.17 of [5].) By Lemma 1.3 (iii), we have $Irr(I_G(\theta_i)|\theta_i) \subset Irr_A(I_G(\theta_i))$ and thus by Lemma 2.3 we have $|Irr(I_G(\theta_i)|\theta_i)| =|Irr(I_G(\theta_i) \cap C|\phi_i)|=|Irr(I_C(\phi_i)|\phi_i)|$. Since $|I_G(\theta_i)\colon G_1|=|G_1I_C(\phi_i)\colon G_1| =|I_C(\phi_i)\colon G_1 \cap C|$, we get $|Irr(I_C(\phi_i)|\phi_i)|=|I_C(\phi_i)\colon G_1 \cap C|$ and hence each element in $Irr(I_C(\phi_i)|\phi_i)$ is an extension of ϕ_i to $I_C(\phi_i)$ and so is modularly irreducible. This applies, in particular, to $(\xi)\pi(I_G(\theta_i), A) \in Irr(I_C(\phi_i)|\phi_i)$. The equality $I_C(\phi_i)=I_C(\hat{\phi}_i)$ implies that

$$\widehat{(\chi)\pi(G, A)}=\widehat{(\xi)\pi(I_G(\theta_i), A)^C}=\widehat{(\xi)\pi(I_G(\theta_i), A)}^C$$

belongs to $IBr(C|\hat{\phi}_i)$. (See also Lemma 3.3 of [6].)
Thus the proof is complete.

Case 2. G/G_1 is a p'-group.

If $\chi \in \Omega_A$, then by Lemma 1.2 (i), $\chi \in Irr(G|\theta_i)$ for some i, $1 \leq i \leq k$. Thus by Lemma 2.3 (i) it follows that $(\chi)\pi(G, A) \in Irr(C|\phi_i)$, so we have $\Lambda \subset \bigcup_{i=1}^{k} Irr(C|\phi_i)$. Conversely if $\mu \in Irr(C|\phi_i)$, then set $\chi=(\mu)\pi^{-1}(G, A) \in Irr_A(G)$ and by Lemma 2.3 (i) again, we have $\chi \in Irr(G|\theta_i)$. Since $\theta_i \in \Omega(G_1)_A$, it follows from Lemma 3.4 (i) that $\chi \in \Omega_A$. Thus $\mu=(\chi)\pi(G, A) \in \Lambda$ and we conclude $\Lambda=\bigcup_{i=1}^{k} Irr(C\mid\phi_i)$. Since $I_C(\phi_i)=I_C(\hat{\phi}_i)$, by Lemma 3.1 $\wedge$ defines a bijection

$$\wedge\colon Irr(C\mid\phi_i)\rightarrow IBr(C\mid\hat{\phi}_i)$$

for each i, $1\leq i\leq k$.

Now if $i\neq j$, then $Irr(C\mid\phi_i)\cap Irr(C\mid\phi_j)$ and $IBr(C\mid\hat{\phi}_i)\cap IBr(C\mid\hat{\phi}_j)$ are both empty. Recall that $\hat{\phi}_1, \cdots, \hat{\phi}_k$ are representatives of C-orbits of $IBr(G_1\cap C)$. Since $IBr(C)=\bigcup_{i=1}^{k} IBr(C\mid\hat{\phi}_i)$, we can conclude that $\wedge$ gives the bijection from Λ onto $IBr(C)$. This completes the proof.

This proposition implies immediately the following.

Corollary 3.7. *Assume Hypothesis* 2.1 *and that G is p-solvable. Let $\Omega\subset Irr(G)$. If Ω has the lifting property with respect to A, then $\wedge$ gives a bijection from $(\Omega_A)\pi(G, A)$ onto $IBr(C)$.*

REMARK. Under the hypotheses in Proposition 3.6 $\Lambda=(\Omega_A)\pi(G, A)$ has the lifting property with respect to a composition series of C obtained as a refinement of $\{G_i\cap C\}_{i=0}^{n}$. This can be shown using Lemma 3.1 and Lemma 3.4 (i). Also if $B\trianglelefteq A$, it can be proved that $(\Omega_A)\pi(G, B)$ has the lifting property with respect to A/B and $\{G_i\cap C_G(B)\}_{i=0}^{n}$. But in general $\pi(G, A)$ does not preserve the lifting property with respect to A. (See Appendix.)

Now assume Hypothesis 2.1 and that G is p-solvable. Suppose $\Omega\subset Irr(G)$ has the lifting property with respect to A and some A-composition series of G. We denote $\wedge^{-1}$ the inverse of the bijection

$$\wedge\colon \Omega\rightarrow IBr(G)\,.$$

Since $\wedge$ obviously preserves the actions of A on Ω and $IBr(G)$, Proposition 3.6 gives us the following diagram of bijections

$$IBr_A(G)\xrightarrow{\wedge^{-1}}\Omega_A\xrightarrow{\pi(G,A)}(\Omega_A)\pi(G,A)\xrightarrow{\wedge}IBr(C)\,.$$

The composition $\tilde{\pi}(G, A)=\wedge^{-1}\pi(G, A)\wedge$ is a bijection from $IBr_A(G)$ onto $IBr(C)$.

From its construction, it appears that $\tilde{\pi}(G, A)$ depends on the choice of Ω. In the rest of this section we shall show that it is independent of the choice of Ω with the lifting propeity with respect to A, if A is solvable. If A is nonsolvable, by the Odd-Order Theorem we may assume $p\neq 2$. When p is odd, we shall prove a stronger result in Section 4, namely, that $\pi(G, A)$ gives a bijection from $\mathcal{Y}(G)\cap Irr_A(G)$ onto $\mathcal{Y}(C)$ (see Theorem 4.1). So we shall have a uniquely defined bijection $\tilde{\pi}(G, A)$.

Proposition 3.8. *Assume Hypothesis* 2.1 *and that* G *is* p-*solvable and* A *is solvable. Let* $B \trianglelefteq A$, $D = C_G(B)$, *and assume that* $\Omega \subset Irr(G)$ *and* $\Lambda \subset Irr(D)$ *both have the lifting property with respect to* A. *Let* $\chi \in \Omega_A$ *and let* ϕ *be the unique element of* $\Lambda_{A/B}$ *such that* $\hat{\phi} = \widehat{(\chi)\pi(G, B)}$. (*Note that by Corollary* 3.7 $(\chi)\pi(G, B)$ *is modularly irreducible.*) *Then* $\widehat{(\chi)\pi(G, A)} = \widehat{(\phi)\pi(D, A/B)}$.

We need a lemma.

Lemma 3.9. *Assume Hypothesis* 2.1 *and that* A *is cyclic. Let* χ, $\psi \in Irr_A(G)$. *If* χ *and* ψ *are both modularly irreducible and* $\hat{\chi} = \hat{\psi}$, *then* $\widehat{(\chi)\pi(G, A)} = \widehat{(\psi)\pi(G, A)}$.

Proof. Let $\Gamma = GA$. We may assume $p \mid |G|$, thus $p \nmid |A|$. Since χ and Ψ are A-invariant, it follows from Lemma 3.1 that

$$\wedge: Irr(\Gamma \mid \chi) \to IBr(\Gamma \mid \hat{\chi}) \text{ and}$$
$$\wedge: Irr(\Gamma \mid \psi) \to IBr(\Gamma \mid \hat{\psi})$$

are bijections. Let χ^* (resp. ψ^*) be the canonical extension of χ (resp. ψ) to Γ. Then we have $Irr(\Gamma \mid \chi) = \{\chi^* \mu \mid \mu \in Irr(A)\}$. Since $\hat{\chi} = \hat{\psi}$, there exists $\mu \in Irr(A)$ such that $\widehat{\chi^* \mu} = \widehat{\psi^*}$. Let a be a generator of A. By Lemma 2.4, there exist $\varepsilon = \pm 1$ and $\varepsilon' = \pm 1$ such that

$$(\chi)\pi(G, A)(g) = \varepsilon \chi^*(ga) \text{ and}$$
$$(\psi)\pi(G, A)(g) = \varepsilon' \psi^*(ga)$$

for all $g \in C$. Note that a is p-regular. Thus we obtain

$$\varepsilon \varepsilon' (\chi)\pi(G, A)(g)\mu(a) = (\psi)\pi(G, A)(g)$$

for every p-regular element $g \in C$. Now evaluation at $g = 1$ yields

$$\varepsilon \varepsilon' (\chi)\pi(G, A)(1)\mu(a) = (\psi)\pi(G, A)(1) .$$

Since $\mu(a)$ is a root of unity, $\varepsilon \varepsilon' \mu(a) = 1$. Thus we obtain

$$\widehat{(\chi)\pi(G, A)} = \widehat{(\psi)\pi(G, A)} .$$

as desired.

Proof of Proposition 3.8. Use induction on $|A|$.

If $A = B$, there is nothing to prove. We may assume $A \neq B$. Let H be a maximal normal subgroup of A containing B. By the inductive hypothesis, we have

$$\widehat{(\chi)\pi(G, H)} = \widehat{(\phi)\pi(D, H/B)} .$$

Also by Corollary 3.7, $(\chi)\pi(G, H)$ and $(\phi)\pi(D, H/B)$ are in $Irr_A(C_G(H))$ and both of them are modularly irreducible. Since A/H is cyclic, it follows from Lemma 3.9 that $(\chi)\pi(G, H)\pi(C_G(H), A/H)$ and $(\phi)\pi(D, H/B)\pi(C_G(H), A/H)$ are equal on the set of p-regular elements of C. Now the proof is completed by Theorem 2.2 (ii).

Theorem 3.10. *Assume Hypothesis* 2.1 *and that G is p-solvable and A is solvable. Then there exists a bijection*

$$\tilde{\pi}(G, A)\colon IBr_A(G) \to IBr(C)$$

which is independent of the choice of Ω with the lifting property with respect to A. And the following hold.

(i) *If $B \trianglelefteq A$, then $\tilde{\pi}(G, A) = \tilde{\pi}(G, B)\tilde{\pi}(C_G(B), A/B)$.*

(ii) *If A is a q-group for a prime q and $\phi \in IBr_A(G)$, then $(\phi)\tilde{\pi}(G, A)$ is the unique irreducible constituent of ϕ_C with multiplicity prime to q.*

Proof. By putting $B=1$ in Proposition 3.8, it follows that $\tilde{\pi}(G, A)$ is independent of the choice of such an Ω. If $B \trianglelefteq A$, it is easily seen by Proposition 3.8 that $\tilde{\pi}(G, A) = \tilde{\pi}(G, B)\tilde{\pi}(C_G(B), A/B)$. Now assume $\phi \in IBr_A(G)$ and fix Ω with the lifting property with respect to A. Then there exists $\chi \in \Omega_A$ such that $\hat{\chi} = \phi$. If A is a q-group, by Theorem 2.2 (iii) we have

$$\chi_C = m(\chi)\pi(G, A) + q\psi ,$$

where m is a positive integer prime to q and ψ is zero or a character of C. Therefore by the definition of $\tilde{\pi}(G, A)$, it follows that

$$\phi_C = m(\phi)\tilde{\pi}(G, A) + q\hat{\psi} ,$$

and the rest of Theorem is obvious.

4. The case: p is odd

In this section, we consider the correspondence of p-modular characters for an odd prime p. First we show the following.

Theorem 4.1. *Assume Hypothesis* 2.1 *and that G is p-solvable. If p is odd, then $\pi(G, A)$ gives a bijection from $\mathcal{Y}_A(G) = \mathcal{Y}(G) \cap Irr_A(G)$ onto $\mathcal{Y}(C)$.*

Before proving the above theorem, we should mention the definition of $\mathcal{Y}(G)$ for an odd prime p. When p is odd, $\mathcal{Y}(G)$ coincides with the set of subnormally p-rational irreducible characters of G. Here a character χ is called subnormally p-rational if upon restriction to every subnormal subgroup, every irreducible constituent of χ is p-rational i.e. has values in some field of the form $\boldsymbol{Q}[\varepsilon]$ where $\varepsilon^n=1$, $p\not| n$.

To prove Theorem 4.1 we need one more lemma about $\pi(G, A)$.

For $\chi \in Irr(G)$, let $\boldsymbol{Q}(\chi)$ be the extension of $\boldsymbol{Q}$ obtained by adjoining the values $\chi(g)$, $g \in G$, to $\boldsymbol{Q}$.

Lemma 4.2. *Assume Hypothesis* 2.1. *Let K be a Galois extension of $\boldsymbol{Q}$ containing a primitive $|G|$-th root of unity. Let $\chi \in Irr_A(G)$ and $\xi=(\chi)\pi(G, A)$. Then $(\chi^\sigma)\pi(G, A)=\xi^\sigma$ for σ in the Galois group of K over $\boldsymbol{Q}$. Moreover $\boldsymbol{Q}(\chi)=\boldsymbol{Q}(\xi)$.*

Proof. Since the actions of A and σ on $Irr(G)$ commute with each other, it follows that $\chi^\sigma \in Irr_A(G)$. Thus $(\chi^\sigma)\pi(G, A)$ is meaningful. First we show $(\chi^\sigma)\pi(G, A)=\xi^\sigma$. When A is solvable, we may assume that $|A|$ is a prime by Theorem 2.2 (ii) and induction on $|A|$. Then it follows from Theorem 2.2 (iii) that ξ^σ is the unique irreducible constituent of χ_C^σ with multiplicity prime to $|A|$. Thus the result follows.

When $|G|$ is odd, by Theorem 2.2 (iv) $\chi_{[G,A]'C}$ has the unique A-invariant irreducible constituent η with odd multiplicity and $(\chi)\pi(G,A)=(\eta)\pi([G,A]'C,A)$. An argement similar to the above one and induction on $|G|$ yield the result.

The rest of Lemma follows from the first statement; the field automorphisms of K that fix χ coincide with those that fix ξ.

Proof of Theorem 4.1. Use induction on $|G|$. Let N be a maximal A-invariant normal subgroup of G.

First we claim $(\mathcal{Y}_A(G))\pi(G, A) \subset \mathcal{Y}(C)$.

Assume $\chi \in \mathcal{Y}_A(G)$. By Lemma 1.2, there exists $\theta \in Irr_A(N)$ such that $[\chi_N, \theta] \neq 0$. Since $\theta \in \mathcal{Y}_A(N)$, it follows from the inductive hypothesis that $(\theta)\pi(N, A) \in \mathcal{Y}(N \cap C)$. Also by Lemma 2.3 (i), we have $(\chi)\pi(G, A) \in Irr(C\,|\,(\theta)\pi(N, A))$. If G/N is a p'-group, then by Lemma 3.4 (i), we have $(\chi)\pi(G, A) \in Irr(C\,|\,(\theta)\pi(N, A)) \subset \mathcal{Y}(C)$ as desired. Now assume that G/N is a p-group. Then $((\theta)\pi(N, A))^C$ has a unique p-rational irreducible constituent. (See Corollary 7.3 of [4].) By Lemma 3.4 (ii), it lies in $\mathcal{Y}(C)$. On the other hand, since χ is p-rational, so is $(\chi)\pi(G, A)$ by Lemma 4.2, Thus we conclude that $(\chi)\pi(G, A)$ is just the element in $\mathcal{Y}(C) \cap Irr(C\,|\,(\theta)\pi(N, A))$.

Now we prove Theorem. Since $\mathcal{Y}(G)$ has the lifting property with respect to A, it follows from Corollary 3.7 that $\wedge$ gives the bijection from $(\mathcal{Y}_A(G))\pi(G,A)$ onto $IBr(C)$. Especially we have $|(\mathcal{Y}_A(G))\pi(G,A)|=|IBr(C)|$. Since $|\mathcal{Y}(C)|=|IBr(C)|$, $(\mathcal{Y}_A(G))\pi(G, A)$ coincides with $\mathcal{Y}(C)$. This completes the proof.

Note that Theorem 4.1 is false if $p=2$. (See Appendix.)

By Theorem 4.1, we can define $\tilde{\pi}(G, A)$ via $\mathcal{Y}(G)$ in the case where p is odd.

Theorem 4.3. *Assume Hypothesis* 2.1 *and that* G *is* p*-solvable for an odd prime* p. *Then there exists a uniquely defined bijection*

$$\tilde{\pi}(G, A)\colon IBr_A(G) \to IBr(C) .$$

Moreover if $B \trianglelefteq A$, *then* $\tilde{\pi}(G, A)=\tilde{\pi}(G, B)\tilde{\pi}(C_G(B), A/B)$.

Proof. We have the following diagram

$$IBr_A(G) \xrightarrow{\wedge^{-1}} \mathcal{Y}_A(G) \xrightarrow{\pi(G,A)} \mathcal{Y}(C) \xrightarrow{\wedge} IBr(C) .$$

Since $\mathcal{Y}(G)$ is a characteristic subset of $Irr(G)$,

$$\tilde{\pi}(G, A) = \wedge^{-1}\pi(G, A)\wedge\colon IBr_A(G) \to IBr(C)$$

is a uniquely defined bijection. The last part of Theorem is clear.

REMARK. In the case where A is solvable and p is odd, Theorem 3.10 permits us to take $\mathcal{Y}(G)$ for Ω, and hence the bijection $\tilde{\pi}(G, A)$ in Theorem 3.10 and Theorem 4.3 are the same.

Under Hypothesis 2.1, we note that application of Lemma 1.1 shows that there is a bijection from the set of A-fixed p-regular conjugacy classes $\mathcal{K}$ of G onto the set of all p-regular conjugacy classes of C sending any such $\mathcal{K}$ into $\mathcal{K}\cap C$.

The final result is analogous to Theorem 13.24 of [5].

Corollary 4.4. *Assume Hypothesis* 2.1 *and that* G *is* p*-solvable. Then the actions of* A *on* $IBr(G)$ *and on the set of* p*-regular conjugacy classes of* G *are permutation isomorphic.*

Proof. The same proof as in Lemma 13.23 and Theorem 13.24 of [5] works for our case.

Appendix

Here we give an example which shows us that Theorem 4.1 is false when $p=2$.

Let q be a prime such that $q\equiv\pm 5 \pmod{12}$. Let E be a nonabelian group of order q^3 and exponent q with $E=\langle x, y\rangle$, $x^q=y^q=[x, y]^q=1$. Define $\gamma\in \mathrm{Aut}(E)$ by $x^\gamma=y^{-1}x$, $y^\gamma=x$ so that $o(\gamma)=6$. Let $G=E\rtimes\langle\gamma^3\rangle$, the semi-direct product, and let $A=\langle\gamma^2\rangle$. Then A acts on G coprimely, centralizing $C=Z(E)\times\langle\gamma^3\rangle$. Let θ be a faithful irreducible character of E. Since θ vanishes on $E \backslash Z(E)$, it is A-invariant. And since $(|E|, |\langle\gamma^3\rangle|)=1$, there exists the canonical extension of θ to G, say χ. Assume $p=2$. Then χ lies in $\mathcal{X}(G)$. For the definition of $\mathcal{X}(G)$, see Definition 2.2 of [6]. Thus χ lies in $\mathcal{Y}_A(G)$. (See Definition 5.1 of [6].) Since $[\chi, \chi]=1$, easy calculation yields $|\chi(\gamma^3)|=1$. Let λ be the unique irreducible constituent of $\chi_{Z(E)}$. Then we have

$$\chi_C = \begin{cases} ((q+1)/2)\lambda+((q-1)/2)\lambda\mu, & \text{if } q\equiv 5 \pmod{12} \\ ((q-1)/2)\lambda+((q+1)/2)\lambda\mu, & \text{if } q\equiv -5 \pmod{12}, \end{cases}$$

where μ is the unique nontrivial character of $\langle\gamma^3\rangle$.

In both cases, we have $3 \nmid [\chi_C, \lambda\mu]$. It follows from Theorem 2.2 (iii) that $(\chi)\pi(G, A)=\lambda\mu$. But since $(\lambda\mu)_{\langle\gamma^3\rangle}=\mu \notin \mathcal{Y}(\langle\gamma^3\rangle)=\{1_{\langle\gamma^3\rangle}\}$ and $\langle\gamma^3\rangle \lhd C$, $\lambda\mu$ does not belong to $\mathcal{Y}(C)$.

Moreover note that the image of $\mathcal{Y}_A(G)$ by $\pi(G, A)$ does not have the lifting property with respect to $\{1\}$, the trivial automorphism of C, although $\mathcal{Y}(G)$ has it with respect to A.

References

[1] G. Glauberman: *Fixed points in groups with operator groups,* Math. Z. **84** (1964), 120–125.

[2] ———: *Correspondence of characters for relatively prime operator groups,* Canad. J. Math. **20** (1968), 1465–1488.

[3] I.M. Isaacs: *Characters of solvable and symplectic groups,* Amer. J. Math. **95** (1973), 594–635.

[4] ———: *Lifting Brauer characters of p-solvable groups,* Pacific J. Math. **53** (1974), 171–188.

[5] ———: Character theory of finite groups, Academic Press, New York, 1976.

[6] ———: *Lifting Brauer characters of p-solvable groups* II, J. Algebra **51** (1978), 476–490.

[7] T.R. Wolf: *Character correspondence in solvable groups,* Illinois J. Math. **22** (1978), 327–340.

[8] ———: *Character correspondences induced by subgroups of operator groups,* J. Algebra **57** (1979), 502–521.

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