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Osaka University
On Essential Components of the Set of Fixed Points

By Shin'ichi Kinoshita

Let $X$ be a compact metric space and let $f$ be a continuous mapping of $X$ into itself. A fixed point $p$ of $f$ was called by M. K. Fort Jr.\(^1\) an essential fixed point of $f$, if for every neighbourhood $U$ of $p$ there exists $\delta > 0$ such that every $g \in X^x$ with $|g - f| < \delta$ has at least one fixed point in $U$. Then for example, the identity mapping of the interval $[0, 1]$ has no essential fixed point. We shall introduce in this note a notion of essential components (see below) of the set of fixed points: thus if $X$ is an absolute retract\(^2\), then every continuous mapping of $X$ into itself has essential components of the set of fixed points and if $X$ is an absolute neighbourhood retract\(^3\), then every continuous mapping of $X$ into itself which is homotopic to a constant mapping has the same property.

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1. Let $X$ be a compact metric space\(^4\) and let $f$ be a mapping\(^5\) of $X$ into itself. Let $f$ have fixed points and let $A$ be the set of all fixed points, $C$ being a component of $A$. Then $C$ will be called an essential component of $A$, if for every open set $U$ which contains $C$ there exists $\delta$ such that every $g \in X^x$ with $|g - f| < \delta$ has at least one fixed point in $U$. We say that $X$ has property $F'$ if every mapping of $X$ into itself has at least one essential component of the set of fixed points.

**Theorem 1.** Property $F'$ is invariant under retraction\(^6\).

Proof. Let $Y$ be a retract of a compact space $X$ having property

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4) In this note we assume that the space is separable metric.
5) In this note every mapping means a continuous mapping.
6) Let $Y$ be a closed subset of $X$. If there exists a mapping $r$ of $X$ onto $Y$ such that $r(x) = x$ for $x \in Y$, then $Y$ is called by K. Borsuk a retract of $X$ and the mapping $r$, a retraction of $X$ onto $Y$. Cf. K. Borsuk, Fund. Math. 17. loc. cit.
and let \( r \) be a retraction of \( X \) onto \( Y \). Let \( f \) be a mapping of \( Y \) into itself. Then \( fr \) is a mapping of \( X \) into itself. Since \( X \) has property \( F' \), there exists an essential component \( C \) of the set of fixed points of \( fr \). Clearly \( C \) is a component of the set of all fixed points of \( f \). If \( U \) is an open subset (of \( Y \)) which contains \( C \), then there exists an open subset \( U' \) (of \( X \)) with \( U' \cdot Y = U \). It follows that for \( U' \) there exists \( \delta > 0 \) such that every \( g' \in X^X \) with \( |g' - fr| < \delta \) has at least one fixed point in \( U' \). Let \( g \) be a mapping of \( Y \) into itself with \( |g - f| < \delta \). Since \( |gr - fr| < \delta \), it follows that \( gr \) has at least one fixed point in \( U' \). Clearly this fixed point is contained in \( Y \). Therefore \( g \) has at least one fixed point in \( U' \cdot Y = U \), and the proof is complete.

**Lemma 1.** The Hilbert cube \( I_\omega \) has property \( F' \).

**Proof.** The Hilbert cube has the fixed point property\(^7\). Let \( f \in I_\omega \) and let \( A \) be the set of all fixed points of \( f \). Let \( A \) be decomposed into components \( C_\alpha \). Then it follows that:

1. \( A = \sum_c C_\alpha \),
2. \( C_\alpha \cdot C_\beta = 0 \) (\( \alpha \neq \beta \)),
3. \( A \) and all \( C_\alpha \) are compact.

If no \( C_\alpha \) is essential component, then for every \( C_\alpha \) there exists an open set \( U_\alpha \) which contains \( C_\alpha \) satisfying the following conditions: for every \( \delta > 0 \) there exists \( g_\alpha \in I_\omega \) with \( |g_\alpha - f| < \delta \) having no fixed point in \( U_\alpha \).

It can easily be seen that there exist two finite open coverings \( \{V_i\} \) and \( \{W_i\} \) (\( i = 1, 2, \ldots, n \)) (of \( A \)) which satisfy the following conditions:

1. \( W_i \subset V_i \),
2. \( V_i \cdot V_j = 0 \) for \( i \neq j \),
3. \( V_i \) contains at least one \( C_\alpha \) with \( U_\alpha \supset V_i \).

Since \( I_\omega - \sum_{i=1}^n W_i \) is compact and \( f \) has no fixed point on it, there exists an \( \alpha > 0 \) such that \( |x - f(x)| > \alpha \) for \( x \in I_\omega - \sum_{i=1}^n W_i \). Since \( V_i \) contains at least one \( C_\alpha \) with \( U_\alpha \supset V_i \), there exists a mapping \( g_\alpha \) with \( |g_\alpha - f| < \alpha \) having no fixed point in \( V_i \).

Using vectorial notation, we construct the mapping \( \varphi \) as follows:

\[
\varphi(x) = f(x) \quad \text{for} \quad x \in I_\omega - \sum_{i=1}^n V_i,
\]

\[
\varphi(x) = g_\alpha(x) \quad \text{for} \quad x \in W_i,
\]

\[
\varphi(x) = \frac{d(x, \overline{W}_i)}{d(x, \overline{W}_i) + d(x, I_\omega - \sum_{i=1}^n V_i)} \cdot f(x) + \frac{d(x, I_\omega - \sum_{i=1}^n V_i)}{d(x, \overline{W}_i) + d(x, I_\omega - \sum_{i=1}^n V_i)} \cdot g_\alpha(x)
\]

for \( x \in V_i - W_i \).

\(7\) See for instance, C. Kuratowski: *Topologie II* (1950), p. 263

\(8\) \( d(x, A) \) means the distance from \( x \) to \( A \).
It is easily seen that \(|\varphi - f| < \alpha\), and consequently \(\varphi \in I_f\) has no fixed point, which is impossible, and the proof is complete.

By Theorem 1 and Lemma 1 it follows immediately the

**Theorem 2.** Every absolute retract\(^9\) has property \(F'\).

2. **Lemma 2.** Let \(X\) be an absolute neighbourhood retract\(^{10}\). If \(f \in X^{I_{\omega}}\), then for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that for every \(g \in X^X\) with \(|g - f'| < \delta\) within \(X^X(f' = f|X)\)\(^{11}\) there exists an extension \(\varphi\) of \(g\) on \(I_\omega\) relative to \(X\) with \(|\varphi - f| < \varepsilon\).

Proof. Let \(X\) be imbedded in \(I_\omega\) and let \(f\) be a mapping of \(I_\omega\) into \(X\). Since \(X\) is an absolute neighbourhood retract, there exist a neighbourhood \(U\) of \(X\) and a retraction \(r\) of \(U\) onto \(X\). For \(\varepsilon/2\) there exists \(\delta' > 0\) such that \(d(x, X) < \delta'\) yields \(|x - r(x)| < \varepsilon/2\).

By a lemma of K. Borsuk\(^{12}\), for \(\delta'\) there exist \(\delta > 0\) such that for every \(g \in X^X\) with \(|g - f'| < \delta\) \((f' = f|X)\) there exists an extension \(\varphi'\) of \(g\) on \(I_\omega\) relative to \(I_\omega\) with \(|\varphi' - f| < \delta\).

Using this \(\delta\), let \(g \in X^X\) with \(|g - f'| < \delta\). Then there exists an extension \(\varphi'\) which satisfies the above condition. Let \(\varphi = r\varphi'\). Then \(|\varphi - f'| < \varepsilon/2\). Since \(|\varphi' - f| < \delta' \leq \delta/2\), it follows \(|\varphi - f| < \varepsilon\) and \(\varphi\) is an extension of \(g\) on \(I_\omega\) relative to \(X\), and the proof is complete.

**Theorem 3.** Let \(X\) be an absolute neighbourhood retract. If \(f \in X^X\) is homotopic to a constant mapping, then \(f\) has at least one essential component of the set of fixed points.

Proof. Let \(X\) be imbedded in \(I_\omega\). If \(f \in X^X\) is homotopic to a constant mapping, then there exists an extension \(\varphi\) of \(f\) on \(I_\omega\) relative to \(X\)\(^{13}\). Since \(I_\omega\) has property \(F'\) by Lemma 1, \(\varphi\) has an essential component \(C\) of the set of fixed points, and \(C\) is at the same time a component of the set of all fixed points of \(f\). Let \(U\) be an open subset (of \(X\)) which contains \(C\). Then there exists an open subset \(U'\) of \(I_\omega\) with

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9) A compact separable metric space is an absolute retract if and only if it is homeomorphic to a retract of \(I_\omega\). K. Borsuk, Fund. Math. 17, loc. cit.

10) A closed subset \(Y\) of \(X\) is a neighbourhood retract of \(X\) if there exists an open set \(U\) which contains \(Y\) and there exists a retraction of \(U\) onto \(Y\). A compact separable metric space \(X\) is an absolute neighbourhood retract if and only if \(X\) is homeomorphic to a neighbourhood retract of \(I_\omega\). K. Borsuk, Fund. Math. 19, loc. cit.

11) \(f|X\) means the partial mapping of \(f\) operating only on \(X\).

12) The lemma of K. Borsuk is as follows: let \(M\) be a separable metric space, \(A\) a closed subset of \(M\) and \(f \in I^M_A\). Then for every \(\varepsilon > 0\) there exists \(\varepsilon > 0\) such that for every \(g \in I^M_A\) with \(|g(x) - f(x)| < \varepsilon\) for \(x \in A\) there exists an extension \(\varphi\) of \(g\) on \(M\) relative to \(I_\omega\) with \(|\varphi - f| < \varepsilon\). K. Borsuk, Fund. Math. 19, loc. cit. p. 227.

$U' \cdot X = U$. It follows that for $U'$ there exists $\delta' > 0$ such that every $\varphi'$ with $|\varphi' - \varphi| < \delta'$ has at least one fixed point in $U'$. For $\delta'$ there exists $\delta > 0$ satisfying the condition of Lemma 2. Then for every $g \in X^x$ with $|g - f| < \delta$ there exists an extension $\varphi'$ of $g$ on $I_\omega$ relative to $X$ with $|\varphi - \varphi'| < \delta'$. Therefore $\varphi'$ has at least one fixed point in $U'$. Since this fixed point of $\varphi'$ is contained in $X$, $g$ has at least one fixed point in $U' \cdot X = U$, and the proof is complete.

**Problem.** Does there exist a space which has the fixed point property but which has not property $F''$?

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