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On Essential Components of the Set of Fixed Points

By Shin'ichi Kinoshita

Let $X$ be a compact metric space and let $f$ be a continuous mapping of $X$ into itself. A fixed point $p$ of $f$ was called by M. K. Fort Jr.\(^1\) an essential fixed point of $f$, if for every neighbourhood $U$ of $p$ there exists $\delta > 0$ such that every $g \in X$ with $|g - f| < \delta$ has at least one fixed point in $U$. Then for example, the identity mapping of the interval $[0, 1]$ has no essential fixed point. We shall introduce in this note a notion of essential components (see below) of the set of fixed points: thus if $X$ is an absolute retract\(^2\), then every continuous mapping of $X$ into itself has essential components of the set of fixed points and if $X$ is an absolute neighbourhood retract\(^3\), then every continuous mapping of $X$ into itself which is homotopic to a constant mapping has the same property.

I express my sincere thanks to Prof. H. Terasaka for his valuable advices.

1. Let $X$ be a compact metric space\(^4\) and let $f$ be a mapping\(^5\) of $X$ into itself. Let $f$ have fixed points and let $A$ be the set of all fixed points, $C$ being a component of $A$. Then $C$ will be called an essential component of $A$, if for every open set $U$ which contains $C$ there exists $\delta$ such that every $g \in X$ with $|g - f| < \delta$ has at least one fixed point in $U$. We say that $X$ has property $F$ if every mapping of $X$ into itself has at least one essential component of the set of fixed points.

**Theorem 1.** Property $F'$ is invariant under retraction\(^6\).

**Proof.** Let $Y$ be a retract of a compact space $X$ having property

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4) In this note we assume that the space is separable metric.
5) In this note every mapping means a continuous mapping.
6) Let $Y$ be a closed subset of $X$. If there exists a mapping $r$ of $X$ onto $Y$ such that $r(x) = x$ for $x \in Y$, then $Y$ is called by K. Borsuk a retract of $X$ and the mapping $r$, a retraction of $X$ onto $Y$. Cf. K. Borsuk, Fund. Math. 17, loc. cit.
and let \( r \) be a retraction of \( X \) onto \( Y \). Let \( f \) be a mapping of \( Y \) into itself. Then \( fr \) is a mapping of \( X \) into itself. Since \( X \) has property \( F'' \), there exists an essential component \( C \) of the set of fixed points of \( fr \). Clearly \( C \) is a component of the set of all fixed points of \( f \). If \( U \) is an open subset (of \( Y \)) which contains \( C \), then there exists an open subset \( U' \) (of \( X \)) with \( U' \cdot Y = U \). It follows that for \( U' \) there exists \( \delta > 0 \) such that every \( g' \in X^x \) with \(|g' - fr| < \delta \) has at least one fixed point in \( U' \). Let \( g \) be a mapping of \( Y \) into itself with \(|g - f| < \delta \). Since \(|gr - fr| < \delta \), it follows that \( gr \) has at least one fixed point in \( U' \). Clearly this fixed point is contained in \( Y \). Therefore \( g \) has at least one fixed point in \( U' \cdot Y = U \), and the proof is complete.

**Lemma 1.** The Hilbert cube \( I^\omega \) has property \( F'' \).

Proof. The Hilbert cube has the fixed point property \(^7\). Let \( f \in I^\omega \) and let \( A \) be the set of all fixed points of \( f \). Let \( A \) be decomposed into components \( C_\alpha \). Then it follows that:

1. \( A = \sum C_\alpha \),
2. \( C_\alpha \cdot C_\beta = 0 \) \((\alpha \neq \beta)\),
3. \( A \) and all \( C_\alpha \) are compact.

If no \( C_\alpha \) is essential component, then for every \( C_\alpha \) there exists an open set \( U_\alpha \) which contains \( C_\alpha \) satisfying the following conditions: for every \( \delta > 0 \) there exists \( g_\alpha \in I^\omega \) with \(|g_\alpha - f| < \delta \) having no fixed point in \( U_\alpha \).

It can easily be seen that there exist two finite open coverings \( \{ V_i \} \) and \( \{ W_i \} \) \((i = 1, 2, \ldots, n)\) (of \( A \)) which satisfy the following conditions:

4. \( \overline{W_i} \subset V_i \),
5. \( V_i \cdot V_j = 0 \) for \( i \neq j \),
6. \( V_i \) contains at least one \( C_\alpha \) with \( U_\alpha \supseteq V_i \).

Since \( I^\omega - \sum W_i \) is compact and \( f \) has no fixed point on it, there exists an \( a > 0 \) such that \(|x - f(x)| > a \) for \( x \in I^\omega - \sum W_i \). Since \( V_i \) contains at least one \( C_\alpha \) with \( U_\alpha \supseteq V_i \), there exists a mapping \( g \), with \(|g - f| < a \) having no fixed point in \( V_i \).

Using vectorial notation, we construct the mapping \( \varphi \) as follows:

\[
\varphi(x) = \begin{cases} f(x) & \text{for } x \in I^\omega - \sum V_i, \\ g(x) & \text{for } x \in W_i, \end{cases}
\]

\[
\varphi(x) = \frac{d(x, \overline{W_i})}{d(x, \overline{W_i}) + d(x, I^\omega - \sum V_i)} f(x) + \frac{d(x, I^\omega - \sum V_i)}{d(x, \overline{W_i}) + d(x, I^\omega - \sum V_i)} g(x) \quad \text{for } x \in \forall V_i - W_i.
\]

\(^7\) See for instance, C. Kuratowski: Topologie II (1950), p. 263

\(^8\) \( d(x, A) \) means the distance from \( x \) to \( A \).
It is easily seen that $|\varphi - f| < \alpha$, and consequently $\varphi \in I^f$ has no fixed point, which is impossible, and the proof is complete.

By Theorem 1 and Lemma 1 it follows immediately the

**Theorem 2.** Every absolute retract has property $F"$.

2. Lemma 2. Let $X$ be an absolute neighbourhood retract. If $f \in X^I\omega$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $g \in X^\varepsilon$ with $|g - f'| < \delta$ within $X^\varepsilon(f' = f|X)$ there exists an extension $\varphi$ of $g$ on $I_\omega$ relative to $X$ with $|\varphi - f| < \varepsilon$.

Proof. Let $X$ be imbedded in $I_\omega$ and let $f$ be a mapping of $I_\omega$ into $X$. Since $X$ is an absolute neighbourhood retract, there exist a neighbourhood $U$ of $X$ and a retraction $r$ of $U$ onto $X$. For $\varepsilon/2$ there exists $\delta' > 0$ such that $d(x, X) < \delta'$ yields $|x - r(x)| < \varepsilon/2$.

By a lemma of K. Borsuk, for $\delta'$ there exist $\delta > 0$ such that for every $g \in X^\varepsilon$ with $|g - f'| < \delta(f' = f|X)$ there exists an extension $\varphi'$ of $g$ on $I_\omega$ relative to $I_\omega$ with $|\varphi' - f| < \delta'$.

Using this $\delta$, let $g \in X^\varepsilon$ with $|g - f'| < \delta$. Then there exists an extension $\varphi'$ which satisfies the above condition. Let $\varphi = r\varphi'$. Then $|\varphi - \varphi'| < \varepsilon/2$. Since $|\varphi' - f| < \delta' \leq \delta/2$, it follows $|\varphi - f| < \varepsilon$ and $\varphi$ is an extension of $g$ on $I_\omega$ relative to $X$, and the proof is complete.

**Theorem 3.** Let $X$ be an absolute neighbourhood retract. If $f \in X^x$ is homotopic to a constant mapping, then $f$ has at least one essential component of the set of fixed points.

Proof. Let $X$ be imbedded in $I_\omega$. If $f \in X^x$ is homotopic to a constant mapping, then there exists an extension $\varphi$ of $f$ on $I_\omega$ relative to $X$. Since $I_\omega$ has property $F"$ by Lemma 1, $\varphi$ has an essential component $C$ of the set of fixed points, and $C$ is at the same time a component of the set of all fixed points of $f$. Let $U$ be an open subset (of $X$) which contains $C$. Then there exists an open subset $U'$ of $I_\omega$ with

9) A compact separable metric space is an absolute retract if and only if it is homeomorphic to a retract of $I_\omega$. K. Borsuk, Fund. Math. 17, loc. cit.
10) A closed subset $Y$ of $X$ is a neighbourhood retract of $X$ if there exists an open set $U$ which contains $Y$ and there exists a retraction of $U$ onto $Y$. A compact separable metric space $X$ is an absolute neighbourhood retract if and only if $X$ is homeomorphic to a neighbourhood retract of $I_\omega$. K. Borsuk, Fund. Math. 19, loc. cit.
11) $f|X$ means the partial mapping of $f$ operating only on $X$.
12) The lemma of K. Borsuk is as follows: let $M$ be a separable metric space, $A$ a closed subset of $M$ and $f \in I^n_M$. Then for every $\varepsilon > 0$ there exists $\varepsilon > 0$ such that for every $g \in I^n_M$ with $|g - f| < \varepsilon$ for $x \in A$ there exists an extension $\varphi$ of $g$ on $M$ relative to $I_\omega$ with $|\varphi - f| < \varepsilon$. K. Borsuk, Fund. Math. 19, loc. cit. p. 227.
It follows that for \( U' \) there exists \( \delta' > 0 \) such that every \( \varphi' \) with \( |\varphi' - \varphi| < \delta' \) has at least one fixed point in \( U' \). For \( \delta' \) there exists \( \delta > 0 \) satisfying the condition of Lemma 2. Then for every \( g \in X^x \) with \( |g - f| < \delta \) there exists an extension \( \varphi' \) of \( g \) on \( I_w \) relative to \( X \) with \( |\varphi - \varphi'| < \delta' \). Therefore \( \varphi' \) has at least one fixed point in \( U' \). Since this fixed point of \( \varphi' \) is contained in \( X \), \( g \) has at least one fixed point in \( U' \cdot X = U \), and the proof is complete.

**Problem.** Does there exist a space which has the fixed point property but which has not property \( F'' \)?

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