

Title	On essential components of the set of fixed points
Author(s)	Kinoshita, Shin'ichi
Citation	Osaka Mathematical Journal. 1952, 4(1), p. 19-22
Version Type	VoR
URL	https://doi.org/10.18910/9530
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

https://ir.library.osaka-u.ac.jp/

The University of Osaka

On Essential Components of the Set of Fixed Points

By Shin'ichi KINOSHITA

Let X be a compact metric space and let f be a continuous mapping of X into itself. A fixed point p of f was called by M. K. Fort Jr.¹⁾ an essential fixed point of f, if for every neighbourhood U of p there exists $\delta > 0$ such that every $g \in X^x$ with $|g-f| < \delta$ has at least one fixed point in U. Then for example, the identity mapping of the interval [0, 1] has no essential fixed point. We shall introduce in this note a notion of essential components (see below) of the set of fixed points : thus if X is an absolute retract²⁾, then every continuous mapping of X into itself has essential components of the set of fixed points and if X is an absolute neighbourhood retract³⁾, then every continuous mapping of X into itself which is homotopic to a constant mapping has the same property.

I express my sincere thanks to Prof. H. Terasaka for his valuable advices.

1. Let X be a compact metric space⁴⁾ and let f be a mapping⁵⁾ of X into itself. Let f have fixed points and let A be the set of all fixed points, C being a component of A. Then C will be called an *essential* component of A, if for every open set U which contains C there exists δ such that every $g \in X^x$ with $|g-f| < \delta$ has at least one fixed point in U. We say that X has property F' if every mapping of X into itself has at least one essential component of the set of fixed points.

Theorem 1. Property F' is invariant under retraction⁶⁾.

Proof. Let Y be a retract of a compact space X having property

- 4) In this note we assume that the space is separable metric.
- 5) In this note every mapping means a continuous mapping.

¹⁾ M. K. Fort Jr.: Essential and nonessential fixed points, Amer. Jour. Math. 72 (1950), pp. 315-322.

²⁾ In the sense of K. Borsuk. See, K. Borsuk: Sur let rétractes, Fund. Math. 17 (1931), pp. 152-170.

³⁾ In the sense of K. Borsuk. See, K. Borsuk: Ueber eine Klasse von lokal zusammenhängenden Räumen, Fund. Math. 19 (1932), pp. 220-242.

⁶⁾ Let Y be a closed subset of X. If there exists a mapping r of X onto Y such that r(x) = x for $x \in Y$, then Y is called by K. Borsuk a retract of X and the mapping r, a retraction of X onto Y. Cf. K. Borsuk, Fund. Math. 17. loc. cit.

F' and let r be a retraction of X onto Y. Let f be a mapping of Y into itself. Then fr is a mapping of X into itself. Since X has property F', there exists an essential component C of the set of fixed points of fr. Clearly C is a component of the set of all fixed points of f. If Uis an open subset (of Y) which contains C, then there exists an open subset U' (of X) with $U' \cdot Y = U$. It follows that for U' there exists $\delta > 0$ such that every $g' \in X^x$ with $|g' - fr| < \delta$ has at least one fixed point in U'. Let g be a mapping of Y into itself with $|g-f| < \delta$. Since $|gr - fr| < \delta$, it follows that gr has at least one fixed point in U'. Clearly this fixed point is contained in Y. Therefore g has at least one fixed point in $U' \cdot Y = U$, and the proof is complete.

Lemma 1. The Hilbert cube I_{ω} has property F'.

Proof. The Hilbert cube has the fixed point property⁷). Let $f \in I_{\omega}^{I_{\omega}}$ and let A be the set of all fixed points of f. Let A be decomposed into components C_{ω} . Then it follows that:

(1) $A = \sum_{\alpha} C_{\alpha}$,

(2) $C_{\alpha} \cdot C_{\beta} = 0 (\alpha \neq \beta),$

(3) A and all C_{α} are compact.

If no C_{α} is essential component, then for every C_{α} there exists an open set U_{α} which contains C_{α} satisfying the following conditions: for every $\delta > 0$ there exists $g_{\alpha} \in I_{\alpha}^{I_{\alpha}}$ with $|g_{\alpha} - f| < \delta$ having no fixed point in U_{α} .

It can easily be seen that there exist two finite open coverings $\{V_i\}$ and $\{W_i\}$ (i = 1, 2, ..., n) (of A) which satisfy the following conditions:

(4) $\overline{W}_i \subset V_i$;

(5) $V_i \cdot V_j = 0$ for $i \neq j$,

(6) V_i contains at least one C_{z_i} with $U_{z_i} > V_i$.

Since $I_{\omega} - \sum_{i=1}^{n} W_i$ is compact and f has no fixed point on it, there exists an a > 0 such that |x - f(x)| > a for $x \in I_{\omega} - \sum_{i=1}^{n} W_i$. Since V_i contains at least one C_{x_i} with $U_{x_i} > V_i$, there exists a mapping g_i with $|g_i - f| < a$ having no fixed point in V_i .

Using vectorial notation, we construct the mapping φ as follows:

$$\begin{split} \varphi(x) &= f(x) \quad \text{for} \quad x \in I_{\omega} - \sum_{i=1}^{n} V_{i} ,\\ \varphi(x) &= g_{i}(x) \quad \text{for} \quad x \in W_{i} ,\\ \varphi(x) &= \frac{d(x, \overline{W}_{i})}{d(x, \overline{W}_{i}) + d(x, I_{\omega} - \sum_{i=1}^{n} V_{i})} f(x) + \frac{d(x, I_{\omega} - \sum_{i=1}^{n} V_{i})}{d(x, \overline{W}_{i}) + d(x, I_{\omega} - \sum_{i=1}^{n} V_{i})} g_{i}(x)^{8)} \\ \text{for} \quad x \in V_{i} - W_{i} . \end{split}$$

7) See for instance, C. Kuratowski: Topologie II (1950), p. 263

⁸⁾ d(x, A) means the distance from x to A.

It is easily seen that $|\varphi - f| < a$, and consequently $\varphi \in I^l$ has no fixed point, which is impossible, and the proof is complete.

By Therem 1 and Lemma 1 it follows immediately the

Theorem 2. Every absolute retract⁹⁾ has property F'.

2. Lemma 2. Let X be an absolute neighbourhood retract¹⁶). If $f \in X^{I_{\infty}}$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $g \in X^{x}$ with $|g-f'| < \delta$ within $X^{x}(f'=f|X)^{(1)}$ there exists an extension φ of g on I_{∞} relative to X with $|\varphi-f| < \varepsilon$.

Proof. Let X be imbedded in I_{ω} and let f be a mapping of I_{ω} into X. Since X is an absolute neghbourhood retract, there exist a neighbourhood U of X and a retraction r of U onto X. For $\varepsilon/2$ there exists $\delta' > 0$ such that $d(x, X) < \delta'$ yields $|x - r(x)| < \varepsilon/2$.

By a lemma of K. Borsuk¹²⁾, for δ' there exist $\delta > 0$ such that for every $g \in X^x$ with $|g-f'| < \delta(f'=f|X)$ there exists an extension φ' of gon I_{ω} relative to I_{ω} with $|\varphi'-f| < \delta'$.

Using this δ , let $g \in X^x$ with $|g-f'| < \delta$. Then there exists an extension φ' which satisfies the above condition. Let $\varphi = r\varphi'$. Then $|\varphi - \varphi'| < \varepsilon/2$. Since $|\varphi' - f| < \delta' \leq \delta/2$, it follows $|\varphi - f| < \varepsilon$ and φ is an extension of g on I_{ω} relative to X, and the proof is complete.

Theorem 3. Let X be an absolute neighbourhood retract. If $f \in X^x$ is homotopic to a constant mapping, then f has at least one essential component of the set of fixed points.

Proof. Let X be imbedded in I_{ω} . If $f \in X^x$ is homotopic to a constant mapping, then there exists an extension φ of f on I_{ω} relative to X^{13} . Since I_{ω} has property F' by Lemma 1, φ has an essential component C of the set of fixed points, and C is at the same time a component of the set of all fixed points of f. Let U be an open subset (of X) which contains C. Then there exists an open subset U' of I_{ω} with

⁹⁾ A compact separable metric space is an absolute retract if and only if it is homeomorphic to a retract of I_{w} . K. Borsuk, Fund. Math. 17, loc. cit.

¹⁰⁾ A closed subset Y of X is a neighbourhood retract of X if there exists an open set U which contains Y and there exists a retraction of U on Y. A compact separable metric space X is an absolute neighbourhood retract if and only if X is homeomorphic to a neighbourhood retract of I_{w} . K. Borsuk, Fund. Math. 19, loc. cit.

¹¹⁾ $f \mid X$ means the partial mapping of f operating only on X.

¹²⁾ The lemma of K. Borsuk is as follows: let M be a separable metric space, A a closed subset of M and $f \in I_{\omega}^{M}$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $g \in I_{\omega}^{A}$ with $|g(x) - f(x)| < \delta$ for $x \in A$ there exists an extension φ of g on M relative to I_{ω} with $|\varphi - f| < \varepsilon$. K. Borsuk, Fund. Math. 19, loc. cit. p. 227.

¹³⁾ K. Borsuk, Fund. Math. 19, 10c. cit. p. 229.

 $U' \cdot X = U$. It follows that for U' there exists $\delta' > 0$ such that every φ' with $|\varphi' - \varphi| < \delta'$ has at least one fixed point in U'. For δ' there exists $\delta > 0$ satisfying the condition of Lemma 2. Then for every $g \in X^x$ with $|g-f| < \delta$ there exists an extension φ' of g on I_{ω} relative to X with $|\varphi - \varphi'| < \delta'$. Therefore φ' has at least one fixed point in U'. Since this fixed point of φ' is contained in X, g has at least one fixed point in U'.

PROBLEM. Does there exist a space which has the fixed point property but which has not property F'?

(Received December 1, 1951)