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REMARKS ON OPEN SURFACES

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0. Introduction

Let X be a smooth complex analytic open surface: that is, X is biholomorphically equivalent to $M \setminus D$, where M is a compact complex variety of dimension 2 and D is a closed analytic subvariety of M . We can assume that M is also smooth.

We shall be interested in the case where X is strongly pseudoconvex (see [6] for definitions). Such an X contains a distinguished compact analytic subset Z , which is the union of all closed analytic subspaces of X of positive dimension. Z is empty if and only if X is Stein.

A famous remark of Serre, in [8], points out that M is not determined up to bimeromorphic equivalence by X . If $X = \mathbb{C}^* \times \mathbb{C}^*$ then M can be rational or elliptic ruled, or (an observation due to Igusa, see [1]) a non-elliptic Hopf surface. Naturally one asks: for what other such X , if any, is M not unique up to bimeromorphic equivalence?

This question and some related ones have been considered by (among others) Tan, in a series of papers ([11], [12], [13], [14]). There it is always assumed that M is minimal. This, however, imposes a further restriction on X . The purpose of this note is to see what happens for general M .

1. A non-minimal example

We give an easy example of a Stein open surface X for which only non-minimal compactifications exist. Let M' be a surface whose universal cover is a bounded domain, say a ball in \mathbb{C}^2 . Then M' is a strongly minimal surface of general type and is hyperbolic in the sense of [4]. Let $\pi: M \rightarrow M'$ be the blow-up of M' in a point p , and let $C = \pi^{-1}(p)$ (so that C is a (-1) -curve in M). Fix some projective embedding of M and let D be a general hyperplane section (so $C \not\subseteq D$). Put $X = M \setminus D$: thus X is affine, and therefore Stein. As we shall see below, the fact that M is of general type implies that it is determined by X up to bimeromorphic equivalence. By the uniqueness of minimal models, M' is the only minimal surface which can possibly contain an open subset biholomorphically equivalent to X . Suppose it does: let $\varphi: X \hookrightarrow M'$ be a

holomorphic embedding. By Theorem VI.4.1 of [4], φ extends to a holomorphic map $\tilde{\varphi}: M \rightarrow M'$. By strong minimality, the bimeromorphic map $\alpha = \pi\tilde{\varphi}^{-1}: M' \dashrightarrow M'$ is biholomorphic. But $\pi = \alpha\tilde{\varphi}$ contracts C : therefore $\tilde{\varphi}$ contracts C , but $\tilde{\varphi}$ is an embedding on X so this is impossible.

A similar argument can be used to give an example of an X which does have a minimal embedding, but only if the boundary D is allowed to have worse than normal crossings singularities. This, too, will force us to consider non-minimal surfaces.

2. Non-algebraic compactifications

We shall say that a surface X is compactifiable if an M and a D such that X is biholomorphically equivalent to $M \setminus D$ exist.

Theorem 2.1. *Let X be a compactifiable Stein surface. Then X has an algebraic compactification.*

Proof. (This is proved in [14] under the assumption that X has a minimal compactification.) Let M be a compactification. By [1] either M is algebraic, in which case there is nothing to prove, or $b_1(M) = 1$ and M has no non-constant meromorphic functions. Assume the latter. Then M contains only finitely many irreducible curves, C_1, \dots, C_k say. Put $C = C_1 \cup \dots \cup C_k$. The boundary $D = M \setminus X$ is a non-empty connected compact curve: since D meets all the curves C_j and there are no others, C is connected.

The intersection form on C must be negative semidefinite. For if there is a divisor C_0 such that $C_0^2 > 0$ then, by Riemann-Roch,

$$h^0(\mathcal{O}_M(nC_0)) + h^0(\mathcal{O}_M(K_M - nC_0)) = n^2 C^2/2 + o(n).$$

So for $n \gg 0$ one of $\mathcal{O}_M(nC_0)$ and $\mathcal{O}_M(K_M - nC_0)$ has two independent sections, and their ratio gives a meromorphic function on M . Furthermore, if the intersection form on C is negative definite then D can be contracted to a (possibly singular) point and Hartogs' Theorem contradicts the holomorphic convexity of X . Therefore there is a divisor D_0 , supported on D , such that $D_0^2 = 0$.

By [15], Lemma 2, we can assume that D_0 is effective and $\text{Supp } D_0 = C$. So $D = C$, so all curves in M , and in particular all (-1) -curves, are contained in the boundary. Hence we can contract the (-1) -curves and assume that M is minimal, which reduces us to the situation of Theorem 3 of [14]. ■

3. Uniqueness questions

Suppose M is algebraic: we ask whether M is unique up to bimeromorphic (or birational) equivalence. In general, of course, it is not. In [13] it is shown that there is usually at most one minimal compactification. We shall

show that there is usually just one compactification M .

We denote by $\kappa_{\text{an}}(X)$ the analytic Kodaira dimension of Sakai ([7]). $\kappa_{\text{an}}(X)$ is a biholomorphic invariant of X . We denote by $\bar{\kappa}(X)$ the logarithmic Kodaira dimension of X (see [2]): this may depend on the choice of compactification. If L is a divisor on M then $\kappa(M, L)$ will be the L -dimension and $\kappa(M)$ the usual Kodaira dimension.

Theorem 3.1. *Let X be a compactifiable strongly pseudoconvex surface and suppose that $\kappa(M) \geq 0$ for some algebraic compactification. Then M is determined by X up to birational equivalence.*

We shall prove this theorem via a series of lemmas, some of which we shall refer to later.

Lemma 3.2. *If $\kappa_{\text{an}}(X) = 2$ then M is unique up to bimeromorphic equivalence.*

Proof. If M_1 and M_2 are compactifications of X then the map $\varphi: X \times X \rightarrow X \times X$ given by $\varphi(x_1, x_2) = (x_2, x_1)$ extends to a bimeromorphic map $\bar{\varphi}: M_1 \times M_2 \rightarrow M_1 \times M_2$. This follows from [7], Proposition 4.3, using the fact that $\kappa_{\text{an}}(M_1 \times M_2) = 4$. For a general point $\xi \in M_2$, the map

$$M_1 \xrightarrow{\sim} M_1 \times \{\xi\} \hookrightarrow M_1 \times M_2 \xrightarrow{\bar{\varphi}} M_1 \times M_2 \xrightarrow{pr_2} M_2$$

is bimeromorphic. In particular, if $\bar{\kappa}(M) = 2$ then M is unique, as was stated in §1 above. ■

Lemma 3.3. (Tan) *If $\kappa(M) \geq 0$ and M is minimal then $\bar{\kappa}(X) = 2$.*

Note that this implies that $\kappa_{\text{an}}(X) = 2$ and therefore that M is unique.

Proof. Suppose $\kappa(M) = \bar{\kappa}(X) = 1$. Since M is minimal, the canonical divisor K is nef. Suppose that $K \cdot D_j > 0$ for some irreducible component D_j of D . Then $(D + rK)^2 > 0$ for $r \gg 0$. If r is such that the r th plurigenus $P_r \geq 2$, then rK is effective, so by Lemma 8.5 of [2] we have $\kappa(M, D + rK) = 2$ and therefore, by Lemma 10.5 of [2], $\bar{\kappa}(X) = 2$. So we can assume that $K \cdot D_j = 0$ for all D_j . Moreover, M is elliptic. D must be contained in the fibres (because of $K \cdot D_j = 0$); but then X contains infinitely many complete curves.

If $K \equiv 0$ we can use Lemma 10.3 of [2] to reduce to the case $K = 0$. If $\bar{\kappa}(X) = 1$ then Theorem 8.6 of [2] shows that X contains infinitely many complete curves; but if M is an abelian surface $h^0(\mathcal{O}_M(2D)) \geq 2$, so $\bar{\kappa}(X) \neq 0$. On the other hand if M is a $K3$ then the boundary is contractible (as in [2], §10.4). ■

REMARK. Tan proves essentially the same lemma in [13] by using the structure theorem in [5] (II.2.3.1). However, there are some details to be checked and the above argument is more elementary.

To conclude the proof of Theorem 3.1 we need to deal with $\kappa(M)=0$ or 1 and M non-minimal. We may assume that Z contains no (-1) -curves (if it does we simply contract them). If $f: M \rightarrow M'$ is the contraction of a (-1) -curve E , we put $D'=f(D)$ and $X'=M' \setminus D'$. Since f is biholomorphic away from E and $X'=f(X \setminus E)$, X' is biholomorphically equivalent to $X \setminus E$. By [9], Lemma 2, $X \setminus E$ is strongly pseudoconvex, so X' is also. If $g: M' \rightarrow M''$ is a birational morphism to a minimal surface, g can be factored into finitely many contractions of (-1) -curves so, by induction, $X''=M'' \setminus g(D)$ is strongly pseudoconvex. By Lemma 3.3, $\bar{\kappa}(X')=2$. By [2], Theorem 11.4 (iii), $\bar{\kappa}(X)=\bar{\kappa}(X')$, so we are done by Lemma 3.2. ■

We can sharpen Theorem 3.1 further. Suppose now that M_i is an algebraic compactification of X_i with boundary $D_i=M_i \setminus X_i$, that X is strongly pseudoconvex and that X is biholomorphically equivalent to X_i for $i=1,2$. After blowing up one point of D_i if necessary (in case it happens that $M_i \cong \mathbf{P}^2$), we can assume that M_i are equipped with morphisms $\pi_i: M_i \rightarrow B_i$, where B_i is a smooth complete curve of genus g_i and the general fibre F_i is a \mathbf{P}^1 . We put $\alpha_i=D_i.F_i$ and let β_i be the number of fibers of π_i contained in D_i . Let $\theta: X_1 \rightarrow X_2$ be biholomorphic.

Theorem 3.4. *Suppose M_1 and M_2 are not bimeromorphically equivalent. Then $g_i \leq 1$, and if $g_i=1$ then $\beta_i=0$ and $\alpha_i=1$ or 2.*

Proof. Suppose first that, say, $g_2 > 1$ or that $g_2=1$ and $\beta_2 > 0$. Then $\pi_2(X_2)$ is $B_2 \setminus (\beta_2 \text{ points})$, which is hyperbolic. By [4], Ch. IV, Corollary 2.6, the map $\pi_2 \theta: F_1 \cap X_1 \rightarrow \pi_2(X_2)$ extends to a map $F_1 \rightarrow B_2$, which is constant by Lüroth's theorem. This determines a map $\hat{\theta}: \hat{B}_1 \rightarrow B_2$, given by $\hat{\theta}(p) = \pi_2 \theta (\pi_1^{-1}(p) \cap X_1)$ and defined on the Zariski-open set $\hat{B}_1 \subseteq B_1$ in which the fibres of π_1 are smooth and not contained in D_1 . It is clear that $\hat{\theta}$ extends to an isomorphism between B_1 and B_2 , so in this case M_1 and M_2 are bimeromorphically equivalent.

Now suppose that for one of the compactifications, M , we have $g=1$ and $\beta=0$, so that B is a complete curve of genus 1, and that $\alpha \geq 3$. I claim that $\bar{\kappa}(X)=2$, so that M is unique by Lemma 3.2. By the open version of $C_{2,1}$ proved in [3] we have

$$\bar{\kappa}(X) \geq \bar{\kappa}(B) + \bar{\kappa}(\mathbf{P}^1 \setminus (\alpha \text{ points})) = 1.$$

If $\bar{\kappa}(X)=1$, we apply the structure theorem in [5], II.2.3.1 already mentioned. Suppose first we are in the quasi-elliptic case, so that there is a birational morphism $f: M \rightarrow M_0$ and a divisor D_0 on M_0 such that $f^*(D_0 + K_{M_0}) = (D + K_M)^+$ (the arithmetically effective part of $D + K_M$). M_0 is equipped with a morphism $\pi_0: M_0 \rightarrow B_0 = B$, and $D_0.F_0=2$ where F_0 is the general fibre of π_0 . If C is a component of D such that $C.F \neq 0$, then C is not rational so f_*C is a component

of D_0 with multiplicity 1 ([5], p. 153). But $f_*F = F_0$, so $2 = D_0.F_0 = D.F = \alpha \geq 3$. This leaves only $\bar{\kappa}(X) = 2$.

If we are in the elliptic case of [5], II.2.3.1, then there is a birational morphism $f: M \rightarrow M_0$ as above, and M_0 is equipped with a morphism $\sigma: M_0 \rightarrow B'$ whose general fibre is elliptic and which contracts no (-1) -curves. Since M_0 is birationally ruled over B there is also a morphism $\pi: M_0 \rightarrow B$, and $B' \cong \mathbf{P}^1$. Then σ factors through $B \times B'$, so M_0 must be minimal (otherwise σ does contract some (-1) -curves).

According to [5], D_0 is a sum of fibres of σ (with multiplicity). So therefore, is $D_0 + K_{M_0}$. The general fibre E of $\sigma: M \rightarrow \mathbf{P}^1$ must meet D . Let C be a component of D such that $E.C > 0$. Since $\text{Supp}(D_0 + K_{M_0})^+$ lies in the fibres of σ , $\text{Supp}(D + K_M)^+$ lies in the fibres of σf . Hence $C \subseteq \text{Supp}(D + K_M)^-$, and in particular it is rational ([5], p. 153). It is therefore contained in a fibre F_0 of $\pi_0 f: M_0 \rightarrow B$. It is not contracted by f , since $C.E > 0$. All other components of F_0 are contracted by f , so they occur in $\text{Supp} K_M$. So the whole of F_0 occurs in $\text{Supp}(D + K_M)$, and indeed in the arithmetically negative part. But $F_0^2 = 0$, and the intersection form on the arithmetically negative part of a divisor is negative definite (see, for instance, [5], p. 37). ■

REMARK. Note that θ itself need not extend to a bimeromorphic map $M_1 \rightarrow M_2$. In general, as is shown in [10], it does not.

This result allows us to give another, partial, extension of a result of Tan to arbitrary smooth surfaces.

Theorem 3.5. (cf. [13], Theorem 4.) *Suppose X is strongly pseudoconvex but not Stein, and an algebraic compactification M exists. Then M is unique up to bimeromorphic equivalence except possibly in the following cases:*

- a) *Z contains a unique smooth elliptic curve E , and there are exactly two algebraic compactifications, M_1 rational and M_2 birationally equivalent to $E \times \mathbf{P}^1$ with $Z_2.F_2 = D_2.F_2 = 1$;*
- b) *Z can be contracted to give finitely many singularities belonging to a restricted class (possibly smooth points) any nonrational compactification M is birationally elliptic ruled, $D.F = 2$ and Z is contained in the singular fibres of M . If in this case D is irreducible, the possible singularities are A_k and D_k .*

Proof. Put $X^* = X \setminus Z$. Both M_1 and M_2 are compactifications of X^* , so by Lemma 3.2 we must have $\bar{\kappa}(X^*) < 2$. By Theorem 3.1, if there is more than one compactification then one of them must be birationally elliptic ruled. Let us abuse notation and call this M , so Z is now a curve in M . The boundary of X^* in M is $D + Z$ so, by the argument used in proving Lemma 3.4, $(D + Z).F \leq 2$. Clearly $D.F > 0$, so we have two cases, $Z.F = 1$ and $Z.F = 0$. In the case $Z.F = 1$ a section E of $M \rightarrow B$ occurs as a component of Z and $D.F = 1$. This is case (a).

Suppose, then, that $Z.F = 0$: thus Z is contained in finitely many fibres, ob-

vously singular ones. A singular fibre F_0 is a tree of rational curves. We contract all (-1) -curves which are contained in either D or Z . If all of Z is then contracted, we are in case (b) with smooth points only.

It is well known that the intersection form on F_0 is negative semidefinite. F_0 certainly contains at least one (-1) -curve: say

$$F_0 = \sum \lambda_i C'_i + \sum \mu_j C_j$$

where the C'_i are (-1) -curves and the C_j are not. Every curve in F_0 either meets D or is contained in Z , because Z is maximal, and the C'_i are not contained in Z nor in D . Hence $D.C'_i \geq 1$. If in addition D is irreducible then

$$2 = D.F_0 = \sum \lambda_i D.C'_i + \sum \mu_j D.C_j \geq \sum \lambda_i,$$

where the inequality comes from the fact that $C_j \not\subset \text{Supp } D$, so that $C_j.D \geq 0$. Furthermore, $p_a(F_0) = 0$ so $F_0.K = -2$. Hence

$$\begin{aligned} -2 &= \sum \lambda_i C'_i.K + \sum \mu_j C_j.K \\ &= -\sum \lambda_i + \sum \mu_j C_j.K \\ &\geq -2 + \sum \mu_j C_j.K. \end{aligned}$$

We have $C_j^2 < -1$, since the intersection form is negative semidefinite and $C_j^2 \neq -1$. (It is easy to see that $C_j^2 \neq 0$.) Also C_j is rational, so $C_j.K > -1$. But $\mu_j \geq 0$ for all j , and $\sum \mu_j C_j.K \leq 0$: hence $C_j.K = 0$ for all j , and the C_j are (-2) -curves. Thus $Z \cap F_0 = \sum C_j$ is a union of (-2) -curves and the intersection form is negative definite. This means that each connected component of Z has a configuration of type A_k, D_k, E_6, E_7 or E_8 . However, if the configuration is E_k then one can easily check that the intersection form on F_0 is not negative semidefinite. ■

Fibres as described in (b) can occur (that is, we cannot eliminate A_k or D_k as we eliminated E_k) but I do not know whether case (b) actually occurs, apart from the smooth case which certainly does.

The restricted class of singularities referred to in (b) is those whose weighted resolution graph can be embedded in the weighted graph of a singular fibre of genus 0 in such a way that its removal leaves at most two connected components. If we know more about D we can say more about Z , by doing calculations similar to those used above for the case D irreducible, but we shall not give any further examples here.

4. Other remarks

The argument in §2 also shows that strongly pseudoconvex surfaces with a non-algebraic, not necessarily minimal compactification are either as described in [11] or modifications of smooth Stein surfaces.

I do not know whether it is true in general that for smooth compactifiable Stein surfaces $\kappa_{\text{an}}(X)=2$ or $-\infty$ (cf. [14]). However, the only smooth curves C with $\kappa_{\text{an}}(C)=0$ are complete elliptic curves, so it follows from the fibration theorem (Theorem 2.3 of [7]) that if $\kappa_{\text{an}}(X)=1$ then X contains infinitely many complete curves. The corresponding result for affines is true: indeed we have the following stronger result.

Corollary 4.1. *Let X be a smooth affine n -dimensional variety. Then $\kappa_{\text{an}}(X)=-\infty$ or n .*

Proof. By Theorem 2.3 of [7] it is enough to show that $\kappa_{\text{an}}(X)\neq 0$ (and proceed by induction on n). By the argument in [14], Proposition 4, it is enough to show that $\kappa(M, D')\neq 0$ for a suitable ample divisor D' supported on D , i.e., that $h^0(\mathcal{O}_M(D'))>1$. This is obvious. ■

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