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<th>Local cohomology and connectedness of analytic subvarietie</th>
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<tr>
<td>Author(s)</td>
<td>Siu, Yum-tong</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 5(2) P.273–P.277</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1968</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/9532">https://doi.org/10.18910/9532</a></td>
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<td>DOI</td>
<td>10.18910/9532</td>
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Suppose $X$ is an analytic subvariety in some open neighborhood $G$ of the origin $0$ in $\mathbb{C}^n$ with $\text{codim}_{G,0}(X)=r$, where $\text{codim}_{G,0}(X)$ denotes the codimension at $0$ of $X$ as a subvariety of $G$. Let $\mathcal{O}$ be the structure sheaf of $\mathbb{C}^n$. Let $H^p_{X,0}(\mathcal{O})$ or simply $H^p_{X,0}$ denote the direct limit of $\{H^p(U-X, \mathcal{O})| U \text{ is an open neighborhood of } 0 \text{ in } G\}$ for $p\geq 1$. ($H^p_{X,0}$ agrees with the stalk at $0$ of the sheaf defined by the $p$-th local cohomology groups at $X$ with coefficients in $\mathcal{O}$, [1], p. 79). We say that $X$ is locally a complete intersection at $0$ if $X$ can be defined locally at $0$ by $r$ holomorphic functions. If $X$ is locally a complete intersection, obviously we have

$$H^p_{X,0} = 0 \quad \text{for } p>r.$$ 

The question naturally arises: to what extent does (1) characterize a local complete intersection? Not much is known about the characterization of local complete intersections. In [3] Hartshorne introduces a concept of connectedness which in our case is equivalent to the following: $X$ is locally connected in codimension $k$ at $0$ if the germ of $X$ at $0$ cannot be decomposed as the union of two subvariety-germs which are both different from the germ of $X$ at $0$ and whose intersection is a subvariety-germ $Y$ with $\text{codim}_{X,0}(Y)>k$. He shows that, if $X$ is locally a complete intersection, then $X$ is locally connected in codimension $1$ at $0$ (and also locally connected in codimension $1$ at $0$ in some properly defined formal sense). In this note we prove that (1) is a stronger necessary condition for local complete intersections than the connectedness condition. The following is our main theorem:

**Theorem 1.** Suppose $q\geq 0$. If $H^p_{X,0}=0$ for $p>q+r$, then $X$ is locally connected in codimension $q+1$ at $0$.

For the proof of Theorem 1 we need the following:

**Lemma 1.** Suppose $Y$ is a 1-dimensional subvariety in some open neighborhood $H$ of $0$ in $\mathbb{C}^n$. Suppose $0$ is the only singular point of $Y$ and $Y$ is locally irreducible at $0$. Then $H^p_{Y,0}=0$. 


Proof. Suppose \( D \) is an arbitrary open neighborhood of 0 in \( H \). By changing linearly the coordinates system of \( C^n \), we can find \( \{z_1, \ldots, z_n\} \subset C^n \) \( \mid z_i \mid < \delta_i, 1 \leq i \leq n \) for some \( \delta_i > 0, 1 \leq i \leq n \), such that the projection \( \pi: C^n \rightarrow C \) defined by \( \pi(z_1, \ldots, z_n) = z_i \) makes \( Y \cap U \) an irreducible analytic cover of \( s \) sheets over \( U_i = \{z_1 \in C \mid |z_1| < \delta_i \} \) with \{0\} as the critical set in \( U_i \) (III, B. 3, [2]) and \( \pi^{-1}(0) \cap Y \cap U = \{0\} \). Let \( \bar{U}_i = \{t \in C \mid |t| < \sqrt{\delta_i} \} \). We are going to define holomorphic functions \( g_k \) on \( \bar{U}_i, 2 \leq k \leq n \), such that

\[
Y \cap U = \{(t^*, g_1(t), \ldots, g_n(t)) \mid t \in \bar{U}_i \}.
\]

Fix \( z^* = (z_1^*, \ldots, z_n^*) \in Y \cap U \) with \( z_i^* \neq 0 \) and fix \( t^* \) with \( (t^*)_i = z_i^* \). Take \( t \in \bar{U}_i - \{0\} \). Let \( \gamma \) be a continuous map from \([0, 1]\) to \( \bar{U}_i - \{0\} \) such that \( \gamma(0) = t^* \) and \( \gamma(1) = t \). Let \( \hat{\gamma} \) be the continuous map from \([0, 1]\) to \( U_1 - \{0\} \) defined by \( \hat{\gamma}(c) = (\gamma(c))_i^* \) for \( c \in [0, 1] \). Then \( \hat{\gamma}(0) = z_1^* \). Since \( Y \cap U - \{0\} \) is a topological covering over \( U_1 - \{0\} \), there is a continuous map \( \hat{\gamma}: [0, 1] \rightarrow Y \cap U - \{0\} \) such that \( \pi \hat{\gamma} = \gamma \) and \( \hat{\gamma}(0) = z_1^* \). Define \( g_k(t) = z_{k, n}, 2 \leq k \leq n \). Set \( g_k(0) = 0, 2 \leq k \leq n \). It is readily verified that \( g_k \), \( 2 \leq k \leq n \), are well-defined and holomorphic. (2) is satisfied, because \( Y \cap U \) is irreducible. Define \( F: C^n \rightarrow C^n \) by \( F(w_1, \ldots, w_n) = ((w_1)^s_1, w_2, \ldots, w_n) \). Let \( F = F^{-1}(Y \cap U) \) and let \( s = F^{-1}(U) \). Let \( e_i, \ldots, e_n \) be all the distinct \( s \)-th roots of unity. Let \( Y_p = \{w \in C^n \mid w_2 = g_k(t), 2 \leq k \leq n \} \). Hence \( Y_p \) is a holomorphic function, \( H^q(U - Y_p, \mathcal{O}) = 0 \) for \( q > n - 1 \) and \( 1 \leq p \leq s \). The following portion of the Mayer-Vietoris sequence is exact:

\[
H^q(U - Y_p, \mathcal{O}) = H^q(U - \bigcup t^*_i Y_i, \mathcal{O}) \rightarrow H^q(U - \bigcup t^*_i Y_i, \mathcal{O}) \rightarrow H^{q+1}(U - \bigcup (t^*_i Y_i), \mathcal{O}), q \geq 0, 1 \leq p < s \].
\]

Since \( H^q(U - \bigcup (t^*_i Y_i), \mathcal{O}) = 0 \) for \( q > n - 1 \) (see Probleme 1, [4] or Th., [5]), by induction on \( p \) we conclude that \( H^q(U - \bigcup t^*_i Y_i, \mathcal{O}) = 0 \) for \( 1 \leq p \leq s \) and \( q \geq n - 1 \). In particular, \( H^q(U - \bar{Y}, \mathcal{O}) = 0 \). Let \( \mathcal{F} \) be the zeroth direct image of \( \mathcal{O} \) under \( F \). Then, since \( H^{q-1}(U - \bar{Y}, \mathcal{O}) = 0 \),

\[
H^{q-1}(U - Y, \mathcal{F}) = 0.
\]

We claim that

\[
\mathcal{F} \simeq \mathcal{O}^s.
\]

Consider the subvariety \( Z = \{z_0, z_1, \ldots, z_n\} | z_1 = (z_0)^s \} \) in \( C^{n+1} \). Let \( \mathcal{O} \) be the structure sheaf of \( Z \). Let \( \theta: C^{n+1} \rightarrow C^n \) be defined by \( \theta(z_0, z_1, \ldots, z_n) = (z_1, \ldots, z_n) \). Let \( T: C^n \rightarrow Z \) be defined by \( T(w_1, \ldots, w_n) = (w_1, (w_1)^s, w_2, \ldots, w_n) \). \( T \) is biholomorphic and \( \theta T = F \). Let \( \mathcal{O} \) be the zeroth direct image of \( \mathcal{O} \) under \( \theta \). To prove (4), we need only prove that \( \mathcal{O} \simeq \mathcal{O}^s \). Suppose \( Q \) is a bounded non-empty Stein open subset in \( C^n \) and \( f \in \Gamma(\theta^{-1}(Q) \cap Z, \mathcal{O}) \). Then \( f = \tilde{f} \theta^{-1}(Q) \cap Z \) for some \( \tilde{f} \in \Gamma(\theta^{-1}(Q), \mathcal{O}) \). By methods analogous to the usual proof of the
Weierstrass division theorem, we obtain 
\[ f = \sum_{i=0}^{s-1} (v_i \circ \theta)(x_0)^i, \]
where \( u \) is a holomorphic function on \( \theta^{-1}(Q) \) and \( v_i, 0 \leq i \leq s-1 \), are holomorphic functions on \( Q \). It is easily seen that \( v_i, 0 \leq i \leq s-1 \), are uniquely determined by \( f \).

It is easily seen that \( v_i \) is a holomorphic function on \( \Omega \sim \Gamma \), and \( v_i \) are uniquely determined by \( f \).

Λ \( \rightarrow K \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \'}
diagram in (5) implies that \( H_{X;0}^n = 0 \). Since \( n-1 > q+r \), \( H_{X;0}^n = 0 \) for some \( p > q+r \).

(b) In the general case, suppose \( H_{X;0}^n = 0 \) for \( p > q+r \). We are going to derive a contradiction. In view of (a) we can assume that the germ of \( Z \) at 0 has positive dimension. Let \( h = \text{codim}_{U;0}(Z) \). Then \( r+q+2 \leq h < n \). After a linear transformation of the coordinates system of \( C^n \) and after a shrinking of \( U \), we can assume that \( Z \cap C^h = \emptyset \), where \( C^h \) is regarded as a linear subspace of \( C^n \) through the embedding sending \((z_1, \ldots, z_h) \in C^h \) to \((z_1, \ldots, z_h, 0, \ldots, 0) \in C^n \). Suppose \( W \) is an arbitrary open neighborhood of 0 in \( U \). Consider the exact sequences

\[
0 \to \mathcal{O} / \sum_{i=h+1}^{n} z_i \mathcal{O} \to \mathcal{O} / \sum_{i=h+1}^{n} z_i \mathcal{O} / \sum_{i=h+1}^{n} z_i \mathcal{O} \to \mathcal{O} / \sum_{i=h+1}^{n} z_i \mathcal{O} \to 0, \quad h+1 \leq k \leq n,
\]

where \( f_k \) is defined by multiplication by \( z_k \) and \( \sum_{i=h+1}^{n} z_i \mathcal{O} = 0 \). These give us exact sequences

\[
H^p(W-X, \mathcal{O} / \sum_{i=h+1}^{n} z_i \mathcal{O}) \to H^p(W-X, \mathcal{O} / \sum_{i=h+1}^{n} z_i \mathcal{O}) \to H^{p+1}(W-X, \mathcal{O} / \sum_{i=h+1}^{n} z_i \mathcal{O}), \quad p \geq 0, \quad h+1 \leq k \leq n.
\]

Passing to direct limits, we have the following exact sequences:

\[
dir. \lim. W H^p(W-X, \mathcal{O} / \sum_{i=h+1}^{n} z_i \mathcal{O}) \]

(6)

\[
dir. \lim. W H^p(W-X, \mathcal{O} / \sum_{i=h+1}^{n} z_i \mathcal{O}), \quad p \geq 0, \quad h+1 \leq k \leq n.
\]

Since \( dir. \lim. W H^p(W-X, \mathcal{O} / \sum_{i=h+1}^{n} z_i \mathcal{O}) = H^{k+1}_{X;0} = 0 \) for \( p \geq q+r \), by (6) and by backward induction on \( h \) we conclude that \( dir. \lim. W H^p(W-X, \mathcal{O} / \sum_{i=h+1}^{n} z_i \mathcal{O}) = 0 \) for \( p \geq q+r \) and \( h+1 \leq k \leq n+1 \). Since for \( p \geq 0 \) \( H^{k+1}_{X;0} = 0 \), we have

\[
H^{k+1}_{X;0} = 0 \quad \text{for} \quad p \geq q+r.
\]

Since no branch-germ of \( X_i \) at 0 contains a branch-germ of \( X_j \) at 0 and vice versa, \( \text{codim}_{U;0}(X_i) < \text{codim}_{U;0}(Z) = h \) for \( i = 1, 2 \). Hence the germ of \( X_i \cap C^h \) at 0 is positive dimensional for \( i = 1, 2 \). We are in the situation of Part (a).

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\[ H^{k+1}_{X;0} = 0 \quad \text{for} \quad p \geq q+r \]

REMARK. The converse of Theorem 1 is not true as is shown in the following example: In \( C^6 \) let \( X_1 = \{(z_1 = z_2 = 0) \cup (z_2 = z_3 = 0) \cup (z_3 = z_4 = 0) \} \cap \{z_5 = 0\} \) and \( X_2 = \{(z_1 = z_2 = 0) \cup (z_1 = z_4 = 0) \cup (z_2 = z_3 = 0) \} \cap \{z_6 = 0\} \). Let \( X = X_1 \cup X_2 \).

For \( i = 1, 2 \), \( X_i \) is of codimension 3 and can be defined by 3 global holomorphic functions, because \( X_1 = \{z_1 z_2 z_3 z_4 = 0, z_2 z_3 = 0, z_3 = 0\} \) and \( X_2 = \{z_1 z_2 z_3 z_4 = 0, z_2 z_3 = 0, z_3 = 0\} \). Hence \( H^{3}_{X;0} = 0 \) for \( p > 3 \) and \( i = 1, 2 \). \( X_1 \cap X_2 = \{(z_1 z_2 z_3 z_4 = 0) \cup (z_1 z_2 z_3 z_4 = 0) \} \cap \{z_5 = 0\} \) is of codimension 4 and is not locally connected in codimension 1 at 0, because \( X_1 \cap X_2 = Y_1 \cup Y_2 \) and \( Y_1 \cap Y_2 = \{0\} \), where \( Y_1 = \{z_1 = z_2 = z_3 = z_4 = 0\} \) and \( Y_2 = \{z_1 = z_2 = z_3 = z_4 = 0\} \). Hence \( H^{4}_{X;0} = 0 \) for some \( p > 4 \). By taking direct limits of Mayor-Vietoris sequences, we obtain exact
sequences $H^p_{X_1;0} \to H^{p+1}_{X_1;0} \otimes H^{p+1}_{X_2;0}$, $p > 0$. Hence $H^p_{X_1;0} \neq 0$ for some $p > 3$. On the other hand, the 6 branch-germs of $X$ are given by $Z_1 = \{z_1 = z_2 = z_5 = 0\}$, $Z_2 = \{z_2 = z_3 = z_5 = 0\}$, $Z_3 = \{z_3 = z_4 = z_5 = 0\}$, $Z_4 = \{z_1 = z_2 = z_6 = 0\}$, $Z_5 = \{z_1 = z_4 = z_6 = 0\}$, and $Z_6 = \{z_3 = z_4 = z_5 = 0\}$. It can be easily verified that we cannot divide these 6 branch-germs into two groups so that the intersection of the union of one group with the union of another group is of dimension < 2. $X$ serves also as an example of a non local complete intersection which is locally connected in codimension 1.

**References**


