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EXTERIOR PRODUCT BUNDLE OVER COMPLEX ABSTRACT WIENER SPACE

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1. Introduction

In this paper, we consider a *complex abstract Wiener space* (CAWS) (B,H,μ), that is a triplet of a complex separable Banach space B, a complex separable Hilbert space H which is densely and continuously imbedded in B and a Borel probability measure μ on B such that

$$(1.1)\quad \int_B \exp(\sqrt{-1}\,\mathrm{Re}_B\langle z, \varphi\rangle_{B^*})\,\mu(dz) = \exp(-\frac{1}{4}\,\|\varphi\|_{H^*}^2) \quad \text{for } \varphi\in B^*\subset H^*.$$

Moreover, we assume that a strictly positive self-adjoint operator A on H^* is given and $B^*\subset C^\infty(A)=\bigcap_{n=1}^{\infty}\mathrm{Dom}(A^n)$. Then we can define $D_A p(z)=(\sqrt{A}\oplus\sqrt{\bar{A}})Dp(z)$ for $p\in\mathcal{P}(B: E)$, E-valued polynomial functional on B.

H-derivative D is a fundamental tool in Malliavin's calculs ([6]), but here we consider D_A instead of D, because we keep quantum field theoretical models in mind. In fact, $\frac{1}{2}D_A^*D_A=d\Gamma(A\oplus\bar{A})$, a free Hamiltonian for a complex Bose field (and its anti-particle field).

Following [3] and [4], we regard B as an infinite dimensional manifold with cotangent space $(H_R^*)^c$ on each $z\in B$. Consequently its exterior product bundle becomes $B\times\Lambda(H_R^*)^c$ and the space of its L^2-sections becomes $L^2(B, \mu: \Lambda(H_R^*)^c)$, i.e. the space of $\Lambda(H_R^*)^c$-valued L^2-functions on B or $L^2(B, \mu)\otimes\Lambda(H_R^*)^c$, a tensor product of the Bosonic Fock space and the Fermionic Fock space. On this space we define an exterior derivative d_A using D_A. Then $\frac{1}{2}(d_A^*d_A+d_Ad_A^*)=d\Gamma(A\oplus\bar{A})\oplus d\Lambda(A\oplus\bar{A})$, a free Hamiltonian for an $N=2$ supersymmetric quantum field.

As in the finite dimensional case, d_A is decomposed as $d_A=\partial_A+\bar{\partial}_A$, and Laplace-Beltrami operators $\square_A$ and $\bar{\square}_A$ are defined as $\square_A=\partial_A^*\partial_A+\partial_A\partial_A^*$ and $\bar{\square}_A=\bar{\partial}_A^*\bar{\partial}_A+\bar{\partial}_A\bar{\partial}_A^*$, respectively. Since $\bar{\partial}_A^2=0$, $\bar{\partial}_A$ defines an elliptic complex and $\bar{\partial}_A$-cohomology groups can be defined as $\mathfrak{H}_A^{p,q}(B)=\mathrm{Ker}(\bar{\partial}_A|\Lambda_2^{p,q}(B))/\mathrm{Im}(\bar{\partial}_A|\Lambda_2^{p,q-1}(B))$, where $\Lambda_2^{p,q}(B)=L^2(B, \mu: \Lambda^{p,q}(H_R^*)^c)$, the space of square integrable (p, q)-forms.

First we show that de Rham-Hodge-Kodaira's decomposition for $\Lambda_2^{p,q}(B)$ holds, that is

$$(1.2)\qquad \Lambda_q^{p,q}(B) = \mathrm{Im}(\bar{\partial}_A \mid \Lambda_2^{p,q-1}(B)) \oplus \mathrm{Im}(\bar{\partial}_A^* \mid \Lambda_2^{p,q}(B)) \oplus \mathfrak{h}_A^{p,q}$$

where $\mathfrak{h}_A^{p,q} = \mathrm{Ker}(\bar{\Box}_A \mid \Lambda_2^{p,q}(B))$, the space of harmonic (p, q)-forms (our discussion is restricted to the L^2-case). From this we conclude that $\mathfrak{H}_A^{p,q}(B) = \mathfrak{h}_A^{p,q}$ and it will be shown by using the expression $\frac{1}{2}\bar{\Box}_A = d\Gamma(\bar{A}) \oplus d\Lambda(\bar{A})$, that $\mathfrak{h}_A^{p,q} = \{0\}$, if $q \leqq 1$ and $\mathfrak{h}_A^{p,0} = \mathrm{Hol}^2(B\colon \Lambda^{p,0}(H_R^*)^c)$, where $\mathrm{Hol}^2(B\colon \Lambda^{o,q}(H_R^*)^c)$ is the set of square integrable holomorphic forms.

We start with a complex separable Hilbert space H, but we regard this as a real separable Hilbert space (this space is denoted by H_R) and consider $(H_R^*)^c$, a complexification of its adjoint space H_R^*. $(H_R^*)^c$ is decomposed as $(H_R^*)^c = H^* \oplus \bar{H}^*$, but the inner product of H^* induced from $(H_R^*)^c$ is slightly different from original one. We sum up these algebraic fundamentals in Appendix A. In Appendix B we state some elementary facts about the Wick product for a complex Gaussian system.

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2. Complex abstract Wiener space

In this section we define a modified Ornstein-Uhlenbeck operator on a CAWS and show that it equals to the free Hamiltonian.

Let (B, H, μ) be a CAWS as in the section 1 and A be a strictly positive self-adjoint operator on H^*. Then it can be easily shown that $\{Z_\theta \mid \theta \in B^*\}$ is a complex Gaussian system satisfying

$$(2.1)\qquad E[|Z_\theta|^2] = \|Z_\theta\|_{H^*}^2 \quad \theta \in H^*,$$

$$(2.2)\qquad E[Z_\theta \bar{Z}_\eta] = (\theta, \eta)_{H^*} \quad \theta, \eta \in H^*,$$

where Z_θ is a complex random variable on B defined as $Z_\theta(z) = {}_B\langle z, \theta\rangle_{B^*}$, $\bar{Z}_\eta$ is a complex conjugate of Z_η and E stands for the integration under μ.

We assume that $B^* \subset C^\infty(A)$ without loss of generality. In fact, let α be a Hilbert-Schmidt operator on H^* and set $K = e^{-A}\alpha$. We define $(u, v)_B = (Ku, Kv)_H$, $\|u\|_B^2 = (u, u)_B^{1/2}$ for $u, v \in H$ and denote the completion of H with respect to $\|\cdot\|_H$ as B. Then $(B, \|\cdot\|_B)$ becomes a complex Banach space and there exists a Borel probability measure μ on B such that

$$(2.3)\qquad \int_B \exp(\sqrt{-1}\,\mathrm{Re}_B\langle z, \varphi\rangle_{B^*})\,\mu(dz) = \exp(-\frac{1}{4}\|\varphi\|_{H^*}^2) \quad \text{for } \varphi \in B^* \subset H^*.$$

and moreover $B^* \subset e^{-A}\alpha(H) \subset C^\infty(A)$.

A (complex-valued) *polynomial functional* on B is a mapping $p: B \to \boldsymbol{C}$ written as

$$p(z) = P(Z_{\theta_1}(z), \cdots, Z_{\theta_n}(z), \bar{Z}_{\theta_1}(z), \cdots, \bar{Z}_{\theta_n}(z)) \tag{2.4}$$

where $n \in \boldsymbol{N}$, $\theta_1, \cdots, \theta_n \in B^*$, P is a polymomial of $2n$-arguments with complex coefficients. If p is written in the form

$$p(z) = P(Z_{\theta_1}(z), \cdots, Z_{\theta_n}(z)) \tag{2.5}$$

p is called a *holomorphic polynomial functional* on B.

We denote by $\mathcal{P}(B: \boldsymbol{C})$ and $\mathcal{P}_h(B: \boldsymbol{C})$ the set of polynomials and holomorphic polynomials on B, respectively. Moreover, for a complex separable Hilbert space E, we set $\mathcal{P}(B: E) = \mathcal{P}(B: \boldsymbol{C}) \otimes E$, $\mathcal{P}_h(B: E) = \mathcal{P}_h(B: \boldsymbol{C}) \otimes E$ (algebraic tensor product) and call them the space of *E-valued polynomial functionals* and *E-valued holomorphic polynomial functionals,* respectively. For $p \in \mathcal{P}(B: E)$, its *H-derivative at* $z \in B$ is defined as follows

$$\langle Dp(z), h \rangle = \frac{d}{dt} p(z+th)|_{t=0} \qquad \text{for} \quad h \in H. \tag{2.6}$$

$Dp(z)$ is an element of $(H_R^*)^c \otimes E$. Since $(H_R^*)^c \otimes E = (H^* \otimes E) \oplus (\bar{H}^* \otimes E)$, we set $\nabla p(z)$ to be an $H^* \otimes E$ component and $\bar{\nabla} p(z)$ to be an $\bar{H}^* \otimes E$ component.

As mentioned in the section 1, we use slightly modified derivative instead of H-derivative as follows,

$$D_A p(z) = (\sqrt{A} \oplus \sqrt{\bar{A}})\, Dp(z) \qquad p \in \mathcal{P}(B: \boldsymbol{C}). \tag{2.7}$$

For the definiton of $\sqrt{\bar{A}}$ see (A.6). We have chosen B so that $B^* \subset C^\infty(A)$, so $Dp(z) \in C^\infty(A)$ and $(\sqrt{A} \oplus \sqrt{\bar{A}}) Dp(z)$ is well defined. $D_A P(z)$ is decomposed as $D_A p(z) = \nabla_A p(z) \oplus \bar{\nabla}_A p(z)$ where $\nabla_A p(z) = \sqrt{A}\, \nabla p(z)$, $\bar{\nabla}_A p(z) = \sqrt{\bar{A}} \bar{\nabla}_A p(z)$. We denote adjoint operators of ∇_A and $\bar{\nabla}_A$ in $L^2(B, \mu: E)$ by ∇_A^* and $\bar{\nabla}_A^*$, respectively. Their explicit formulas for Wick polynomials are given as follows.

Proposition 2.1. *For* $\theta_1, \cdots, \theta_n, \eta_1, \cdots, \eta_m, \zeta \in B^*$, *it holds that*

$$\nabla_A : Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m} : = \sum_{j=1}^{n} : Z_{\theta_1} \cdots \hat{Z}_{\theta_j} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m} : \sqrt{A}\theta_j \tag{2.8}$$

$$\bar{\nabla}_A : Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\theta_1} \cdots \bar{Z}_{\eta_m} : = \sum_{j=1}^{m} : Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \hat{\bar{Z}}_{\eta_j} \cdots \bar{Z}_{\eta_m} : \overline{\sqrt{A}\theta_j} \tag{2.9}$$

$$\nabla_A^* : Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m} : \zeta = 2 : Z_{\sqrt{A}\zeta} Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m} : \tag{2.10}$$

$$\bar{\nabla}_A^* : Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m} : \bar{\zeta} = 2 : \bar{Z}_{\sqrt{A}\zeta} Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m} : \tag{2.11}$$

Proof. As in the real case, it can be easily shown that

$$\nabla_A P(Z_\theta \bar{Z}_\theta) = \sum_{j=1}^{n} \frac{\partial P}{\partial z_j}(Z_\theta \bar{Z}_\theta)\sqrt{A}\theta_j$$

$$\bar{\nabla}_A P(Z_\theta \bar{Z}_\theta) = \sum_{j=1}^{n} \frac{\partial P}{\partial \bar{z}_j}(Z_\theta \bar{Z}_\theta)\overline{\sqrt{A}\theta_j}$$

$$\nabla_A^* P(Z_\theta \bar{Z}_\theta)\zeta = -2\sum_{j=1}^{n} \frac{\partial P}{\partial z_j}(Z_\theta \bar{Z}_\theta)(\zeta, \sqrt{A}\theta_j)_{H^*} + 2\, Z_{\sqrt{A}\zeta} P(Z_\theta \bar{Z}_\theta)$$

$$\bar{\nabla}_A^* P(Z_\theta \bar{Z}_\theta)\bar{\zeta} = -2\sum_{j=1}^{n} \frac{\partial P}{\partial \bar{z}_j}(Z_\theta \bar{Z}_\theta)(\sqrt{A}\theta_j, \zeta)_{H^*} + 2\bar{Z}_{\sqrt{A}\zeta}(P(Z_\theta \bar{Z}_\theta)$$

where $P(Z_\theta, \bar{Z}_\theta) = P(Z_{\theta_1}, \cdots, Z_{\theta_n}, \bar{Z}_{\theta_1}, \cdots, \bar{Z}_{\theta_n}) \in \mathcal{P}(B: \boldsymbol{C})$, $\zeta \in B^*$ (see e.g. [6]). Combining this with (2.2) (B.3)~(B.6), we can prove (2.8)~2.11). $\square$

Therefore ∇_A^* and $\bar{\nabla}_A^*$ are densely defined operators, so ∇_A and $\bar{\nabla}_A$ are closable and we denote their closures by the same symbols.

Next we obtain the kernel of $\bar{\nabla}_A$.

Proposition 2.2. *It holds that*

$$\operatorname{Ker}(\bar{\nabla}_A) = \operatorname{Hol}^2(B: E) \tag{2.12}$$

where $\operatorname{Hol}^2(B: E)$ *is the closure of* $\mathcal{P}_h(B: E)$ *in* $L^2(B, \mu: E)$.

Proof. We give a proof for $E = \boldsymbol{C}$. General cases can be proved similarly. First we introduce some notations. Let $\{\theta_n\}_{n=1}^{\infty}$ be an ONB of H^*.

$$\mathfrak{A} = \{n = (n_j)_{j=1}^{\infty} \in \boldsymbol{Z}_+^N \mid \sum_{j=1}^{\infty} n_j < \infty\}, \quad \boldsymbol{Z}_+ = \{0, 1, 2, 3, \cdots\},$$

$$W_{n,m} = \prod_{j=1}^{\infty} (n_j!\, m_j!)^{-1/2} : \prod_{j=1}^{\infty} Z_{\theta_j}^{n_j} \bar{Z}_{\theta_j}^{m_j} :, \quad n, m \in \mathfrak{A},$$

$$\mathfrak{A}_N = \{n = (n_j)_{j=1}^{\infty} \in \mathfrak{A} \mid n_j = 0 \quad \text{if } j > N\},$$

$$L_N = [W_{n,m} \mid n, m \in \mathfrak{A}_N]^{-\|\cdot\|_2},$$

$P_N: L^2(B, \mu) \to L_N$ orthogonal projection,

$p_N: \bar{H}^* \to [\theta_1, \cdots, \theta_N]$ orthogonal projection,

where $[\cdot]$ stands for the linear span and $-\|\cdot\|_2$ means the closure in $L^2(B, \mu)$. $\{W_{n,m}\}_{n,m \in \mathfrak{A}}$ forms an ONB of $L^2(B, \mu)$, so P_N converges strongly to the identity and it holds that

$$P_N \circ \bar{\nabla}_A^* W_{n,m} \bar{\theta}_j = \sum_{k=1}^{N} 2(\theta_k, A\theta_j)_{H^*}(m_k+1)^{1/2} W_{n,m+\varepsilon_k} \qquad n, m \in \mathfrak{A}_N \tag{2.13}$$

where $\varepsilon_k = (0, \cdots, 0 \overset{\overset{k}{\vee}}{1}, 0, \cdots) \in \mathfrak{A}$, and moreover $P_N \otimes p_N \circ \bar{\nabla}_A = \bar{\nabla}_A \circ P_N$.

If $F \in \operatorname{Ker}(\bar{\nabla}_A) = \operatorname{Im}(\bar{\nabla}_A^*)^{\perp}$, then for $n, m \in \mathfrak{A}_N$, $j \in \{1 \cdots N\}$,

$$\begin{aligned}0 &= (\overline{\nabla}_A F, W_{n,m}\bar{\theta}_j) = (P_N \otimes p_N \circ \overline{\nabla}_A F, W_{n,m}\bar{\theta}_j) \\ &= (\overline{\nabla}_A \circ P_N F, W_{n,m}\bar{\theta}_j) = (F, P_N \circ \overline{\nabla}_A^* W_{n,m}\bar{\theta}_j) \\ &= 2\sum_{k=1}^{N} (A\theta_j, \theta_k)_{H^*}(m_k+1)^{1/2}(F, W_{n,m+\varepsilon_k})\end{aligned}$$

Since A is strictly positive, we have

$$(F, W_{n,m+\varepsilon_k}) = 0\,, \quad n, m \in \mathfrak{A}_N\,, \quad k \in \{1 \cdots N\}, \quad N \in \boldsymbol{N}\,.$$

Thus we have $F \in [W_{n,0} | n \in \mathfrak{R}]^{-\|\cdot\|_2} = \overline{\mathcal{P}_h(B\colon \boldsymbol{C})}^{\|\cdot\|_2} = \mathrm{Hol}^2(B\colon \boldsymbol{C})$ and hence $\mathrm{Ker}\,(\overline{\nabla}_h) \subset \mathrm{Hol}^2(B\colon \boldsymbol{C})$.

Conversely it is easy to see that $\mathrm{Hol}^2(B\colon \boldsymbol{C}) \subset \mathrm{Ker}\,(\overline{\nabla}_A)$. This completes the proof. □

We set

$$L_A = -\nabla_A^* \nabla_A \qquad L_{\bar{A}} = -\overline{\nabla}_A^* \overline{\nabla}_A^*\,. \tag{2.14}$$

Then L_A and $L_{\bar{A}}$ are negative self-adjoint operators on $L^2(B, \mu)$. Let us show that L_A and $L_{\bar{A}}$-correspond to the Hamiltonian for complex Bosons and their anti-particles, respectively.

DEFINITION 2.3. *Bosonic second quantized operator* of A and $\bar{A}$ on $L^2(B, \mu)$ is defined on the Wick polynomials as follows

$$d\Gamma(A)\colon Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}\colon = \sum_{j=1}^{n} \colon Z_{\theta_1} \cdots Z_{A\theta_j} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}\colon \tag{2.15}$$

$$d\Gamma(\bar{A})\colon Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}\colon = \sum_{j=1}^{m} \colon Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{A\eta_j} \cdots \bar{Z}_{\eta_m}\colon \tag{2.16}$$

where $\theta_1, \cdots, \theta_n, \eta_1, \cdots, \eta_m \in B^*$. $d\Gamma(A)$ and $d\Gamma(\bar{A})$ are essentially self-adjoint on the space of the Wick polynomials and we denote its closure by the same symbol (see e.g. [2]).

Theorem 2.4. *It holds that*

$$L_A = -2d\Gamma(A) \qquad L_{\bar{A}} = -2d\Gamma(\bar{A})\,. \tag{2.17}$$

Proof. To prove (2.17), it is enough to show that

$$L_A p = -2\, d\Gamma(A) p \qquad L_{\bar{A}} p = -2\, d\Gamma(\bar{A}) p$$

for a Wick polynomial $p = \colon Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}\colon$. By Proposition 2.1,

$$\begin{aligned}&L_A\colon Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}\colon = -\nabla_A^* \nabla_A\colon Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}\colon \\ &= -\sum_{j=1}^{n} \nabla_A^*\colon Z_{\theta_1} \cdots \hat{Z}_{\theta_j} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}\colon \sqrt{A}\theta_j = -2\sum_{j=1}^{n} \colon Z_{A\theta_j} Z_{\theta_1} \cdots \hat{Z}_{\theta_j} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}\colon \\ &= -2\, d\Gamma(A)\colon Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}\colon.\end{aligned}$$

The latter can be proved similarly. □

3. An exterior product bundle

Let us define an exterior product bundle over a CAWS. To do this, let $\Lambda(H_R^*)^c=\bigoplus_{n=0}^{\infty}\Lambda^n(H_R^*)^c$ where $\Lambda^n(H_R^*)^c$ is an anti-symmetric part of n-tensor product of $(H_R^*)^c$ and its inner product is given by

$$(3.1)\qquad (\omega,\eta)=\frac{1}{n!}(\omega,\eta)_{\otimes^n(H_R^*)^c}\qquad \text{for}\quad \omega,\eta\in\Lambda^n(H_R^*)^c,$$

where $(\cdot)_{\otimes^n(H_R^*)^c}$ is the natural inner product on $\otimes^n(H_R^*)^c$. We define an exterior product of $\omega\in\Lambda^n(H_R^*)^c$ and $\eta\in\Lambda^n(H_R^*)^c$ by

$$(3.2)\qquad \omega\wedge\eta=\frac{(n+m)!}{n!m!}\mathcal{A}_{n+m}\omega\otimes\eta$$

where $\mathcal{A}_{n+m}$ is an $(n+m)$-th normalized anti-symmetrization defined by

$$(3.3)\qquad \mathcal{A}_n(\omega_1\otimes\cdots\otimes\omega_n)=\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\omega_{\sigma_1}\otimes\cdots\otimes\omega_{\sigma_n}.$$

We set $\Lambda^{p,q}(H_R^*)^c=\Lambda^pH^*\wedge\Lambda^q\bar{H}^*$. Then

$$(3.4)\qquad \Lambda^n(H_R^*)^c=\bigoplus_{p+q+n}\Lambda^{p,q}(H_R^*)^c.$$

Exterior derivative $d_A=\partial_A+\bar{\partial}_A$ on polynomial functionals is defined as follows

$$(3.5)\qquad d_A\omega=(n+1)\mathcal{A}_{n+1}D_A\omega$$

$$(3.6)\qquad \partial_A\omega=(n+1)\mathcal{A}_{n+1}\nabla_A\omega$$

$$(3.7)\qquad \bar{\partial}_A\omega=(n+1)\mathcal{A}_{n+1}\bar{\nabla}_A\omega$$

for $\omega\in\mathcal{P}(B:\Lambda^n(H_R^*)^c)$. We denote adjoint operators of ∂_A and $\bar{\partial}_A$ in $L^2(B,\mu:\Lambda(H_R^*)^c)$ by ∂_A^* and $\bar{\partial}_A^*$, respectively. Then it holds as in the real case ([3])

$$(3.8)\quad \partial_Af(z)\theta_1\wedge\cdots\wedge\theta_p\wedge\bar{\eta}_1\wedge\cdots\wedge\bar{\eta}_q=\nabla_Af(z)\wedge\theta_1\wedge\cdots\wedge\theta_p\wedge\bar{\eta}_1\wedge\cdots\wedge\bar{\eta}_q$$

$$(3.9)\quad \bar{\partial}_Af(z)\theta_1\wedge\cdots\wedge\theta_p\wedge\bar{\eta}_q\wedge\cdots\wedge\bar{\eta}_q=\bar{\nabla}_Af(z)\wedge\theta_1\wedge\cdots\wedge\theta_p\wedge\bar{\eta}_1\wedge\cdots\wedge\bar{\eta}_q$$

$$(3.10)\quad \partial_A^*f(z)\theta_1\wedge\cdots\wedge\theta_p\wedge\bar{\eta}_1\wedge\cdots\wedge\bar{\eta}_q=\sum_{j=1}^{p}(-1)^{j-1}\nabla_A^*(f(z)\theta_j)\theta_1\wedge\cdots\wedge\theta_j\wedge \\ \wedge\theta_p\wedge\bar{\eta}_1\wedge\cdots\wedge\bar{\eta}_q$$

$$(3.11)\quad \bar{\partial}_A^*f(z)\theta_1\wedge\cdots\wedge\theta_p\wedge\bar{\eta}_1\wedge\cdots\wedge\bar{\eta}_q=\sum_{j=1}^{q}(-1)^{p+j-1}\bar{\nabla}_A^*(f(z)\bar{\eta}_j)\theta_1\wedge\cdots\wedge\theta_p \\ \wedge\bar{\eta}_1\wedge\cdots\wedge\hat{\bar{\eta}}_j\wedge\cdots\wedge\bar{\eta}_q$$

where $f\in\mathcal{P}(B: \boldsymbol{C})$, $\theta_1, \cdots, \theta_p, \eta_1, \cdots, \eta_q \in B^*$. Thus ∂_A^* and $\bar{\partial}_A^*$ are densely defined operators and so ∂_A and $\bar{\partial}_A$ are closable. We denote their closures by the same symbols. Then we easily have the following.

Proposition 3.1. *It holds that*

$$d_A^2=0 \tag{3.12}$$

and

$$\partial_A^2=0 \qquad \bar{\partial}_A^2=0 \qquad \partial_A\bar{\partial}_A+\bar{\partial}_A\partial_A=0 \tag{3.13}$$

$$\partial_A^{*2}=0 \qquad \bar{\partial}_A^{*2}=0 \qquad \partial_A^*\bar{\partial}_A^*+\bar{\partial}_A^*\partial_A^*=0\,. \tag{3.14}$$

Laplace-Beltrami operators $\square_A$ and $\bar{\square}_A$ are defined as follows

$$\square_A=\partial_A\partial_A^*+\partial_A^*\partial_A\,, \qquad \bar{\square}_A=\bar{\partial}_A\bar{\partial}_A^*+\bar{\partial}_A^*\bar{\partial}_A\,. \tag{3.15}$$

Then $\square_A$ and $\bar{\square}_A$ are positive self-adjoint operators on $\mathrm{Dom}(\partial_A\partial_A^*)\cap\mathrm{Dom}(\partial_A^*\partial_A)$ and $\mathrm{Dom}(\bar{\partial}_A\bar{\partial}_A^*)\cap\mathrm{Dom}(\bar{\partial}_A^*\bar{\partial}_A)$, respectively ([1]). We will show that $\square_A$ and $\bar{\square}_A$ correspond to the free Hamiltonian of supersymmetric particle field and its antiparticle field, respectively.

DEFINITION 3.2. *Fermionic second quantized operators* of A and $\bar{A}$ respectively, on $\Lambda(H_R^*)^c$ are defined as follows

$$d\Lambda(A)\theta_1\wedge\cdots\wedge\theta_p\wedge\bar{\eta}_1\wedge\cdots\wedge\bar{\eta}_q=\sum_{j=1}^{p}\theta_1\wedge\cdots\wedge\theta_j\wedge\cdots\wedge A\theta_p \wedge\bar{\eta}_1\wedge\cdots\wedge\bar{\eta}_q \tag{3.16}$$

$$d\Lambda(\bar{A})\theta_1\wedge\cdots\wedge\theta_p\wedge\bar{\eta}_1\wedge\cdots\wedge\bar{\eta}_q=\sum_{j=1}^{q}\theta_1\wedge\cdots\wedge\theta_p\wedge\bar{\eta}\wedge\cdots\wedge \bar{A}\bar{\eta}_j\wedge\cdots\wedge\bar{\eta}_q \tag{3.17}$$

where $\theta_1, \cdots, \theta_p, \eta_1, \cdots, \eta_q\in B^*$. Then $d\Lambda(A)$ and $d\Lambda(\bar{A})$ are essentially self-adjoint on $\bigoplus_{n=0}^{\infty}\Lambda^n(B^*\oplus\bar{B}^*)$ (algebraic sense) ([2]). We denote their closures by the same symbols.

Theorem 3.3. *It holds that*

$$\square_A=-L_A+2\,d\Lambda(A)=2(d\Gamma(A)+d\Lambda(A)) \tag{3.18}$$

$$\bar{\square}_A=-L_{\bar{A}}+2\,d\Lambda(\bar{A})=2(d\Gamma(\bar{A})+d\Lambda(\bar{A})) \tag{3.19}$$

Proof. To prove (3.18), it is enough to show that

$$\begin{aligned}&\square_A: Z_{\theta_1}\cdots Z_{\theta_n}\bar{Z}_{\eta_1}\cdots\bar{Z}_{\eta_m}: \omega_1\wedge\cdots\wedge\omega_p\wedge\bar{\xi}_1\wedge\cdots\wedge\bar{\xi}_q\\ &=2(d\Gamma(A)+d\Lambda(A)): Z_{\theta_1}\cdots Z_{\theta_n}\bar{Z}_{\eta_1}\cdots\bar{Z}_{\eta_m}: \omega_1\wedge\cdots\wedge\omega_p\wedge\bar{\xi}_1\wedge\cdots\wedge\bar{\xi}_q\end{aligned}$$

for $\theta_1, \cdots, \theta_n, \eta_1, \cdots, \eta_m, \omega_1, \cdots, \omega_p, \xi_1, \cdots, \xi_q \in B^*$.

$$\partial_A \partial_A^*: Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}: \omega_1 \wedge \cdots \wedge \omega_p \wedge \bar{\xi}^1 \wedge \cdots \wedge \bar{\xi}_q$$

$$= \partial_A \sum_{j=1}^{p} (-1)^{j-1} \nabla_A^*(: Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}: \omega_j) \omega_1 \wedge \cdots \wedge \hat{\omega}_j \wedge \cdots \wedge \omega_p \wedge \bar{\xi}_1 \wedge \cdots \wedge \bar{\xi}_q$$

$$= \partial_A \sum_{j=1}^{p} (-1)^{j-1} 2: Z_{\sqrt{A}\omega_j} Z_{\theta_1} \cdots Z_{\eta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}: \omega_1 \wedge \cdots \wedge \hat{\omega}_j \wedge \cdots \wedge \omega_p \wedge \bar{\xi}_1 \wedge \cdots \wedge \bar{\xi}_q$$

$$= 2 \sum_{j=1}^{p} : Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}: \omega_1 \wedge \cdots \wedge A\omega_j \wedge \cdots \wedge \omega_p \wedge \bar{\xi}_1 \wedge \cdots \wedge \bar{\xi}_q$$

$$+ \sum_{k=1}^{n} (-1)^{j-1}: Z_{\sqrt{A}\omega_j} Z_{\theta_1} \cdots \hat{Z}_{\theta_k} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}: \sqrt{A}\theta_k \wedge \omega_1 \wedge \cdots \wedge \omega_p \wedge \cdots \wedge \omega_p \wedge \bar{\xi}_1 \wedge \cdots \wedge \bar{\xi}_q \,.$$

$$\partial_A^* \partial_A: Z_{\theta_1} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}: \omega_1 \wedge \cdots \wedge \omega_p \wedge \bar{\xi}_1 \wedge \cdots \wedge \bar{\xi}_q$$

$$= \partial_A^* \sum_{k=1}^{n} : Z_{\theta_1} \cdots \hat{Z}_{\theta_k} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}: \sqrt{A}\theta_k \wedge \omega_1 \wedge \cdots \wedge \omega_p \wedge \bar{\xi}_1 \wedge \cdots \wedge \bar{\xi}_q$$

$$= \sum_{k=1}^{n} \nabla_A^*(: Z_{\theta_1} \cdots \hat{Z}_{\theta_k} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}: \sqrt{A}\theta_k) \omega_1 \wedge \cdots \wedge \omega_p \wedge \bar{\xi}_1 \wedge \cdots \wedge \bar{\xi}_q$$

$$+ \sum_{k=1}^{n} \sum_{j=1}^{p} (-1)^j \nabla_A^*(: Z_{\theta_1} \cdots \hat{Z}_{\theta_k} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}: \omega_j) \sqrt{A}\theta_k \wedge \omega_1 \wedge \cdots \wedge \hat{\omega}_j \wedge \cdots \wedge \omega_p \wedge \bar{\xi}_1 \wedge \cdots \wedge \bar{\xi}_q$$

$$= 2 \sum_{k=1}^{n} : Z_{\theta_1} \cdots Z_{A\theta_k} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}: \omega_1 \wedge \cdots \wedge \omega_p \wedge \bar{\xi}_1 \wedge \cdots \wedge \bar{\xi}_q$$

$$+ 2 \sum_{k=1}^{n} \sum_{j=1}^{p} (-1)^j: Z_{\sqrt{A}\omega_j} Z_{\theta_1} \cdots \hat{Z}_{\theta_k} \cdots Z_{\theta_n} \bar{Z}_{\eta_1} \cdots \bar{Z}_{\eta_m}: \sqrt{A}\theta_k \wedge \omega_1 \wedge \cdots \wedge \hat{\omega}_j \wedge \cdots \wedge \omega_p \wedge \bar{\xi}_1 \wedge \cdots \wedge \bar{\xi}_q \,.$$

Thus (3.20) holds. (3.19) can be proved similarly. □

4. $\bar{\partial}_A$-cohomology group of a complex abstract Wiener space

In this section we shall define $\bar{\partial}_A$-cohomology group and determine their structure. First we shall define (p, q)-harmonic forms and prove de Rham-Hodge-Kodaira's decomposition. From this decomposition it is clear that $\bar{\partial}_A$-cohomology groups are isomorphic to the spaces of harmonic forms and so their structure can be determined completely.

DEFINITION 4.1. We set

$$\mathfrak{H}_A^{p,q} = \operatorname{Ker}(\bar{\square}_A^{p,q}) \tag{4.1}$$

and call its element a *harmonic* (p, q)*-form*, where $\bar{\square}_A^{p,q}$ is the restriction of $\bar{\square}_A$ to $\Lambda_2^{p,q}(B) = L^2(B, \mu: \Lambda^{p,q}(H_R^*)^c)$.

We shall determine the structure of $\mathfrak{h}_A^{p,q}$

Proposition 4.2.

$$\mathfrak{h}_A^{p,q} = \begin{cases} \{0\} & \text{for } q \geq 1\,, \\ \mathrm{Hol}^2(B\colon \Lambda^{p,0}(H_R^*)^c) & \text{for } q = 0\,. \end{cases} \tag{4.2}$$

Proof. For $q \geq 1$, from (3.19),

$$\square_A^{p,q} = -L_A + 2\, d\Gamma(A)_q\,.$$

Thus $\square_A^{p,q}$ is a strictly positive definite self-adjoint operator and $\mathrm{Ker}(\square_A^{p,q}) = \{0\}$. For $q=0$, from (3.19) and (2.12)

$$\mathrm{Ker}(\square_A^{p,q}) = \mathrm{Ker}(L_A) = \mathrm{Ker}(\nabla_A) = \mathrm{Hol}^2(B, \mu\colon \Lambda^{p,0}(H_R^*)^c). \quad \square$$

Now we can show de Rham-Hodge-Kodaira's decomposition. It is easy to show the following lemma, so we omit the proof.

Lemma. *Let $\mathcal{H}$ be a complex separable Hilbert space, A be a self-adjoint operator on $\mathcal{H}$ and $\sigma(|A|)$ be the spectrum of $|A|$. If $\sigma(|A|)\backslash\{0\} \subset [m, \infty)$ for a positive constant m, then A has a closed range.*

Theorem 4.3. *$\Lambda_2^{p,q}(B)$ is orthogonally decomposed as follows*

$$\Lambda_2^{p,q}(B) = \mathrm{Im}(\bar\partial_A^{p,q-1}) \oplus \mathrm{Im}(\bar\partial_A^{*p,q}) \oplus \mathfrak{h}_A^{p,q} \tag{4.3}$$

where $\bar\partial_A^{p,q}$ is the restriction of $\bar\partial_A$ to $\Lambda_2^{p,q}(B)$ and $\bar\partial_A^{*p,q}$ is the restriction of $\bar\partial_A^*$ to $\Lambda_2^{p,q+1}(B)$. We set $\mathrm{Im}(\bar\partial_A^{p,q-1}) = \{0\}$ *if* $q=0$.

Proof. From Theorem 3.3, $\sigma(\bar\square_A^{p,q})\backslash\{0\} \subset [m, \infty)$ where $m = \inf \sigma(A) > 0$. Thus from the above lemma,

$$\Lambda_2^{p,q}(B) = \mathrm{Ker}(\bar\square_A^{p,q}) \oplus \mathrm{Im}(\bar\square_A^{q,q}) = \mathfrak{h}_A^{p,q} \oplus \mathrm{Im}(\bar\square_A^{p,q})\,.$$

For $q=0$, $\bar\square_A^{p,0} = \bar\partial_A^{*p,0} \bar\partial_A^{p,0}$, thus $\mathrm{Im}(\bar\square_A^{p,0}) \subset \mathrm{Im}(\bar\partial_A^{*p,0})$. On the other hand, since $\Lambda_2^{p,0}(B) = \mathrm{Ker}(\bar\partial_A^{p,0}) \oplus \overline{\mathrm{Im}(\bar\partial_A^{*p,0})}$ and $\mathrm{Ker}(\bar\square_A^{p,0}) = \mathrm{Ker}(\bar\partial_A^{p,0})$, we have $\mathrm{Im}(\bar\square_A^{p,0}) = \overline{\mathrm{Im}(\bar\partial_A^{*p,0})}$. Therefore, $\mathrm{Im}(\bar\square_A^{p,0}) = \mathrm{Im}(\bar\partial_A^{*p,0})$ and $\Lambda_2^{p,0}(B) = \mathfrak{h}_A^{p,0} \oplus \mathrm{Im}(\bar\partial_A^{*p,0})$.

Next we show (4.3) for $q \geq 1$. We note $\bar\square_A^{p,q} = \bar\partial_A^{*p,q} \bar\partial_A^{p,q} + \bar\partial_A^{p,q-1} \bar\partial_A^{*p,q-1}$ and hence $\mathrm{Im}(\bar\square_A^{p,q}) \subset \mathrm{Im}(\bar\partial_A^{*p,q}) \oplus \mathrm{Im}(\bar\partial_A^{p,q-1})$. On the other hand, since $\Lambda_2^{p,q}(B) = \mathrm{Ker}(\bar\partial_A^{p,q}) \cap \mathrm{Ker}(\bar\partial_A^{*p,q-1}) \oplus \overline{\mathrm{Im}(\bar\partial_A^{*p,q})} \oplus \overline{\mathrm{Im}(\bar\partial_A^{p,q-1})}$ and $\mathrm{Ker}(\bar\square_A^{p,q}) = \mathrm{Ker}(\bar\partial_A^{p,q}) \cap \mathrm{Ker}(\bar\partial_A^{*p,q-1})$, we have $\mathrm{Im}(\bar\square_A^{p,q}) = \overline{\mathrm{Im}(\bar\partial_A^{*p,q})} \oplus \overline{\mathrm{Im}(\bar\partial_A^{p,q-1})}$. Therefore, $\mathrm{Im}(\bar\square_A^{p,q}) = \mathrm{Im}(\bar\partial_A^{*p,q}) \oplus \mathrm{Im}(\bar\partial_A^{p,q-1})$ and $\Lambda_2^{p,q}(B) = \mathfrak{h}_A^{p,q} \oplus \mathrm{Im}(\bar\partial_A^{*p,q}) \oplus \mathrm{Im}(\bar\partial_A^{p,q-1})$. $\square$

We define $\bar\partial_A$-cohomology group as follows

$$\mathfrak{H}_A^{p,q}(B) = \mathrm{Ker}(\bar\partial_A^{p,q}) / \mathrm{Im}(\bar\partial_A^{p,q-1})\,. \tag{4.4}$$

From Theorem 4.3, $\mathrm{Ker}(\bar{\partial}_A^{p,q})=\mathrm{Im}(\bar{\partial}_A^{*p,q})^{\perp}=\mathrm{Im}(\bar{\partial}_A^{p,q-1})\oplus\mathfrak{h}_A^{p,q}$. Therefore $\mathfrak{H}_A^{p,q}(B)=\mathfrak{h}_A^{p,q}$ and thus the following theorem can be obtained.

Theorem 4.4. *It holds that*

$$\mathfrak{H}_A^{p,b}(B)=\begin{cases}\{0\} & \text{for } q\geq 1\,,\\ \mathrm{Hol}^2(B\colon \Lambda^{p,0}(H_R^*)^c) & \text{for } q=0\,.\end{cases} \tag{4.5}$$

Appendix A The fundamentals concerning the complexification of a complex separable Hilbert space

Let H be a complex separable Hilbert space with inner product $(\ )_H$ and $\{e_n\}_{n=1}^{\infty}$ be its ONB. The adjoint space of H, denoted by H^*, is a space of $\boldsymbol{C}$-linear continuous functionals on H and becomes a complex separable Hilbert space with the following inner rpoduct:

$$(\theta,\eta)_{H^*}=\sum_{n=1}^{\infty}\langle\theta, e_n\rangle\overline{\langle\eta, e_n\rangle} \qquad \text{for } \theta, \eta\in H^*. \tag{A.1}$$

H becomes a real separable Hilbert space with respect to the following inner product,

$$(x, y)_R=\mathrm{Re}(x, y)_H\,. \tag{A.2}$$

We denote this real Hilbert space by H_R. H_R has a natural complex structure J defined by $Jx=\sqrt{-1}x$ for $x\in H_R$. Then it holds that $J^2=-1$, J is skew-adjoint and $\{e_n, Je_n\}_{n=1}^{\infty}$ is an ONB of H_R.

The adjoint space of H_R, denoted by H_R^*, is a space of $\boldsymbol{R}$-linear continuous functionals on H_R and becomes a real separable Hilbert space with respect to the following inner product:

$$(\varphi, \psi)_{H_R^*}=\sum_{n=1}^{\infty}\{\langle\varphi, e_n\rangle\langle\psi, e_n\rangle+\langle\varphi, Je_n\rangle\langle\psi, Je_n\rangle\} \qquad \text{for } \varphi, \psi\in H_R^*\,. \tag{A.3}$$

A complex structure J' on H_R^* is defined by $\langle J'\varphi, x\rangle=\langle\varphi, Jx\rangle$ for $\varphi\in H_R^*$, $x\in H_R$.

Let $(H_R^*)^c=H_R\otimes\boldsymbol{C}$, the complexification of H_R^*. An inner product on $(H_R^*)^c$ is given by $(\varphi\otimes z, \psi\otimes w)_{(H_R^*)^c}=(\varphi, \psi)_{H_R^*}z\bar{w}$ for $\varphi, \psi\in H_R^*$, $z, w\in\boldsymbol{C}$, which is extended by the $\boldsymbol{R}$-linearlity in each argument. Then $(H_R^*)^c$ becomes a complex separable Hilbert space with respect to this inner product. $(H_R^*)^c$ is naturally regarded as a space of $\boldsymbol{C}$-valued $\boldsymbol{R}$-linear functionals on H_R by $\langle\varphi\otimes z, x\rangle=\langle\varphi, x\rangle z$. Then its inner product is also given by

$$(\xi, \eta)_{(H_R^*)^c}=\sum_{n=1}^{\infty}\{\langle\xi, e_n\rangle\overline{\langle\eta, e_n\rangle}+\langle\xi, Je_n\rangle\overline{\langle\eta, Je_n\rangle}\} \qquad \text{for } \xi, \eta\in(H_R^*)^c. \tag{A.4}$$

$\boldsymbol{R}$-linear operator J' can be extended to a $\boldsymbol{C}$-linear operator on $(H_R^*)^c$ by $J'(\varphi\otimes z)=(J'\varphi)\otimes z$. We note that $J'^2=-1$ and J' is skew-adjoint on $(H_R^*)^c$. Thus $(H_R^*)^c$ is orthogonally decomposed as a sum of $\mathrm{Ker}(J'-\sqrt{-1})$ and $\mathrm{Ker}(J'+\sqrt{-1})$, where $\mathrm{Ker}(J'-\sqrt{-1})=H^*$, the space of $\boldsymbol{C}$-linear continuous functionals on H_R and $\mathrm{Ker}(J'+\sqrt{-1})=\bar{H}^*$, the space of anti $\boldsymbol{C}$-linear continuous functionals on H_R.

Complex conjugate on $(H_R^*)^c$ is given by $\overline{\varphi\otimes z}=\varphi\otimes\bar{z}$. Then $\langle\overline{\varphi\otimes z}, x\rangle=\overline{\langle\varphi\otimes z, x\rangle}$ for $x\in H_R$, so if $\theta\in H^*$, then $\bar{\theta}\in\bar{H}^*$ and vice versa.

We note difference between the inner product on H^* induced from $(H_R^*)^c$ and the original one. For $\kappa, \eta\in H^*$,

$$(\theta, \eta)_{(H_R^*)^c}=\sum_{n=1}^{\infty}\{\langle\theta, e_n\rangle\overline{\langle\eta, e_n\rangle}+\langle\theta, Je_n\rangle\overline{\langle\eta, Je_n\rangle}\} \tag{A.5}$$

$$=2\sum_{n=1}^{\infty}\{\langle\theta, e_n\rangle\overline{\langle\eta, e_n\rangle}\}=2(\theta, \eta)_{H^*}.$$

Thus if $\{\theta_n\}_{n=1}^{\infty}$ is an ONB of H^*, then $\{\frac{1}{\sqrt{2}}\theta_n, \frac{1}{\sqrt{2}}\bar{\theta}_n\}_{n=1}^{\infty}$ becomes an ONB of $(H_R^*)^c$.

For an operator C on H^*, we define an operator $\bar{C}$ on $\bar{H}^*$ as follows:

$$\langle\bar{C}\zeta, x\rangle=\overline{\langle C\zeta, x\rangle} \qquad \text{for } \zeta\in\bar{H}^*, x\in H_R. \tag{A.6}$$

Complex conjugate defines anti-unitary isomorphism from H^* to $\bar{H}^*$. Thus C and $\bar{C}$ are anti-uintarily isomorphic and if C is self-adjoint, then $\bar{C}$ is also self-adjoint and they are isomorphic.

Appendix B Complex Gaussian random variables and Wick product

Let $Z=X+\sqrt{-1}Y$ be a complex random variable with mean 0. We call Z a *complex Gaussian random variable* if X and Y are independent and identically distributed Gaussian random variables. This is equivalent to stating that $E[\exp(\sqrt{-1}\mathrm{Re}(aZ))]=\exp(-\frac{1}{4}|a|^2E[Z\bar{Z}])$ for any $a\in\boldsymbol{C}$.

Complex random variables $Z_1\cdots Z_n$ are called *jointly complex Gaussian random variables* if for any $\alpha_1\cdots\alpha_n\in\boldsymbol{C}$, $\alpha_1Z_1+\cdots+\alpha_nZ_n$ becomes a complex Gaussian random variable.

Proposition B.1. *Let $Z_1, \cdots, Z_n$ $W_1, \cdots, W_m$ be jointly complex Gaussian random variables. Then it holds that*

$$E[Z_1\cdots Z_n\bar{W}_1\cdots\bar{W}_m]=0 \qquad \text{if } n\neq m, \tag{B.1}$$

$$E[Z_1\cdots Z_n\bar{W}_1\cdots\bar{W}_n]=\sum_{\sigma\in\mathfrak{S}_n}E[Z_1\bar{W}_{\sigma_1}]\cdots E[Z_n\bar{W}_{\sigma_n}], \tag{B.2}$$

where $\mathfrak{S}_n$ denotes the permutation group on n letters.

For jointly complex Gaussian random variables $Z_1, \cdots, Z_n, W_1, \cdots, W_m$, we define their *Wick product*: $Z_1 \cdots Z_n \bar{W}_1 \cdots \bar{W}_m$: by induction with respect to (n, m) as follows,

$$\text{(B.3)} \quad \begin{aligned} &:Z_1 \cdots Z_n \bar{W}_1 \cdots \bar{W}_m: \\ &= Z_n :Z_1 \cdots Z_{n-1} \bar{W}_1 \cdots \bar{W}_m: - \sum_{j=1}^{m} E[Z_n \bar{W}_j] :Z_1 \cdots Z_{n-1} \bar{W}_1 \cdots \hat{\bar{W}}_j \cdots \bar{W}_m: \end{aligned}$$

$$\text{(B.4)} \quad \begin{aligned} &:Z_1 \cdots Z_n W_1 \cdots W_m: \\ &= W_m :Z_1 \cdots Z_n \bar{W}_1 \cdots \bar{W}_{m-1}: - \sum_{k=1}^{n} E[Z_k \bar{W}_m] :Z_1 \cdots \hat{Z}_k \cdots Z_n \bar{W}_1 \cdots \bar{W}_{m-1}: \end{aligned}$$

where $\hat{\alpha}$ denotes α is deleted. From this definition we can show that for jointly complex Gaussian random variables $Z_1 \cdots Z_\nu$,

$$\text{(B.5)} \quad \frac{\partial}{\partial Z_j} :Z_1^{n_1} \cdots Z_\nu^{n_\nu} \bar{Z}_1^{m_1} \cdots \bar{Z}_\nu^{m_\nu}: = n_j :Z_1^{n_1} \cdots Z_j^{n_j - 1} \cdots Z_\nu^{n_\nu} \bar{Z}_1^{m_1} \cdots \bar{Z}_\nu^{m_\nu}: ,$$

$$\text{(B.6)} \quad \frac{\partial}{\partial \bar{Z}_j} :Z_1^{n_1} \cdots Z_\nu^{n_\nu} Z_1^{m_1} \cdots Z_\nu^{m_\nu}: = m_j :Z_1^{n_1} \cdots Z_\nu^{n_\nu} \bar{Z}_1^{m_1} \cdots \bar{Z}_j^{m_j - 1} \cdots \bar{Z}_\nu^{m_\nu}: ,$$

$$\text{(B.7)} \quad E[:Z_1^{n_1} \cdots Z_\nu^{n_\nu} \bar{Z}_1^{m_1} \cdots \bar{Z}_\nu^{m_\nu}:] = 0$$

and moreover the following can be proven.

Proposition B.2. (a) *For jointly complex Gaussian random variables $Z_1^{(1)}, \cdots Z_{n_1}^{(1)}, W_1^{(1)}, \cdots, W_{m_1}^{(1)}, Z_1^{(2)}, \cdots, Z_{n_2}^{(2)}, W_1^{(2)}, \cdots, W_{m_2}^{(2)}$*

$$\text{(B.8)} \quad E[:Z_1^{(1)} \cdots Z_{n_1}^{(1)} \bar{W}_1^{(1)} \cdots \bar{W}_{m_1}^{(1)}: \ :Z_1^{(2)} \cdots Z_{n_2}^{(2)} \bar{W}_1^{(2)} \cdots \bar{W}_{m_2}^{(2)}:] = 0$$

if $n_1 \neq n_2$ or $m_1 \neq m_2$,
(b) *For jointly complex Gaussian random variables $Z_1, \cdots, Z_\nu$ such that $(Z_i, Z_j)_{L^2} = \delta_{i,j}$ for $1 \leq i, j \leq \nu$*

$$\text{(B.9)} \quad \begin{aligned} &(:Z_1^{n_1} \cdots Z_\nu^{n_\nu} \bar{Z}_1^{m_1} \cdots \bar{Z}_\nu^{m_\nu}:, :Z_1^{l_1} \cdots Z_\nu^{l_\nu} \bar{Z}_1^{k_1} \cdots \bar{Z}_\nu^{k_\nu})_{L^2} \\ &= \delta_{n_1, l_1} \cdots \delta_{n_\nu, l_\nu} \delta_{m_1, k_1} \cdots \delta_{m_\nu, k_\nu} n_1! \cdots n_\nu! m_1! \cdots m_\nu! \end{aligned}$$

where $(X, Y)_{L^2} = E[X \bar{Y}]$ for complex random variables X and Y.

The proof is similar to the real case. See [5].

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