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## THE RESIDUAL LIMIT SETS AND THE GENERATORS OF FINITELY GENERATED KLEINIAN GROUPS

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### 1. Introduction

1.1. Let  $G$  be a Kleinian group and denote by  $\Omega(G)$  and  $\Lambda(G)$  the region of discontinuity and the limit set of  $G$ , respectively. Throughout this paper, a Kleinian group means a non-elementary one. The residual limit set of  $G$ , which is denoted by  $\Lambda_0(G)$ , is the subset of  $\Lambda(G)$  which consists of all the points not lying on the boundary of any component of  $\Omega(G)$ . Although the study of Kleinian groups has long history, the residual limit sets were not treated or were thought to be empty, until in 1971 Abikoff showed the existence of Kleinian groups with the non-empty residual limit sets [1]. In his paper [2], Abikoff also studied the properties of residual limit sets and showed their non-emptiness for all finitely generated Kleinian groups except for those of two classes which have clearly the empty residual limit set; one is a class of function groups and the other is a class of  $Z_2$ -extensions of quasi-Fuchsian groups.

In this paper we shall show the importance of the residual limit sets by proving the following.

**Theorem 1.1.** *Let  $G$  be a finitely generated Kleinian group and let  $S$  be a finite set of generators of  $G$ . If  $G$  is neither a function group nor a  $Z_2$ -extension of a quasi-Fuchsian group, then  $S$  can be changed into a set of generators  $S_0$  of  $G$  with the following properties:*

- i) *each element of  $S_0$  is loxodromic and its fixed points lie on  $\Lambda_0(G)$ , and*
- ii) *the number of elements of  $S_0$  is not greater than that of  $S$ .*

Among the sets of generators of a finitely generated group, there is a set, the number of elements of which is minimum. We shall call it the minimal set of generators. Choosing  $S$  in Theorem 1.1 to be the minimal set of generators, we have the following.

**Corollary 1.2.** *Among the minimal sets of generators of a finitely generated Kleinian group  $G$  with the non-empty residual limit set, there is a set consisting of only loxodromic elements with the fixed points on  $\Lambda_0(G)$ .*

1.2. This paper is arranged with respect to steps of the proof of Theorem 1.1. In §2, we list up some known results which we shall need later and then define and discuss the rotation order of some loxodromic element. We change  $S$  into  $S_0$  in three steps; in §3 into a set which consists of the loxodromic elements only, in §4 into another set which consists of the loxodromic elements only and contains at least one element which has the fixed points on  $\Lambda_0(G)$  and in §5 into the desired  $S_0$ . In each step the changed set is a set of generators of  $G$  and the number of the elements of the set is not greater than that of the original set. In §6,  $S_0$  is studied in detail for non-web groups. The author wishes to express his deep gratitude to professor T. Kuroda for his advices.

## 2. Known results and rotation order of a loxodromic element

2.1. Let  $G$  be a finitely generated Kleinian group and let  $\Delta$  be a component of  $G$ . The component subgroup  $G_\Delta$  for  $\Delta$  is the maximal subgroup of  $G$  which leaves  $\Delta$  invariant. For component subgroups of  $G$ , the followings are known.

**Theorem 2.1** [3].  *$G_\Delta$  is a finitely generated function group with  $\Delta$  as an invariant component.*

**Theorem 2.2** [4]. *If  $G_\Delta$  has an invariant component different from  $\Delta$ , then  $G_\Delta$  is a quasi-Fuchsian group with the invariant Jordan curve  $\partial\Delta = \Lambda(G_\Delta)$ .*

From these theorems we have the following.

**Corollary 2.3.** *Let  $\Delta'$  be a component of  $G_\Delta$  which is different from  $\Delta$ . Then the component subgroup  $G_{\Delta'}$  for  $\Delta'$  of  $G_\Delta$  is a quasi-Fuchsian group with the invariant Jordan curve  $\partial\Delta' = \Lambda(G_{\Delta'})$ .*

The Jordan curve  $\partial\Delta'$  in this corollary is called a separator of  $G$  and the set of all such separators of  $G$  is called the set of separators of  $G$ .

**Lemma 2.4** [2]. *Separators do not cross each other.*

**Lemma 2.5** [2]. *If  $\infty \in \Omega(G)$ , then the diameters of separators of  $G$  form a null sequence.*

For common subgroups of component subgroups of  $G$  and for common boundary points of components of  $G$ , the followings are known.

**Theorem 2.6** [5,7]. *Let  $\{\Delta_1, \Delta_2, \dots, \Delta_n\}$  be an arbitrary collection of components of  $G$ . Then  $\Lambda(\cap_{i=1}^n G_{\Delta_i}) = \cap_{i=1}^n \partial\Delta_i$ . If  $n \geq 3$ , then  $\cap_{i=1}^n \partial\Delta_i$  consists of at most two points.*

**Theorem 2.7** [6]. *Let  $\Delta', \Delta''$  be the non-invariant components of  $G_\Delta$ . Then  $\partial\Delta' \cap \partial\Delta''$  consists of at most one point. If it is not empty, then the point is*

*the fixed point of a parabolic element of  $G$ .*

2.2. Here we shall recall the auxiliary domains. Let  $\Delta_i, \Delta_j$  be the components of  $G$ . Let  $\Delta'_{ij}$  be the component of  $G_{\Delta_i}$  containing  $\Delta_j$ . The complement of the closure of  $\Delta'_{ij}$  is called the auxiliary domain of  $\Delta_i$  with respect to  $\Delta_j$  and it is denoted by  $D_{ij}$  and, if there is no confusion, we write  $D$  instead of  $D_{ij}$ . By definition, the boundary  $\partial D$  is a separator of  $G$ . For auxiliary domains, we have followings.

**Lemma 2.8** [6]. *Let  $\Delta_i$  and  $\Delta_j$  ( $\neq \Delta_i$ ) be the components of  $G$ . Then  $D_{ij} \supset \Delta_i$ ,  $\partial D_{ij} \subset \partial \Delta_i$ ,  $D_{ij} \cap D_{ij} = \emptyset$  and  $\partial \Delta_i \cap \partial \Delta_j = \partial D_{ij} \cap \partial D_{ji}$ .*

**Lemma 2.9** [7]. *Let  $\{\Delta_1, \Delta_2, \dots, \Delta_n\}$  ( $n > 2$ ) be an arbitrary collection of components of  $G$ . If  $\bigcap_{i=1}^n \partial \Delta_i$  consists of two points, then  $D_{ij} = D_{ik}$  for any integers  $i, j, k$ .*

If  $\Delta_i$  is a component of  $G$  containing  $\infty$ , then the auxiliary domain  $D_{ji}$  of any component  $\Delta_j$  of  $G$  with respect to  $\Delta_i$  is bounded. By Lemma 2.8, the diameter of  $\Delta_j$  is identical with that of  $D_{ji}$ . By Theorem 2.6, the set  $\partial D_{ji}$  can be the subset of boundaries of at most two components of  $G$ . Hence by Lemma 2.5, we have the following.

**Lemma 2.10.** *If there is a component  $\Delta$  of  $G$  containing  $\infty$ , then the diameters of components (excluding  $\Delta$ ) of  $G$  form a null sequence.*

2.3. For loxodromic elements of  $G$  and for component subgroups, the following is known.

**Theorem 2.11** [5]. *Let  $\gamma$  be a loxodromic element of  $G$  with a fixed point on the boundary of a component  $\Delta$  of  $F$ . Then there is a positive integer  $r$  such that  $\gamma^r \in G_\Delta$ . Hence the other fixed point of  $\gamma$  also lies on the boundary of the same component  $\Delta$ .*

The minimum of  $r$  in the above theorem is called the rotation order of  $\gamma$  for  $\Delta$ .

**Lemma 2.12.** *Let  $\gamma$  be a loxodromic element of  $G$ . If one fixed point of  $\gamma$  lies on a separator of  $G$ , then the other fixed point of  $\gamma$  also lies on the same separator.*

*Proof.* Let  $\partial \Delta'$  be a separator, on which one fixed point of  $\gamma$  lies, and let  $\Delta$  be a component such that  $\Delta'$  is a component of  $G_\Delta$ . Since  $\partial \Delta' \subset \partial \Delta$ , we see by Theorem 2.11 that both fixed points of  $\gamma$  lie on  $\partial \Delta$  and that  $\gamma^r \in G_\Delta$  for the rotation order  $r$  of  $\gamma$  for  $\Delta$ . Hence we see by using Theorem 2.11 again that both fixed points of  $\gamma$  lie on  $\partial \Delta'$ .

**Lemma 2.13.** *Let  $\gamma$  be a loxodromic element of  $G$  with a fixed point on the*

boundary of a component  $\Delta$  of  $G$ . Assume that the rotation order  $r$  of  $\gamma$  for  $\Delta$  is greater than 1. If  $D$  denotes the auxiliary domain of  $\Delta$  with respect to  $\gamma(\Delta)$ , then the followings hold:

- i) the fixed points of  $\gamma$  lie on  $\partial D$ ,
- ii)  $\gamma^i(D)$  is identical with the auxiliary domain of  $\gamma^i(\Delta)$  with respect to  $\Delta$ ,  $i \not\equiv r \pmod{r}$ , and
- iii)  $\gamma^i(D) \cap \gamma^j(D) = \emptyset$  for integers  $i, j \not\equiv i \pmod{r}$ .

Proof. i) Let  $D_1$  be the auxiliary domain of  $\gamma(\Delta)$  with respect to  $\Delta$ . Since  $\gamma$  has a fixed point on the boundary of  $\Delta$ , one fixed point of  $\gamma$  lies on  $\partial\Delta \cap \partial\gamma(\Delta)$ . By Lemma 2.8, we see that it also lies on  $\partial D \cap \partial D_1$ . Hence  $\gamma$  has one fixed point on the separator  $\partial D$ . Therefore the assertion follows from Lemma 2.12.

ii) Let  $D_i$  be the auxiliary domain of  $\gamma^i(\Delta)$  with respect to  $\Delta$ ,  $1 \leq i < r$ . Since  $\gamma^i(D) \supset \gamma^i(\Delta)$  and  $\partial\gamma^i(D) \subset \partial\gamma^i(\Delta)$ , we see that the outside of  $\gamma^i(D)$  is a component of the complement of  $\gamma^i(\Delta)$ . We assert that  $\gamma^i(D) \cap D = \emptyset$ . In fact, if  $i=1$ , then evidently  $\gamma(D) \cap D = \emptyset$ . If  $1 < i$ , then  $\gamma^i(\Delta)$  is contained in a component of  $G_\Delta$  different from  $\Delta$ , and hence, if  $\gamma^i(D) \cap D \neq \emptyset$ , then  $\gamma^i(D) \subset D$ , so  $\gamma^i(\Delta)$  lies in a non-invariant component of  $G_\Delta$  which is different from the one containing  $\gamma(\Delta)$ . Since  $\partial\gamma^i(\Delta) \cap \partial\gamma(\Delta)$  contains at least two fixed points of  $\gamma$ , this contradicts Theorem 2.7. Hence, in any case, we have the assertion that  $\gamma^i(D) \cap D = \emptyset$ . Therefore, the outside of  $\gamma^i(D)$  is the component of the complement of  $\gamma^i(\Delta)$  which contains  $\Delta$ . So we have  $\gamma^i(D) = D_i$ .

iii) In the case of  $r=2$ , the assertion follows from Lemma 2.8 and ii). If  $r>2$ , then by Lemma 2.9 and ii) we see that  $\gamma^i(D)$  (or  $\gamma^j(D)$ ) is the auxiliary domain of  $\gamma^i(\Delta)$  (or  $\gamma^j(\Delta)$ ) with respect to  $\gamma^j(\Delta)$  (or  $\gamma^i(\Delta)$ ). Hence the assertion follows from Lemma 2.8.

**Theorem 2.14.** *Let  $\gamma$  and  $\Delta_1, \Delta_2$  be a loxodromic element and two components of  $G$ , respectively. If the fixed points of  $\gamma$  lie on the common boundary of  $\Delta_1$  and  $\Delta_2$ , then the rotation order of  $\gamma$  for  $\Delta_1$  is identical with that of  $\gamma$  for  $\Delta_2$ .*

Proof. Assume that the rotation order of  $\gamma$  for  $\Delta_1$  be  $r \geq 2$ . If  $\Delta_2 = \gamma^i(\Delta_1)$  for some integer  $i$ , then we see at once that the rotation order of  $\gamma$  for  $\Delta_2$  is  $r$ . Therefore we assume that  $\Delta_2 \neq \gamma^i(\Delta_1)$  for any integer  $i$ . Let  $D_{12}$  (or  $D_{21}$ ) be the auxiliary domain of  $\Delta_1$  (or  $\Delta_2$ ) with respect to  $\Delta_2$  (or  $\Delta_1$ ). Then by Lemma 2.8, we see  $D_{12} \cap D_{21} = \emptyset$ . Let  $D$  be the auxiliary domain of  $\Delta_1$  with respect to  $\gamma(\Delta_1)$ . Then by Lemma 2.9, we see  $D_{12} = D$ . Hence by Lemma 2.13 and Lemma 2.9, we see  $\gamma^i(D_{12}) \cap D_{21} = \emptyset$  for any integer  $i$ . Hence we can find two integers  $i$  and  $j$  such that the component of the complement of the closure of  $\gamma^i(D_{12}) \cup \gamma^j(D_{12})$  including  $D_{21}$  does not include any  $\gamma^k(D_{12})$ ,  $1 \leq k < r$ . Since  $\gamma$  is an orientation preserving homeomorphism,  $\gamma^l(D_{21})$  lies between  $\gamma^{i+l}(D_{12})$  and  $\gamma^{j+l}(D_{12})$  for any integer  $l$ . This and Lemma 2.13 imply that the rotation

order  $r'$  of  $\gamma$  for  $\Delta_2$  is not less than  $r$ . Similarly, we see  $r' \leq r$ . Hence we have  $r' = r$ . As a consequence of this, we see that the rotation order of  $\gamma$  for  $\Delta_1$  is 1 if and only if that of  $\gamma$  for  $\Delta_2$  is 1. Thus we have our theorem.

For the common subgroup  $\cap_{i=1}^n G_{\Delta_i}$  in Theorem 2.6 we have the following.

**Corollary 2.15.** *Let  $\gamma$  and  $\{\Delta_1, \Delta_2, \dots, \Delta_n\}$  be a loxodromic element and an arbitrary collection of components of  $G$ , respectively. If the fixed points of  $\gamma$  lie on  $\cap_{i=1}^n \partial\Delta_i$  and if  $\gamma \in G_{\Delta_i}$ , then  $\gamma \in \cap_{i=1}^n G_{\Delta_i}$ .*

*Proof.* If  $\gamma \in G_{\Delta_i}$ , then the rotation order of  $\gamma$  for  $\Delta_i$  is 1. By Theorem 2.14, we see that the rotation order of  $\gamma$  for any  $\Delta_j$  is 1. Hence our assertion follows.

For later use we also need a following form of Theorem 2.14.

**Lemma 2.16.** *Let  $D$  be an arbitrary auxiliary domain of a component  $\Delta$  of  $G$ . If a loxodromic element  $\gamma$  of  $G_\Delta$  has a fixed point on  $\partial D$ , then  $\gamma(D) = D$ .*

*Proof.* Let  $\Delta'$  be a component of  $G_\Delta$  whose complement is the closure of  $D$ . Then the fixed points of  $\gamma$  lie on  $\partial\Delta'$ . Applying Theorem 2.14 to  $\Delta$  and  $\Delta'$ , we have  $\gamma(\Delta') = \Delta'$ , so that  $\gamma(D) = D$ .

### 3. Loxodromic generators

3.1. In the following three sections including this section, we assume that  $G$  satisfies the condition of Theorem 1.1. In this § we shall change a finite set  $S$  of generators of a given finitely generated Kleinian group  $G$  in Theorem 1.1 into the set of generators consisting of loxodromic elements only. Our process is repetition of the following three kinds of operations;  $\gamma_i$  is changed into one or  $\gamma_i^{-1}$ ,  $\gamma_i\gamma_j$  and  $\gamma_j\gamma_i$ , where  $\gamma_i, \gamma_j$  are elements of  $S$  or of the changed sets by this process. This operation does not increase the number of elements of the set of generators and the changed set is clearly a set of generators of the same group.

Let  $S = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ . Assume that there are elliptic elements in  $S$  with the same fixed points. Since  $G$  is Kleinian, we can replace them by a single elliptic element of  $G$  so that the changed set is also a set of generators of  $G$  and the number of elements of this changed set is not greater than that of  $S$ . Hence we may assume that  $S$  does not contain any two elliptic elements with the same fixed points. We consider three cases.

3.2. The case (I) where  $S$  contains at least one loxodromic element: Without loss of generality we may assume that  $\gamma_1$  is loxodromic and its matrix representation has the form  $\begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix}$ ,  $|k| > 1$ . Consider an elliptic or a parabolic element  $\gamma_i \in S$  with matrix representation  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $ad - bc = 1$ . We con-

sider the element of the form  $\gamma_1^m \gamma_i$  with the trace  $ak^m + dk^{-m}$ , where  $m$  is an integer. If  $a \neq 0$  (or  $d \neq 0$ ), then we can take  $m$  so large (or small) that  $|ak^m + dk^{-m}| > 2$ . Hence  $\gamma_1^m \gamma_i$  is loxodromic. With such an  $m$  we replace  $\gamma_i$  by  $\gamma_1^m \gamma_i$  and after carrying out the above procedure for all such  $\gamma_i$ , we denote the new set of generators by the same  $S$ . If  $a = d = 0$ , then  $\gamma_i$  is an elliptic element of order 2 and changes 0,  $\infty$  into each other. Consider another loxodromic element  $\gamma_j (\neq \gamma_1)$  of  $S$  whose fixed points are different from 0 and  $\infty$ . Since  $G$  is non-elementary, the existence of such a  $\gamma_j$  in  $S$  is assured. First we change  $\gamma_j$  into  $\gamma_1^m \gamma_j \gamma_1^{-m}$  with so large integer  $m$  that  $|\xi_j \xi_j'| > |b|^2$ , where  $\xi_j, \xi_j'$  are the fixed points of  $\gamma_1^m \gamma_j \gamma_1^{-m}$ , which we also denote by the same  $\gamma_j$ . Let  $A$  be a linear transformation which maps  $\xi_j$  and  $\xi_j'$  to 0 and  $\infty$ , respectively. Then the conjugations of  $\gamma_i, \gamma_j$  by  $A$  have the forms

$$\begin{aligned} \gamma_i^* &= A \gamma_i A^{-1} = \begin{pmatrix} \frac{1}{D} & \frac{-\xi_j}{D} \\ \frac{1}{D} & \frac{-\xi_j'}{D} \end{pmatrix} \begin{pmatrix} 0 & b \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{-\xi_j'}{D} & \frac{\xi_j}{D} \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{-(b + \xi_j \xi_j' b^{-1})}{D^2} & \frac{b + \xi_j^2 b^{-1}}{D^2} \\ \frac{-(b + (\xi_j')^2 b^{-1})}{D^2} & \frac{b + \xi_j \xi_j' b^{-1}}{D^2} \end{pmatrix} \end{aligned}$$

and

$$\gamma_j^* = A \gamma_j A^{-1} = \begin{pmatrix} l & 0 \\ 0 & 1/l \end{pmatrix}, \quad |l| \neq 1,$$

respectively, where  $D = (\xi_j - \xi_j')^{1/2}$ . Since  $(b + \xi_j \xi_j' b^{-1}) / (\xi_j - \xi_j') \neq 0$ , we see that  $(\gamma_j^*)^m \gamma_i^*$  is loxodromic for some integer  $m$  and hence  $\gamma_j^m \gamma_i$  is also loxodromic for some integer  $m$ . We replace  $\gamma_i$  by  $\gamma_j^m \gamma_i$ .

3.3. The case (II) where  $S$  contains at least one parabolic element: Without loss of generality we may assume that  $\gamma_1$  is parabolic and its matrix representation has the form  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Since  $G$  is non-elementary, there is an element  $\gamma_i$  of  $S$  with the matrix representation  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $c \neq 0$ . We consider the element of the form  $\gamma_1^m \gamma_i$  with an integer  $m$ . Since the trace of  $\gamma_1^m \gamma_i$  equals  $a + d + cm$ , we see that for a sufficiently large  $m$ ,  $\gamma_1^m \gamma_i$  is loxodromic. We replace  $\gamma_i$  by  $\gamma_1^m \gamma_i$ . Then  $S$  reduces to a set of generators in the previous case (I).

3.4. The case (III) where  $S$  consists of elliptic elements only: We shall first prove the following two lemmas.

**Lemma 3.1.** *Let  $\gamma$  and  $\delta$  be linear transformations with the matrix representations of the forms  $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ ,  $0 < |\theta| < \pi$ , and  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $ad - bc = 1$ , respectively. If*

$a+d$  is real and if  $d \neq \bar{a}$ , then  $\gamma\delta$  is loxodromic.

Proof. Set  $a = a_1 + ia_2$  and  $d = d_1 + id_2$  with real numbers  $a_1, a_2, d_1, d_2$ . Then  $d_2 = -a_2$  and  $a_1 \neq d_1$ . Hence we see that the trace of  $\gamma\delta$  is not real. Therefore,  $\gamma\delta$  is loxodromic.

**Lemma 3.2.** *Let  $\gamma$  and  $\delta$  be elliptic. If all four fixed points of them do not lie on a line nor on a circle, then  $\gamma\delta$  is loxodromic.*

Proof. Without loss of generality we may assume that the fixed points of  $\gamma$  are 0 and  $\infty$  and that  $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $ad - bc = 1$ ,  $b \neq 0$  and  $c \neq 0$ . Set  $a = a_1 + ia_2$  and  $d = d_1 + id_2$  with real numbers  $a_1, a_2, d_1, d_2$ . Since  $\delta$  is elliptic, we see that  $d_2 = -a_2$ . If  $d = \bar{a}$ , then, by writing  $c = |c|e^{i\theta}$ , we see that the fixed points of  $\delta$  are

$$\begin{aligned} \frac{a-d \pm \sqrt{(a+d)^2 - 4}}{2c} &= \frac{2a_2i \pm \sqrt{4 - (a+d)^2}i}{2|c|e^{i\theta}} \\ &= \frac{2a_2 \pm \sqrt{4 - (a+d)^2}}{2|c|} e^{i(\pi/2 - \theta)}, \end{aligned}$$

because  $(a+d)^2 < 4$ . Hence all fixed points of  $\gamma$  and  $\delta$  lie on the line which passes through 0 and makes an angle  $\pi/2 - \theta$  with the real axis. This contradicts our assumption. Hence we obtain that  $d \neq \bar{a}$ . By Lemma 3.1,  $\gamma\delta$  is loxodromic.

If  $S$  contains two elements  $\gamma_i, \gamma_j$  whose all four fixed points do not lie on a circle nor on a line, then Lemma 3.2 implies that the changing  $\gamma_i$  into  $\gamma_i\gamma_j$  takes  $S$  into a set of generators in the case (I). On the other hand, we shall see in the following that, under our assumption that  $G$  is neither a function group nor a  $Z_2$ -extension of a quasi-Fuchsian group,  $S$  or its changed set by our operation contains such  $\gamma_i$  and  $\gamma_j$  as stated above.

For the purpose, we assume that, for any two elements of  $S$  (or its changed set by our operations), all their fixed points lie on a circle or a line. Since this property is invariant under the conjugation by a linear transformation, we may assume that the fixed points of  $\gamma_1$  are 0 and  $\infty$ . Let  $L_i$  be a line on which the fixed points of  $\gamma_i$  lie. Since  $G$  is non-elementary, there is an element  $\gamma_2$  of  $S$  which does not leave  $\infty$  invariant. Then  $L_2$  passes through 0. If  $\gamma_i$  ( $\neq \gamma_1$ ) has  $\infty$  as a fixed point, then  $L_i$  must be identical with  $L_2$ .

3.5. We first treat the case where there is an element  $\gamma_{j_0}$  of  $S$  with  $L_{j_0} \neq L_2$ . Then  $\gamma_{j_0}$  has the finite fixed points and  $L_{j_0}$  passes through 0.

**Lemma 3.3.** *Under these circumstances, each element of  $S$  except for  $\gamma_1$  has the finite and non-zero fixed points.*



Proof. If  $\gamma_2$  has 0 as a fixed point, then  $\gamma_{j_0}$  fixed above must have 0 as a fixed point. This is also true for all  $\gamma_j$  with  $L_j \neq L_2$ . Further, since  $\gamma_{j_0}$  has another fixed point different from  $\infty$  as already mentioned, we see  $\gamma_j$  satisfying  $L_j = L_2$  must have 0 as a fixed point. This shows that every element of  $S$  and of  $G$  has 0 as the fixed point, so that  $G$  is elementary, a contradiction. Hence the fixed points of  $\gamma_2$  are different from 0. So each  $\gamma_j$  with  $L_j \neq L_2$  and, in particular,  $\gamma_{j_0}$  has not 0 as a fixed point. Therefore  $\gamma_j$  ( $j \neq 1$ ) satisfying  $L_j = L_2$  has not 0 or  $\infty$  as a fixed point. Thus we have our lemma.

Let  $\xi_2, \xi_2'$  be the fixed points of  $\gamma_2$ . Without loss of generality we may assume that  $|\xi_2 \xi_2'| = 1$ . By an elementary geometric consideration we see that if the line segment  $\xi_2 \xi_2'$  includes (or does not include) 0, then the line segment  $\xi_{j_0} \xi_{j_0}'$  includes (or does not include) 0 and  $|\xi_{j_0} \xi_{j_0}'| = 1$ , where  $\xi_{j_0}, \xi_{j_0}'$  are the fixed points of  $\gamma_{j_0}$ . Hence, it is not difficult to see that in both cases these are also true for each  $\gamma_j$  ( $j \geq 2$ ) of  $S$ . In the case where the line segment  $\xi_2 \xi_2'$  does not contain 0, this implies that the fixed points of each element of  $S$  lie in the mirror images with respect to the circle  $C = \{z \mid |z| = 1\}$ , so that  $C$  is invariant under the action of each element of  $S$ , hence, of  $G$ . Hence  $\Lambda(G) \subset C$ . This contradicts our assumption that  $G$  is neither a function group nor a  $Z_2$ -extension of a quasi-Fuchsian group. Hence this case does not occur. Before going to treat the case where the line segment  $\xi_2 \xi_2'$  includes 0, we show the following.

**Lemma 3.4.** *Let  $\gamma_1$  and  $\gamma_2$  be elliptic transformations. If  $\gamma_j$  has the fixed points  $r_j e^{i\Theta_j}$  and  $-r_j^{-1} e^{i\Theta_j}$  ( $j=1, 2$ ) and if these four points lie on a line or a circle, then  $\gamma_1 \gamma_2$  is the identity or an elliptic transformation with the fixed points of the similar forms, where, if  $r_j = 0$ , then  $r_j^{-1} e^{i\Theta_j}$  means  $\infty$ .*

Proof. If  $\gamma_1$  and  $\gamma_2$  have the same fixed points, then the assertion is clear. Hence we assume that the fixed points of  $\gamma_1$  are different from those of  $\gamma_2$ . If the fixed points of  $\gamma_1$  are finite, we may assume that  $\Theta_1 = 0$  and we consider a transformation

$$A: z \mapsto \frac{1}{r_1} \frac{z - r_1}{z + r_1^{-1}}.$$

Then  $A\gamma_1 A^{-1}$  has 0 and  $\infty$  as the fixed points and the fixed points of  $A\gamma_2 A^{-1}$  lie on a line passing through 0 and separate 0 and  $\infty$  on the line. If  $r_2 = 0$  or  $=\infty$ , then the fixed points of  $A\gamma_2 A^{-1}$  are  $-r_1$  and  $r_1^{-1}$ . If  $0 < r_2 < \infty$ , then the fixed points of  $A\gamma_2 A^{-1}$  are

$$\frac{1}{r_1} \frac{r_2 e^{i\Theta_2} - r_1}{r_2 e^{i\Theta_2} + r_1^{-1}} \quad \text{and} \quad \frac{1}{r_1} \frac{-r_2^{-1} e^{i\Theta_2} - r_1}{-r_2^{-1} e^{i\Theta_2} + r_1^{-1}},$$

and clearly the absolute value of the product of these two numbers is equal to 1.

Thus we may assume without loss of generality that

$$\gamma_1 = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix}$$

and

$$\gamma_2 = \begin{pmatrix} \frac{r_2^{-1}e^{i\theta_2} + r_2e^{-i\theta_2}}{D} & \frac{-e^{i\theta_2} + e^{-i\theta_2}}{D} \\ \frac{-e^{i\theta_2} + e^{-i\theta_2}}{D} & \frac{r_2e^{i\theta_2} + r_2^{-1}e^{-i\theta_2}}{D} \end{pmatrix},$$

where  $D = r_2 + r_2^{-1}$  and  $\theta_2$  is not a multiple of  $\pi$ . Then the matrix representation of  $\gamma_1\gamma_2$  is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{r_2^{-1}e^{i(\theta_1+\theta_2)} + r_2e^{-i(\theta_1-\theta_2)}}{D} & \frac{-e^{i(\theta_1+\theta_2)} + e^{i(\theta_1-\theta_2)}}{D} \\ \frac{-e^{-i(\theta_1-\theta_2)} + e^{-i(\theta_1+\theta_2)}}{D} & \frac{r_2e^{-i(\theta_1-\theta_2)} + r_2^{-1}e^{-i(\theta_1+\theta_2)}}{D} \end{pmatrix}$$

and the trace of  $\gamma_1\gamma_2$  is

$$a + d = 2 \frac{\cos(\theta_1 + \theta_2) + r_2^2 \cos(\theta_1 - \theta_2)}{1 + r_2^2}.$$

Hence we have  $-2 \leq \text{trace } \gamma_1\gamma_2 \leq 2$ . The equalities occur only when  $\cos(\theta_1 + \theta_2) = \cos(\theta_1 - \theta_2) = \pm 1$ . These imply that  $\theta_1 + \theta_2 = k\pi$  and  $\theta_1 - \theta_2 = k\pi + 2m\pi$ , where  $k, m$  are integers, and hence  $\theta_1$  and  $\theta_2$  are multiples of  $\pi$ . Hence  $\gamma_1$  and  $\gamma_2$  are the identity transformations. Therefore, the equalities do not occur and  $\gamma_1\gamma_2$  is elliptic. We see easily that the product of the fixed points of  $\gamma_1\gamma_2$  has the absolute value 1. In order to complete the proof of our lemma we have only to show that the line segment connecting the fixed points of  $\gamma_1\gamma_2$  includes 0. It is easy to see that the ratio of the fixed points of  $\gamma_1\gamma_2$  is real. Hence it suffices to show that the absolute value of the difference of the fixed points is greater than that of the sum of them, or equivalently, to show  $|(a+d)^2 - 4| > |a-d|^2$ , which can be easily verified. Hence the line segment connecting the fixed points of  $\gamma_1\gamma_2$  includes 0. Thus we have completed the proof of our lemma.

Now we return to the case where the line segment  $\xi_2\xi_2'$  includes 0. As was already mentioned, the line segment  $\xi_j\xi_j'$  has the same property ( $j \geq 2$ ), where  $\xi_j$  and  $\xi_j'$  are the fixed points of  $\gamma_j \in S$ . By Lemma 3.4, we see that any product of a finite number of elements of  $S$  is an elliptic transformation or the identity, so that  $G$  is a finite group, a contradiction. Hence this case also

does not occur. Therefore, we have shown that if there is an element  $\gamma_i$  of  $S$  with  $L_i \neq L_2$ , then  $S$  contains two elements whose four fixed points do not lie on a circle nor a line.

3.6. We next treat the case where the fixed points of each element of  $S$  lie on  $L_2$ . If the order of each element of  $S$  is two, then  $L_2$  is invariant under the actions of  $S$  and of  $G$ , so that  $\Lambda(G) \subset L_2$ . This contradicts our assumption that  $G$  is neither a function group nor a  $Z_2$ -extension of a quasi-Fuchsian group. Hence there is an element of  $S$  whose order is greater than two. We may assume that the order of  $\gamma_1$  is greater than two. We shall show that  $S$  contains at least one more element which is different from  $\gamma_1$  and  $\gamma_2$ . Assume contrary that  $S = \{\gamma_1, \gamma_2\}$ . Since  $G$  is non-elementary, the fixed points of  $\gamma_2$  are finite and different from 0. If the line segment connecting the fixed points of  $\gamma_2$  does not contain 0, then there is a circle with the center 0 being invariant under  $\gamma_2$ . This circle is also invariant under  $\gamma_1$ . Hence the limit set of  $G$  is contained in the circle, a contradiction. If the line segment connecting the fixed points of  $\gamma_2$  contains 0, then, by Lemma 3.4, we see that  $G$  is elementary, a contradiction. Thus we have shown that  $S$  contains an element  $\gamma_i$  ( $i > 2$ ) with the fixed points on  $L_2$ . We change  $S$  into  $\{\gamma_1, \gamma_2, \dots, \gamma_1\gamma_i\gamma_1^{-1}, \dots, \gamma_n\}$ . Since the order of  $\gamma_1$  is greater than 2, the line on which the fixed points of  $\gamma_1\gamma_i\gamma_1^{-1}$  ( $i > 2$ ) lie is different from  $L_2$ . Hence this case reduces to the case discussed already.

Therefore the case where each element of  $S$  is elliptic can be reduced to the case where  $S$  contains at least one loxodromic element. Thus we have completed to change  $S$  into the set of generators of  $G$ , which consists of loxodromic elements only and the number of elements of which is not greater than that of  $S$ .

#### 4. Loxodromic elements with the fixed points on $\Lambda_0(G)$

4.1. Let  $S$  be a set of generators of  $G$  consisting of loxodromic elements only. As the second step of the proof of Theorem 1.1 we shall change  $S$  into a set of generators of  $G$ , which consists of loxodromic elements only and containing at least one element with fixed points on  $\Lambda_0(G)$  and the number of elements of which is not greater than that of  $S$ . Without loss of generality we may assume that  $\infty \in \Omega(G)$ . We shall first prove the following four lemmas.

**Lemma 4.1.** *Let  $\gamma_i$  and  $\gamma_j$  be loxodromic elements of  $G$  with no common fixed point and let  $\xi_j, \xi_j'$  be the repelling and the attractive fixed points of  $\gamma_j$ , respectively. Then, for a sufficiently large integer  $m$ ,  $\gamma_i\gamma_j^m$  is loxodromic and the repelling and the attractive fixed points of  $\gamma_i\gamma_j^m$  converge to  $\xi_j$  and to  $\gamma_i(\xi_j')$ , respectively, as  $m$  tends to  $\infty$ .*

**Proof.** For an arbitrary positive number  $\varepsilon > 0$ , there is a neighbourhood

$U$  of  $\gamma_i(\xi_j')$  such that  $U \cap \gamma_i^{-1}(U) = \emptyset$ ,  $\bar{U} \cap \xi_j = \emptyset$  and the diameter of  $U$  is smaller than  $\varepsilon$ . Let  $m$  be an integer and let  $C_m$  be the isometric circle of  $\gamma_j^m$ . For a sufficiently large  $m$ , we see that the diameter of  $C_m$  is smaller than  $\varepsilon$  and that  $C_m$  is contained in an open disc with center  $\xi_j$  and radius  $\varepsilon$  and  $\gamma_j^m(C_m) \subset \gamma_i^{-1}(U)$ . Then  $\gamma_i\gamma_j^m$  sends the exterior of  $C_m$  into  $U$ . Hence, for a large  $m$ ,  $\gamma_i\gamma_j^m$  is loxodromic and the distance between the repelling (or the attractive) fixed point of  $\gamma_i\gamma_j^m$  and  $\xi_j$  (or  $\gamma_i(\xi_j')$ ) is smaller than  $\varepsilon$ , which are to be shown.

**Lemma 4.2.** *Let  $\gamma_i$  and  $\gamma_j$  be loxodromic elements of  $G$  whose fixed points are different from each other and do not lie on  $\Lambda_0(G)$ . If there is no component of  $G$ , on whose boundary all the fixed points of  $\gamma_i$  and  $\gamma_j$  lie, then there is an integer  $m$  such that  $\gamma_i\gamma_j^m$  is a loxodromic element with the fixed points on  $\Lambda_0(G)$ .*

*Proof.* Let  $\xi_j$  and  $\xi_j'$  be the repelling and the attractive fixed points of  $\gamma_j$ , respectively. Then, by our assumption and by Theorem 2.11,  $\xi_j$  and  $\xi_j'$  lie on the boundary of a component of  $G$ . Since  $G$  is Kleinian,  $\gamma_i(\xi_j') \neq \xi_j$ , so that  $d = |\xi_j - \gamma_i(\xi_j')| > 0$ . By Lemma 2.10, there is a finite number of components of  $G$  whose diameters exceed  $d/2$ . Let  $\delta_1$  (or  $\delta_2$ ) be the minimum of the distances between  $\xi_j$  (or  $\gamma_i(\xi_j')$ ) and the components whose diameters exceed  $d/2$  and whose boundaries do not contain  $\xi_j$  (or  $\gamma_i(\xi_j')$ ). Let  $\delta$  be a positive number smaller than  $\min(\delta_1, \delta_2, d/4)$ . Lemma 4.1 implies that we can find an integer  $m$  sufficiently large such that the distances between  $\xi_j$  and the repelling fixed point of  $\gamma_i\gamma_j^m$  and between  $\gamma_i(\xi_j')$  and the attractive fixed point of  $\gamma_i\gamma_j^m$  are smaller than  $\delta$  and such that  $\gamma_i\gamma_j^m$  is loxodromic. If there is a component on whose boundary the fixed points of  $\gamma_i\gamma_j^m$  lie, then we see from the definition of  $\delta$  that  $\xi_j$  and  $\gamma_i(\xi_j')$  must lie on the boundary of that component. Since  $\xi_j$  and  $\gamma_i(\xi_j')$  are the fixed points of  $\gamma_j$  and  $\gamma_i\gamma_j\gamma_i^{-1}$ , respectively, we see by Theorem 2.11 that  $\xi_j'$  and  $\gamma_i(\xi_j)$  also lie on the same boundary. Hence, by Theorem 2.6, there are at most two components of  $G$  on whose boundaries the fixed points of  $\gamma_i\gamma_j^m$  lie. Let  $\Delta$  be such a one. Then the rotation order of  $\gamma_i\gamma_j^m$  for  $\Delta$  is at most 2. Let  $r$  be the rotation order of  $\gamma_j$  for  $\Delta$  and take  $m$  as a multiple of  $r$ . Then rotation order of  $\gamma_i\gamma_j^m$  for  $\Delta$  must be 2. In fact, otherwise,  $\gamma_i\gamma_j^m \in G_\Delta$  or  $\gamma_i \in G_\Delta$ , so that the fixed points of  $\gamma_i$  lie on the boundary of  $\Delta$ , which contradicts our assumption. Hence  $\gamma_i\gamma_j^m\gamma_i\gamma_j^m \in G_\Delta$  or  $\gamma_i\gamma_j^m\gamma_i \in G_\Delta$ . On the other hand, since  $\gamma_i\gamma_j^r\gamma_i^{-1}$  is an element of  $G_{\gamma_i(\Delta)}$  and has the fixed points on the boundary of  $\Delta$ , we see by Theorem 2.14 that  $\gamma_i\gamma_j^r\gamma_i^{-1} \in G_\Delta$ . Hence we have  $\gamma_i^2 \in G_\Delta$  so that the fixed points of  $\gamma_i$  lie on the boundary of  $\Delta$ . This contradicts our assumption that four fixed points of  $\gamma_i, \gamma_j$  do not lie on boundary of a single component of  $G$ . Therefore, for a large integer  $m$  there is no component on whose boundary the fixed points of  $\gamma_i\gamma_j^m$  lie. Thus  $\gamma_i\gamma_j^m$  is a loxodromic element with the fixed points on  $\Lambda_0(G)$ .

**Lemma 4.3.** *Let  $\gamma_i$  and  $\gamma_j$  be loxodromic elements of  $G$ , let  $\Delta$  be a compo-*

ment of  $G$  on whose boundary the fixed points of  $\gamma_i$  lie and let  $D$  be the auxiliary domain of  $\Delta$  with respect to  $\gamma_i(\Delta)$ . If the rotation order of  $\gamma_i$  for  $\Delta$  is greater than 1 and if the fixed points of  $\gamma_j$  lie in  $D$ , then, for a large integer  $m$ ,  $\gamma_i\gamma_j^m$  is a loxodromic element with the fixed points on  $\Lambda_0(G)$ .

*Proof.* Let  $\xi_j, \xi_j'$  be the repelling and the attractive fixed points of  $\gamma_j$ , respectively. By Lemma 2.13,  $D$  and  $\gamma_i(D)$  lie outside of each other and contain the points  $\xi_j$  and  $\gamma_i(\xi_j')$ , respectively. Lemma 4.1 shows that, for a large integer  $m$ , the repelling and the attractive fixed points of  $\gamma_i\gamma_j^m$  lie in  $D$  and in  $\gamma_i(D)$ , respectively. Hence the fixed points of  $\gamma_i\gamma_j^m$  are separated by the separator  $\partial D$ . Therefore we see easily that the fixed points of  $\gamma_i\gamma_j^m$  lie on  $\Lambda_0(G)$  (cf. [1]).

**Lemma 4.4.** *Let  $\gamma_i$  and  $\gamma_j$  be loxodromic elements of  $G$ , let  $\Delta$  be a component of  $G$  on whose boundary the fixed points of  $\gamma_i$  lie and let  $D$  be the auxiliary domain of  $\Delta$  with respect to  $\gamma_i(\Delta)$ . If the rotation order of  $\gamma_i$  for  $\Delta$  is greater than 2 and if the fixed points of  $\gamma_j$  lie on  $\partial D$  and are different from the fixed points of  $\gamma_i$ , then, for a large integer  $m$ ,  $\gamma_i\gamma_j^m$  is a loxodromic element with the fixed points on  $\Lambda_0(G)$ .*

*Proof.* We assume that the conclusion of the lemma is false. By Lemma 2.13, we see that  $D, \gamma_i(D), \dots, \gamma_i^{r-1}(D)$  lie outside of each other and have the fixed points of  $\gamma_i$  as the common boundary points, where  $r$  is the rotation order of  $\gamma_i$  for  $\Delta$ . As we have seen in the proof of Lemma 4.2, for a sufficiently large integer  $m$ , the components which have the fixed points of  $\gamma_i\gamma_j^m$  on the boundaries, must have the four fixed points of  $\gamma_j$  and  $\gamma_i\gamma_j\gamma_i^{-1}$  on the boundaries. Let  $C_1$  (or  $C_2$ ) be the subarc of  $\partial D$  which has the end points at the fixed points of  $\gamma_i$  and on which the repelling fixed point of  $\gamma_j$  lies (or does not lie). If the attractive fixed point of  $\gamma_j$  lies on  $C_2$ , then the component which has the fixed points of  $\gamma_i\gamma_j^m$  on the boundary, must be identical with  $\Delta$ , so that the rotation order of  $\gamma_i\gamma_j^m$  for  $\Delta$  is 1. Take  $m=sk$  with an integer  $k$ , where  $s$  is the rotation order of  $\gamma_j$  for  $\Delta$ . Then  $\gamma_i\gamma_j^m = \gamma_i(\gamma_j^s)^k \in G_\Delta$  or  $\gamma_i \in G_\Delta$ , which contradicts the assumption that the rotation order of  $\gamma_i$  for  $\Delta$  is greater than 2. Hence the attractive fixed point of  $\gamma_j$  lies on  $C_1$ . Here we shall say that  $\gamma_i(C_1)$  (or  $\gamma_i(C_2)$ ) faces to  $C_2$  (or  $C_1$ ) if the component of the complement of  $\gamma_i(C_1) \cup C_2$  (or  $\gamma_i(C_2) \cup C_1$ ) containing  $D$  includes all  $\gamma_i^l(D)$ ,  $0 \leq l \leq r-1$ . Note that  $\gamma_i(C_1)$  and  $\gamma_i(C_2)$  are the subboundaries of  $\gamma_i(D)$  with the common terminal points being the fixed points of  $\gamma_i$ . We see easily that if  $\gamma_i(C_1)$  (or  $\gamma_i(C_2)$ ) does not face to  $C_2$  (or  $C_1$ ), then there is no component of  $G$  on whose boundary the fixed points of  $\gamma_i\gamma_j^m$  lie. If  $\gamma_i(C_1)$  faces to  $C_2$ , then, since the fixed points of  $\gamma_i\gamma_j\gamma_i^{-1}$  lie on  $\gamma_i(C_1)$ , the component which has the fixed points of  $\gamma_i\gamma_j^m$  on the boundary, must be identical with  $\Delta$ . So we have the same contradiction stated just above. Hence  $\gamma_i(C_2)$  must face to  $C_1$ , so that the component which has the fixed points of  $\gamma_i\gamma_j^m$  on the boundary, must be identical with  $\gamma_i(\Delta)$  and the rotation order of

$\gamma_i \gamma_j^m$  for  $\gamma_i(\Delta)$  is 1. Then  $\gamma_i^{-1} \gamma_i \gamma_j^m \gamma_i \in G_\Delta$  or  $\gamma_j^m \gamma_i \in G_\Delta$ . Taking  $m = sk$  with integers  $k$  and  $s$  being the rotation order of  $\gamma_i$  for  $\Delta$ , we have the same contradiction as before. Thus we have proved our lemma.

4.2. If there are elements of  $S$  with the common fixed points, then we shall replace one of them by another one as follows: Let  $\gamma_i$  and  $\gamma_j$  be elements of  $S$  with the common fixed points and let  $\gamma_k$  be an element of  $S$  with no common fixed point with  $\gamma_i$ . Lemma 4.1 implies that, for a large integer  $m$ ,  $\gamma_i \gamma_k^m$  is a loxodromic element with no common fixed point with  $\gamma_j$  and  $\gamma_k$ . The new set,  $S$  with the replacement of  $\gamma_i$  by  $\gamma_i \gamma_k^m$ , has the same property as  $S$  stated at the beginning of this §. Making such replacements, if necessary, we may assume that the fixed points set of each element of  $S$  is different from that of any other element of  $S$ . If there are elements  $\gamma_i, \gamma_j$  of  $S$  satisfying one of the conditions of Lemmas 4.2~4.4, then we change  $\gamma_i$  into  $\gamma_i \gamma_j^m$  with a suitable integer  $m$  so that we have the desired conclusion in this §. Hence we assume that there are no elements  $\gamma_i, \gamma_j$  in  $S$  satisfying one of the conditions of Lemmas 4.2~4.4. So we may consider the case where, for any two elements of  $S$ , there is a component of  $G$  on whose boundary all the fixed points of them lie.

4.3. We shall first treat the case where there is a component  $\Delta$  of  $G$  on whose boundary the fixed points of each element of  $S$  lie. Since  $G$  is not a function group, there is at least one element, say  $\gamma_i$ , of  $S$  whose rotation order for  $\Delta$  is not 1. Let  $D$  be the auxiliary domain of  $\Delta$  with respect to  $\gamma_i(\Delta)$ . Since, by Lemma 2.12 and Lemma 2.8, each element of  $S$  has both fixed points in  $D$  or on  $\partial D$  simultaneously and since we have assumed that there are no elements  $\gamma_i, \gamma_j$  in  $S$  satisfying one of conditions of Lemma 4.2~4.4, we see that each element of  $S$  has both fixed points on  $\partial D$  and that the rotation order of  $\gamma_i$  for  $\Delta$  is 2. Let  $\gamma_j$  be an element of  $S$  different from  $\gamma_i$ . By the same reasoning as above, the rotation order of  $\gamma_j$  for  $\Delta$  is equal to 1 or 2. We assert that if, for any large integer  $m$ , the fixed points of  $\gamma_i \gamma_j^m$  do not lie on  $\Lambda_0(G)$ , then either  $\gamma_j(D) = D$  and  $\gamma_j(\gamma_i(D)) = \gamma_i(D)$  or  $\gamma_j(D) = \gamma_i(D)$  and  $\gamma_j(\gamma_i(D)) = D$ .

To prove our assertion, we recall the following facts: As we have seen in the proof of Lemma 4.2, the components, whose boundary contains the fixed points of  $\gamma_i \gamma_j^m$  for a large integer  $m$ , have four fixed points of  $\gamma_j$  and  $\gamma_i \gamma_j \gamma_i^{-1}$  on the boundary and the rotation order of  $\gamma_i \gamma_j^m$  for such a component is at most two. Moreover, as we have seen in the proof of Lemma 4.4, such a component is  $\Delta$  or  $\gamma_i(\Delta)$ .

First we consider the case where the rotation order of  $\gamma_j$  for  $\Delta$  is 1. If the fixed points of  $\gamma_i \gamma_j^m$  lie on  $\partial \Delta$ , then  $\gamma_i \gamma_j^m \gamma_i \gamma_j^m(\Delta) = \Delta$  or  $\gamma_i \gamma_j^m \gamma_i(\Delta) = \Delta$ . If the fixed points of  $\gamma_i \gamma_j^m$  lie on  $\partial \gamma_i(\Delta)$ , then  $\gamma_i \gamma_j^m \gamma_i \gamma_j^m \gamma_i(\Delta) = \gamma_i(\Delta)$  or  $\gamma_j^m \gamma_i \gamma_j^m \gamma_i(\Delta) = \Delta$  or, equivalently,  $\gamma_i \gamma_j^m \gamma_i(\Delta) = \Delta$ . In both cases we have that  $\gamma_i \gamma_j^m \gamma_i(\Delta) = \Delta$ . Since  $\gamma_i^2(\Delta) = \Delta$ , we see that  $\gamma_j^m \gamma_i(\Delta) = \gamma_i^{-1}(\Delta) = \gamma_i(\Delta)$ , so that

$\gamma_j$  has the fixed points on  $\partial\gamma_i(\Delta)$ . By Theorem 2.14, we see that the rotation order of  $\gamma_j$  for  $\gamma_i(\Delta)$  is 1. Applying Lemma 2.16 to  $\Delta$  (or  $\gamma_i(\Delta)$ ) and the auxiliary domain  $D$  (or  $\gamma_i(D)$ ) of  $\Delta$  (or  $\gamma_i(\Delta)$ ), we obtain that  $\gamma_j(D)=D$  and  $\gamma_j(\gamma_i(D))=\gamma_i(D)$ .

Next we consider the case where the rotation order of  $\gamma_j$  for  $\Delta$  is 2. Let  $m$  be an odd number. If the fixed points of  $\gamma_i\gamma_j^m$  lie on  $\partial\Delta$ , then  $\gamma_i\gamma_j^m(\Delta)=\gamma_i\gamma_j(\Delta)\neq\gamma_i(\Delta)$  so that  $\gamma_i\gamma_j^m(\Delta)=\Delta$ . Hence we obtain from  $\gamma_i\gamma_j(\Delta)=\Delta$  that  $\gamma_j(\Delta)=\gamma_i(\Delta)$ . If the fixed points of  $\gamma_i\gamma_j^m$  lie on  $\partial\gamma_i(\Delta)$ , then either  $\gamma_i\gamma_j^m\gamma_i(\Delta)=\gamma_i(\Delta)$  or  $\gamma_i\gamma_j^m\gamma_i(\Delta)=\Delta$  holds. The latter case corresponds to the case where the rotation order of  $\gamma_i\gamma_j^m$  for  $\gamma_i(\Delta)$  is 2. So, if this case occurs, we see  $\gamma_i\gamma_j^m\gamma_i(\Delta)=\gamma_j(\Delta)$  for an odd  $m$ , which is absurd. Hence the latter case does not occur. Therefore, it holds that  $\gamma_i\gamma_j^m\gamma_i(\Delta)=\gamma_i(\Delta)$ . This implies that  $\gamma_j^m\gamma_i(\Delta)=\Delta$  so that  $\gamma_j\gamma_i(\Delta)=\Delta$  and  $\gamma_j(\Delta)=\gamma_i(\Delta)$ . In both cases we have  $\gamma_j(\Delta)=\gamma_i(\Delta)$ . Since the auxiliary domain of  $\Delta$  with respect to  $\gamma_j(\Delta)$  is then identical with  $D$ , we obtain  $\gamma_j(D)=\gamma_i(D)$  and  $\gamma_j\gamma_i(D)=\gamma_j^2(D)=D$  by Lemma 2.13 and Lemma 2.16. Thus our assertion is established.

By what just has been proved, we see that if, for each element  $\gamma_j$  of  $S$  and for any large integer  $m$ , the fixed points of  $\gamma_i\gamma_j^m$  does not lie on  $\Lambda_0(G)$ , then  $\partial D \cup \partial\gamma_i(D)$  is invariant under  $S$  and hence under  $G$ , so that  $\Lambda(G)=\partial D \cup \partial\gamma_i(D)$ . But, we shall show that this does not occur. Since  $D \cap \gamma_i(D)=\emptyset$  by Lemma 2.13, the equality  $\Lambda(G)=\partial D \cup \partial\gamma_i(D)$  implies  $D \cap \Lambda(G)=\emptyset$ . Therefore, we have  $D=\Delta$  by Lemma 2.8, so that  $G_\Delta$  is a quasi-Fuchsian group with the invariant curve  $\partial D$ . If  $\partial D=\partial\gamma_i(D)$ , then  $\Lambda(G)=\partial\Delta$ . This contradicts our assumption that  $G$  is not a  $Z_2$ -extension of a quasi-Fuchsian group. Hence  $\partial D \neq \partial\gamma_i(D)$ . Since  $\Lambda(G)$  and  $\partial D$  are invariant under  $G_\Delta$ , the set  $\partial\gamma_i(D) \setminus \partial D$  is invariant under  $G_\Delta$ . Let  $p$  be a point on  $\partial D$  not lying on  $\partial\gamma_i(D)$  and let  $d$  be the distance between  $p$  and  $\partial\gamma_i(D)$ . It is well known that there is a loxodromic element  $\gamma$  of  $G_\Delta$  such that the distance between  $p$  and the attractive fixed point of  $\gamma$  is smaller than  $d$ . Then, for a sufficiently large integer  $m$ , the distance between  $p$  and  $\gamma^m(\partial\gamma_i(D) \setminus \partial D)$  is smaller than  $d$ . Since  $\gamma^m(\partial\gamma_i(D) \setminus \partial D)=\partial\gamma_i(D) \setminus \partial D$ , we have a contradiction. Thus the equality  $\Lambda(G)=\partial D \cup \partial\gamma_i(D)$  does not occur. Therefore, there are an element  $\gamma_j$  of  $S$  and an integer  $m$  such that  $\gamma_i\gamma_j^m$  is a loxodromic element with the fixed points on  $\Lambda_0(G)$ . Thus we can change  $S$  into the desired one in the beginning of this section.

4.4. Now we shall treat the case where there is no component of  $G$  on whose boundary the fixed points of all elements of  $S$  lie. First we shall show that there are elements  $\gamma_i, \gamma_j, \gamma_k$  of  $S$  and components  $\Delta_i, \Delta_j, \Delta_k$  of  $G$  such that the fixed points of  $\gamma_i, \gamma_j$  and  $\gamma_k$  lie on  $(\partial\Delta_j \cap \partial\Delta_k) \setminus \partial\Delta_i, (\partial\Delta_k \cap \partial\Delta_j) \setminus \partial\Delta_i$  and  $(\partial\Delta_i \cap \partial\Delta_j) \setminus \partial\Delta_k$ , respectively. Let  $\Delta$  be a component of  $G$  on whose boundary the fixed points of  $\gamma_1, \gamma_2 \in S$  lie. By Theorem 2.6, there is, except for  $\Delta$ , at most one component  $\Delta'$  on whose boundary the fixed points of  $\gamma_1$  and  $\gamma_2$  lie.

If  $\Delta'$  does not exist, then let  $\gamma_i = \gamma_1$ ,  $\gamma_j = \gamma_2$ ,  $\gamma_k$  be an element of  $S$  whose fixed points do not lie on  $\partial\Delta$ ,  $\Delta_i$  (or  $\Delta_j$ ) be a component of  $G$  on whose boundary the fixed points of  $\gamma_j$ ,  $\gamma_k$  (or  $\gamma_k$ ,  $\gamma_i$ ) lie, and  $\Delta_k = \Delta$ . If  $\Delta'$  exists and if there is an element  $\gamma_l$  of  $S$  whose fixed points do not lie on  $\partial\Delta \cup \partial\Delta'$ , then let  $\gamma_i = \gamma_1$ ,  $\gamma_j = \gamma_2$ ,  $\gamma_k = \gamma_l$ ,  $\Delta_i$  (or  $\Delta_j$ ) be a component of  $G$  on whose boundary the fixed points of  $\gamma_2$ ,  $\gamma_l$  (or  $\gamma_1$ ,  $\gamma_l$ ) lie, and  $\Delta_k = \Delta$ . If  $\Delta'$  exists and if the fixed points of all elements of  $S$  lie on  $\partial\Delta \cup \partial\Delta'$ , then there are elements  $\gamma_p, \gamma_q \in S$  whose fixed points lie on  $\partial\Delta' \setminus \partial\Delta$  and on  $\partial\Delta \setminus \partial\Delta'$ , respectively. Let  $\Delta''$  be a component of  $G$  on whose boundary the fixed points of  $\gamma_p$  and  $\gamma_q$  lie. It is easy to see that  $\Delta'' \neq \Delta$  and  $\Delta'' \neq \Delta'$ . Hence the fixed points of  $\gamma_1$  and  $\gamma_2$  do not lie on  $\partial\Delta''$  simultaneously. Let  $\gamma_i$  be either  $\gamma_1$  or  $\gamma_2$  whose fixed points do not lie on  $\partial\Delta''$ ,  $\gamma_j = \gamma_p$ ,  $\gamma_k = \gamma_q$ ,  $\Delta_i = \Delta''$ ,  $\Delta_j = \Delta$  and  $\Delta_k = \Delta'$ . It is easy to see that, in each case stated just above, these  $\gamma_i, \gamma_j, \gamma_k, \Delta_i, \Delta_j$  and  $\Delta_k$  have the desired property.

Next, by using  $\gamma_i, \gamma_j, \gamma_k$ , we shall change  $S$  into a set of generators of  $G$ , which satisfies the property stated in the beginning of this section. Let  $D_{pq}$  be the auxiliary domain of  $\Delta_p$  with respect to  $\Delta_q$ ,  $p, q = i, j, k$ . Since  $\partial\Delta_j \cap \partial\Delta_k$  contains both fixed points of  $\gamma_i$ , it follows from Theorem 2.7 that  $\Delta_j$  and  $\Delta_k$  are not included in the distinct components of  $G_{\Delta_i}$ . Hence we see that  $D_{ij} = D_{ik}$ . By the same reasoning as above, we see that  $D_{ji} = D_{jk}$  and  $D_{ki} = D_{kj}$ . For simplicity, we shall denote  $D_{pq}$  by  $D_p$ ,  $p = i, j, k$ . Let  $\xi_p$  and  $\xi_p'$  be the attractive and the repelling fixed points of  $\gamma_p$ , respectively,  $p = i, j, k$ . Then, we see by Lemma 2.8 that  $\xi_i$  and  $\xi_i'$  lie on  $(\partial D_j \cap \partial D_k) \setminus \partial D_i$ . Let  $r$  be the rotation order of  $\gamma_i$  for  $\Delta_j$ . Clearly  $\gamma_i^r \in G_{\Delta_j}$ . Then by Theorem 2.14,  $\gamma_i^r \in G_{\Delta_k}$ . By Lemma 2.16, we see that  $\gamma_i^r(D_j) = D_j$  and  $\gamma_i^r(D_k) = D_k$ . We consider the element of the form  $\gamma_i^m \gamma_k \gamma_i^{-rm}$  with a positive integer  $m$ . Since  $\gamma_i^m(\partial D_j) = \partial D_j$  and  $\gamma_i^m(\partial D_k) = \partial D_k$ , we see that, for a sufficiently large  $m$ ,  $\gamma_i^m(\partial D_i)$  lies near  $\xi_i$  and meets to  $\partial D_j$  (or  $\partial D_k$ ) at  $\gamma_i^m(\xi_k)$  (or  $\gamma_i^m(\xi_j)$ ), and that the fixed points of  $\gamma_i^m \gamma_k \gamma_i^{-rm}$  lie on  $(\partial D_j \cap \gamma_i^m(\partial D_i)) \setminus \partial D_k$ . On the other hand, we can easily verify that, for any integer  $l$ , there is a Jordan curve lying in  $D_j \cup D_k \cup \gamma_i^l(D_i) \cup \{\xi_i, \gamma_i^l(\xi_j), \gamma_i^l(\xi_k)\}$ . Therefore, there is no component of  $G$  on whose boundary the fixed points of both  $\gamma_j$  and  $\gamma_i^m \gamma_k \gamma_i^{-rm}$  lie. Hence  $\gamma_j$  and  $\gamma_i^m \gamma_k \gamma_i^{-rm}$  satisfy the assumption of Lemma 4.2. Changing  $\gamma_k$  into  $\gamma_i^m \gamma_k \gamma_i^{-rm}$  and applying Lemma 4.2, we can change  $S$  into a desired set of generators of  $G$ .

4.5. We can easily see that the results in this section give an alternative proof of the following.

**Theorem [2].** *Let  $G$  be a finitely generated Kleinian group. Then  $\Lambda_0(G) = \emptyset$  if and only if  $G$  is either a function group or a  $Z_2$ -extension of a quasi-Fuchsian group.*



## 5. Final step of the proof of Theorem 1.1

5.1. Let  $S$  be a set of generators of  $G$  which consists of loxodromic elements only and one of which has the fixed points on  $\Lambda_0(G)$ . We shall change  $S$  into  $S_0$  in Theorem 1.1. Without loss of generality we may assume that  $\infty \in \Omega(G)$  and  $\gamma_1 \in S$  has the fixed points on  $\Lambda_0(G)$ . Let  $\xi_1$  and  $\xi_1'$  be the repelling and the attractive fixed points of  $\gamma_1$ , respectively, and let  $\gamma_i$  be an element of  $S$  whose fixed points do not lie on  $\Lambda_0(G)$ . By Lemma 4.1, for a sufficiently large integer  $m$ ,  $\gamma_i \gamma_1^m$  is loxodromic and the repelling and the attractive fixed points of  $\gamma_i \gamma_1^m$  lie near  $\xi_1$  and  $\gamma_i(\xi_1')$ , respectively. Let  $d$  be the distance between  $\xi_1$  and  $\gamma_i(\xi_1')$ . By Lemma 2.10, there is a finite number of components of  $G$  whose diameters are greater than  $d/2$ . Let  $\delta$  be the minimum of the distances of  $\xi_1$  or  $\gamma_i(\xi_1')$  from the components whose diameters are greater than  $d/2$ . Since  $\xi_1$  and  $\gamma_i(\xi_1')$  are the points on  $\Lambda_0(G)$ ,  $\delta$  is positive. Let  $m$  be so large that the distance between  $\xi_1$  (or  $\gamma_i(\xi_1')$ ) and the repelling (or the attractive) fixed point of  $\gamma_i \gamma_1^m$  is smaller than  $\delta$ . Then there is no component of  $G$  whose diameter is greater than  $d/2$  and on whose boundary the fixed points of  $\gamma_i \gamma_1^m$  lie. By Theorem 2.11, we see that there is no component on whose boundary the fixed points of  $\gamma_i \gamma_1^m$  lie. Hence, for a large integer  $m$ ,  $\gamma_i \gamma_1^m$  is a loxodromic element with the fixed points on  $\Lambda_0(G)$ . Changing each  $\gamma_i$  of  $S$ , whose fixed points do not lie on  $\Lambda_0(G)$ , into  $\gamma_i \gamma_1^m$ , we obtain the desired  $S_0$ . Since our operations do not increase the number of elements of the set of generators, the second property of  $S_0$  is clear. Thus we have completed the proof of Theorem 1.1.

## 6. Non-web groups

6.1. Among the set of finitely generated Kleinian groups with the non-empty residual limit sets there is a class of web groups. A finitely generated (non-elementary) Kleinian group  $G$  is called a web group if, for each component  $\Delta$  of  $G$ , the component subgroup  $G_\Delta$  is quasi-Fuchsian [2]. Usually those group which are themselves quasi-Fuchsian are excluded from the class. If  $G$  is a finitely generated Kleinian group with the non-empty residual limit set and is not a web group, then there is a subset  $L_1(G)$  of  $\Lambda_0(G)$  consisting of the points, to each of which there is a converging nest sequence of the separators of  $G$  [2]. A sequence  $\{C_m\}_{m=1}^\infty$  of Jordan curves, which converges to a point  $p$ , is called a nest sequence if  $p \notin C_m$  and  $C_{m+1}$  separates  $p$  from  $C_m$  for every  $m$ . In this § we shall improve Theorem 1.1 and Corollary 1.2 for those groups  $G$  with non-empty sets  $L_1(G)$ .

6.2. Later we need the followings.

**Lemma 6.1.** *Let  $G$  be a finitely generated Kleinian group and let  $D$  be a Jordan domain whose boundary is a separator of  $G$ . Assume that the fixed points*

of a loxodromic element  $\gamma$  of  $G$  lie on  $\Lambda_0(G)$ . If  $D \subset \gamma(D)$ , then the fixed points of  $\gamma$  lie on  $L_1(G)$ .

**Proof.** The assumption that  $D \subset \gamma(D)$  implies that the repelling and the attractive fixed points of  $\gamma$  lie in  $\bar{D}$  and in the complement of  $\gamma(D)$ , respectively. Since the fixed points of  $\gamma$  do not lie on  $\partial D$ , they are separated by a separator  $\partial D$ . Then  $\{\gamma^m(\partial D)\}_{m=1}^\infty$  (or  $\{\gamma^{-m}(\partial D)\}_{m=1}^\infty$ ) forms a nest sequence of separators converging to the attractive (or the repelling) fixed point of  $\gamma$ . Hence the fixed points of  $\gamma$  lie on  $L_1(G)$ .

**Lemma 6.2.** *Let  $G$  be a finitely generated Kleinian group and let  $\Delta$  and  $\gamma$  be a component and a loxodromic element  $G$ , respectively. Assume that the fixed points of  $\gamma$  lie on  $\Lambda_0(G) \setminus L_1(G)$  and denote by  $D$  and  $D'$  the auxiliary domains of  $\Delta$  and of  $\gamma(\Delta)$  with respect to  $\gamma(\Delta)$  and  $\Delta$ , respectively. Then  $\gamma(D) = D'$  so that  $D \cap \gamma(D) = \emptyset$ .*

**Proof.** Since both  $\gamma(D)$  and  $D'$  contain  $\gamma(\Delta)$ , we have only to prove that  $\gamma(\partial D) = \partial D'$ . If it is not true, then  $\gamma(\partial D) \cap D' \neq \emptyset$  and  $\gamma(\partial D)$  lies in  $\bar{D}'$ , because  $\bar{D}' \supset \gamma(\bar{\Delta})$ . Hence either  $\gamma(\bar{D})$  or the exterior of  $\gamma(D)$  is contained in  $\bar{D}'$ . If  $\gamma(\bar{D}) \subsetneq \bar{D}'$ , then there are points of  $\partial D' \setminus \gamma(\bar{D})$  ( $\subset \gamma(\bar{\Delta}) \setminus \gamma(\bar{D})$ ). This contradicts the fact that  $\gamma(\bar{\Delta}) \subset \gamma(\bar{D})$ . Hence the exterior of  $\gamma(D)$  is contained in  $\bar{D}'$ . Therefore the exterior of  $D'$  is contained in  $\gamma(D)$ . Since  $D \cap D' = \emptyset$ , we have  $D \subset \gamma(D)$ . By Lemma 6.1, the fixed points of  $\gamma$  lie on  $L_1(G)$ , a contradiction. Hence we have  $\gamma(\partial D) = \partial D'$  and our lemma.

**Lemma 6.3.** *Let  $G$  be a finitely generated Kleinian group and let  $\Delta$  and  $\gamma$  be a component and a loxodromic element of  $G$ , respectively. If the fixed points of  $\gamma$  lie on  $\Lambda_0(G) \setminus L_1(G)$ , then  $\gamma^{-1}(\Delta)$  is contained in the component of  $G_\Delta$  which contains  $\gamma(\Delta)$ .*

**Proof.** Let  $D$  be the auxiliary domain of  $\Delta$  with respect to  $\gamma(\Delta)$ . Note that the exterior of  $D$  is a component of  $G_\Delta$ . We shall show that both  $\gamma(\Delta)$  and  $\gamma^{-1}(\Delta)$  are contained in the exterior of  $D$ . By Lemma 6.2, we have only to show this for  $\gamma^{-1}(\Delta)$ . If it is not true, then  $\gamma^{-1}(\Delta) \subset D$ . If  $\gamma^{-1}(D) \subset D$ , then by Lemma 6.1 we see that the fixed points of  $\gamma^{-1}$  lie on  $L_1(G)$ , which contradicts the assumption of the lemma. Hence  $\gamma^{-1}(D) \not\subset D$ . On the other hand,  $\gamma^{-1}(\Delta) \subset D$  implies  $\partial \gamma^{-1}(D) \subset \bar{D}$ . This implies that  $\gamma^{-1}(D)$  contains the exterior of  $D$ . Hence by Lemma 6.2,  $\gamma^{-1}(D) \supset \gamma(D)$  or  $D \subset \gamma^{-2}(D)$ . By Lemma 6.1, the fixed points of  $\gamma$  lie on  $L_1(G)$ , a contradiction. Hence  $\gamma^{-1}(\Delta)$  is contained in the exterior of  $D$ . Thus we have our lemma.

6.3. Now we shall prove the following.

**Theorem 6.4.** *Let  $G$  be a finitely generated Kleinian group and let  $S$  be a*

finite set of generators of  $G$ . If  $G$  is neither a function group nor a web group, then  $S$  can be changed into a set of generators  $S_1$  of  $G$  with the following properties:

- i) each element of  $S_1$  is loxodromic and its fixed points lie on  $L_1(G)$ , and
- ii) the number of elements of  $S_1$  is not greater than that of  $S$ .

To prove our theorem, we first change  $S$  into  $S_0$  which has the properties i) and ii) in Theorem 1.1. We shall next change  $S_0$  by our operation stated in §3.1 into a set which consists of loxodromic elements only and contains at least one element with the fixed points on  $L_1(G)$ . Assume that each element of  $S_0$  has the fixed points on  $\Lambda_0(G) \setminus L_1(G)$ . Then we assert that there are elements  $\gamma_i, \gamma_j$  of  $S_0$  and a component  $\Delta$  of  $G$  such that the components  $\gamma_i(\Delta)$  and  $\gamma_j(\Delta)$  lie in the distinct components of the component subgroup  $G_\Delta$ .

In order to prove this assertion we assume that there is no triple  $(\gamma_i, \gamma_j, \Delta)$  with the property stated just above. Let  $\Delta$  be a component of  $G$ , for which the component subgroup  $G_\Delta$  is not quasi-Fuchsian. Then, by Theorem 2.2, each component of  $G_\Delta$  which is different from  $\Delta$  is a non-invariant component of  $G_\Delta$ . Let  $\Delta'$  be the component of  $G_\Delta$  which contains the component  $\gamma_1(\Delta)$  of  $G$ ,  $\gamma_1 \in S_0$ . Then, from the assumption just stated above,  $\Delta'$  contains each component  $\gamma_i(\Delta)$  of  $G$ ,  $\gamma_i \in S_0$ . Let  $D$  be the auxiliary domain of  $\Delta$  with respect to  $\gamma_i(\Delta)$ ,  $\gamma_i \in S_0$ . Clearly  $\partial D = \partial \Delta'$ . Since  $\Delta'$  is a non-invariant component of  $G_\Delta$ , there are a component  $\Delta'' (\neq \Delta')$  of  $G_\Delta$  and an element  $g$  of  $G_\Delta$  such that  $g(\Delta') = \Delta''$ . It is easy to see that  $\Delta'' \subsetneq D$  and  $\partial \Delta'' \cap D \neq \emptyset$ . Let  $\delta = g\gamma_1 g$ . Then  $\delta$  maps  $\Delta$  to a component  $\delta(\Delta) = \Delta^*$  of  $G$  lying in  $\Delta''$  and we have  $\delta(D) \cap D \neq \emptyset$ . Let  $D^*$  be the auxiliary domain of  $\Delta^*$  with respect to  $\Delta$ . We can see that  $\delta(\partial D) \cap g\gamma_1(D) = g\gamma_1(\partial \Delta'') \cap g\gamma_1(D) \neq \emptyset$ . On the other hand, we have easily  $g\gamma_1(D) = D^*$ . So we obtain  $\delta(\partial D) \neq \partial D^*$ . Therefore,  $\delta(D)$  is not contained in  $D$  and  $\delta(D)$  does not contain  $D$ . Since  $\delta$  is an element of  $G$ , we can represent it by elements of  $S_0$  as  $\delta = \delta_m \delta_{m-1} \cdots \delta_1$ , where  $\delta_i$  ( $i=1, 2, \dots, m$ ) is an element of  $S_0$  or its inverse and  $\delta_i \delta_{i-1}$  is not identity ( $2 \leq i \leq m$ ). We set  $\varepsilon_k = \delta_k \delta_{k-1} \cdots \delta_1$  ( $1 \leq k \leq m$ ).

Lemma 6.2 implies  $\varepsilon_1(D) \cap D = \emptyset$ . It may happen for some  $k$  ( $2 < k \leq m$ ) that  $\varepsilon_{k-1}(D) \cap D \neq \emptyset$ . By noting Lemma 2.4, we see that following three cases may occur:

$$(1)_{k-1}: \varepsilon_{k-1}(D) \subset D,$$

$$(2)_{k-1}: \varepsilon_{k-1}(D) \supset D,$$

and

$$(3)_{k-1}: D^c \subset \varepsilon_{k-1}(D), \text{ where } D^c \text{ is the complementary set of } D.$$

We also denote by  $(0)_k$  the property  $\varepsilon_k(D) \cap D = \emptyset$ .

[I] The property  $(1)_{k-1}$  implies the property  $(0)_k$ . In fact,  $(1)_{k-1}$  means  $\varepsilon_{k-1}(D) \subset D$ , so we have  $\varepsilon_k(D) = \delta_k(\varepsilon_{k-1}(D)) \subset \delta_k(D)$ . On the other hand, Lemma 6.2 and Lemma 6.3 yield  $\delta_k(D) \cap D = \emptyset$ . Hence  $\varepsilon_k(D) \cap D = \emptyset$ .

[II] The property  $(2)_{k-1}$  implies the property  $(0)_k$ . In fact, the property

$(0)_1$ :  $\varepsilon_1(D) \cap D = \emptyset$  shows  $\varepsilon_1(\Delta) \cap D = \emptyset$ . Hence  $\delta_1(\varepsilon_1(\Delta)) \cap \delta_1(D) = \delta_1(\varepsilon_1(\Delta) \cap D) = \emptyset$ . Lemma 6.3 and the assumption for elements of  $S_0$  and for components of  $G$  imply that  $\varepsilon_2(\Delta) \cap \varepsilon_1(D) = \emptyset$ . By the same reasoning, we can see that  $\varepsilon_2(\Delta) \cap \varepsilon_1(D) = \emptyset$  implies  $\varepsilon_3(\Delta) \cap \varepsilon_2(D) = \emptyset$ . Repeating this procedure, we obtain  $\varepsilon_k(\Delta) \cap \varepsilon_{k-1}(D) = \emptyset$ . Therefore,  $\varepsilon_{k-1}(D)$  is an auxiliary domain of  $\varepsilon_{k-1}(\Delta)$  with respect to  $\varepsilon_k(\Delta)$ . Lemma 6.2 yields  $\varepsilon_k(D) \cap \varepsilon_{k-1}(D) = \emptyset$ . Since  $\varepsilon_{k-1}(D) \supset D$ , we have  $\varepsilon_k(D) \cap D = \emptyset$ .

[III] If the property  $(0)_{k-1}$  holds, then the property  $(3)_k$  does not hold  $(1 < k \leq m)$ . In fact,  $(0)_{k-1}$  implies  $D^c \supset \varepsilon_{k-1}(D)$ , which contradicts  $(3)_k$ .

Now we recall that the property  $(0)_1$  holds. The propositions [I], [II] and [III] show that  $(3)_m$  does not occur. So we see that the one of two relations  $\delta(D) \subset D$  and  $\delta(D) \supset D$  must hold, because  $\delta(D) \cap D \neq \emptyset$ . This contradicts the fact obtained already. Thus we have the assertion that there is a triple  $(\gamma_i, \gamma_j, \Delta)$  such that  $\gamma_i(\Delta)$  and  $\gamma_j(\Delta)$  lie in the distinct components of the component subgroups  $G_\Delta$ .

6.4. Let  $D_i$  and  $D_j$  be the auxiliary domains of  $\gamma_i(\Delta)$  and  $\gamma_j(\Delta)$  with respect to  $\Delta$ , respectively. Since they are included in the distinct components of  $G_\Delta$ , we see by Theorem 2.7 that  $D_i \cap D_j = \emptyset$  and that  $\partial D_i \cap \partial D_j$  consists of at most one point. We shall show that  $\gamma_i \gamma_j^{-1}$  is loxodromic and its repelling and attractive fixed points lie in  $D_j$  and in  $D_i$ , respectively. Since two fixed points of  $\gamma_i \gamma_j^{-1}$  are separated from each other by a separator of  $G$ , the fixed points of  $\gamma_i \gamma_j^{-1}$  lie on  $L_1(G)$ . Since  $\gamma_i(\Delta) \subset D_i$  and  $\gamma_j(\Delta) \subset D_j$ , it suffices to show that  $\gamma_i \gamma_j^{-1}(\partial D_j) \neq \partial D_i$  and that if  $\partial D_i \cap \partial D_j = \{p\}$ , then  $\gamma_i \gamma_j^{-1}(p) \neq p$ . In fact, from these properties, we see easily that  $\gamma_i \gamma_j^{-1}(D_j) \not\equiv D_j$  and that  $\gamma_i \gamma_j^{-1}$  can be neither parabolic nor elliptic so that Lemma 6.1 implies the assertion. Let  $D_i^*$  and  $D_j^*$  be the auxiliary domains of  $\Delta$  with respect to  $\gamma_i(\Delta)$  and  $\gamma_j(\Delta)$ , respectively. Then  $\gamma_i(D_i^*) = D_i$ ,  $\gamma_j(D_j^*) = D_j$  and  $\partial D_i^* \neq \partial D_j^*$ . Lemma 6.2 shows  $\gamma_i(D_i^*) = \partial D_i$  and  $\gamma_j(\partial D_j^*) = \partial D_j$  and we see that  $\gamma_i \gamma_j^{-1}(\partial D_j) = \gamma_i(\partial D_j^*) \neq \gamma_i(\partial D_i^*) = \partial D_i$ . If  $\partial D_i \cap \partial D_j = \{p\}$ , then, by Theorem 2.7,  $p$  is the fixed point of a parabolic element of  $G$ . Since  $\gamma_j$  is loxodromic, we see  $\gamma_j^{-1}(p) \neq p$  and  $\gamma_j^{-1}(p) \in \partial D_j^*$ . Hence, by Theorem 2.7, we have  $\gamma_j^{-1}(p) \notin \partial D_i^*$ . Hence we have  $\gamma_i \gamma_j^{-1}(p) \notin \partial D_i$ , so that  $\gamma_i \gamma_j^{-1}(p) \neq p$ .

Thus we have shown that  $\gamma_i \gamma_j^{-1}$  is loxodromic and has the fixed points on  $L_1(G)$ . Changing  $\gamma_i$  by  $\gamma_i \gamma_j^{-1}$ , we obtain the desired set of generators, which we shall denote by  $S_0^*$ .

Lastly we shall change  $S_0^*$  into  $S_1$ . Without loss of generality we may assume that  $\infty \in \Omega(G)$  and that the fixed points of  $\gamma_1$  lie on  $L_1(G)$ . Let  $\xi_1$  and  $\xi_1'$  be the repelling and the attractive fixed points of  $\gamma_1$ , respectively. Then there is a nest sequence of separators of  $G$  which converges to  $\xi_1$ . Let  $\gamma_i$  be an element of  $S_0^*$  and let  $m$  be an integer such that  $\gamma_i \gamma_1^m$  is loxodromic and that the fixed points of  $\gamma_i \gamma_1^m$  are separated by a separator. The existence of such

$m$  is assured by Lemma 4.1. Then clearly  $\gamma_i\gamma_1^m$  is the desired loxodromic element. Changing each element  $\gamma_i$  of  $S_0^*$  which has the fixed points on  $\Lambda_0(G)\setminus L_1(G)$  by the element of the form  $\gamma_i\gamma_1^m$ , we obtain the desired  $S_1$  and complete the proof of Theorem 6.4.

6.5. Choosing  $S$  in Theorem 6.4 to be the minimal set of generators, we have the following.

**Corollary 6.5.** *Among the minimal sets of generators of a finitely generated Kleinian group  $G$  which is neither a function group nor a web group, there is a set consisting of only loxodromic elements with the fixed points on  $L_1(G)$ .*

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