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SOLVABILITY OF GROUPS OF ORDER $2^a p^b$

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1. Introduction

At the beginning of this century Burnside proved his famous $p^a q^b$-theorem by the help of character theory. Group-theoretic proof of the theorem was given by Goldschmidt [2] for odd primes in 1970.

The object of this paper is to give a simple group-theoretic proof of the following

Theorem.[1]   *Groups of order $2^a p^b$ are solvable.*

Lemma 1, 4 and 5 are due to Goldschmidt [2].   Notation used here follows Gorenstein [3].

2. Preliminaries

Lemma 1. *Suppose $\mathfrak{P}$ is a p-subgroup of the p-solvable group $\mathfrak{G}$. Then $0_{p'}(N_{\mathfrak{G}}(\mathfrak{P})) \subseteq 0_{p'}(\mathfrak{G})$*

Proof.   See Goldschmidt [2], lemma 2.
Next lemma plays an important role in this paper.

Lemma 2. *Suppose $\mathfrak{G}$ is a p-group and $\mathfrak{H}$ is a subgroup of $\mathfrak{G}$. Then $\mathfrak{H} \trianglelefteq \mathfrak{G}$ or $N_{\mathfrak{G}}(\mathfrak{H}) \supseteq \mathfrak{H}^x (\neq \mathfrak{H})$ for some $X \in \mathfrak{G}$.*

Proof.   Let Δ be a $\mathfrak{G}$-conjugate class containing $\mathfrak{H}$.   If $|\Delta| \neq 1$, then $\mathfrak{H}$ acts on $\Delta - \{\mathfrak{H}\}$ by conjugation.   Since $p \nmid |\Delta - \{\mathfrak{H}\}|$, $\mathfrak{H}$ fixes some element $\mathfrak{H}^{x^{-1}}$. Then $\mathfrak{H} \subseteq N_{\mathfrak{G}}(\mathfrak{H}^{x^{-1}})$ and hence $\mathfrak{H}^x \subseteq N_{\mathfrak{G}}(\mathfrak{H})$.

Lemma 3. (Suzuki-Thompson)   *Suppose Δ is a conjugate class of a group $\mathfrak{G}$. If any two elements of Δ generate a p-group, then $\Delta \subseteq 0_p(\mathfrak{G})$.*

Proof.   See [3], 3.8.2.

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3. The Minimal counter example

In this section let $\mathfrak{G}$ be a minimal counter example to the theorem. It is immediate to show that $\mathfrak{G}$ is simple and any proper subgroup of $\mathfrak{G}$ is solvable.

Let r be either prime divisor of $|\mathfrak{G}|$.

Lemma 4. *A sylow r-subgroup of $\mathfrak{G}$ normalizes no non-identity r'-subgroup of $\mathfrak{G}$.*

Proof. See Goldschmidt [2], Lemma 3.

Lemma 5. (Bender) *Suppose $\mathfrak{M}$ is a maximal subgroup of $\mathfrak{G}$. Then the Fitting subgroup of $\mathfrak{M}$ is an r-group.*

Proof. We set $\mathfrak{F} = F(\mathfrak{M})$, the Fitting subgroup of $\mathfrak{M}$. Let $\mathfrak{F} = \mathfrak{F}_2 \times \mathfrak{F}_p$ be the primary decomposition, and $\mathfrak{Z} = Z(\mathfrak{F}) = \mathfrak{Z}_2 \times \mathfrak{Z}_p$, the center of $\mathfrak{F}$.

Suppose lemma 5 is false, then $\mathfrak{F}_2 \neq 1$, $\mathfrak{F}_p \neq 1$. We first prove the next assertion [A].

[A] $\mathfrak{F}_r$ has two distinct subgroups of order r, for some $r \in \{2, p\}$.

Suppose [A] is false, then $\mathfrak{F}_p$ is cyclic, and $\mathfrak{F}_2$ is cyclic or a quaternion group.

(i) In the case $\mathfrak{F}_2$ is cyclic.

Let $\mathfrak{P}$ be a Sylow p-subgroup of $\mathfrak{M}$. Since $\mathfrak{P}/C_{\mathfrak{P}}(\mathfrak{F}_2)$ is a 2-group, $\mathfrak{P} = C_{\mathfrak{P}}(\mathfrak{F}_2)$. Then $Z(\mathfrak{P}) \subseteq C_{\mathfrak{M}}(\mathfrak{F})$, and hence $Z(\mathfrak{P}) \subseteq \mathfrak{F}_p$ by Fitting's theorem. (See [3], 6.1.3.) Since $\mathfrak{F}_p$ is cyclic, $Z(\mathfrak{P})$ is a characteristic subgroup of $\mathfrak{F}_p$. Then $\mathfrak{M} = N_{\mathfrak{G}}(Z(\mathfrak{P}))$ and $\mathfrak{P}$ is a Sylow p-subgroup of $\mathfrak{G}$, contrary to lemma 4.

(ii) In the case $\mathfrak{F}_2$ is a quaternion group.

Let $\mathfrak{Q}$ be a Sylow 2-group of $\mathfrak{M}$. Since $\mathfrak{Q}/C_{\mathfrak{Q}}(\mathfrak{F}_p)$ is abelian, $\mathfrak{Q}' \subseteq C_{\mathfrak{Q}}(\mathfrak{F}_p)$. Then $Z(\mathfrak{Q}) \cap \mathfrak{Q}' \subseteq \mathfrak{F}_2$. $Z(\mathfrak{Q}) \cap \mathfrak{Q}'$ contains a unique subgroup $\mathfrak{H}$ of order 2. So $\mathfrak{H}$ is a chatacteristic subgroup of $\mathfrak{Q}$. Since $\mathfrak{M} = N_{\mathfrak{G}}(\mathfrak{H}) \supseteq N_{\mathfrak{G}}(\mathfrak{Q})$, it follows that $\mathfrak{Q}$ is a Sylow 2-subgroup of $\mathfrak{G}$. A contradiction.

By (i) and (ii), we have [A].

Next we prove the following statement [B].

[B] Let $\overline{\mathfrak{M}}$ be a maximal subgroup of $\mathfrak{G}$ containing $\mathfrak{Z}$. Then $\overline{\mathfrak{M}} = \mathfrak{M}$

Let $\overline{\mathfrak{F}} = F(\overline{\mathfrak{M}}) = \overline{\mathfrak{F}}_2 \times \overline{\mathfrak{F}}_p$ be the Fitting subgroup of $\overline{\mathfrak{M}}$ and $\overline{\mathfrak{Z}} = \overline{\mathfrak{Z}}_2 \times \overline{\mathfrak{Z}}_p$ be the centre of $\overline{\mathfrak{F}}$. Since $\mathfrak{Z}_2 \times \mathfrak{Z}_p$ is contained in $\overline{\mathfrak{M}}$, $O_p(N_{\overline{\mathfrak{M}}}(\mathfrak{Z}_2)) \subseteq \overline{\mathfrak{F}}_p = O_p(\overline{\mathfrak{M}})$ by lemma 1. Now $\mathfrak{Z}_p$ is a normal subgroup of $N_{\overline{\mathfrak{M}}}(\mathfrak{Z}_2)$ we have $\mathfrak{Z}_p \subseteq O_p(N_{\overline{\mathfrak{M}}}(\mathfrak{Z}_2))$. Then $[\mathfrak{Z}_p, \overline{\mathfrak{F}}_2] = 1$. So $\overline{\mathfrak{F}}_2 \subseteq N_{\mathfrak{G}}(\mathfrak{Z}_p) = \mathfrak{M}$. In the same way, we have $\overline{\mathfrak{F}}_p \subseteq \mathfrak{M}$. Then in the same way as above we have $\overline{\mathfrak{F}}_2 \subseteq O_2(N_{\mathfrak{M}}(\overline{\mathfrak{F}}_p)) \subseteq \mathfrak{F}_2$. Interchanging $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ in the above argument, we obtain $\mathfrak{F}_2 \subseteq \overline{\mathfrak{F}}_2$. Then $\mathfrak{F}_2 = \overline{\mathfrak{F}}_2$ and we have $\overline{\mathfrak{M}} = \mathfrak{M}$. Thus [B] holds.

Now we prove lemma 5. By [A] we may assume that $\mathfrak{F}_r$ contains an abelian subgroup $\mathfrak{A}$ of type (r, r). Let $\mathfrak{R}$ be a Sylow r-subgroup of $\mathfrak{M}$. If $\mathfrak{K}$ is an r'-subgroup of $\mathfrak{G}$ normailzed by $\mathfrak{R}$, then $\mathfrak{K} = \prod_{X \in A - \{1\}} C_{\mathfrak{K}}(X)$. (See [3], 5.3.16.) Since

$C_{\Re}(X) \subseteq C_{\mathfrak{G}}(X)$ and $C_{\mathfrak{G}}(X) \supseteq \mathfrak{Z}$, $C_{\Re}(X) \subseteq \mathfrak{M}$ by [B]. It follows $\Re \subseteq \mathfrak{M}$. Then $\mathfrak{F}_{r'}$ is the unique maximal r'-subgroup of $\mathfrak{G}$ normalized by $\Re$. So $N_{\mathfrak{G}}(\Re) \subseteq N_{\mathfrak{G}}(\mathfrak{F}_{r'}) = \mathfrak{M}$. Then $\Re$ is a Sylow r-subgroup of $\mathfrak{G}$. A contradiction.

<div align="right">q.e.d.</div>

Lemma 6. $\mathfrak{G}$ *contains a maximal subgroup* $\mathfrak{M}$ *which satisfies the following condition;*

$$\mathfrak{M} \cap Z(\mathfrak{P}) \neq 1, \quad \mathfrak{M} \cap Z(\mathfrak{Q}) \neq 1$$

for some Sylow p-subgroup $\mathfrak{P}$ *and Sylow 2-subgroup* $\mathfrak{Q}$ *of* $\mathfrak{G}$.

Proof. Let $\mathfrak{Q}$ be a Sylow 2-subgroup of $\mathfrak{G}$ and X be an invloution contained in $Z(\mathfrak{Q})$. Suppose Δ is a conjugate class of $\mathfrak{G}$ containing X. By lemma 3, Δ contains two elements X_1, X_2 such that $\langle X_1, X_2 \rangle$ is not a 2-group. Since $\langle X_1, X_2 \rangle$ is a dihedial group, $|X_1 \cdot X_2|$ is not a power of 2. Then $\langle X_1 \cdot X_2 \rangle$ contains a unique subgroup $\mathfrak{H}$ of order P. Let $\mathfrak{M}$ be a maximal subgroup containing $N_{\mathfrak{G}}(\mathfrak{H})$. It is immediate to show that $\mathfrak{M}$ satisfies the condition of the lemma.

<div align="right">q.e.d.</div>

Proof of the theorem. Let $\mathfrak{M}$ be a maximal subgroup of $\mathfrak{G}$ which satisfies the condition of lemma 6. By lemma 5 $F(\mathfrak{M})$ is an r-group. Let G be an element of $\mathfrak{M}$ contained in the centre of some Sylow r'-subgroup $\bar{\Re}$ of $\mathfrak{G}$, and let $\Re$ be a Sylow r-subgroup of $\mathfrak{G}$ containing $\mathfrak{F}_r = F(\mathfrak{M})$. Since $\mathfrak{M} = N_{\mathfrak{G}}(\mathfrak{F}_r)$, it follows $Z(\Re) \subseteq \mathfrak{F}_r$ by Fitting. Then $\Re_0 = \langle Z(\Re)^X : X \in \langle G \rangle \rangle \subseteq \mathfrak{F}_r$ and hence it is an r-group normalized by G. Let Ω be a complete $\mathfrak{G}$-conjugate class containing $Z(\Re)$ and $\Omega = \Omega_1 + \cdots + \Omega_s$ be a disjoint sum of $\langle G \rangle$-orbits. Let $\Re_i$ be a group generated by Ω_i. For some element $Y \in \Re$, $Z(\Re)^Y \in \Omega_i$, then $\Omega_i = \langle Z(\Re)^{YX} : X \in \langle G \rangle \rangle = \langle Z(\Re)^{XY} ; X \in \langle G \rangle \rangle$. It follows that $\Re_i = \Re_0^Y$. Then $\Re_i$ is an r-group normalized by $G^Y = G$ for $i = 1, \cdots, S$. So there exist $\Omega_{i_1}, \cdots, \Omega_{i_l}$ such that the group generated by $\Omega_{i_1} \cup \cdots \cup \Omega_{i_l}$ is an r-group normalized by G. ($l \geq 1$) Let l be maximal. We may assume $\{i_1 \cdots i_l\} = \{1, \cdots, l\}$ and $\Re = \langle \Omega_1 \cup \cdots \cup \Omega_l \rangle$. It is trivial to show that $N_{\mathfrak{G}}(\Re) \ni G$. Let $\Re_0$ be a Sylow r-subgroup of $\mathfrak{G}$ containing $\Re$. By lemma 2, $\Re \trianglelefteq \Re_0$ or $N_{\mathfrak{G}}(\Re) \supseteq \Re^X (\neq \Re)$ for some $X \in \Re_0$. If $\Re \trianglelefteq \Re_0$, then $N_{\mathfrak{G}}(\Re)$ contains a complete conjugate class of $\mathfrak{G}$ containing G. A contradiction. If $N_{\mathfrak{G}}(\Re) \supseteq \Re^X (\neq \Re)$, then since $\Omega_1^X \cup \cdots \cup \Omega_l^X \nsubseteq \Re$, there exists some element Y of $\Re$ such that $Z(\Re)^Y \subseteq \Re^X$ and $Z(\Re)^Y \nsubseteq \Re$. Suppose $Z(\Re)^Y$ is an element of Ω_i. ($i > l$), then $\Re_i \subseteq N_{\mathfrak{G}}(\Re)$ from $N_{\mathfrak{G}}(\Re) \ni G$ and $N_{\mathfrak{G}}(\Re) \supseteq Z(\Re)^Y$. Now $\Re \cdot \Re_i$ is an r-group normalized by G and generated by $\Omega_1 \cup \cdots \cup \Omega_l \cup \Omega_i$, contrary to our choice of $\Re$. Thus we proved the theorem.

<div align="right">q.e.d.</div>

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