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THE HILBERT SCHEME OF ELLIPTIC CURVES AND REFLEXIVE SHEAVES ON FANO 3-FOLDS

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Abstract

We study the cohomological properties of reflexive rank 2 sheaves on smooth projective threefolds. Applications are given to the relationship between moduli of reflexive sheaves and Hilbert schemes of associated elliptic curves on Fano threefolds.

1. Introduction

We work over an algebraically closed field of characteristic 0.

In this paper we continue the study of reflexive sheaves on projective threefolds [5], [6], [7]. Recall that a coherent sheaf $\mathcal{F}$ is *torsion-free* if the natural map of $\mathcal{F}$ to its double-dual $h\colon \mathcal{F} \to \mathcal{F}^{**}$ is injective, and that $\mathcal{F}$ is *reflexive* if h is an isomorphism. Recall the following *Serre correspondence* for reflexive sheaves:

Theorem 1 ([4, 4.1]). *Let X be a smooth projective threefold, M an invertible sheaf with $\mathrm{H}^1(X, M^*) = \mathrm{H}^2(X, M^*) = 0$. There is a one-to-one correspondence between*
(1) *pairs $(\mathcal{F}, s)$ where $\mathcal{F}$ is a rank 2 reflexive sheaf on X with $\det\mathcal{F} = M$ and $s \in \Gamma(\mathcal{F})$ is a section whose zero scheme has codimension 2*
(2) *pairs (Y, ξ) where Y is a closed Cohen-Macaulay curve in X, generically a local complete intersection, and $\xi \in \Gamma(Y, \omega_Y^\circ \otimes \omega_X^* \otimes M^*)$ is a section which generates the sheaf $\omega_Y^\circ \otimes \omega_X^* \otimes M^*$ except at finitely many points.*
Furthermore, $\mathrm{c}_3(\mathcal{F}) = 2p_a(Y) - 2 + \mathrm{c}_1(X)\mathrm{c}_2(\mathcal{F}) - \mathrm{c}_1(\mathcal{F})\mathrm{c}_2(\mathcal{F})$.

Note that if $\mathcal{F}$ is locally free, then the corresponding curve Y is *subcanonical*.

In Section 2 we give examples of the relationship between the structure and the cohomology of a reflexive sheaf $\mathcal{F}$. In particular, we give some cohomological criteria to determine when a reflexive sheaf is actually locally free—this is equivalent to the vanishing of the third Chern class and is needed in the last section. In Section 3, we investigate the influence of global sections on the cohomology of $\mathcal{F}$ and in our main result (Theorem 7) we give an application of these results to the case of elliptic curves on Fano threefolds. The main result is another example of the relationship between the moduli space of vector bundles and the Hilbert scheme of curves on a threefold [6].

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2. Local freeness via cohomology

As a simple example of the connection between the cohomology and the structure of $\mathcal{F}$, recall that for a locally free sheaf $\mathcal{F}$ and a very ample line bundle L, if $h^i(X, \mathcal{F} \otimes L^n) = 0$ for $i = 1, 2$, $n \in \mathbb{Z}$, we say $\mathcal{F}$ is *L-aCM* (arithmetically Cohen-Macaulay), as in this case the associated curve Y is arithmetically Cohen-Macaulay (in the embedding by L) if and only if $\mathcal{F}$ is L-aCM ([1], [2]). If $\mathcal{F}$ is reflexive we have:

Proposition 2. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on a smooth projective 3-fold X, L an ample invertible sheaf. If $H^2(X, \mathcal{F} \otimes L^n) = 0$ for all $n \ll 0$, then $\mathcal{F}$ is locally free. In particular, $\mathcal{F}$ reflexive and L-aCM implies that $\mathcal{F}$ is locally free.*

Proof. This is essentially [4, 2.5.1], where it is shown that $H^2(X, \mathcal{F} \otimes L^n) = c_3(\mathcal{F})$ for all $n \ll 0$, and that $c_3(\mathcal{F}) = 0$ if and only if $\mathcal{F}$ is locally free. □

REMARK 3. It is shown in [8] that there are aCM curves on the general sextic threefold which are not subcanonical. If C is such a curve, then by Proposition 2 no reflexive sheaf $\mathcal{F}$ associated to C by the Serre correspondence is aCM.

As a second example, we look at a case where the Riemann-Roch formula becomes especially simple.

Proposition 4. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on a smooth projective 3-fold X with $c_1(\mathcal{F}) = c_1(\omega_X)$. Then $c_3(\mathcal{F}) = 2\,h^2(X, \mathcal{F}) - 2\,h^1(X, \mathcal{F})$. Hence the following are equivalent*:
(1) $h^2(X, \mathcal{F}) \le h^1(X, \mathcal{F})$.
(2) $h^2(X, \mathcal{F}) = h^1(X, \mathcal{F})$.
(3) $\chi(X, \mathcal{F}) = 0$.
(4) *$\mathcal{F}$ is locally free.*

Proof. From the usual spectral sequence, we have the exact 5-term sequence

$$0 \to H^1(X, \mathcal{F}^* \otimes \omega_X) \to H^2(X, \mathcal{F})^* \to H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \omega_X))$$
$$\to H^2(X, \mathcal{F}^* \otimes \omega_X) \to H^1(X, \mathcal{F})^* \to 0.$$

Further, by [4, 2.6] we have $c_3(\mathcal{F}) = h^0(X, \mathcal{E}xt^1(\mathcal{F}, \omega_X))$. Our hypotheses imply that $\mathcal{F}^* \otimes \omega_X = \mathcal{F}$ and the statements immediately follow. □

In general, we have the following:

DEFINITION 5. Let X be a projective threefold. A rank 2 reflexive sheaf $\mathcal{F}$ has *canonical parity* if $\det \mathcal{F} \otimes \omega_X^* = L^{\otimes 2}$ for some invertible sheaf L.

By Proposition 4 we have

Corollary 6. *Let $\mathcal{F}$ be a rank* 2 *reflexive sheaf with canonical parity on a smooth projective* 3*-fold X. Then $\mathcal{F}$ is locally free if and only if*

$$\chi(X, \mathcal{F} \otimes (\det \mathcal{F}^* \otimes \omega_X)^{1/2}) = 0.$$

3. Sheaves associated to elliptic curves

Our main application is to the relation between the moduli space of torsion free sheaves and the Hilbert scheme of curves.

Theorem 7. *Let $C \subset X$ be an irreducible local complete intersection curve with $p_a(C) = 1$ on a smooth projective threefold with $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$, $\mathrm{H}^2(X, \omega_X^*) = 0$, and $\mathrm{h}^0(C, \omega_X \otimes \mathcal{O}_C) = 0$ (e.g. X is Fano). Via the Serre correspondence one can associate to C a rank* 2 *vector bundle $\mathcal{F}$ with $\det \mathcal{F} = \omega_X^*$.*

If $\mathcal{F}$ is simple then $\mathrm{h}^1(C, N_{C/X}) \geq \mathrm{ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$, and $\mathrm{h}^0(C, N_{C/X}) - \mathrm{ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \geq \mathrm{h}^0(X, \mathcal{I}_C \otimes \omega_X^*) - \mathrm{h}^1(X, \mathcal{I}_C \otimes \omega_X^*)$.

If we also have $\mathrm{h}^1(X, \mathcal{I}_C \otimes \omega_X^*) = 0$, then $\mathrm{h}^1(C, N_{C/X}) = \mathrm{ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ and $\mathrm{h}^0(C, N_{C/X}) - \mathrm{ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) = \mathrm{h}^0(X, \mathcal{I}_C \otimes \omega_X^*)$.

Proof (Cf. [6, Proposition 23]). Because $\det \mathcal{F} = \omega_X^*$, $\mathcal{F}^*$ has canonical parity and so $\mathrm{h}^1(X, \mathcal{F}^*) = \mathrm{h}^2(X, \mathcal{F}^*)$ by Proposition 4. We have the sequence

$$0 \to \omega_X \to \mathcal{F}^* \to \mathcal{I}_C \to 0.$$

Now, $\mathrm{H}^1(X, \omega_X) = \mathrm{H}^2(X, \omega_X) = 0$ by hypothesis, hence $\mathrm{H}^1(X, \mathcal{F}^*) = \mathrm{H}^1(X, \mathcal{I}_C) = 0$.

By hypothesis, $\mathrm{H}^i(X, \omega_X) = 0$ for $0 \leq i \leq 2$ and $\mathrm{h}^0(C, \omega_X \otimes \mathcal{O}_C) = 0$, therefore $\mathrm{h}^i(X, \mathcal{I}_C \otimes \omega_X) = 0$ for $i = 0, 1$. From the sequence

$$0 \to \omega_X^{\otimes 2} \to \mathcal{F}^* \otimes \omega_X \to \mathcal{I}_C \otimes \omega_X \to 0$$

we see $\mathrm{H}^1(X, \mathcal{F}^* \otimes \omega_X) = \mathrm{H}^1(X, \omega_X^{\otimes 2})$. However, $\mathrm{H}^2(X, \mathcal{F}) = \mathrm{H}^1(X, \mathcal{F}^* \otimes \omega_X)^*$ by Serre duality and $\mathrm{H}^1(X, \omega_X^{\otimes 2}) = \mathrm{H}^2(X, \omega_X^*) = 0$ by hypothesis, hence $\mathrm{H}^2(X, \mathcal{F}) = 0$.

The short exact sequence in the first paragraph gives $\mathrm{H}^3(X, \mathcal{F}^*) = \mathrm{H}^0(X, \omega_X \otimes \mathcal{F}) = \mathrm{H}^0(X, \mathcal{F}^*) = 0$. Then tensoring that sequence with $\mathcal{F}$ yields $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) = \mathrm{H}^i(X, \mathcal{I}_C \otimes \mathcal{F})$.

Tensoring the standard sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

with $\mathcal{F}$ yields

$$\begin{aligned} 0 &\to \mathrm{H}^0(X, \mathcal{I}_C \otimes \mathcal{F}) \to \mathrm{H}^0(X, \mathcal{F}) \to \mathrm{H}^0(C, N_{C/X}) \\ &\to \mathrm{H}^1(X, \mathcal{I}_C \otimes \mathcal{F}) \to \mathrm{H}^1(X, \mathcal{F}) \to \mathrm{H}^1(C, N_{C/X}) \\ &\to \mathrm{H}^2(X, \mathcal{I}_C \otimes \mathcal{F}) \to 0. \end{aligned}$$

Noting that $\mathrm{h}^0(X, \mathcal{I}_C \otimes \mathcal{F}) = \mathrm{ext}^0_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) = 1$ by simplicity, we have

$$\mathrm{h}^1(\mathcal{F}) - \mathrm{h}^0(\mathcal{F}) + 1 = \mathrm{h}^1(N_{C/X}) - \mathrm{h}^0(N_{C/X}) + h^1(\mathcal{I}_C \otimes \mathcal{F}) - h^2(\mathcal{I}_C \otimes \mathcal{F}).$$

By Serre duality, $\mathrm{H}^i(X, \mathcal{F}) = \mathrm{H}^{3-i}(X, \mathcal{F}^* \otimes \omega_X)^*$ and so again from the sequence

$$0 \to \omega_X^{\otimes 2} \to \mathcal{F}^* \otimes \omega_X \to \mathcal{I}_C \otimes \omega_X \to 0$$

we see

$$\mathrm{h}^1(\mathcal{F}) - \mathrm{h}^0(\mathcal{F}) + 1 = -\,\mathrm{h}^0(\omega_X^*) + \mathrm{h}^0(\omega_X^* \otimes \mathcal{O}_C) + \mathrm{h}^1(\omega_X^*).$$

The long exact cohomology sequence above immediately implies that $\mathrm{h}^1(C, N_{C/X}) \geq \mathrm{ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$. Combining this with the two equalities above gives

$$\begin{aligned} \mathrm{h}^0(N_{C/X}) - \mathrm{ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) &\geq \mathrm{h}^0(\omega_X^*) - \mathrm{h}^0(\omega_X^* \otimes \mathcal{O}_C) - \mathrm{h}^1(\omega_X^*) \\ &= \mathrm{h}^0(X, \mathcal{I}_C \otimes \omega_X^*) - \mathrm{h}^1(X, \mathcal{I}_C \otimes \omega_X^*). \end{aligned}$$

For the last statement, note that if $\mathrm{h}^1(X, \mathcal{I}_C \otimes \omega_X^*) = 0$ then the sequence

$$0 \to \mathcal{O}_X \to \mathcal{F} \to \mathcal{I}_C \otimes \omega_X^* \to 0$$

implies that $\mathrm{H}^1(X, \mathcal{F}) = 0$. Now the long exact cohomology sequence above gives us $\mathrm{h}^1(C, N_{C/X}) = \mathrm{ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$, and $\mathrm{h}^0(C, N_{C/X}) - \mathrm{ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) = \mathrm{h}^0(X, \mathcal{I}_C \otimes \omega_X^*)$ since we no longer have an inequality above. □

It is easy to see that the hypotheses of Theorem 7 are not vacuous:

EXAMPLE 8. If $C \subset X \subset \mathbb{P}^4$ is a nondegenerate elliptic normal curve on a smooth hypersurface X of degree $2 \leq d \leq 4$, then the associated rank two vector bundle is stable (hence simple).

Similarly, if $C \subset X \cap \mathbb{P}^3 \subset \mathbb{P}^4$ is a degenerate elliptic normal (in $\mathbb{P}^3$) curve and X is a smooth hypersurface of degree 4, then the associated rank two vector bundle is stable.

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