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## ON A CLASS OF C*-ALGEBRAS

# gENERATED BY PARTIAL ISOMETRIES 

## Masatoshi FUJII

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## INTRODUCTION

This thesis is devoted to the study of $C^{*}-a l g e b r a s O_{n}$ and $O_{A}$ which are typical examples of simple $C^{*}$-algebras.

These C*-algebras were first considered by Cuntz [12], and Cuntz and Krieger [171 respectively. In 1977, Cuntz introduced $O_{n}$ as a $C^{*}$-algebra generated by isometries $S_{1}, \ldots, S_{n}$ acting on a Hilbert space such that $\sum_{i} S_{i} S_{i}{ }^{*}=1$. He proved that the isomorphism class of $O_{n}$ does not depend on the choice of generators and $O_{n}$ is simple. $O_{n}$ is an example of nuclear C*-algebras which are not strongly amenable. Now one of the great developments in the $C^{*}$-algebra theory in $1970^{\prime}$ s is the extension theory due to Brown, Douglas and Fillmore, simply known as the BDF theory. This theory was followed up by the K-theory for $C^{*}$-algebras. In the BDF theory, the $C^{*}$-algebra $C(T)$ of all continuous functions on the unit circle $T$ in the plane is adopted as a model. Since $C(T)$ is naturally regarded as $O_{1}$, we may use $O_{n}$ as an available model for the BDF theory and K-theory. As a matter of fact, $O_{n}$ was the first example of non-commutative $C *-a l g e b r a s$ taken up in the BDF theory. $O_{n}$ and $O_{m}$ are not stably isomorphic if $n \neq m$, because the weak extension group of $O_{n}$ is isomorphic to $z^{n} /(n-1) z^{n}$, cf. [38] and [41].

Afterwards, Cuntz and Krieger have generalized the Cuntz algebra $O_{n}$. Let $A=(A(i, j))$ be an $n x n$ matrix whose
entries $A(i, j)$ are 0 or 1 and not all zero in any row nor in any column. Then a $C^{*}$ - algebra $O_{A}$ is generated by partial isometries $S_{1}, \ldots, S_{n}$ acting on a Hilbert space satisfying the conditions
(A) $\quad S_{i} * S_{j}=0 \quad(i \neq j)$, and $S_{i} * S_{i}=\Sigma_{j} A(i, j) S_{j} S_{j} *$ for $i=1, \ldots, n$. Under a suitable condition on the matrix A, the isomorphism class of $O_{A}$ does not depend on the choice of generators. We call $O_{A}$ the Cuntz-Krieger algebra (associated with $A$ ). Note that $O_{A}=O_{n}$ if $A$ is the $n \times n$ matrix whose entries are all 1. A $C *-a l g e b r a O_{A}$ is associated with the topological Markov chain ( $X_{A}, \sigma_{A}$ ).

Now, it is known that a matrix $A$ determining the $C^{*}-a l g e-$ bra $O_{A}$ corresponds to a digraph $G$ as its adjacency matrix. Therefore we can attempt a graph theoretic approach to $O_{A}$. This method was initiated by Enomoto and Watatani [29], and it plays one of the central roles in our study of $O_{A}$.

This thesis consists of four chapters. We explain briefly the contents of each chapter.

In the first chapter, we will be concerned with automorphisms on $O_{n}$. In [1], Archbold considered the 'flip-flop' automorphism $\theta$ of $O_{2}=C *\left(S_{1}, S_{2}\right)$ determined by

$$
\theta\left(S_{1}\right)=S_{2} \text { and } \theta\left(S_{2}\right)=S_{1},
$$

which is an analog of the flip-flop automorphism on tensor products. He proved that $\theta$ is outer. This was generalized by Enomoto, Takehana and Watatani [26] as a representation to
automorphisms on $O_{n}$ of the symmetric group $S(n)$ with degree n. Furthermore they considered a similar representation of the unitary group $U(n)$ of all $n \times n$ unitary matrices; for $u=$ $\left(u_{i j}\right) \varepsilon U(n)$

$$
\alpha_{u}\left(S_{k}\right)=\varepsilon_{j} u_{j k} S_{j} \quad(k=1, \ldots, n)
$$

By the uniqueness theorem on $O_{n}, \alpha_{u}$ can be extended to an automorphism on $O_{n}$ and they proved that the action $\alpha$ is outer.

Now $O_{n}$ can be regarded as a semigroup version of the group von Neumann algebra $R\left(G_{n}\right)$ of a free group $G_{n}$ on $n$ generators. Phillips [40] and Choda [9] showed that $R\left(G_{n}\right)$ is isomorphic to the crossed product of $R\left(G_{k(n-1)+1}\right)$ by a single automorphism with period $k$. Choda [9] also determined the fixed point algebra of $R\left(G_{2}\right)$ under an automorphism with period k.

Now we shall determine the fixed point algebras of $O_{n}$ under certain periodic automorphisms :

Let $z$ be a primitive $k$-th root of 1 and $z 1 \varepsilon U(n)$. Then the fixed point algebra of $O_{n}$ under $\alpha_{z 1}$ is generated by a UHF-algebra $F_{n}$ of type $n^{\infty}$ and $S_{1}{ }^{k}$, where $S_{1}$ is a generator of $O_{n}$. Furthermore, the fixed point algebra is also a Cuntz algebra $\mathrm{O}_{\mathrm{n}} \mathrm{k}$.

Since the matrix $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ corresponding to the 'flip-flop' automorphism of $0_{2}$ is unitarily equivalent to $\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)$, we consider the fixed point algebra of $\mathrm{O}_{2}=\mathrm{C}^{*}\left(\mathrm{~S}_{1}, \mathrm{~S}_{2}\right)$ under $\alpha_{u}$, such that $u=\left(\begin{array}{ll}1 & 0 \\ 0 & z\end{array}\right)$ with a primitive $k$-th root $z$ of 1 . We see that it is the subalgebra generated by $S_{1}, S_{2}^{k}$ and
$\left\{S_{2}{ }^{j} S_{1} S_{2}{ }^{*} ; j=1, \ldots, k-1\right\}$. In particular, the fixed point algebra under the 'flip-flop' automorphism is isomorphic to the $C^{*}$-algebra generated by $S_{1}, S_{2}{ }^{2}$ and $S_{2} S_{1} S_{2}{ }^{*}$.

Here we have a problem; whether $\alpha_{-1}$ and the 'flip-flop' automorphism $\theta$ on $O_{2}$ are conjugate or not ? Since $S_{2} S_{1} S_{2}{ }^{*}$ is not an isometry, the fixed point algebra $\mathrm{O}_{2}{ }^{\theta}$ seems to be not of type of $O_{n}$. As a matter of fact, $O_{2}{ }^{\theta}$ is a CuntzKrieger algebra. It will become clear in Chapter III that the fixed point algebras under $\alpha_{-1}$ and $\theta$ are not stably isomorphic, so our problem is solved negatively.

In the last part of this chapter, we shall investigate the relation between the spectrum $\sigma(u)$ of $u \in U(n)$ and $\sigma\left(\alpha_{u}\right)$ of $\alpha_{u}$ in the Banach algebra of all bounded linear maps on $O_{A}$. We prove that $\sigma\left(\alpha_{u}\right)$ is the closed subgroup of the unit circle $T$ in the plane generated by $\sigma(u)$, and for any closed subgroup $G$ of $T$, there is a $u \varepsilon U(n)$ such that $\sigma\left(\alpha_{u}\right)=G$.

In the second chapter, we shall discuss extensions of $O_{n}$ and $O_{A}$ by the compacts. In [11], Coburn studied the $C *-a l g e-$ bra generated by an isometry acting on a Hilbert space $H$. He proved that the $C^{*}-a l g e b r a$ generated by a simple unilateral shift $U_{+}$on $H$ contains the ideal $K(H)$ of all compact operators on $H$ and
(1)

$$
\mathrm{O} \longrightarrow \mathrm{~K}(\mathrm{H}) \longrightarrow \mathrm{C}^{*}\left(\mathrm{U}_{+}\right) \longrightarrow \mathrm{C}(\mathrm{~T}) \longrightarrow 0
$$

is exact, that is, $C *\left(U_{+}\right)$is an extension of $C(T)$ by $K(H)$. In the BDF theory, we know that the extension group Ext $C(T)$ coincides with the additive group $Z$ of all integers under the
correspondence $n=-\operatorname{ind} U_{+}^{(n)}$, where ind $S$ stands For the index of a Fredholm operator $S$.

Cuntz [12] proved further that if $P_{n}$ is the $C *$-algebra generated by isometries $T_{1}, \ldots, T_{n}$ on $H$ such that $I$ $\Sigma_{i} \mathrm{~T}_{i} \mathrm{~T}_{\mathrm{i}}{ }^{*}$ is a non-zero projection, then

$$
\begin{equation*}
0 \longrightarrow \mathrm{~K}(\mathrm{H}) \longrightarrow \mathrm{P}_{\mathrm{n}} \longrightarrow 0_{\mathrm{n}} \longrightarrow 0 \tag{2}
\end{equation*}
$$

is exact. Ext $O_{n}=Z$ was proved by Paschke and Salinas [38].
In our discussion, we shall first point out that an extension of $O_{n}$ can be reduced to one of $C(T)$ via a unilateral shift. Then we give a complete set of representatives for extensions of $\mathrm{O}_{\mathrm{n}}$.

Next we shall discuss extensions of $O_{A}$. One of our objectives is to find a condition for that $\alpha_{u}$ defined in Chapter I can be extended to an automorphism on $O_{A}$. Now, Evans [30] and Katayama have independently realized a C*-algebra $P_{n}$ as a 'tensor algebra' on the full Fock space $F(H)$, and constructed a unitary $F(u)$ on $F(H)$ for each $u \varepsilon U(n)$. In this realization, the automorphism $\bar{\alpha}_{u}$ on $P_{n}$ implemented by $F(u)$ corresponds to the automorphism $\alpha_{u}$ on $o_{n}$. We will here construct a subspace $L_{A}$ of $F(H)$ associated with an $n \times n$ matrix $A$, and the $C *$-algebra $P_{A}$ generated by the compression to $L_{A}$ of the creation operators on $F(H)$ is an extension of $O_{A}$ by $K(H)$. Also we shall consider conditions on $u \varepsilon U(n)$ that $L_{A}$ reduces $F(u)$ and $F(u) \mid L_{A}$ implements an automorphism $\alpha_{u}$ on $O_{A}$. As an application, we have a good characterization of $O_{n}$; if for all $u \in U(n) \quad \alpha_{u}$ can be extended to an automorphism on $O_{A}=C^{*}\left(S_{1}, \cdots, S_{n}\right)$, then $O_{A}$
$=O_{n}$. To make these discussions, the graph theoretic approach is very useful. $O_{A}$ will be sometimes written as $O_{G}$ if $G$ is the graph with the adjacency matrix $A$.

Another extension of $\mathrm{O}_{\mathrm{A}}$ can be obtained by using the concept of adjoint graphs in the graph theory. We shall prove that $O_{G}=O_{G *}$ when $G^{*}$ is the adjoint graph of $G$. As a consequence, we shall see that the reduced $C^{*}$-algebra generated by the free category of a digraph $G$ is an extension of $O_{G}$ by $K(H)$.

The main purpose of the third chapter is to classify CuntzKrieger algebras $O_{A}$ for $A$ 's with irreducible $3 \times 3$ matrices. This will be done in section 3 . The irreducibility of $A$ implies the simplicity of $O_{A}$. We will make an effective use of the K-theory in our classification problem.

We give attention to the 'position' of the unit 1 of a
unital $C^{*-a l g e b r a} B$ in the corresponding $K_{0}$-group $K_{0}(B)$. It will be called the marker of $B$ and denoted by mark( $B$ ). It is obvious that if unital C*-algebras B and C are isomorphic, then $K_{o}(B)=K_{o}(C)$ and mark $(B)=\operatorname{mark}(C)$, but this fact is very important for the classification. Actually, we shall prove that the following statements are equivalent for $3 \times 3$ irreducible matrices $A$ and $B$;
(1) $O_{A}$ is isomorphic to $O_{B}$,
(2) $K_{O}\left(O_{A}\right)=K_{O}\left(O_{B}\right)$ and $\operatorname{mark}\left(O_{A}\right)=\operatorname{mark}\left(O_{B}\right)$, and
(3) $A$ is primitively equivalent to $B$.

As a preparatory task for this, we listed up all the
strongly connected digraphs with 3 vertices and $3 \times 3$ irreducible matrices. We get 29 different matrices. Then we introduced a transformation of matrices by which $O_{A}$ is left isomorphic. Primitive equivalence also introduced among matrices and this equivalence too makes corresponding algebras isomorphic. The relation $(3) \rightarrow(1) \rightarrow(2)$ follows from these facts, and (2) $\rightarrow(3)$ is then checked one by one.

Next we shall discuss how to change the marker under the tensor product with a matrix algebra $M_{k}$. As a corollary, we have another proof of a result on $O_{n}$ due to Paschke and Salinas [38].

We also define the explosion as a generalization of the adjoint of a digraph. This again leaves isomorphic the corresponding algebras. Using these notions, we can complete to give representatives in the classification. We also discuss the value $\operatorname{det}(1-A)$ because it is very important in the theory of Markov chains $\left(X_{A}, \sigma_{A}\right)$, and show that $O_{A}$ and $O_{B}$ are isomorphic for strongly shift equivalent matrices $A$ and B under some additional assumptions.

Concluding this chapter, we shall point out that any finitely generated abelian group can be expressed as the weak extension group and also $K_{o}$-group of a simple Cuntz-Krieger algebra. Additionally, we shall discuss the periodicity of the weak extension group of ${ }^{O_{A}}$ associated with random walks.

In the final chapter, we shall study the existence of KMS states for gauge action on $O_{A}$. Here we note that the proof of
the uniqueness theorem on $\mathrm{O}_{\mathrm{A}}$ is based on the existence of the gauge automorphism $\alpha_{t}(t \varepsilon R)$ on $O_{A}$ such that

$$
\alpha_{t}\left(S_{j}\right)=e^{i t^{S}}{ }_{j} \quad j=1, \ldots, n
$$

where $R$ is the group of real numbers. The action $\alpha$ is called the gauge action. Olesen and Pedersen [33] proved that the $C^{*}$-dynamical system ( $O_{n}, R, \alpha$ ) admits a $\beta-K M S$ state if and only if $\beta=\log n$, and the corresponding KMS state is unique. Furthermore, the weak closure of the GNS representation of $O_{n}$ by the unique KMS state is a factor of type $I I I_{1 / n}$. On the other hand, there exist matrices $A$ and $B$ such that the spectral radii $r(A)$ and $r(B)$ are different though $O_{A}$ and $O_{B}$ are isomorphic. So we want to find out a condition that spectral radii coincide.

We shall generalize the theorem of Olesen and Pedersen on $\left(O_{A}, R ; \alpha\right)$. We remark that $r(A)=n$ if $A$ is the $n \times n$ matrix whose entries are all 1 , that is, $A$ corresponds to $O_{n}$. Now we prove that if $A$ is irreducible, then $\left(O_{A}, R, \alpha\right)$ admits a $\beta$-KMS state if and only if $\beta=\log r(A)$, and the corresponding KMS state is unique. It seems to be interesting that the Perron-Frobenius theorem for positive matrices is applied in this proof. Furthermore we obtain that the weak closure of the GNS representation of $O_{A}$ by the state is a factor of type III $1 / r(A) d(A)$, where $d(A)$ is the period of $A$. Therefore, since $\log r(A)$ is the topological entropy $h\left(\sigma_{A}\right)$, the pair $\left(h\left(\sigma_{A}\right), d(A)\right)$ is an invariant for the conjugacy of C*-dynamical system $\left(O_{A}, R, \alpha\right)$. In other words, the equivalence of the subshift $\sigma_{A}$ as a measure preserving transfor-

```
mation is an invariant for the conjugacy because (log r(A),
d(A)) is a complete invariant for of}\mp@subsup{A}{A}{}\mathrm{ as a measure preserving
transformation. Finally, we discuss a relation between KMS
states and eigenvalues of positive maps in a general setting.
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## CHAPTER I AUTOMORPHISMS ON $O_{n}$

I-1. Action of $U(n)$ on $O_{n}$.
A $C^{*}$-algebra $O_{n}$ considered by Cuntz is generated by isometries $S_{1}, \ldots, S_{n}$ acting on a Hilbert space such that $\Sigma_{i} S_{i} S_{i}^{*}=1$. He proved the following uniqueness theorem on $O_{n}$ :

The uniqueness theorem. The isomorphism class of $O_{n}$ does not depend on the choice of generators.

That is, if $\left\{T_{1}, \ldots, T_{n}\right\}$ is another family of isometries such that $\Sigma_{i} T_{i} T_{i}^{*}=1$, then there is a canonical isomorphism of $C *\left(S_{1}, \ldots, S_{n}\right)$ onto $C *\left(T_{1}, \ldots, T_{n}\right)$, where $C^{*}(S)$ is the $C^{*}$-algebra generated by $S$. In other words, if. $\left\{T_{1}, \ldots, T_{n}\right\}$ is as in above, then the map $S_{i} \longrightarrow T_{i}(1 \leq i \leq n)$ can be extended to an isomorphism of $C^{*}\left(S_{1}, \ldots, S_{n}\right)$ onto $C^{*}\left(\mathrm{~T}_{1}, \ldots, \mathrm{~T}_{\mathrm{n}}\right)$.

Inspired by the flip-flop automorphism of tensor products, Archbold [1] considered the 'flip-flop' automorphism $\theta$ of $\mathrm{O}_{2}$ $=C *\left(S_{1}, S_{2}\right)$ determined by

$$
\theta\left(S_{1}\right)=S_{2} \quad \text { and } \theta\left(S_{2}\right)=S_{1}
$$

He proved that $\theta$ is outer. This might be the first application of the uniqueness theorem.

Following after Archbold, Enomoto, Takehana and Watatani [26] showed that the symmetric group $S(n)$ has a represen-
tation as a subgroup of outer automorphisms on $O_{n}$ for $n \geq 2$. Furthermore they extended it as follows; the group $U(n)$ of $\mathrm{n} x \mathrm{n}$ unitary matrices is faithfully represented as a subgroup of outer automorphisms on $O_{n}$ by

$$
a_{u}\left(S_{k}\right)=\Sigma_{j} u_{j k} S_{j} \quad(k=1, \ldots, n)
$$

for unitary $u=\left(u_{j k}\right)$. If we take $u=\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$, then $\alpha_{u}$ is the 'flip-flop' automorphism on $\mathrm{O}_{2}$.

Though Archbold considered $O_{n}$ from the view point of tensor products, it can be regarded as a semigroup version of the group von Neumann algebra $R\left(G_{n}\right)$ of a free group $G_{n}$ on $n$ generators, cf. [27]. Phillips [40] and Choda [9] showed that $R\left(G_{n}\right)$ is isomorphic to the crossed product of $R\left(G_{k(n-1)+1}\right)$ by a single automorphism with period k. And Choda determined the fixed point algebra of $R\left(G_{2}\right)$ under a single automorphism with peroid $k$ by $R\left(G_{k+1}\right)$. Moreover, the fixed point algebra under the gauge automorphism group $\alpha_{T}$ is determined by Olesen and Pedersen.

In the below, we shall investigate the fixed point algebra of $O_{n}$ under a periodic automorphism $\alpha_{u}$ for $u \varepsilon U(n)$.

I-2. Fixed point algebras.
First of all, we shall explain notation. Let $W_{n}^{k}(k=$ $1,2, \ldots)$ be the set of all k-tuples $(j(1), \ldots, j(k))$ with $1 \leq j(i) \leq n$ and let $W_{n}{ }^{0}=\{0\}$. Let $\left\{S_{1}, \ldots, S_{n}\right\}$ be a family of generators of $O_{n}$. Then we put

$$
S_{\mu}=S_{j(1)} S_{j(2)} \cdots S_{j(k)}
$$

for $\mu=(j(i), \ldots, j(k)) \varepsilon W_{n}^{k}$ and $S_{o}=1$. Let $F_{n}^{k}$ be the C*-algebra generated by $\left\{S_{\mu} S_{U}{ }^{*} ; \mu, v \varepsilon W_{n} k_{\}}\right.$and $F_{n}^{0}=$ \{1\}. Finally let $F_{n}$ be the $C^{*}$-algebra generated by $\left\{F_{n}{ }^{k}\right.$; $k=0,1,2, \ldots\}$.

Theorem 1.1. Let $z$ be a primitive $k$-th root of 1 and $\alpha=\alpha_{z 1}$ the automorphism on $O_{n}$ induced by $z 1 \varepsilon U(n)$. Then the fixed point algebra $B$ of $\alpha$ is the $C^{*}$-algebra generated by $F_{n}$ and $S_{1}{ }^{k}$.

Proof. If $1(\mu)=l(u)$ where $l(\gamma)$ is the length of $\gamma$, then $S_{\mu} S_{v}{ }^{*} \varepsilon B$. Since $F_{n}$ is generated by $\left\{S_{\mu} S_{v}{ }^{*} ; I(\mu)=I(v)\right\}$ and $S_{1}{ }^{k} \varepsilon B, B$ includes the $C^{*}$-algebra $C$ generated by $F_{n}$ and $S_{1}{ }^{k}$. Conversely, let $x \varepsilon B$ and $\varepsilon>0$. Then there is $y$ in the *-algebra $Q_{n}$ generated by $\left\{S_{1}, \ldots, S_{n}\right\}$ such that $\|x-y\|<\varepsilon$. It is known that $y$ has a unique representation;

$$
y=\varepsilon_{1}^{m} S_{1}^{* i} a_{-i}+a_{0}+\varepsilon_{1}^{m} a_{i} s_{1}^{i}
$$

where $a_{i} \varepsilon Q_{n} \cap F_{n}$. Putting $\beta=\left(\alpha^{k-1}+\alpha^{k-2}+\ldots+\alpha+1\right) / k$, every $a_{i}$ is fixed by $B$. Since

$$
0=z^{i k}-1=\left(z^{i}-1\right)\left(z^{i(k-1)}+z^{i(k-2)}+\cdots+z^{i}+1\right)
$$

for $i=1,2, \ldots$, we have

$$
\begin{aligned}
B(y) & =(1 / k) \Sigma_{i=1}^{m} \Sigma_{j=0}^{k-1} z^{* i j} S_{1} *^{i} a_{-i}+a_{0}+(1 / k) \Sigma_{i=1}^{m} \sum_{j=0}^{k-1} z^{i j} a_{i} S_{1}^{i} \\
& =\varepsilon_{i=m k} S_{1}{ }^{i} a_{-i}+a_{0}+\Sigma_{i=m k} a_{i} S_{1}^{i}
\end{aligned}
$$

so that $B(y) \in C$. On the other hand, we have

$$
\begin{aligned}
\|x-\beta(y)\| & \leq\left(\left\|x-\alpha^{k-1}(y)\right\|+\left\|x-\alpha^{k-2}(y)\right\|+\cdots+\|x-y\|\right) / k \\
& =\left(\left\|\alpha^{k-1}(x-y)\right\|+\left\|\alpha^{k-2}(x-y)\right\|+\cdots+\|x-y\|\right) / k \\
& =\|x-y\|<\varepsilon
\end{aligned}
$$

The following theorem shows that the fixed point algebra of an automorphism $\alpha$ in the above is also a Cuntz algebra.

Theorem 1.2. Let $\alpha$ be as in Theorem 1.1. Then the fixed point algebra $B$ is isomorphic to the Cuntz algebra $O_{n k}$.

Proof. First of all, we shall prove that $B$ coincides with $C=C^{*}\left(S_{\mu} ; I(\mu)=k\right)$. Since $B=C^{*}\left(F_{n}, S_{1}{ }^{k}\right)$ by Theorem 1.1 and $B \geqslant C$ clearly, it suffices to show that $F_{n} \subseteq C$, that is, $S_{\mu} S_{u}^{*} \in C$ if $I(\mu)=I(v)$. We may assume that $I(\mu)=$ $1(u)<k$. Then the length of $S_{\mu} S_{u}{ }^{*}$ is enlarged as follows:

$$
\begin{aligned}
S_{\mu} S_{v}^{*} & =S_{\mu}\left(\Sigma_{i} S_{i} S_{i}^{*}\right) S_{U}^{*} \\
& =\Sigma{ }_{i}\left(S_{\mu} S_{i}\right)\left(S_{u} S_{i}\right)^{*} \\
& =\Sigma \sum_{i, j}\left(S_{\mu} S_{i} S_{j}\right)\left(S_{U} S_{i} S_{j}\right)^{*}
\end{aligned}
$$

Thus $S_{\mu} S_{U}{ }^{*}$ is expressed as a finite sum of $\left\{S_{\gamma} S_{\delta}{ }^{*} ; l(\gamma)=\right.$ $I(\delta)=k\}$ by repeating this calculation.

To prove that $C$ is isomorphic to $O_{n k}$, we shall show that $\left\{S_{\mu} ; 1(\mu)=k\right\}$ is a family of generators of $O_{n k}$ such that $\Sigma_{\mu} \mathrm{S}_{\mu} \mathrm{S}_{\mu}{ }^{*}=1$ by induction. If $\mathrm{k}=2$, then

$$
\begin{aligned}
\Sigma_{\mu} S_{\mu} S_{\mu}^{*} & =\Sigma_{i, j}\left(S_{i} S_{j}\right)\left(S_{i} S_{j}\right) * \\
& =S_{1}\left(\Sigma_{i} S_{i} S_{i}^{*}\right) S_{1}^{*}+\ldots+S_{n}\left(\Sigma_{i} S_{i} S_{i}^{*}\right) S_{n}^{*} \\
& =S_{1} S_{1}^{*}+\cdots+S_{n} S_{n}^{*} \\
& =1
\end{aligned}
$$

Suppose that it is true for $k=p$. Then we have

$$
\Sigma l(\mu)=p+1 S_{\mu} S_{\mu}^{*}=\sum_{i=1}^{n} S_{i} T S_{i}^{*}
$$

where $T=\Sigma 1(\mu)=p S_{\mu} S_{\mu}{ }^{*}$. By the assumption of induction, we have $T=1$ as desired.

Remark. Let $z$ be a complex number such that $|z|=1$ and $z^{k} \neq 1$ for all $k$. Then the fixed point algebra of $\alpha_{z 1}$ is $F_{n}$ by a result of Olesen and Pedersen [36; lemma 1].

Next we shall discuss the fixed point algebra of automorphism on $O_{2}$ induced by $\left(\begin{array}{ll}1 & 0 \\ 0 & z\end{array}\right)$.

Theorem 1.3. Let $z$ be a complex number with period $k$ and $\alpha$ the automorphism on $\mathrm{O}_{2}$ induced by $\left(\begin{array}{ll}1 & 0 \\ 0 & z\end{array}\right)$. Then the fixed point algebra $B$ is the $C^{*}$-algebra $C$ generated by $S_{1}$, $S_{2}^{k}$ and $\left\{S_{2}{ }^{j} S_{1} S_{2}^{* j} ; j=1, \ldots, k-1\right\}$.

Proof. For given $x \in B$ and $\varepsilon>0$, there is $y \varepsilon Q=Q_{2}$ such that $\|x-y\|<\varepsilon$. Putting $w=\left(y+\alpha(y)+\ldots+\alpha^{k-1}(y)\right) / k$, we have $w \varepsilon B \cap Q$ and $\|x-w\|<\varepsilon$ as in the proof of Theorem 1.1. So it follows that $B \cap Q$ is dense in $B$. Since $C \subseteq B$, it suffices to show that $B \cap Q \subseteq C$. Every element y $\varepsilon$ $Q$ has a unique representation; $y=\varepsilon_{i=1}^{m} S_{1}{ }^{i} a_{-i}+a_{0}+\sum_{i=1}^{m} a_{i} S_{1}{ }^{i}$, where $a_{i} \varepsilon Q \cap F_{2}$. Since $\alpha(y)=y$ iff $\alpha\left(a_{i}\right)=a_{i}$ for all $i$, we may confine ourselves to consider elements in $F_{2}$. It is clear that $S_{\mu} S_{v}^{*} \varepsilon F_{2}$ is fixed by $\alpha$ iff $m(\mu)=m(v) \bmod k$, where $m(\gamma)$ denotes the number of $S_{2}$ in $S_{\gamma}$. If $m(\mu)<k$, then we have

$$
\begin{aligned}
S_{\mu}= & S_{1}{ }^{i(0)} S_{2}{ }^{j(1)} S_{1}{ }^{i(1)} S_{2}^{j(2)} \ldots S_{2}^{j(r)} S_{1}^{i(r)} \\
= & S_{1}^{i(0)}\left(S_{2}^{j(1)} S_{1}^{i(1)} S_{2}^{* j(1)}\right)\left(S_{2}^{j(1)+j(2)} S_{S_{1}} S_{2}^{* j(1)+j(2)}\right) \\
& \ldots\left(S_{2}^{j(1)+j(2)+\ldots+j(r-1)} S_{1}^{i(r-1)} S_{2}^{*}{ }^{j(1)+\ldots+j(r)}\right) \\
& S_{2}^{j(1)+\ldots+j(r)} S_{1}^{i(r)},
\end{aligned}
$$

so that $S_{\mu}=X_{2}{ }^{m(\mu)} S_{1}{ }^{i}$ for some $X \in C$ and integer i. In particular, if $m(\mu)=k$, then $S_{\mu} \varepsilon C$ since $k=j(1)+\ldots$ $+j(r)$. In general, if $m(\mu)=h \bmod k$, then $S_{\mu}=X S_{2}{ }^{h} S_{1}{ }^{i}$ for some $X \in C$ and some integer i. Therefore, if $m(\mu)=h$ $\bmod k$ and $h<k$, then

$$
S_{\mu} S_{v}^{*}=X S_{2}^{h} S_{1}^{i}\left(Y S_{2}^{h} S_{1}^{j}\right)^{*}=X\left(S_{2}^{h} S_{1}^{i} S_{2}^{* h}\right)\left(S_{2}^{h} S_{1} *^{j} S_{2} *^{h}\right) Y^{*},
$$

so that $S_{\mu} S_{U}{ }^{*} \varepsilon C$. Since $B \cap Q$ is generated by $F_{2}$ and $S_{1}{ }^{k}$ we have $B \cap Q \subseteq C$. Hence it follows that $B=C l B \cap Q \subseteq C$ where $c l$ means the norm closure, so that $B=C$.

The matrix $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ corresponding to the 'flip-flop' automorphism introduced by Archbold is unitarily equivalent to $\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)$. Therefore we have the following.

Corollary 1.4. The fixed point algebra of the 'flip-flop' automorphism on $\mathrm{O}_{2}$ is isomorphic to the $\mathrm{C}^{*}$-algebra generated by $S_{1}, S_{2}{ }^{2}$ and $S_{2} S_{1} S_{2}{ }^{*}$.

Remark. The fixed point algebra $\mathrm{O}_{2}{ }^{(2)}$ of $\mathrm{O}_{2}$ under (2) $=Z / 2 Z$ is isomorphic to $O_{4}$ by Theorem 1.2. However, the crossed product $\mathrm{O}_{2} \times(2)$ of $\mathrm{O}_{2}$ by (2) is not isomorphic to $\mathrm{O}_{4}$. In fact, if $\mathrm{O}_{2} \mathrm{x}(2) \cong \mathrm{O}_{4}$, then $\mathrm{O}_{2}$ is included in $\mathrm{O}_{4}$ with the same unit, which is false. Furthermore, $O_{4} x(2)$ is not isomorphic to $\mathrm{O}_{2}$. Actually, if $\mathrm{O}_{4} \mathrm{x}(2) \cong \mathrm{O}_{2}$, then $\mathrm{O}_{4} \otimes \mathrm{M}_{2} \cong \mathrm{O}_{2} \mathrm{x}$ (2) by Takai's duality theorem for $\mathrm{C}^{*}$-algebras [42]. Since $\mathrm{O}_{4} \otimes \mathrm{M}_{2} \cong \mathrm{O}_{4}$, we have $\mathrm{O}_{4} \cong \mathrm{O}_{2} \times$ (2), which is a contradiction.

I-3. Spectra of automorphisms.
Now we shall investigate the relation between the spectrum $\sigma(u)$ of $u \in U(n)$ and the spectrum $\sigma\left(\alpha_{u}\right)$ of $\alpha_{u}$ in the Banach algebra of all bounded linear maps on $O_{n}$.

Theorem 1.5. (i) If $u \in U(n)$, then $\sigma\left(\alpha_{u}\right)$ is the closed subgroup $G$ of the unit circle $T$ in the plane generated by $\sigma(u)$. (ii) For a closed subgoup $G$ of $T$, there is $u \in U(n)$ such that $\sigma\left(\alpha_{u}\right)=G$.

Proof. (i) We may assume that $u$ is diagonal with eigenvalues $\left(t_{1}, \ldots, t_{n}\right)$. Suppose that $u$ is periodic with period $k$. Then $G$ is generated by $\left\{t_{1}, \ldots, t_{n}\right\}$. If $z \in G$, then there are integers $s(1), \ldots, s(n)$ such that $z=t_{1} s(1)$ $\ldots t_{n} s(n)$ and $0 \leq s(i) \leq k-1$ for all i. If we put $R=S_{1}^{s(1)} \ldots S_{n}^{s(n)}$,
then $R \neq 0$ and $\alpha_{u}(R)=z R$. Since $z \varepsilon \sigma\left(\alpha_{u}\right)$, we have $G \subseteq \sigma\left(\alpha_{u}\right)$. Conversely, since $G=\{\exp (2 \pi i m / k) ; 0 \leq m \leq k-1\}$ and $\left(\alpha_{u}\right)^{k}=1$, we have $\sigma\left(\alpha_{u}\right) \subseteq G$ by the spectral mapping theorem. If $u$ is aperiodic, then $G=T$. Since $\alpha_{u}\left(S_{1}{ }^{m}\right)=$ $t_{1}{ }^{m} S_{1}{ }^{m}$ for all $m$, we have $\sigma\left(\alpha_{u}\right)=T=G$.
(ii) If $G=\{\exp (2 \pi i m / k) ; 0 \leq m \leq k-1\}$ for some $k$, then $\sigma\left(\alpha_{u}\right)=G$ for $u=\exp (2 \pi i / k)$. If $G=T$, then we take $u=z 1$ for some aperiodic $z$.

## Chapter II Extensions of $O_{A}$

II-1. Extensions of $O_{n}$.
The purpose of this section is to show that extensions of On are reduced to the extensions of $C(T)$ via a unilateral shift. In [11], Coburn studied the $C^{*}$-algebra generated by an isometry acting on a Hilbert space $H$. He proved that if $U_{+}$ is a simple unilateral shift on $H$, then the $C^{*}$-algebra $C^{*}\left(U_{+}\right)$ generated by $U_{+}$contains the ideal $K(H)$ of all compact operators on $H$ and
(1) $\mathrm{O} \longrightarrow \mathrm{K}(\mathrm{H}) \longrightarrow \mathrm{C}^{*}\left(\mathrm{U}_{+}\right) \longrightarrow \mathrm{C}(\mathrm{T}) \longrightarrow 0$
is exact, where $C(T)$ is the $C^{*}$-algebra of all continuous functions on $T$. In other words, $C^{*}\left(U_{+}\right)$is an extension of $C(T)$ by $K(H)$. In the $B D F$ theory [8], the extension group Ext $C(T)$ coincides with the additive group $Z$ of all integers. Here we mention a proof of this fact: Let $\pi$ be the quotient map of $B(H)$ onto $Q(H)=B(H) / K(H)$, where $B(H)$ is the algebra of all bouded linear operators on H. An operator $S$ is essentially normal if $\pi(S)$ is normal. A typical example of an essentially normal operator is a simple unilateral shift $U_{+}$. The task to determine Ext $C(T)$ is identified with the classification of essentially normal operators with es-
sential spectrum T. Moreover, it is known in [8; Theorem 3.1] that such an operator $S$ is unitarily equivalent to $U_{+}^{k}+K$ (resp. $U_{+}^{(0)}+K$ and $U_{+}^{*}{ }^{k}+K$ ) if indS $=-k<0$ (resp. ind $S$
$=0$ and ind $S=k>0$ ), where $U_{+}(0)$ is a simple bilateral shift, $K$ is compact and ind $S$ is the index of $S$. Hence a family of $C^{*}$-algebras $\left\{C^{*}\left(U_{+}(k)\right)+K(H)\right\}$ is a complete set of representatives for the extensions of $C(T)$, and the identifying map with Ext $C(T)$ and $Z$ is - ind $U_{+}{ }^{(k)}=k$. We remark that $C *\left(U_{+}\right)+K(H)=C^{*}\left(U_{+}\right)$by (1).

Let $P_{n}$ be the $C^{*}$-algebra generated by isometries $T_{1}, \ldots$, $T_{n}$ such that $1-\Sigma_{i} T_{i} T_{i}^{*}$ is a non-zero projection. Then

$$
0 \longrightarrow K(H) \longrightarrow P_{n} \longrightarrow 0_{n} \longrightarrow 0
$$

is exact according to [12; 3.1]. Later, Enomoto, Takehana and Watatani realized $P_{n}$ as the $C^{*}$-algebra $C^{*}{ }_{r}\left(G_{n}{ }^{+}\right)$generated by the left regular representations of a free semigroup $G_{n}{ }^{+}$ on $n$ generators. And they proved that $P_{n}$ is unique up to isomorphism as well as $O_{n}$. Cuntz stated in [12; Remark 1 in § 3] that it seems to be interesting to study more general extensions of $O_{n}$ by the compacts. Paschke and Salinas proved that Ext $O_{n}=Z$ by using an index of extensions and showed implicitly that a family $\left\{P_{n}\right\}$ is a complete set of representatives for extensions of $O_{n}$ corresponding to the negative integers.

The fact that Ext $O_{n}=Z$ reminds us an analogy with Ext $C(T)=Z$. We shall give attention to the first isometry $S_{1}$ among the generators of $O_{n}$. A proof of Ext $O_{n}=Z$ will be obtained by using the $C^{*}$-algebras $P_{n} k$ generated by $\left\{U_{+}{ }^{(k)} S_{1}\right.$, $\left.S_{2}, \ldots, S_{n}\right\}$ and $K(H)$, where $U_{+}$is a simple unilateral shift on ran $S_{1}$ and $V^{(k)}=V^{k}(k \geq 0)$ and $V^{*} k(k<0)$. As a matter of fact, $P_{n}^{k}$ is corresponding to an integer $k=$

- ind $U_{+}^{(k)}$, which is the same as the case of $C(T)$.

As usual, an extension of a unital separable C*-algebra B is a *-monomorphism of $B$ into $Q(H)$. Paschke and Salinas used the following index $m$ for extensions of $o_{n}$ to prove Ext $O_{n}=2$ : Let $\tau$ be an extension of $O_{n}=C *\left(S_{1}, \ldots, S_{n}\right)$ and $v_{\tau}$ the matrix in $Q(H \oplus \ldots \oplus H)=Q(H) \otimes M_{n}$ with zeros in the second through $n$-th row and with $\tau\left(S_{1}\right) \ldots \tau\left(S_{n}\right)$ in the first row. Then $v_{\tau}$ is isometric and $v_{\tau} v_{\tau}{ }^{*}=\pi\left(P_{H}\right)$, where $P_{H}$ is the projection of $H \oplus \ldots \oplus H$ onto $H \oplus O \oplus \ldots \oplus 0$. So there is a partial isometry $V=V_{\tau}$ on $H \oplus \ldots \oplus H$ such that $\pi(V)=V_{\tau}$ and $V V^{*} \leq P_{H}$ [38; Lemma 1.11. They put $m(\tau)$ $=\operatorname{dim}\left(1-V^{*} V\right)-\operatorname{dim}\left(P_{H}-V^{*}\right)$. Note that $m(\tau)=$ ind $V$ as an operator of $H \oplus \ldots \oplus H$ into $H \oplus O \oplus \ldots \oplus 0$, and so $m(\tau)$ is well-defined. It is known that $m(\tau)=m(\tau)$ if $\tau$ and $\tau^{\prime}$ are strongly equivalent, and that $m(\tau)=0$ iff $\tau$ is trivial. Since $m\left(\tau \oplus \tau^{\prime}\right)=m(\tau)+m\left(\tau^{\prime}\right)$, $m$ is a homomorphism of Ext $O_{n}$ into $Z$. The fact to be established is that $m$ is onto. Now we shall give a proof to this fact by using a unilateral shift $U_{+}$.

Theorem 2.1. Ext $O_{n}=Z$.

Proof. It suffices to show that $m$ is onto. For the sake of simplicity, we consider the case of $\mathrm{O}_{2}=\mathrm{C}^{*}\left(\mathrm{~S}_{1}, \mathrm{~S}_{2}\right)$. Let $V_{+}=U_{+} \oplus 0$ on $H=r a n S_{1} \oplus\left(r a n S_{1}\right)^{+}$. Let us put $P_{2}^{k}=$ $C^{*}\left(V_{+}{ }^{(k)} S_{1}, S_{2}\right)+K(H)$. Then it follows from the uniqueness theorem on $\mathrm{O}_{2}$ that $\mathrm{P}_{2}^{\mathrm{k}} / \mathrm{K}(\mathrm{H})$ is isomorphic to $\mathrm{O}_{2}$ via the
quotient map $\pi$. We define the extension ${ }^{\tau} k$ for each integer $k$ by $\tau_{k}\left(S_{1}\right)=\pi\left(V_{+}{ }^{(k)} S_{1}\right)$ and $\tau_{k}\left(S_{2}\right)=\pi\left(S_{2}\right)$. Since

$$
v_{k}=\left(\begin{array}{cc}
{ }^{\tau} k\left(S_{1}\right) & { }^{\tau} k\left(S_{2}\right) \\
0 & 0
\end{array}\right)=\left[\begin{array}{cc}
\pi\left(v_{+}^{(k)} S_{1}\right) & \pi\left(S_{2}\right) \\
0 & 0
\end{array}\right)
$$

we have $\pi\left(v_{k}\right)=v_{k}$ for $v_{k}=\left(\begin{array}{cc}v_{+}^{(k)} S_{1} & S_{2} \\ 0 & 0\end{array}\right)$ and

$$
\begin{aligned}
m\left(\tau_{k}\right)= & \operatorname{dim}\left(1-V_{k}{ }^{*} V_{k}\right)-\operatorname{dim}\left(P_{H}-V_{k} V_{k}^{*}\right) \\
= & \operatorname{dim}\left(P_{H}-\left(V_{+}(k) S_{1}\right) *\left(V_{+}(k) S_{1}\right)\right) \\
& -\operatorname{dim}\left(P_{H}-\left(V_{+}(k) S_{1}\right)\left(V_{+}(k) S_{1}\right) *\right) \\
= & -k .
\end{aligned}
$$

Hence this implies that $m$ is onto.

We remark that $P_{1}$ is the Coburn algebra and $C(T)$ is regarded as $O_{1}$. Putting $P_{1}^{k}=C^{*}\left(U_{+}^{(K)} U\right)+K(H)$, we can prove that Ext $C(T)=Z$. Actually, $P_{1}{ }^{k} / K(H)$ is isomorphic to $C(T)=C^{*}(U)$, where $U$ is a simple bilateral shift. If $\tau_{k}$ is the extension defined by $\left.\tau_{k}(U)=\pi\left(U_{+}{ }^{(k)}\right)_{U}\right)$, then $m\left(\tau_{k}\right)$ $=-k=$ ind $U_{+}{ }^{(k)}$. It is easily seen that $\left\{P_{n} k_{k}\right.$ is a complete set of representatives for extensions of $O_{n}$.

II-2. Extensions of $O_{A}$ - tensor representation.
Cuntz and Krieger 617J constructed a new C*-algebra ${ }^{O_{A}}$ which is associated with a topological Markov chain ( $X_{A}, \sigma_{A}$ ). Let $A=(A(i, j))$ be an $n \times n$ matrix such that $A(i, j)=0$ or 1 and every row and column is non-zero. A C*-algebra ${ }^{\circ} O_{A}$ is
generated by non-zero partial isometries $S_{1}, \ldots, S_{n}$ acting on a Hilbert space satisfying the condition
(A) $\quad S_{i}{ }^{*} S_{j}=0$ for $i \neq j$, and $S_{i}{ }^{*} S_{i}=\sum_{j} A(i, j) S_{j} S_{j}{ }^{*}$ for all i. They proved:

The uniqueness theorem. The isomorphism class of $O_{A}$ does not depend on the choice of generators if $A$ satisfies the condition (I), see Lemma 2.1. Furthermore, if $A$ is irreducible, i.e., it is not a permutation and for each $i$ and $j$ there is a $k$ such that $A^{k}(i, j)>0$, then $O_{A}$ is uniquely determined and is simple.

We attempt a graph theoretic approach to Cuntz-Kriegr algebras. A digraph $G$ is an ensemble of a finite set $V(G)$ of vertices $1,2, \ldots, n$ and a finite set $E(G)$ of edges which are ordered pairs (i, $j$ ) of vertices. It is known that a digraph $G$ is represented by an adjacency matrix $A$ with 0 and 1 as entries: $A(i, j)=1$ if $(i, j) \varepsilon E(G)$ and $A(i, j)$ $=0$ if not. Thus we identify a digraph with its adjacency matrix.

Now a path from $j$ to $i$ in $G$ is a finite sequence of edges $\left\{\left(i_{k-1}, i_{k}\right)\right\}$ such that $i_{1}=i$ and $i_{m}=j$. A vertex has an m-cycle if there is a path $\left\{\left(i_{k-1}, i_{k}\right)\right\}$ from $i$ to $i$ with $i_{k} \neq i$ for $2 \leq k \leq m-1$. Particularly, a 1-cycle is called a loop. We note that a vertex $i$ has at least two different cycles if and only if i $\varepsilon \varepsilon_{O}$, where ${ }^{\Sigma}{ }_{O}$ is refered to [17]. Hence the condition (I) of Cuntz and Krieger is
rephrased as follows:

Lemma 2.1 A digraph $G$ satisfies the condition (I) if and only if for each $i \in V(G)$ there is a path from $j$ to $i$ for some $j \in \varepsilon_{0}$.

This reformulation is very useful. For a digraph $G$ satisfying the condition (I), $O_{G}=O_{A}$ is unique up to isomorphism. In the below, we always assume that a matrix $A$ and a digraph $G$ satisfy the condition (I).

Now let $P_{n}$ be as in the above an extension of $O_{n}$ by the compacts. Evans [30] and Katayama showed independently that $P_{n}$ is realized as a 'tensor algebra' on the full Fock space $F(H)$, which is analogous to the construction of the CAR algebra on the anti-symmetric Fock space. Furthermore they constructed a unitary $F(u)$ on $F(H)$ for $u \in U(n)$. Then $\bar{\alpha}_{u}$ on $P_{n}$ implemented by $F(u)$ corresponds to the automorphism $\alpha_{u}$ on $O_{n}$ discussed in the preceding chapter.

In this section, we shall construct a subspace $L_{A}$ of $F(H)$ associated with a matrix $A$ and consider the $C^{*}$-algebra $P_{A}$ on $L_{A}$ generated by the compressions to $L_{A}$ of the creation operators on $F(H)$. We shall see that $P_{A}$ is an extension of $O_{A}$ by the compacts.

For an $n$-dimensional Hilbert space $H$, let $H_{m}=\otimes^{m} H$ be the $m$-fold tensor product and $F(H)=\Sigma_{m=0}^{\infty} \oplus H_{m}$ the Fock space,
where $H_{O}$ is the 1-dimensional Hilbert space spanned by the Fock vacuum unit vector $\Omega$. For $f \varepsilon H$, there is a bounded operator $O(f)$ on $F(H)$ such that

$$
o(f) \Omega_{0}=f, \quad o(f)\left(f_{1} \otimes \ldots \otimes f_{m}\right)=f \otimes f_{1} \otimes \ldots \otimes f_{m},
$$

and

$$
o(f) * \Omega=0, \quad o(f) *\left(f_{1} \otimes \ldots \otimes f_{m}\right)=\left(f_{1}, f\right) f_{2} \otimes \ldots \otimes f_{m} .
$$

Then the $C^{*}$-algebra generated by $\{O(f) ; f \varepsilon H\}$ is isomorphic to $P_{n}$ which is called the Clifford C*-algebra in [45].

Now we shall consider two subspaces of $F(H)$. Let $\left\{e_{1}\right.$, ... , $\left.e_{n}\right\}$ be an orthonormal basis of $H$. Let $L_{m}$ be the subspace of $H_{m}$ spanned by

$$
\left\{e_{i(1)} \otimes \ldots e_{i(m)} ; A(i(k), i(k+1))=1 \text { for } 1 \leq k \leq m-1\right\}
$$

and $L_{A}=\Sigma_{m=0}^{\infty} \mathrm{L}_{\mathrm{m}}$, where $\mathrm{L}_{\mathrm{O}}=\mathrm{H}_{\mathrm{O}}$ and $\mathrm{L}_{1}=\mathrm{H}_{1}=\mathrm{H}$. Let $\mathrm{M}_{\mathrm{m}}$ be the subspace of $H_{m}$ spanned by

$$
\left\{e_{i(1)} \otimes \ldots \otimes e_{i(m)} ; \pi \prod_{k=1}^{m-1} A(i(k), i(k+1))=0\right\}
$$

and $M_{A}=\Sigma_{m=0}^{\infty} \oplus M_{m}$, where $M_{O}=M_{1}=\{0\}$. Then $F(H)=L_{A} \oplus M_{A}$ and $L_{A}$ is called the sub-Fock space associated with $A$. Let us put $S_{i}=P_{L_{A}} o\left(e_{i}\right) \mid L_{A}$ for $1 \leq i \leq n$, where $P_{L_{A}}$ is the projection onto $L_{A}$. Then we denote by $P_{A}$ the $C^{*}$-algebra generated by $\left\{S_{i} ; 1 \leq i \leq n\right\}$.

Theorem 2.2. The $C^{*}$-algebra $P_{A}$ acts irreducibly on $L_{A}$ and contains the compacts $K\left(L_{A}\right)$. Moreover $P_{A}$ is an extension of $O_{A}$ by $K\left(L_{A}\right)$, that is,

$$
\begin{equation*}
0 \longrightarrow K\left(L_{A}\right) \longrightarrow P_{A} \longrightarrow O_{A} \longrightarrow 0 \tag{3}
\end{equation*}
$$

is exact.

To prove Theorem 2.2, we need following lemmas.

Lemma 2.3. Notation as in the above. Then each $S_{k}$ is partial isometry such that $\mathrm{S}_{\mathrm{k}}{ }^{*} \mathrm{~S}_{\mathrm{k}} \Omega=\Omega, \mathrm{S}_{\mathrm{k}} \mathrm{S}_{\mathrm{k}}{ }^{* \Omega}=0$,

$$
S_{k} * S_{k}\left(e_{i(1)} \otimes \cdots \otimes e_{i(m)}\right)=A(k, i(1)) e_{i(1)} \otimes \cdots \otimes e_{i(m)}
$$

and
$S_{k} S_{k}{ }^{*}\left(e_{i(1)} \otimes \cdots \otimes e_{i(m)}\right)=\delta(k, i(1)) e_{i(1)} \otimes \cdots \otimes e_{i(m)}$ for all $g=e_{i(1)} \otimes \ldots \otimes e_{i(m)} \in L_{A}$, where $\delta(k, i)$ is Kronecker's delta.

Proof. We put $P=P_{L_{A}}, P_{k}=S_{k} S_{k}{ }^{*}$ and $Q_{k}=S_{k}{ }^{*} S_{k}$ for $1 \leq k \leq n$. Then we have

$$
Q_{k} \Omega=P \circ\left(e_{k}\right) * P O\left(e_{k}\right) \Omega=P \circ\left(e_{k}\right) * P e_{k}=P o\left(e_{k}\right) * e_{k}=P \Omega=\Omega
$$

Since $O\left(e_{k}\right)^{*} \Omega=0$, it follows that $P_{k} \Omega=0$.
Next we have

$$
\begin{aligned}
Q_{k} e_{i} & =\operatorname{Po}\left(e_{k}\right) * \operatorname{Po}\left(e_{k}\right) e_{i}=\operatorname{Po}\left(e_{k}\right) * P\left(e_{k} \otimes e_{i}\right) \\
& =\operatorname{Po}\left(e_{k}\right) * A(k, i) e_{k} \otimes e_{i} \\
& =A(k, i) P e_{i}=A(k, i) e_{i},
\end{aligned}
$$

so that $Q_{k} g=A(k, i) g$ for $g \varepsilon L_{A}$. Since $o\left(e_{k}\right) * e_{i}=\delta(k, i) e_{i}$ we have $P_{k} e_{i}=\delta(k, i) e_{i}$, which implies $P_{k} g=\delta(k, i(1)) g$.

Lemma 2.4. If E is the projection onto $\mathrm{L}_{\mathrm{O}}=\mathrm{H}_{\mathrm{O}}$, then

$$
\begin{equation*}
S_{k}{ }^{*} S_{k}=\Sigma_{j} A(k, j) S_{j} S_{j}^{*}+E . \tag{4}
\end{equation*}
$$

Proof. Let $P_{k}$ and $Q_{k}$ be as in above. Then we have

$$
\left(\Sigma_{j} A(k, j) P_{j}+E\right) \Omega=\Omega=Q_{k} \Omega
$$

by Lemma 2.3. Next we have

$$
\begin{aligned}
& \left(\Sigma{ }_{j} A(k, j) P_{j}+E\right) e_{i}=\Sigma{ }_{j} A(k, j) P_{j} e_{i} \\
& \quad=\sum_{j} A(k, j) \delta(j, i) e_{i}=A(k, i) e_{i}=Q_{k} e_{i} .
\end{aligned}
$$

Hence it implies that $\left(\Sigma{ }_{j} A(k, j) P_{j}+E\right) g=Q_{k} g$ for $g \varepsilon L_{A}$.

Remark. It follows from (4) that $E$ is in $P_{A}$ and (5)

$$
\Sigma_{j} P_{j}+E=1 \quad \text { on } \quad L_{A}
$$

Proof of Theorem 2.2. First of all, we shall prove that $P_{A} x$ is dense in $L_{A}$ for all $0 \neq x \in L_{A}$. Since $x \neq 0$, there is $m$ such that the direct summand $X_{m}$ of $x$ on $L_{m}$ is nonzero. If $x_{m}=\Sigma x_{m}(i(1), \ldots, i(m)) e_{i(1)} \otimes \cdots e_{i(m)}$, where $\Sigma$ is taken over $(i(1), \ldots, i(m))$ such that $\prod_{k=1}^{m-1} A(i(k)$, $i(k+1))=1$, then there is $\mu=(i(1), \ldots, i(m))$ such that $x_{m}(\mu) \neq 0$. Since $S_{\mu}^{*} x=x_{m}(\mu) \Omega+y$ for some $y \varepsilon \sum_{h=1}^{\infty} \oplus L_{h}$, we have $\operatorname{ES}_{\mu} * \mathrm{X}=\mathrm{X}_{\mathrm{m}}(\mu) \Omega \neq 0$. Furthermore, for any $z=e_{j(1)^{\otimes}}$ $\cdots \otimes e_{j(h)}{ }^{\varepsilon} L_{A}$ we have

$$
x_{m}(\mu)^{-1} S_{j(1)} \cdots S_{j(h)}^{E S_{\mu}^{*} x=e_{j(1)} \otimes \ldots \otimes e_{j(h)}, ~}
$$

which implies that $P_{A} x$ is dense in $L_{A}$. Since $E$ is rank one, $P_{A}$ contains the compact operators $K\left(L_{A}\right)$.

Let $\pi$ be the quotient map of $B\left(L_{A}\right)$ onto $Q\left(L_{A}\right)$. Then $\pi(E)=0$. Noting that the range of $P_{i}$ is infinite dimensional by the condition (I), we put $T_{k}=\pi\left(S_{k}\right) \neq 0$ for $1 \leq k \leq n$. Then $\pi\left(P_{A}\right)$ is generated by partial isometries $T_{1}, \ldots, T_{n}$, and

$$
T_{k}^{*} T_{k}=\Sigma_{j} A(k, j) T_{j} T_{j}^{*} \quad \text { and } \quad \Sigma_{j} T_{j} T_{j}^{*}=1
$$

by (4) and (5). Therefore $\pi\left(\mathrm{P}_{A}\right)$ is isomorphic to $\mathrm{O}_{\mathrm{A}}$.

II-3. Extensions of $O_{A}$ - adjoint graphs.
According to [3], the adjoint graph $G *$ of a digraph $G$ is defined to be a digraph whose vertices $u_{1}, \ldots, u_{m}$ represent the edges of $G$ and which has an edge $u_{i} \leftarrow u_{j}$ if $i_{2}=$ $j_{1}$ where $u_{i}=\left(i_{1}, i_{2}\right)$ and $u_{j}=\left(j_{1}, j_{2}\right)$. In this section, we shall prove that $O_{G}=O_{G *}$. By using this, we shall give another extension of $\mathrm{O}_{\mathrm{A}}$.

First of all, we show an example of an adjoint graph:


Now we shall make sure that the adjoint graph of a digraph with the condition (I) satisfies (I) also.

Lemma 2.5. If a digraph $G$ satisfies the condition (I), then so does the adjoint $G^{*}$ of $G$.

Proof. It suffices to show that for each ( $\left.i_{1}, i_{2}\right) \varepsilon V\left(G^{*}\right)$ there is a vertex $(r, s) \varepsilon \varepsilon_{0}$ having a path $P\left(\left(i_{1}, i_{2}\right),(r, s)\right)$ in $G^{*}$. Since $G$ satisfies the condition (I), a vertex $i_{2}$ of $G$ has a path $P\left(i_{2}, i_{0}\right)$ for some $i_{O} \varepsilon \varepsilon_{O}$, which induces a path $P\left(\left(i_{1}, i_{2}\right),\left(i_{2}, i_{3}\right), \ldots,\left(i_{k-1}, i_{k}\right),\left(i_{k}, i_{O}\right)\right)$ in $G^{*}$. on the other hand, since $i_{O} \varepsilon \varepsilon_{0}$, there are two different cycles $E$ and $F$ in $G$ such that $i_{O} \varepsilon V(E) \cap V(F)$, so there is a path $P\left(\left(i_{k}, i_{0}\right),(r, s)\right)$. Hence there is a path $P\left(\left(i_{k}, i_{O}\right),(r, s)\right)$ and
(r,s) $\varepsilon \varepsilon_{0}$, that is, $G^{*}$ satisfies (I).

Now we shall realize edges of $G$ as partial isometries $T_{i, j}=S_{i} P_{j}$ in $O_{G}$, where $P_{j}=S_{j} S_{j}{ }^{*}$.

Lemma 2.6. Let $O_{A}=C^{*}\left(S_{1}, \ldots, S_{n}\right)$ and $T_{i, j}=S_{i} P_{j}$. then $T_{i, j}=0$ if and only if $A(i, j)=0$.

Proof. Note that $T_{i, j}=0$ iff $P_{j} S_{i}{ }^{*} S_{i} P_{j}=0$, or equivalently $S_{i} * S_{i} P_{j}=0$. If $A(i, j)=1$, then $S_{i}{ }^{*} S_{i}=$ $\sum_{k} A(i, k) P_{k} \geq P_{j}$. Therefore we have $S_{i}{ }^{*} S_{i} P_{j}=P_{j}$, so that $T_{i, j}=0$. Conversely, if $A(i, j)=0$, then

$$
S_{i}{ }^{S_{i} P_{j}}=\Sigma{ }_{k} A(i, k) P_{k} P_{j}=A(i, j) P_{j}=0 .
$$

Hence it implies $T_{i, j}=0$.

Theorem 2.7. $O_{G}{ }^{*}=O_{G}$.

Proof. Since ${ }^{O_{G}}$ coincides with the $C^{*}$-algebra $B$ generated by $\left\{T_{i, j} ; A(i, j)=1\right\}$, we shall show that $B$ is the Cuntz Krieger algebra $O_{G *}$, that is, a family $\left\{T_{i, j} ; A(i, j)=1\right\}$ satisfies the condition (A) and $\Sigma_{i, j} T_{i, j} T_{i, j}{ }^{*}=1$. By Lemma 2.6 we have
$\Sigma_{A(i, j)=1} T_{i, j} T_{i, j}^{*}=\Sigma_{i, j} S_{i} S_{j} S_{j}{ }^{*} S_{i}{ }^{*}=\Sigma{ }_{i} S_{i}\left(\Sigma{ }_{j} P_{j}\right) S_{i}{ }^{*}=1$.
Next, if $(i, j) \neq(p, q) \in E(G)$, then $i \neq p$ or $j \neq q$. If
$i \neq p$, then $T=T_{i, j} T_{i, j}{ }^{*} T_{p, q} T_{p, q}{ }^{*}=0$. On the other hand, if $i=p$ and $j \neq q$, then $T=S_{i} P_{j} S_{i}{ }^{*} S_{i} P_{q}{ }^{T} p, q{ }^{*}=S_{i} P_{j} P_{q}{ }^{T} p, q{ }^{*}=0$ because $P_{j} P_{q}=0$. Finally, by the definition of the adjoint
graph and Lemma 2.5 we have

$$
\begin{aligned}
& \Sigma_{(i, j)} A^{*}((p, q),(i, j)) T_{i, j} T_{i}, j^{*} \\
= & \Sigma_{A}(q, j)=1 A^{*}((p, q),(q, j)) S_{q} P_{j} S_{q}^{*} \\
= & \Sigma_{j} T_{q, j} S_{q}^{*}=S_{q}\left(\Sigma{ }_{j} P_{j}\right) S_{q}{ }^{*}=P_{q} . \\
\text { Since } \quad & T_{p, q}{ }^{*} T_{p, q}=P_{q} S_{p}{ }^{*} S_{p} P_{q}=P_{q}, \text { it follows that }\left\{T_{i, j} ;\right. \\
A(i, j)= & 1\} \text { satisfies the condition (A). }
\end{aligned}
$$

After [32], we say that a category $D(G)$ is the free category of a digraph $G$ if $D(G)$ is a category whose morphisms consist of all paths in $G$ and whose objects consist of $V(G)$. Let $s(g)$ be the source of $g \varepsilon D(G)$ and $t(g)$ the target of $g$. Let $l^{2}(D(G))$ be the Hilbert space of all square summable sequences on $D(G)$ with the orthonormal basis $\left\{e_{d}\right.$; $d \varepsilon D(G)\}$, where $e_{d}(g)=\delta_{d, g}$ for $g \varepsilon D(G)$. For each i $\varepsilon V(G)$ let $H_{i}$ be the subspace of $I^{2}(D(G))$ spanned by $\left\{e_{d} ; d \varepsilon\right.$ $D(G), S(d)=i f$. Now we shall define the left regular representation $u$ of $D(G)$ on $l^{2}(D(G))$. For each $g \varepsilon D(G)$, a partial isometry $u_{g}$ on $l^{2}(D(G))$ is defined by $u_{g} e_{h}=e_{g h}$ if $s(g)=t(h)$ and $u_{g} e_{h}=0$ if not. Let. $C^{*}{ }_{r}(G)$ denote the C*-algebra generated by $\left\{u_{g} ; g \varepsilon D(G)\right\}$. Since $u_{g}{ }^{*} u_{h}=e_{k}$ if $h=g k$ for some $k$ and $u_{g}{ }^{*} e_{h}=0$ if not, every $H_{i}$ is invariant under $C^{*}{ }_{r}(G)$. So, putting $\rho_{i}(a)=a \mid H_{i}$ for $a \varepsilon$ $C^{*}{ }_{r}(G)$ and $i \in V(G)$, then $\rho_{i}$ is a representation of $C^{*}{ }_{r}(G)$ on $H_{i}$ and $\oplus{ }_{i \varepsilon V(G)}{ }_{i}$ is the identity representation of $C^{*}{ }_{r}(G)$ on $I^{2}(D(g))$.
$C^{*}{ }_{r}(G)$ and $\rho_{i}\left(C^{*}{ }_{r}(G)\right)$ contains the compacts $K\left(H_{i}\right)$. Furthermore, if $G$ satisfies the condition (I), then

$$
0 \longrightarrow K\left(H_{i}\right) \longrightarrow o_{i}\left(C^{*}{ }_{r}(G)\right) \longrightarrow 0_{G} \longrightarrow 0
$$

is exact.

Proof. By the definition of $u_{g}$, we have $u_{g}{ }^{*} u_{g} e_{h}=e_{h}$ if $s(g)=t(h)$ and $u_{g}{ }^{*} u_{g} e_{h}=0$ if not, and $u_{g} u^{*}{ }^{*} e_{b}=e_{b}$ if $b=g h$ for some $h$ and $u_{g} u_{g}{ }^{*} e_{b}=0$ if not. Therefore $u_{g}{ }^{*} u_{g}$ (resp. $u_{g} u_{g}{ }^{*}$ ) is the projection on $\left[e_{h} ; s(g)=t(h)\right]$ (resp. [egh; $h \in D(G)]$ ), where $[M]$ denotes the subspace spanned by $M$. Since $P=1-{ }^{\Sigma} t \in E(G)^{u_{t}}{ }_{t}{ }^{*}$ is the projection on $\left[e_{j} ; j \varepsilon V(G) J\right.$, it follows that $\rho_{i}(P)$ is the projection $\left[e_{i}\right]$ for every $i \varepsilon V(G)$.

To show the irreducibility of $\rho_{i}$, we shall prove that $\rho_{i}\left(C^{*}{ }_{r}(G)\right) x$ is dense in $H$ for all non-zero $x \varepsilon H_{i}$. Let $x=\Sigma_{S}(b)=i x(b) e_{b} \varepsilon H_{i}$. Then there is $g \varepsilon D(G)$ such that $s(g)=i$ and $x(g) \neq 0$. Since $u_{g}{ }^{*} x=\Sigma x(g h) e_{h}$ where $\Sigma$ is taken over $h$ such that $s(h)=i$ and $s(g)=t(h)$, we have Pu $_{g}{ }^{*} x=x(g i) e_{i}=x(g) e_{i} \neq 0$. Moreover, if $k \in D(G)$ and $s(k)$ $=\mathrm{i}$, then

$$
\rho_{i}\left(u_{k}\right) \rho_{i}(P) \rho_{i}\left(u_{k}\right) * x=u_{k} P u_{k}^{*} x=x(g) u_{k} e_{i}=x(g) e_{k}
$$

Hence it follows that $\rho_{i}$ is irreducible on $H_{i}$. Since $\rho_{i}(P)$ is rank one, $\rho_{i}\left(C^{*}(G)\right)$ contains $K\left(H_{i}\right)$.

Let $\pi$ be the quotient map of $B\left(H_{i}\right)$ onto $Q\left(H_{i}\right)$. It is clear that $\pi \rho_{i}\left(C^{*}{ }_{r}(G)\right)$ is generated by partial isometries $\left\{T_{g} ; g \in E(G)\right\}$, where $T_{g}=\pi \rho_{i}\left(u_{g}\right)$. So we shall show that $C^{*}\left(T_{g} ; g \varepsilon E(G)\right)$ is the $C^{*}$-algebra $O_{G *}$. Since $\rho_{i}(P)$ is
rank one, we have $\Sigma T \mathrm{~T}_{\mathrm{g}} \mathrm{T}^{*}=1$. Furthermore $\rho_{\mathrm{i}}\left(u_{g}\right){ }^{*} \rho_{\mathrm{i}}\left(u_{g}\right)$ (resp. $\left.\rho_{i}\left(u_{g}\right) \rho_{i}\left(u_{g}\right)^{*}\right)$ is the projection on $\left[e_{h} ; s(g)=t(h)\right.$, $s(h)=i]$ (resp. [e ; $k=g h$ for some $h \varepsilon D(G)$ with $s(h)=$ i]). It follows that $T_{g}^{* T} \mathrm{~T}_{\mathrm{g}}=\Sigma_{h \in E(G)} A^{*}(\mathrm{~g}, \mathrm{~h}) \mathrm{T}_{\mathrm{h}} \mathrm{T}_{\mathrm{h}}{ }^{*}$, where $\mathrm{A}^{*}$ is the adjacency matrix of $G^{*}$. Hence it implies that $C^{*}\left(T_{g}\right.$; $g \varepsilon E(G))=O_{G *}$, so that $\rho_{i}\left(C^{*}{ }_{r}(G)\right) / K\left(H_{i}\right)=O_{G}$ by the preceding theorem.

## II-4. Applications to automorphisms on $\mathrm{O}_{\mathrm{A}}$.

In the first chapter, we have discussed a representation of the unitary group $U(n)$ into the outer automorphisms on $O_{n}$. Unfortunately, for general $O_{A}=C^{*}\left(T_{1}, \ldots, T_{n}\right)$ there are unitaries $u$ such that $\alpha_{u}\left(T_{i}\right)=\Sigma{ }_{k} u_{k i} T_{k}$ cannot be extended to automorphisms on $O_{A}$. For example, if $A=\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$, then $a_{u}$ can be extended to an automorphism on $O_{A}$ if and only if $u$ is diagonal. As applications of extensions in II-2, we shall characterize unitary matrices such that $\alpha_{u}$ can be extended to automorphisms on $\mathrm{O}_{\mathrm{A}}$.

Let $H$ be an n-dimensional Hilbert space with an orthonormal basis $\left\{e_{i}\right\}$. For each $u \in U(n)$, let us put $U_{0}=I$ on $H_{o}=H, U_{m}=x^{m} u$ on $H_{m}=x^{m} H$ for $m \geq 1$, and $F(u)=$ $\Sigma_{m}+U_{m}$ on the full Fock space $F(H)$. Then $F(u)$ is unitary. Evans and Katayama showed that $F(u)$ implements an automorphism $\bar{\alpha}_{u}$ on $P_{A}=C^{*}\left(o\left(e_{i}\right) ; 1 \leq i \leq n\right)$ such that

$$
\bar{\alpha}_{u}\left(o\left(e_{i}\right)\right)=F(u) o\left(e_{i}\right) F(u)^{*}=\varepsilon_{k} u_{k i} o\left(e_{k}\right)
$$

First of all, we shall consider a condition on $u \in U(n)$ such
that the sub-Fock space $L_{A}$ associated with $A$ reduces $F(u)$.

Lemma 2.9. Let $A=(A(i, j))$ be an $n x n$ matrix and $u$ $=\left(u_{i j}\right) \varepsilon U(n)$ such that $A(i, j)=0$ and $A(k, m)=1$ imply $u_{k i} u_{m j}=0$ for all $i, j, k, m$. Then the sub-Fock space $L_{A}$ associated with $A$ is reducing for $F(u)$.

Proof. It is obvious that $U_{0} M_{0}=U_{0}\{O\}=\{0\}$ and $U_{1} M_{1}=$

$$
\begin{aligned}
& U_{1}\{0\}=\{0\} \text {. If } e_{i} \otimes e_{j} \varepsilon M_{2}, i . e ., A(i, j)=0, \text { then } \\
& U_{2}\left(e_{i} \otimes e_{j}\right)=(u \otimes u)\left(e_{i} \otimes e_{j}\right)=\left(\Sigma_{k} u_{k i} e_{k}\right)\left(\Sigma{ }_{m} u_{m j} e_{m}\right) \\
& =\Sigma_{A(k, m)=0} u_{k i} u_{m j} e_{k} \otimes e_{m}+\Sigma_{A(k, m)=1} u_{k i} u_{m j} e_{k} \otimes e_{m} \\
& =\varepsilon_{A(k, m)=0} u_{k i} u_{m j} e_{k} \otimes e_{m} \varepsilon M_{2}
\end{aligned}
$$

by the assumption. Similarly we have $U_{m} M_{m} \subseteq M_{m}$ for $m \geq 3$.

Theorem 2.10. Let $O_{A}=C^{*}\left(T_{1}, \ldots, T_{n}\right)$. Then the following statements are equivalent for $u \varepsilon U(n)$;
(1) $\alpha_{u}\left(T_{i}\right)=\Sigma_{k} u_{k i} T_{i}$ can be extended to an automorphism on $O_{A}$,
(2) (1-A(i,j))A(k,m) $u_{k i} u_{m j}=0$ for all $i, j, k, m$, and
(3) $A(i, j)=0$ and $A(k, m)=1$ imply $u_{k i} u_{m j}=0$ for all i,j,k,m.

Proof. It is clear that (2) and (3) are equivalent by noting the case that $A(i, j)=0$ and $A(k, m)=1$. Suppose that (1) is hold. If $A(i, j)=0$, then $T_{i} T_{j}=0$, so that

$$
\begin{aligned}
0 & =\alpha_{u}\left(T_{i} T_{j}\right)=\alpha_{u}\left(T_{i}\right) \alpha_{u}\left(T_{j}\right)=\left(\Sigma_{k} u_{k i} T_{k}\right)\left(\Sigma_{m} u_{m j} T_{m}\right) \\
& =\Sigma_{k, m} u_{k i} u_{m j} T_{k} T_{m}=\Sigma_{A(k, m)=1} u_{k i} u_{m j} T_{k} T_{m} .
\end{aligned}
$$

Since $\left\{T_{k} T_{m} ; A(k, m)=1\right\}$ is Iinearly independent, we have $u_{k i} u_{m j}=0$ if $A(k, m)=1$, which implies (3).

Conversely, suppose that $u$ satisfies (3). By Lemma 2.9, a unitary $F(u) \mid L_{A}$ implements an automorphism $\gamma_{u}$ on $P_{A}$. For generators $S_{i}=P o\left(e_{i}\right) \mid L_{A}$ of $P_{A}$ where $P=P_{L_{A}}$, we have

$$
\begin{aligned}
Y_{u}\left(S_{i}\right) & =F(u) S_{i} F(u) *\left|L_{A}=F(u) P o\left(e_{i}\right) P F(u) *\right| L_{A} \\
& =P F(u) \circ\left(e_{i}\right) F(u) *\left|L_{A}=P \Sigma_{k} u_{k i} \circ\left(e_{k}\right)\right| L_{A} \\
& =\Sigma_{k} u_{k i} S_{k} .
\end{aligned}
$$

Since $\gamma_{u}\left(K\left(L_{A}\right)\right)=K\left(L_{A}\right), \gamma_{u}$ induces an automorphism $\alpha_{u}$ on $O_{A}$ such that $\alpha_{u}(\pi(X))=\pi\left(\gamma_{u}(X)\right)$ for $X \in P_{A}$ by Theorem 2.8, where $\pi$ is the quotient map of $P_{A}$ onto $O_{A}$ and $T_{i}=\pi\left(S_{i}\right)$. Moreover we have

$$
\alpha_{u}\left(T_{i}\right)=\alpha_{u}\left(\pi\left(S_{i}\right)\right)=\pi\left(\gamma_{u}\left(S_{i}\right)\right)=\pi\left(\Sigma_{k} u_{k i} S_{k}\right)=\Sigma_{k} u_{k i} T_{k}
$$

Corollary 2.11. Let $O_{A}=C^{*}\left(T_{i} ; 1 \leq i \leq n\right)$. Then $O_{A}=$ $O_{n}$ if and only if $\alpha_{u}$ can be extended to an automorphism on $O_{A}$ for all $u \in U(n)$.

Proof. Suppose that $\alpha_{u}$ can be extended to an automorphism on $O_{A}$ for all $u \varepsilon U(n)$. Then (2) in Theorem 2.10 holds true for all $u \varepsilon U(n)$. For $n \geq 3$, let $q$ be the matrix whose entries are $1 / n$, and $r=2 q-1$. Then we have

$$
(1-A(i, j)) A(k, m)=0
$$

for $1 \leq i, j, k, m \leq n$. Since $A(k, m)=1$ for some $k$ and $m$, it follows that $1-A(i, j)=0$ for all $i$ and $j$, so that $O_{A}=O_{n}$. If $n=2$, then we consider $r=\left(\begin{array}{cc}1 & -1 \\ 1 & 1\end{array}\right) / \sqrt{ } 2$.

Incidentally, we shall discuss outerness of automorphisms
on $O_{A}$. So we introduce a notion on vertices of digraphs. Vertices $i$ and $j$ of a digraph $G$ are equivalent if $A(i, k)$ $=A(j, k)$ and $A(k, i)=A(k, j)$ for all $k \varepsilon V(G)$. Typical examples are as follows;


Then 1 and 2 are equivalent.

Corollary 2.12. Let $G$ be a digraph with $n$ vertices such that $1, \ldots, m$ are equivalent. If $u=\left(u_{i j}\right)$ is a unitary matrix such that. $u_{i j}=\delta_{i j}$ for $m+1 \leq i, j \leq n$, then $\alpha_{u}$ can be extended to an automorphism on $O_{A}$. Furthermore, if $G$ is strongly connected, then ${ }^{\alpha} u$ is outer except $u=1$.

Proof. It suffices to show that $A(i, j)=1$ or $A(k, p)=$ 0 if $u_{k i} u_{p j}=0$. Note that $k=i$ or $1 \leq k, i \leq m$ if $u_{k i}$ $\neq 0$. So we must consider the following four cases; (i) $k=i$ and $p=j$, (ii) $k=i$ and $1 \leq p, j \leq m,(i i i) p=j$ and $1 \leq k, i \leq m$, and (iv) $1 \leq k, i \leq m$ and $1 \leq p, j \leq m$.
(i) implies that $A(i, j)=A(k, p)$. (ii) implies that $A(k, p)=A(i, p)$ and $A(i, p)=A(i, j)$ by the equivalence of $p$ and $j$. Similarly (iii) implies that $A(i, j)=A(k, p)$. Finally.(iv) implies that $A(k, p)=A(k, j)=A(i, j)$. Hence we have $A(i, j)=A(k, p)$ for all cases, so that $A(i, j)=1$ or $A(k, p)$ $=0$.

```
    Suppose that G is strongly connected. To prove that au
is outer, we may assume that }u\mathrm{ is diagonal and }\mp@subsup{u}{11}{}\not=1\mathrm{ . It
follows from [29; Remark in § 3] that }\mp@subsup{\alpha}{u}{}\mathrm{ is outer if 1 has
a loop. If 1 has no loop, then 1 has a q-cycle {(i ik-1, , i
such that i}\mp@subsup{i}{k}{}\not=\mp@subsup{i}{p}{}\mathrm{ for k f p and m+1 s i
sq-1. (See the above examples.) Since }\mp@subsup{u}{11}{}\not=1\mathrm{ and }\mp@subsup{u}{jj}{}=
for m+1 \leq j \leqn, it follows that }\mp@subsup{\alpha}{u}{}\mathrm{ is outer.
```

$$
\text { Chapter III K-theory for } O_{A}
$$

III-1. Prologue.
Cuntz and Krieger proved that the weak extension group Ext ${ }^{W} O_{A}$ is isomorphic to $Z^{n} /(1-A) Z^{n}$, the Bowen-Franks invariant for a subshift $\sigma_{A}$. And Cuntz $[14]$ showed that $K_{o}\left(O_{A}\right)$ is isomorphic to $Z^{n} /\left(1-t_{A}\right) Z^{n}$. In addition, it is known that $K_{o}(B)$ is realized as $B^{P} / \approx$ for any unital purely infinite simple $C^{*}$-algebra $B$, where $\approx$ is the von Neumann equivalence among the non-zero projections $B^{P}$ in $B$, so that we identify the corresponding class in $K_{o}$-group with the von Neumann equivalence class $I P I \approx$ of $P \in B^{P}$. Moreover $O_{A}$ is unital, purely infinite and simple for irreducible $A$. We here remark that $A$ is irreducible if and only if the corresponding digraph $G$ of $A$ is strongly connected, i.e., for any vertices $i \neq j$ of $G$ there are paths $P(i, j)$ and $P(j, i)$.

Now we shall introduce a new invariant for unital $C^{*}$-algebras: Let $B$ be a $C^{*}$-algebra with unit 1 . Then $\mathbb{I 1 I}$ stands for the corresponding class in $K_{o}(B)$ for 1 . For $g, h \in K_{o}(B)$ we write $g \sim h$ if $g=\alpha(h)$ for some automorphism $\alpha$ of $K_{o}(B)$. Putting $K_{o}(B)^{-}=K_{o}(B) / \sim$, the marker of $B$ is the equivalence class $\mathbb{I I \mathbb { I } ^ { - }}$ of $\mathbb{I} 1 \mathbb{1}$. In particular, since $K_{o}(B)$ is identified with $B^{P} / \approx$ for a unital purely infinite simple C*-algebra $B$, we have $\operatorname{mark}(B)=\llbracket 1 \mathbb{1}_{\approx}$.

The following theorem is evident but very important:

Theorem 3.1. If $B$ and $C$ are unital $C^{*}$-algebras which are isomorphic, then

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K
```

Here we show simple examples to apply Theorem 1.1.

Example 3.1.1. Let $A=\left(\begin{array}{lll}1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0\end{array}\right)$ and $B=t_{A}$, whose corre sponding digraphs are as follows;


It is easily seen that $G$ and $H$ are strongly connected and $K_{O}\left(O_{G}\right)=K_{O}\left(O_{H}\right)=Z_{2}=(Z / 2 Z)$. Therefore $K_{O}\left(O_{G}\right)^{-}=K_{O}\left(O_{H}\right)^{-}=$ $Z_{2}{ }^{-}=\{\overline{0}, \overline{1}\}$. Since $G$ and $H$ are strongly connected, these C*-algebras are unital purely infinite and simple. If $O_{G}=$ $C^{*}\left(S_{1}, S_{2}, S_{3}\right)$ and $P_{i}=S_{i} S_{i}$ for $i=1,2,3$, then

$$
\left(\begin{array}{l}
\mathbb{P} P_{1} \mathbb{I} \\
\mathbb{I} P_{2} \mathbb{I} \\
\mathbb{I P} P_{3} \mathbb{I}
\end{array}\right)=\left(\begin{array}{lll}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 0 & 0
\end{array}\right)\left(\begin{array}{l}
\mathbb{I} P_{1} \mathbb{I} \\
\mathbb{I P} P_{2} \mathbb{I} \\
\mathbb{I P} P_{3} \mathbb{I}
\end{array}\right)=\left(\begin{array}{l}
\mathbb{I} P_{1} \mathbb{I}+\mathbb{I} P_{2} \mathbb{I}+\mathbb{P} P_{3} \mathbb{I} \\
\mathbb{I} P_{1} \mathbb{I}+\mathbb{I} P_{2} \mathbb{I}+\mathbb{I} P_{3} \mathbb{I} \\
\mathbb{I} P_{1} \mathbb{I}
\end{array}\right)
$$

so that $\mathbb{I} \mathbb{I}=\mathbb{I} P_{1} \mathbb{I}=\mathbb{I} P_{2} \mathbb{I}$ and $\mathbb{I} P_{3} \mathbb{I}=\mathbb{I} P_{1} \mathbb{I}$. Hence $\mathbb{I} 1 \mathbb{I}$ must be a generator of $Z_{2}$, that is, $\operatorname{mark}\left(O_{G}\right)=\overline{1}$. On the other hand, we have $\operatorname{mark}\left(O_{H}\right)=\bar{O}$. Actually $\mathbb{I} 1 \mathbb{I}=\mathbb{L} P_{1} \mathbb{I}$ and $\mathbb{I} P_{1} \mathbb{I}$ $+\mathbb{I P} P_{2} \mathbb{I}=\mathbb{I P} P_{2} \mathbb{I}$, so that $\mathbb{I} \mathbb{I}=\mathbb{I} P_{1} \mathbb{I}$ is neutral in $K_{o}\left(O_{H}\right)$. By Theorem 3.1, $\mathrm{O}_{\mathrm{G}}$ and $\mathrm{O}_{\mathrm{H}}$ are non-isomorphic.

Next we shall consider the case that $K_{o}(B)=Z$.

Example 3.1.2. For each non-negative integer $n \varepsilon Z^{-}$, there is a Cuntz-Krieger algebra $O_{G}$ such that $K_{o}\left(O_{G}\right)=Z$ and $\operatorname{mark}\left(O_{G}\right)=n$.

For $n \geq 1$, let $A(n)$ be a matrix with degree $n+4$;

$$
A(n)=\left(\begin{array}{llllll}
1 & 1 & 1 & 1 & & \\
1 & 1 & 0 & 0 & & \\
1 & 0 & 1 & 1 & & \\
0 & 1 & 0 & 1 & \ddots & \\
& \vdots & & \ddots & \\
& \vdots & & \ddots & \\
1 & 0 & & \ddots & 1 \\
1 & & & 1
\end{array}\right) \quad \text { and } \quad A(0)=\left(\begin{array}{lll}
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{array}\right) .
$$

Then

$$
K_{o}\left(O_{G(n)}\right)=Z \text { and } \operatorname{mark}\left(O_{G(n)}\right)=n \text { for } n \geq 0
$$

III-2. Transfered graphs.
In order to classify simple Cuntz-Krieger algebras $O_{A}$ (for $3 \times 3$ matrices $A$ ), we shall introduce transfered graphs of digraphs. First of all, we begin with the following simplest example:

Example 3.2.1. Let $\mathrm{O}_{2}=\mathrm{C}^{*}\left(\mathrm{~S}_{1}, \mathrm{~S}_{2}\right), \mathrm{T}_{1}=\mathrm{S}_{1}$ and $\mathrm{T}_{2}=$ $\mathrm{S}_{2} \mathrm{~S}_{1}{ }^{*}$. Then $\mathrm{C}^{*}\left(\mathrm{~T}_{1}, \mathrm{~T}_{2}\right)$ is isomorphic to $\mathrm{O}_{\mathrm{B}}$, where $\mathrm{B}=\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$ and $C *\left(T_{1}, T_{2}\right)=C^{*}\left(S_{1}, S_{2}\right)$ as a set, that is, $O_{2}=O_{B}$.

The above example inspires us the following definition. Let $G$ be a digraph and $\Gamma^{-}(i)=\{j \varepsilon V(G) ; j \longrightarrow i\}$, where $i \longrightarrow j$ stands for the edge (i,j).

Definition 3.2. Suppose that $\Gamma^{-}(k)=r^{-}(m)$ for some $k \neq$ $m \in V(G)$. Then the transfered graph $H=G(k \rightarrow m)$ from $k$ to $m$ is defined by $V(H)=V(G)$ and $E(H)=(E(G) \backslash f(m, i) \varepsilon E(G)$; i $\varepsilon V(G)\}) \cup\{(m, k)\}$, that is, take away all edges whose targets are $m$ and add the edge $k \rightarrow m$.

The adjacency matrix $B$ of $G(k \rightarrow m)$ is determined as
follows: Let $A_{i}$ be the i-th row vector of $A$. Then $r^{-}(k)=$ $r^{-}(m)$ means $A_{k}=A_{m}$. We then put

$$
B(i, j)= \begin{cases}A(i, j) & \text { for } i \neq m, \\ \delta_{k, j} & \text { for } i=m .\end{cases}
$$

For the sake of convenience, we denote it by

$$
A \underset{A_{k} \longrightarrow A_{m}}{\Longrightarrow} B .
$$

So Example 3.2.1 is changed in the following form;

Example 3.2.2.


In general, we obtain that the transfered graph preserves isomorphisms between Cuntz-Krieger algebras.

Theorem 3.3. Let $H=G(k \rightarrow m)$ be the transfered graph of a digraph $G$ from $k$ to $m$. Then $O_{H}$ is isomorphic to $O_{G}$.

Proof. Let $A$ and $B$ be the adjacency matrices of $G$ and $H$ respectively, and $O_{A}=C *\left(S_{1}, \ldots, S_{n}\right)$. Since $k \neq m$
by definition, we assume that $k=2$ and $m=1$. Now we put
$T_{1}=S_{1} S_{2} *$ and $T_{i}=S_{i}$ for $2 \leq i \leq n$. Since $A_{1}=A_{2}$ where $A_{i}$ is the i-th row vector of $A$, we have $S_{1} * S_{1}=S_{2} * S_{2}$ and so

$$
T_{1} T_{1}^{*}=S_{1} S_{2}^{*} S_{2} S_{1}^{*}=S_{1}\left(S_{1} * S_{1}\right) S_{1}{ }^{*}=S_{1} S_{1}{ }^{*},
$$

so that $T_{i} T_{i}{ }^{*}=S_{i} S_{i}{ }^{*}$ for $I \leq i \leq n$. Furthermore, since
$B(i, j)=A(i, j)$ for $i \neq 1$ and $B(1, j)=\delta_{2, j}$, we rave

$$
\begin{aligned}
T_{1}{ }^{*} T_{1} & =S_{2} S_{1} * S_{1} S_{2}^{*}=S_{2} S_{2}{ }^{*} S_{2} S_{2}^{*}=S_{2} S_{2}^{*}=\Sigma_{j} \delta_{2, j} S_{j} S_{j}^{*} \\
& =\varepsilon_{j} B(1, j) S_{j} S_{j}^{*}=\Sigma_{j} B(1, j) T_{j} T_{j}^{*},
\end{aligned}
$$

and for $2 \leq i \leq n$

$$
T_{i}{ }^{*} T_{i}=S_{i}{ }^{*} S_{i}=\varepsilon_{j} A(i, j) S_{j} S_{j}^{*}=\varepsilon_{j} B(i, j) T_{j} T_{j}^{*} .
$$

Hence $C^{*}\left(T_{1}, \cdots, T_{n}\right)$ is isomorphic to the Cuntz-Krieger algebra $O_{B}$. It is clear that $C *\left(T_{1}, \ldots, T_{n}\right) \subseteq C^{*}\left(S_{1}, \ldots, S_{n}\right)$. On the other hand, since

$$
T_{1} T_{2}=S_{1} S_{2} * S_{2}=S_{1} S_{1} * S_{1}=S_{1},
$$

we have $C^{*}\left(S_{1}, \ldots, S_{n}\right)=C *\left(T_{1}, \ldots, T_{n}\right)$ as a set. Since $C^{*}\left(S_{1}, \ldots, S_{n}\right)$ does not depend on the choice of generators, $C^{*}\left(T_{1}, \ldots, T_{n}\right)$ is isomorphic to $O_{B}$.

Next we shall generalize the above transfered graph of a digraph.

Definition 3.4. Let $A$ be an $n \times n$ matrix, and $E_{i}=$ $\left(0, \ldots, 0, \frac{1}{1}, 0, \ldots, 0\right)$ for $1 \leq i \leq n$. Suppose that

$$
A_{p}=E_{k(1)}+\cdots+E_{k(r)}+A_{m(1)}+\cdots+A_{m(s)}
$$

for some $k(1), \ldots, k(r), m(1), \ldots, m(s)$ which are mutually different and $p \notin\{m(1), \ldots, m(s)\}$. Then an $n \times n$ matrix $B$ is defined by

$$
B(i, j)= \begin{cases}A(i, j) & \text { for } i \neq p \\ 1 & \text { for } i=p \text { and } j \in\{k(1), \ldots, k(r), m(1), \ldots, m(s)\} \\ 0 & \text { otherwise },\end{cases}
$$

and $B$ is called to be primitively transfered from $A$, in symbol, $A \xlongequal[\text { prim }]{ } B$, or more precisely,
(*)
$\xrightarrow[E_{k(1)}+\cdots+E_{k(r)}+A_{m(1)}+\cdots+A_{m(s)} \longrightarrow A_{p}]{ }$

The primitive transformation ' $\underset{\text { prim }}{ }$ ' generates the following equivalence relation which is called the primitive equivalence; $A_{\text {prim }} \quad B$ if and only if there are matrices $C_{1}$, ... , $C_{q}$ such that


Example 3.2.3.


Here we have a generarization of Theorem 3.3.

Theorem 3.5. If $A$ is primitively equivalent to $B$, then $O_{A}$ is isomorphic to $O_{B}$.

Proof. We assume (*) and $p=1$. Let $O_{A}=C^{*}\left(S_{1}, \ldots, S_{n}\right)$, $P_{i}=S_{i} S_{i}{ }^{*}$ and $Q_{i}=S_{i}{ }^{*} S_{i}$. Then we put

$$
T_{1}=S_{1}\left(P_{k(1)}+\cdots+P_{k(r)}+S_{m(1)}{ }^{*}+\cdots+S_{m(s)}{ }^{*}\right)
$$

and $T_{i}=S_{i}$ for $i \neq 1$. Since

$$
A_{1}=E_{k(1)}+\cdots+E_{k(r)}+A_{m(1)}+\cdots+A_{m(s)}
$$

it follows that

$$
Q_{1}=P_{k(1)}+\cdots+P_{k(r)}+Q_{m(1)}+\cdots+Q_{m(s)} .
$$

Then $\left\{P_{k(i)}, Q_{m(j)} ; 1 \leq i \leq r, 1 \leq j \leq s\right\}$ is a family of orthogonal projections. Furthermore, since $k(1), \ldots, k(r)$, $m(1), \ldots, m(s)$ are mutually different, a family $\mathcal{I P}_{\mathrm{k}(\mathrm{i})}, \mathrm{P}_{\mathrm{m}(\mathrm{j})}$ ; $1 \leq i \leq r, 1 \leq j \leq s\}$ is orthogonal. Hence we have

$$
\mathrm{T}_{1} \mathrm{~T}_{1} *=\mathrm{S}_{1}\left(\mathrm{~S}_{1} * \mathrm{~S}_{1}\right) \mathrm{S}_{1},
$$

so that $T_{i} T_{i}^{*}=P_{i}$ for $1 \leq i \leq n$. On the other hand,

$$
T_{i}{ }^{*} T_{i}=\Sigma_{j} B(1, j) P_{j}=\varepsilon_{j} B(1, j) T_{j} T_{j} *
$$

and for $i \neq 1$

$$
T_{i}{ }^{*} T_{i}=Q_{i}=\varepsilon_{j} A(i, j) P_{j}=\varepsilon_{j} B(i, j) T_{j} T_{j}{ }^{*}
$$

Therefore $C^{*}\left(T_{1}, \ldots, T_{n}\right)$ is isomorphic to $O_{B}$. It is clear that $C^{*}\left(T_{1}, \ldots, T_{n}\right) \subseteq C^{*}\left(S_{1}, \ldots, S_{n}\right)$. Since $1 \neq m(j)$ for $1 \leq j \leq s$ by definition, we have

$$
\begin{aligned}
& T_{1}\left(T_{k(1)^{T}} T_{\left.k(1)^{*}+\cdots+T_{k(r)} T_{k(r)}+T_{m(1)^{+}}+\cdots+T_{m(s)}\right)}^{=S_{1}\left(P_{k(1)}+\cdots+P_{k(r)}+Q_{m(1)}+\cdots+Q_{m(s)}\right)}\right. \\
& =S_{1}\left(S_{1}{ }^{*} S_{1}\right) \\
& =S_{1}
\end{aligned}
$$

so that $C^{*}\left(T_{1}, \ldots, T_{n}\right)=C^{*}\left(S_{1}, \ldots, S_{n}\right)$ as a set. Hence it implies that $O_{A}$ is isomorphic to $O_{B}$. Since $O_{A}$ does not depend on the choice of generators, $O_{B}$ does not depend on them either.

III-3. Classifications of ${ }^{0}$.
Now we shall classify Cuntz-Krieger algebras $O_{A}$ for $3 \times 3$ irreducible matrices $A$ and pose a classification table expressed by the corresponding digraphs.

Theorem 3.6. Let $A$ and $B$ be $3 \times 3$ irreducible matrices. Then the followings are equivalent:
(1) $O_{A}$ is isomorphic to $O_{B}$,
(2) $\mathrm{K}_{\mathrm{O}}\left(\mathrm{O}_{\mathrm{A}}\right)=\mathrm{K}\left(\mathrm{O}_{\mathrm{B}}\right)$ and $\operatorname{mark}\left(\mathrm{O}_{\mathrm{A}}\right)=\operatorname{mark}\left(\mathrm{O}_{\mathrm{B}}\right)$, and
(3) $A$ is primitively equivalent to $B$.

Proof. By theorems 3.1 and 3.3, it suffices to show that (2) implies (3). By using a computer, we listed up all strongly connected digraphs with 3 vertices satisfying the condition (I). (Note that $A$ is irreducible if and only if the corresponding digraph $G$ of $A$ is strongly connected.) Then these digraphs are classified by $K_{0}$ and marker of $O_{A}$, which are shown in the following classification table. The final step of the proof is to show that digraphs with the same $K_{0}$ and marker in the table are primitively equivalent. This can be checked one by one.

The classification table of $C_{A}$ for $3 \times 3$ irreducible matrices.


The case that $K_{0}\left(O_{G}\right)=0$.


The case that $K_{0}\left(O_{G}\right)=Z_{2}$ and $\operatorname{mark}\left(O_{G}\right)=\overline{0}$.


The case that $K_{o}\left(O_{G}\right)=Z_{2}$ and $\operatorname{mark}\left(O_{G}\right)=\overline{1}$.


The case that $K_{0}\left(O_{G}\right)=Z_{3}$ and $\operatorname{mark}\left(O_{G}\right)=\overline{1}$.


The case that $K_{O}\left(O_{G}\right)=Z$ and $\operatorname{mark}\left(O_{G}\right)=\overline{0}$.


This completes the proof except to determine the representatives, which will be done in the following sections 4 and 5.

Remark. In the first chapter, we have discussed fixed point algebras of periodic automorphisms on $O_{n}$ and determined the one of the 'flip-flop' automorphism $\theta$ on $O_{2}$ considered by Archbold. Now the fixed point algebra $C *\left(S_{1}, S_{2}{ }^{2}, S_{2} S_{1} S_{2}{ }^{*}\right)$ of $\theta$ is the Cuntz-Krieger algebra $O_{A}$ such that $A(3,1)=0$ and $A(i, j)=1$ for otherwise $i, j$. Therefore it follows from the classification table that $\mathrm{K}_{\mathrm{O}}\left(\mathrm{O}_{\mathrm{A}}\right)=0$. On the other hand, it is known that the fixed point algebra of $\mathrm{O}_{2}$ under $\alpha_{-1}$. is isomorphic to $O_{4}$, so that $Z_{3}$ is its $K_{0}$-group. Hence they are not conjugate.

III-4. Tensor products of $0_{A}$ by matrix algebras.
Paschke and Salinas [38] studies the tensor product of $O_{n}$ by the matrix algebra $M_{k}$, and proved that $O_{n}$ and $O_{n} \otimes M_{k}$ are non-isomorphic if $k$ and $n-1$ are relatively prime. In this section, we shall investigate transferences of markers under the tensor product by $M_{k}$. Let us define $k \cdot x^{-}$by $(k x)^{-}$ for $x^{-} \in K_{o}(B)^{-}$and an integer $k$. (Since $k x=x+\ldots+x$ ( $k$ times), $k \cdot x^{-}$does not depend on representatives of $x^{-}$, that is, $k \cdot x^{-}$is well-defined.)

Theorem 3.7. For a unital $C^{*}$-algebra $B, \operatorname{mark}\left(B \otimes M_{k}\right)=$ $k \cdot \operatorname{mark}(B)$.

Proof. Note that $M_{k} \otimes K(H)=K\left(\sum_{1}^{k} \oplus H_{j}\right)$ is spatially isomorphic to $K(H)$ by an isomorphism $\phi^{-1}$, where $H_{i}=H$. Let $e$ be a one-dimensional projection in $K(H)$. Then $\phi(e)$ is one-dimensional, so that we may assume that $\phi(e)=e \oplus 0 \oplus$ ... $\oplus$ - Since
$\mathbb{I} 1 \otimes(1 \oplus \ldots \oplus 1) \otimes \mathbb{I} \approx=\mathbb{I} 1 \otimes(\mathrm{e} \oplus \ldots \oplus \mathrm{e}) \mathbb{I} \approx$

$$
\begin{aligned}
& =\mathbb{I} \otimes(\mathrm{e} \oplus \mathrm{O} \oplus \ldots \oplus \mathrm{C}) \mathbb{I} \approx+\ldots+\mathbb{H} 1 \otimes(\mathrm{O} \oplus \ldots \oplus \mathrm{O} \oplus \mathrm{e} \mathbb{I} \approx \\
& =\mathrm{k} \cdot \mathbb{I} 1 \otimes \phi(e) \mathbb{I} \approx=\mathrm{k} \cdot \mathbb{I} 1 \otimes \mathrm{e} \mathbb{I} \approx,
\end{aligned}
$$

we have $\operatorname{mark}\left(B \otimes M_{k}\right)=k \cdot \operatorname{mark}(B)$.

Corollary 3.8. If $K_{0}(B)=Z_{n}$ and $\operatorname{mark}(B)=\overline{1}$, then $\operatorname{mark}\left(B \otimes M_{k}\right)=\bar{k}$ for $2 \leq n \leq \infty$, where $Z_{\infty}=Z$.

Corollary 3.9. (Paschke-Salinas) If $k$ and $n-1$ are not
relatively prime, then $O_{n} \otimes M_{k}$ and $O_{n}$ are non-isomorphic.

Proof. It is known that $K_{0}\left(O_{n}\right)=Z_{n-1}$ and the equivalence class of 1 is a generator of $K_{o}\left(O_{n}\right)$, [15; 3.7]. Note that $k$ is a generator of $Z_{m}$ if and only if $\alpha(1)=k$ for some $\alpha$ E Aut $Z_{m}$, i.e., $\overline{1}=\bar{k}$. Hence we have $\operatorname{mark}\left(O_{n}\right)=\overline{1}$. Suppose that $O_{n} \otimes M_{k}$ is isomorphic to $O_{n}$. Since $\overline{\mathrm{k}}=\operatorname{mark}\left(\mathrm{O}_{\mathrm{n}} \otimes \mathrm{M}_{\mathrm{k}}\right)=\operatorname{mark}\left(\mathrm{O}_{\mathrm{n}}\right)=\overline{1}$,
$k$ is a generator of $Z_{n-1}$. Therefore there is an integer $j$ such that $j k=1 \bmod n-1$. Furthermore, since $j k+a(n-1)=$ 1 for some $a \varepsilon Z, k$ and $n-1$ are relatively prime.

Remark. We point out that $O_{A} \otimes M_{K}$ is also a CuntzKrieger algebra. Actually, since $O_{A} \otimes M_{k}$ is generated by

$$
\left.i\left(\begin{array}{l}
S_{i} \\
\end{array}\right),\left(\begin{array}{l}
0 \\
P_{i} \\
\end{array}\right), \ldots,\left(\begin{array}{l} 
\\
P_{i} 0
\end{array}\right) ; 1 \leq i \leq n \cdot\right\}
$$

we have

$$
\mathrm{B}=\left(\begin{array}{llll}
0 & & & \mathrm{~A} \\
1 & \ddots & & \\
1 & \ddots & & \\
& \ddots & \\
& \ddots & \\
& & & 1
\end{array}\right)
$$

As another application, we consider the inclusion among Cuntz algebras $O_{n}$.

Theorem 3.10. $O_{m}$ is included in $O_{n}$ containing the unit if and only if $m=n+(n-1) k$ for some integer $k \geq 0$.

Proof. Assume that $O_{n}$ includes $O_{m}$. Then it follows from [13; Remark 7] that $m \leq n$. Furthermore, we have $n \mathbb{I I I I}=$ $\mathbb{C 1} \mathbb{I}=\mathbb{M} \mathbb{1} \mathbb{1}$ in $K_{0}\left(O_{n}\right)$, so that $(m-n) \mathbb{I} \mathbb{I}$ is the neutrai element in $K_{0}\left(O_{n}\right)=Z_{n-1}$. Then $m-n=k(n-1)$ for some $k$. We prove the converse by induction. Let $O_{n}=C *\left(S_{1}, \ldots\right.$, $S_{n}$ ). The case $k=0$ is trivial. If $m=n+(n-1)$, then we put. $T_{j}=S_{1} S_{j}$ for $j=1, \ldots, n$ and so $C^{*}\left(T_{j}, S_{k} ; 1 \leq j\right.$ $\leq n, 2 \leq k \leq n)$ is isomorphic to $O_{m}$. Next, if $m=n+2(n-I)$ then we put $U_{i}=T_{1} S_{i}$ for $i=1, \ldots, n$ and also $C *\left(U_{i}\right.$, $T_{j}, S_{k} ; 1 \leq i \leq n, 2 \leq j, k \leq n$ ) is isomorphic to $O_{m}$. We can construct $C^{*}$-algebras isomorphic to $O_{m}$ in such a way.

III-5. Explosions of digraphs.
The adjoint graph $G^{*}$ of a digraph $G$ is defined to be a digraph whose vertices $u_{1}, \ldots, u_{m}$ represent the edges of $G$ and which has an edge $u_{i} \longleftarrow u_{j}$ if $i_{2}=j_{1}$, where $u_{i}=$ $\left(i_{1}, i_{2}\right)$ and $u_{j}=\left(j_{1}, j_{2}\right)$. We shall generalize the adjoint of a digraph in order to determine completely the repesentatives in the preceding classification of $O_{A}$. This process will be called explosion.

Definition 3.11. Let $G$ be a digraph. Suppose that the number of $\Gamma^{-}(i)$ is greater than 2 for some $i \varepsilon V(G)$. (For simplicity, assume that $i=1$.) Decompose $r^{-}(1)=V U W$ such that $1 \in V$ if $1 \varepsilon \Gamma^{-}(1)$. Then the explosion $H$ of $G$ at 1 (with respect to $V$ and $W$ ) is defined as follows;

$$
\begin{aligned}
V(H)= & (V(G) \backslash\{1\}) \cup\left\{v_{0}, w_{0}\right\}, \text { and } \\
E(H)= & (E(G) \backslash\{(1, j),(k, 1) ; j, k \varepsilon V(G)\}) \\
& \cup\left\{\left(v_{0}, V\right),\left(w_{0}, w\right) ; V \varepsilon V \backslash\{1\}, w \in W\right\} \\
& \cup\left\{\left(i, v_{0}\right),\left(i, w_{0}\right) ;(i, 1) \varepsilon E(G)\right\},
\end{aligned}
$$

and if $1 \in r^{-}(1),\left\{\left(v_{0}, v_{0}\right),\left(v_{0}, w_{0}\right)\right\}$ is added to the set on the right hand side. This operation is called as explosion, and every digraph obtained by repeating explosions is called an explosion of $G$.

Example 3.5.1. Let $G$ be a digraph;


Then the explosion $H$ of $G$ at 1 is the following;


Moreover, it is easily seen that the explosion of $H$ at 3 is the adjoint $G^{*}$ of $G$.

More generally, it is obvious that the adjoint of a digraph $G$ is an explosion of $G$, and the adjoint operation multiplies the number of vertices.

Now, by using explosions, we can increase the number of vertices by one. As an application, the classification problem of $O_{A}$ for $n \times n$ matrices $A$ is included in one of $O_{B}$ for $(n+1) \times(n+1)$ matrices $B$ by the following theorem, whose idea is the same as Theorem 2.7.

Theorem 3.12. If a digraph $H$ is an explosion of a digraph $G$, then $O_{H}$ is isomorpnic to $O_{G}$.

Proof. We may assume that $H$ is the explosion of $G$ at 1 and $\Gamma^{-}(1)=V \cup W$ such as $1 \varepsilon V$ if $1 \varepsilon \Gamma^{-}(1)$. Let $O_{G}$ $=C^{*}\left(S_{1}, \ldots . S_{n}\right), \quad P_{i}=S_{i} S_{i}^{*}$ and $P_{Y}=\varepsilon_{i \varepsilon Y} P_{i}$. Then, if we put $T_{V}=S_{1} P_{V}, T_{W}=S_{1} P_{W}$ and $T_{k}=S_{k}$ for $2 \leq k \leq n$, then we have

$$
T_{V} V_{V}^{*}+T_{W} T_{W}^{*}=S_{1}\left(P_{V}+P_{W}\right) S_{1}^{*}=S_{1}\left(S_{1}^{*} S_{1}\right) S_{1} *=P_{1}
$$

so that

$$
T_{V} T_{V}^{*}+T_{W} T_{W}^{*}+\sum_{k=2}^{n} T_{k} T_{k}^{*}=\sum_{k} P_{k}
$$

Furthermore, since $T_{V}{ }^{*} T_{V}=P_{V}$ and $T_{W}{ }^{*} T_{W}=P_{W}$, the family of partial isometries $T_{V}, T_{W}, T_{2}, \ldots, T_{n}$ satisfies the condition (A). Hence the $C^{*}$-algebra generated by them is the Cuntz-Krieger algebra $O_{H}$ and is included in $O_{G}$. Since $S_{1}=$ $T_{V}+T_{W}$, it coincides with $O_{G}$, so that $O_{H}$ is isomorphic to ${ }^{0}{ }_{G}$.

The following corollary shows that there are many cuntsKrieger algebras isomorphic to $\mathrm{O}_{2}$.

degree $n$, then ${ }^{O} C(n)$ is isomorphic to $O_{2}$.

Proof. The case of $n=2$ is Example 3.1.1. Consider the adjoint graph of $C(n)$ and its transfered graph inductively;


Theorems 3.12 and 3.3 implies that ${ }^{O_{C(n)}}$ is isomorphic to $O_{2}$.

Concluding this section, we shall complete the representafives in the table by applying Theorems 3.3 and 3.11. The above corollary proves the case that $K_{0}\left(O_{A}\right)=0$. Next we shall prove the case that $K_{0}\left(O_{A}\right)=Z_{2}$ and $\operatorname{mark}\left(O_{A}\right)=\bar{O}$. Let $O_{3}=C^{*}\left(S_{1}, S_{2}, S_{3}\right)$. Then $O_{3} \otimes M_{k}$ is generated by

$$
\left(\begin{array}{ll}
S_{1} & 0 \\
0 & 0
\end{array}\right),\left(\begin{array}{ll}
S_{2} & 0 \\
0 & 0
\end{array}\right),\left(\begin{array}{ll}
0 & S_{3} \\
0 & 0
\end{array}\right) \text { and }\left(\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right)
$$

so that $O_{3} \otimes M_{2}$ is isomorphic to $O_{B}$, where $B=\left(\begin{array}{llll}1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0\end{array}\right)$.

We have


On the other hand, $D$ is the explosion of $G$ at 2 , where G;


In the case that $K_{0}\left(O_{A}\right)=Z_{3}$ and $\operatorname{mark}\left(O_{A}\right)=\overline{1}$, if $A$ is the $4 \times 4$ matrix whose entries are 1, i.e., $0_{A}=O_{4}$, then


$$
A_{1} \rightarrow A_{2} \quad B_{1} \rightarrow B_{3} \quad C_{1} \rightarrow C_{4} \quad D_{4} \rightarrow D_{3}
$$

and moreover $E$ is the explosion of $H$ at 3 , where
H;

$\left(\begin{array}{lll}1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0\end{array}\right)$.

Finally, the case that $K_{O}\left(O_{A}\right)=Z_{4}$ and $\operatorname{mark}\left(O_{A}\right)=\overline{2}$ is stated in [17].

III-6. Shift equivalence and determinant.
A matrix $A$ is strongly shift equivalent to a matrix $B$ if there are matrices $R$ and $S$ such that $A=R S$ and $B=$ SR, cf. [37]. If $A$ and $B$ are strongly shift equivalent, then $O_{A}$ and $O_{B}$ are stably isomorphic [17]. While we have the following example by the classification table: There are strongly shift equivalent matrices $A$ and $B$ such that $O_{A}$
is not isomorphic to $O_{B}$. As a matter of fact, let $A=R S$ and $B=S R$ where

$$
R=\left(\begin{array}{lll}
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 1
\end{array}\right) \quad \text { and } \quad S=\left(\begin{array}{lll}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 1
\end{array}\right) .
$$

Then $K_{0}\left(O_{A}\right)=K_{o}\left(O_{B}\right)=Z_{2}$. On the other hand, since mark $\left(O_{A}\right)$ $=\bar{I}$ and $\operatorname{mark}\left(O_{B}\right)=\overline{0}, O_{A}$ is not isomorphic to $O_{B}$.

The following theorem shows that, under an additional assumption, $O_{A}$ and $O_{B}$ are isomorphic for strongly shift equivalent matrices $A$ and $B$.

Theorem 3.14. Let $R$ and $S$ be matrices such that
${ }^{\Sigma}{ }_{i} R(i, j)=1$ for all $j$, and $R S$ and $S R$ satisfy the condition (I). Then $O_{R S}$ and $O_{S R}$ are isomorphic.

Proof. Let $R$ (resp. S) be an $n \times m$ (resp. $m \times n$ ) matrix and put $A=R S$ and $B=S R$. Let $H_{j}(j=1, \ldots, m)$ and $K_{i}$ ( $\mathrm{i}=1, \ldots, \mathrm{n}$ ) be infinite dimensional Hilbert spaces, and $P_{j}\left(\right.$ resp. $\left.Q_{i}\right)$ the projection of $H=\Sigma \oplus H_{j}$ (resp. $K=\Sigma \oplus K_{i}$ ) onto $H_{j}\left(\right.$ resp. $\left.K_{i}\right)$. Take partial isometries $U_{i}$ and $V_{j}$ of $K$ into $H$ such that
(*) $\quad U_{i} U_{i}{ }^{*}=Q_{i}, \quad U_{i}{ }^{*} U_{i}=\Sigma{ }_{j} R(i, j) P_{j}, \quad$ and
(**) $\quad V_{j} V_{j}{ }^{*}=P_{j}, \quad V_{j}{ }^{*} V_{j}=\Sigma_{k} S(j, k) Q_{k}$.
And let $C$ (resp. D) be the $C^{*}$-algebra generated by $\left\{U_{i} V_{j}\right.$; $1 \leq i \leq n, 1 \leq j \leq m\} \quad\left(r e s p . ~\left\{V_{j} U_{i} ; 1 \leq i \leq n, 1 \leq j \leq m\right\}\right)$. Then we shall prove that $C$ (resp. D) is isomorphic to ${ }^{O_{A}}$ (resp. $O_{B}$ ) and $C$ is isomorhic to $D$.

If we put $T_{i}=U_{i}\left(\Sigma_{j} V_{j}\right)$, then $T_{i} T_{i}{ }^{*}=Q_{i}$ and

$$
\begin{aligned}
T_{i}^{*} T_{i} & =\Sigma_{j} R(i, j)\left(\Sigma_{j} V_{j}\right) * P_{j}\left(\Sigma_{j} V_{j}\right)=\Sigma_{j} R(i, j) V_{j} * V_{j} \\
& =\Sigma_{k}\left(\Sigma_{j} R(i, j) S(j, k)\right) Q_{k}=\Sigma_{k} A(i, k) Q_{k}
\end{aligned}
$$

for all i. Hence it implies that the $C^{*}$-algebra $C^{*}\left(T_{i} ; 1 \leq i\right.$ $\leq n$ ) is isomorphic to $O_{A}$. On the other hand, since $V_{j}{ }^{*} V_{j}=$ ${ }^{\Sigma} k_{k} S(j, k) Q_{k}=\Sigma{ }_{k} S(j, k) T_{k} T_{k}{ }^{*}$ and $T_{i} V_{j}{ }^{*} V_{j}=R(i, j) U_{i} V_{j}$, we have $C=C^{*}\left(T_{i} ; 1 \leq i \leq n\right)$ as a set, so that $C$ is isomorphic to $O_{A}$. Similarly $D$ is isomorphic to $O_{B}$.

By the assumption of $R$ and (*), $W=\sum_{i} U_{i}$ is an isometry from $H$ onto $K$. Since

$$
\left(W^{*} U_{i} V_{j} W\right)\left(W^{*} U_{i} V_{j} W\right) *=R(i, j) P_{j} \quad \text { and } \quad W^{*} U_{i} V_{j} W\left(U_{i} * U_{i}\right)=V_{j} U_{i}
$$

by (*) and (**), we have
$C^{*}\left(W^{*} U_{i} V_{j} W ; 1 \leq i \leq n, 1 \leq j \leq m\right)=C^{*}\left(V_{j} U_{i} ; 1 \leq i \leq n, 1 \leq j \leq m\right)$ as a set. Therefore $C$ and $D$ are isomorphic.

Next we shall discuss an topological invatiant det(1-A). It is known that, identifying a digraph with its adjacency matrix as usual, for a digraph $G$,

$$
\operatorname{det}\left(x-G^{*}\right)=x^{k} \operatorname{det}(x-G)
$$

where $k=V\left(G^{*}\right)^{=}-V(G)^{=}$and $M^{=}$is the cardinal number of M. A key of a proof is to find matrices $A$ and $B$ such that $G^{*}=A B$ and $G=B A$. Inspired by this, we shall reformulate explosions of digraphs. Here a matrix $A$ is represented by $\left(a_{i j}\right)$.

Definition 3.15. Let $G=\left(a_{i j}\right)$ be an $n x n$ matrix (digraph) with $\Gamma^{-}(1)^{-} \geq 2$. Then a digraph $H$ is the explosion
of $G$ at 1 (with respect to $V_{1}$ and $V_{2}$ ) if $H=G_{0} E_{0}$ such that $G_{o}\left(r e s p . E_{0}\right)$ is an ( $n+1$ ) $\times n$ (resp. $n \times(n+1)$ ) matrix expressed by
where $a_{p j}=\left\{\begin{array}{ll}1 & \text { if } j \varepsilon V_{1}, \\ 0 & \text { if not, }\end{array} \quad\right.$ and $a_{q j}= \begin{cases}1 & \text { if } j \varepsilon V_{2}, \\ 0 & \text { if not. }\end{cases}$

Lemma 3.16. Definitions 3.11 and 3.15 are identical. Moreover, if notation is as in above, then $E=E_{o} G_{0}$.

Proof. We represent the original explosion of $G$ as its adjacency matrix;

Thus elementary calculations lead us the conclusion.

The lemma gives us another proof of Theorem 3.11 by joining Theorem 3.14.

$$
\text { By the way, it is proved that }|\operatorname{det}(1-A)| \text { is a stable }
$$

invariant for $O_{A}$. And Cuntz conjectures that $\operatorname{det}(1-A)$ is a stable invariant for $O_{A}$. Now we have the following results on $\operatorname{det}(1-A)$.

Theorem 3.17. If $H$ is an explosion of a digraph $G$, then

$$
\operatorname{det}(x-H)=x \operatorname{det}(x-G),
$$

and so $\operatorname{det}(1-H)=\operatorname{det}(1-G)$.

Proof. Since $V(H)^{=}=V(G)^{=}+1$, the statement follows from the preceding lemma.

Theorem 3.18. If $H$ is a transfered graph of $G$, then $\operatorname{det}(1-H)=\operatorname{det}(1-G)$.

Proof. Suppose that $G \xlongequal[E_{K}+A_{M} \longrightarrow A_{1}]{ } H$, where $E_{K}=\sum_{i=1}^{r} E_{k(i)}$ and $A_{M}=\Sigma_{j=1}^{s} A_{m(j)}$. Since $A_{1}=E_{K}+A_{M}$, it follows from the definition of transfered graphs that

$$
\begin{aligned}
\operatorname{det}(1-G) & =\operatorname{det}\left(\left(\begin{array}{l}
E_{1} \\
E_{2} \\
\vdots \\
E_{n}
\end{array}\right)-\left(\begin{array}{c}
E_{K}+A_{M} \\
A_{2} \\
\vdots \\
A_{n}
\end{array}\right)\right)=\operatorname{det}\left(\begin{array}{c}
E_{1}-E_{K}-E_{M} \\
E_{2}-A_{2} \\
\vdots \\
E_{n}-A_{n}
\end{array}\right) \\
& =\operatorname{det}\left(\left(\begin{array}{c}
E_{1} \\
E_{2} \\
\vdots \\
E_{n}
\end{array}\right)-\left(\begin{array}{c}
E_{K}+E_{M} \\
A_{2} \\
\vdots \\
A_{n}
\end{array}\right)\right)=\operatorname{det}(1-H) .
\end{aligned}
$$

III-7. Weak extension groups of $O_{A}$.
Let $Q(H)$ be the Calkin algebra on an infinite dimensional separable Hilbert space $H$ and $\pi$ the quotient map of $B(H)$ onto $Q(H)$. For a separable unital $C^{*}-a l g e b r a \quad B$, let ext $(B)$ be the set of all unital *-monomorphisms (extensions) of B into $Q(H)$. Extensions $\tau$ and $\sigma$ are weakly equivalent if there is a unitary $u \in Q(H)$ such that $\tau(x)=u \sigma(x) u^{*}$ for all $x \in B$. Let Ext (B) denote the set of all weak equivalence classes in ext( $B$ ), which is called the weak extension group of B. Cuntz and Krieger determined the weak extension group of $O_{A}$ by the Bowen-Franks invariant $Z^{n} /(1-A) Z^{n}$.

In this section, we shall prove that any finitely generated abelian group is represented by the weak extension group of a simple Cuntz-Krieger algebra.

Theorem 3.19. Let $H$ be a finitely generated abelian group. Then there is a simple Cuntz-Kreiger algebra $O_{A}$ such that $\operatorname{Ext}^{W} \mathrm{O}_{\mathrm{A}}=\mathrm{H}$.

Now it is known that every finitely generated abelian group $H$ is represented;

$$
H=Z \oplus \cdots \oplus Z \oplus Z_{n(1)}{ }^{\oplus} \cdots \oplus Z_{n(m)}
$$

where $Z_{n}=Z / n Z$. So we shall devide into several cases. In the beginning, we shall consider the simple case $H=Z$, which is a key in the proof. It is known that Ext $^{W_{O}}{ }_{n+1}=Z_{n}$. However, we shall pose another Cuntz-Krieger algebras $O_{A}$ with the same property. We omit often $O$ entries of matrices
in the below.

Lemma 3.20. Let $G(n)$ be the digraph whose (adjacency) matrix is of degree $n+1$ and expressed by

$$
\left(\begin{array}{cccccc}
0 & & & & 0 & 1 \\
1 & \cdot & & & \vdots & \vdots \\
& \cdot & \cdot & & \vdots & \vdots \\
& & \cdot & \cdot & \cdot & \vdots \\
& & & \cdot & 0 & \cdot \\
& & & & 1 & 1
\end{array}\right)
$$

Then $E^{E x t}{ }^{W_{O}}(n)=Z_{n}$ for $n \geq 1$. Particularly, Ext ${ }^{W_{O}}{ }_{G(1)}$ is trivial.

Proof. Since $G(n)$ has an $(n+1)$-cycle and the vertex $n+1$ has a loop, $G(n)$ satisfies the condition (I) and is strongly connected. It implies that $O_{G(n)}$ is simple. We have also

By the Elementarteilersatz $[42 ;$ §118], it follows that

$$
z^{n+1} /(1-G(n)) z^{n+1}=Z_{n}
$$

so that $\operatorname{Ext}^{W_{O}}{ }_{G(n)}=Z_{n}$.

Next we shall consider the case $H=Z \oplus \ldots \oplus Z^{H} \oplus Z_{n}$.

Lemma 3.21. Let $G(k \mid n)$ be the digraph whose matrix is of degree $k+n+1$ and expressed by

$$
\left(\begin{array}{cccc}
1 & & & 1 \\
\ddots & & 0 \\
& \ddots & & \vdots \\
1 & & 1 & 0 \\
\vdots & \vdots & & \\
\vdots & \cdots(n) & \\
0 \cdots & & &
\end{array}\right)
$$

Then $E x t{ }^{W} O_{G(k \mid m)}=Z \oplus \ldots \oplus Z \oplus Z_{n}$, where $Z \oplus \ldots \oplus Z$ is $k-$ copies of $Z$. Particularly, $E x t^{W_{O}}{ }_{G(k \mid 1)}=Z \oplus \ldots \oplus Z$.

Proof. Since $A(i, k+n)=1$ for $1 \leq i \leq k, A(k+n, j)=1$ for $1 \leq j \leq k$ and the strongly connected digraph $G(n)$ satis fies the condition (I), $G(k \mid n)$ satisfies the condition (I) and is strongly connected. So ${ }^{O_{G}}(k \mid n)$ is simple. Moreover we have

$$
\left(\begin{array}{ccc}
1 & & -1 \\
\ddots & & \vdots \\
& 1 & \\
& -1 \\
& 1 & \\
& & \ddots
\end{array}\right)(G(k \mid n)-1)\left(\begin{array}{ccc}
1 & & \\
\\
& & \ddots
\end{array}\right)
$$

By Lemma 3.20, it implies that $\operatorname{Ext}^{W_{O}}{ }_{G(k \mid n)}=Z \oplus \ldots \oplus Z \oplus Z_{n}$.

For the case $H=Z_{m} \oplus Z_{n}$, we shall apply Lemma 3.20 again.

Lemma 3.22. Let $G(m, n)$ be the digraph which is expressed as follows. Then Ext ${ }^{W_{O}}{ }_{G(m, n)}=Z_{m} \oplus Z_{n}$.
$\left(\begin{array}{ccc} & 1 & \\ G(m) & \vdots & \\ & 0 & \\ & 1 & \\ & \vdots & \\ & \vdots & G(n) \\ & 0 & \end{array}\right)$

Proof. Since $A(m+1, m+2)=1=A(m+2, m+1)$, $G(m, n)$ satisfies (I) and is strongly connected, so that $O_{G(m, n)}$ is simple by similar calculations, we have

It follows from Lemma 3.20 that $\operatorname{Ext}^{W} O_{G(m, n)}=Z_{m} \oplus Z_{n}$.

Here we shall remark that $\operatorname{Ext}^{W_{0}}{ }_{G(p, m, n)}=Z_{p} \oplus Z_{m} \oplus Z_{n}$ if we define $G(p, m, n)$ anologously. Now let us join Lemmas 3.21 and 3.22. Let $G(k \mid m, n)$ be the digraph expressed by

$$
\begin{aligned}
& \left(\begin{array}{ccccc}
1 & & & \vdots & 0 \\
\vdots
\end{array}\right] \\
& \text { - } 60 \text { - }
\end{aligned}
$$

Then we have

On the other hand, $G(k \mid m, n)$ satisfies $(I)$ and is strongly connected by $A(k, k+m)=1=A(k+m, k)$. So it follows from Lemma 3.21 that $E \operatorname{Ext}^{W} O_{G(k \mid m, n)}=Z \oplus \ldots \oplus Z \oplus Z_{m} \oplus Z_{n}$. Thus it is easily seen that there is a strongly connected digraph $G$ $=G(k \mid n(1), n(2), \ldots, n(m))$ such that $E x t{ }^{W} O_{G}=Z \oplus \ldots \oplus \mathcal{Z} \oplus$ $Z_{n(1)} \oplus \ldots \oplus Z_{n(m)}$, which completes the proof of Theorem 3.19.

Finally we shall discuss the periodicity of weak extension groups of Cuntz-Krieger algebras associated with random walks. We consider the following example associated with a random walk PPRW reflecting at both boundaries, cf. [35].

Example 3.7.1. Let $A(n)(n \geq 2)$ be the digraph;


Then $E x{ }^{W^{W}} O_{A(n)}=Z$ for $n=3 m$ and $O$ for otherwise.
In fact, it is proved that $\operatorname{Ext}^{W} \mathrm{O}_{\mathrm{B}(\mathrm{n}+1)}=\operatorname{Ext}^{\mathrm{W}} \mathrm{O}_{\mathrm{A}(\mathrm{n})}=$ Ext ${ }^{W}{ }^{6} O_{B(n-2)}$, where $B(n)$ is the digraph;

$$
1 \rightleftarrows 2 \rightleftarrows \cdots \rightleftarrows n-1 \rightleftarrows n \text { n }
$$

Since $A(n)$ (resp. $B(n)$ ) is expressed by

$$
\left(\begin{array}{llll}
1 & 1 . & & \\
1 & 0 & \ddots & \\
& \ddots & \\
& \ddots & \ddots & \\
& & \ddots & 1 \\
& & & 1
\end{array}\right) \quad\left(\begin{array}{llll}
1
\end{array}\right) \quad\left(\begin{array}{lll}
0 & 1 & \\
1 & \ddots & \\
& \ddots & \\
& \ddots & \\
& & \ddots \\
& 1 \\
& & 1
\end{array}\right)
$$

we have

$$
\left(\begin{array}{llll}
1 & & & \\
1 & \ddots & \\
& & \ddots_{1}
\end{array}\right)^{(B(n+1)-1)}\left(\begin{array}{llll}
1 & 1 & & \\
& & \ddots & \\
& & & \ddots
\end{array}\right)=\left(\begin{array}{ll}
-1 & \\
& A(n)-1
\end{array}\right)
$$

Therefore $\operatorname{Ext}^{W} \mathrm{O}_{\mathrm{B}(\mathrm{n}+1)}=\operatorname{Ext}^{W_{O}} \mathrm{~A}(\mathrm{n})$.
Next, if we put
then we have

$$
I_{n}(A(n)-1) J_{n}=\left(\begin{array}{rrrrl}
1 & & & & \\
-1 & & & & \\
& -1 & 1 & & \\
& 1 & \ddots & \\
& \ddots & \ddots & \\
& & \ddots & -1 & 1 \\
& & & 1 & 0
\end{array}\right)=\left(\begin{array}{lll}
1 & & \\
& & \\
& & \\
& & \\
& & \\
& &
\end{array}\right.
$$

so that $\operatorname{Ext}^{W} O_{A(n)}=\operatorname{Ext}^{W} O_{B(n-2)}$.

Next we shall replace the edges of a polygon by directed edges $\rightleftarrows$, which is associated with a random walk CRW on a circle, cf. [33].

Example 3.7.2. Let $C(n)(n \geq 3)$ be the digraph;


It is somewhat surprising that $\operatorname{Ext}^{W} \mathrm{O}_{\mathrm{C}(\mathrm{n})}$ is periodic with period 6: The weak extension group of ${ }^{O_{C(3)}}$ (resp. $0_{C(4)}$, $\cdots$


Let us put
and $B_{i+1}=B_{i} J_{i}$ for $1 \leq i \leq n-3$. If we put $b_{i}{ }^{(k)}=B_{i}(k, i)$
and $c_{i}{ }^{(k)}=B_{i}(k, n-i)$ for $k=1,2$ and $1 \leq i \leq n-3$, then $B_{i+1}(k, n-(i+1))=c_{i}{ }^{(k)}-b_{i}{ }^{(k)}+1$ and $b_{i+1}^{(k)}=c_{i}^{(k)}$. Hence we have
(B)

$$
b_{i+2}^{(k)}=b_{i+1}^{(k)}-b_{i}^{(k)}+1
$$

for $k=1,2$ and $1 \leq i \leq n-4$. Furthermore, since we have

$$
\mathrm{B}_{\mathrm{n}-3}=\left(\begin{array}{ccccc}
b_{n-3}{ }^{(1)} & 1 & b_{n-2^{(1)}} & & \\
b_{n-3^{(2)}} & 0 & b_{n-2}^{(1)} & * & \\
1 & 1 & 0 & & \\
& & & -1 \cdot & \\
& & & v_{-1}
\end{array}\right)
$$

it follows that

$$
\mathrm{B}_{n-2}=\mathrm{B}_{n-3^{J}{ }_{n-3}=\left(\begin{array}{llll}
b_{n-2}^{(1)} & b_{n-2}^{(1)}-b_{n-3}{ }^{(1)}+1 & & * \\
b_{n-2}^{(2)} & b_{n-2}^{(2)}-b_{n-3}^{(2)} & & \\
& & -1 & \\
& & &
\end{array}\right) \text {, } 1}
$$

so that $\operatorname{Ext}^{W}{ }^{W} C(n)$ depends only on the matrix

$$
\left(\begin{array}{ll}
b_{n-2}^{(1)} & b_{n-2} \\
b_{n-2}^{(1)}-b_{n-3}^{(1)} & b_{n-2}^{(2)}-b_{n-3}^{(2)}
\end{array}\right)
$$

On the other hand, if $d_{i+1}=d_{i}-d_{i-1}$, then

$$
\begin{aligned}
d_{i} & =d_{i-1}-d_{i-2}=\left(d_{i-2}-d_{i-3}\right)-d_{i-2}=-d_{i-3} \\
& =-d_{i-4}+d_{i-5}=-\left(d_{i-5}-d_{i-6}\right)+d_{i-5}=d_{i-6}
\end{aligned}
$$

so that $d_{n}$ is of period 6. Since $d_{i}=b_{i}-1$ satisfies that $d_{i+1}=d_{i}-d_{i-1}$ by (B), $\left\{b_{n}\right\}$ is of period 6 and so is $E x t^{W_{O}^{O}}{ }_{C(n)}$. In addition, since $b_{1}^{(1)}=b_{2}^{(1)}=b_{1}^{(2)}=1$ and $b_{2}^{(2)}=2$, the equation $(B)$ implies the conclusion.

Finally we shall give an example of a sequence of digraphs $S(n)$ such that $E \operatorname{Ex}^{W}{ }^{W} S(n)$ is not periodic. Replacing the edges of an $n$-simplex $\Delta(n)(n \geq 3)$ by directed edges $\longrightarrow$, we obtain the digraph $S(n)$.

Example 3.7.3. Let $S(n)$ be the digraph whose matrix $A$ is given by $A(i, j)=1-\delta_{i, j}$. Then $\operatorname{Ext}^{W} O_{S(n)}=Z_{2}^{n-2} \oplus Z_{2 n-4}$.

As a matter of fact, we have

$$
\begin{aligned}
& =\left(\begin{array}{llll}
1 & & & \\
& 2 & & \\
& & \ddots & \\
& & & 2 \\
& & & 4-2 n
\end{array}\right) \text {. }
\end{aligned}
$$

## Chapter IV KMS states on $\mathrm{O}_{\mathrm{A}}$

IV-1. KMS states.
A proof of the uniqueness theorem on $O_{A}$ is based on the existence of the gauge automorphism $\alpha_{t}\binom{t}{\varepsilon}$ on $O_{A}$ by $\alpha_{t}\left(S_{j}\right)=e^{i t_{S}}{ }_{j} \quad$ for $1 \leq j \leq n$,
where $R$ is the group of real numbers. The action $\alpha$ is called the gauge action on $O_{A}$. Olesen and Pedersen [36] proved the following theorem on the $C^{*}$-dynamical system $\left(\mathrm{O}_{\mathrm{A}}\right.$, R, a), cf. also [30] :

Theorem 4.1. The $C^{*}$-dynamical system ( $\left.O_{A}, R, a\right)$ admits a $\beta-K M S$ state if and only if $\beta=\log n$, and the corresponding KMS state is unique.

Now we remark that if $A(i, j)=1$ for $1 \leq i, j \leq n$, then $O_{A}=O_{n}$ and the spectral radius $r(A)$ of $A$ is just $n$.

Under these situation, we shall give a natural generalization of Theorem 4.1. As a matter of fact, we shall point out that the Perron-Frobenius theorem for positive matrices is applicable to the existence of KMS states on the $C^{*}$-dynamical system ( $\left.O_{A}, R, \alpha\right)$. More precisely,

Theorem 4.2. If $A$ is irreducible, then ( $\left.O_{A}, R, \alpha\right)$ admits a $\beta$-KMS state if and only if $B=\log r(A)$, and the corre-
sponding KMS state is unique.

The topological entropy of a subshift $\sigma_{A}$ is defined by $h\left(\sigma_{A}\right)=\log r(A)$. Therefore, the above theorem shows that the topological entropy of a subshift is the value $\beta$ which gives a unique $B-K M S$ state for $\left(O_{A}, R, \alpha\right)$, and consequently, if $C^{*}-$ dynamical systems $\left(O_{A}, R, \alpha\right)$ and $\left(O_{B}, R, \alpha\right)$ are conjugate, and $A$ and $B$ are irreducible, then their topological entropies coincide.

Incidentally, the period of $A$ will be concerned with a factor representation of type III $_{\lambda}$ in the following section, in which it will be proved that the period is also a conjugacy invariant for $\left(O_{A}, R, a\right)$. It is known that a pair of the topological entropy and the period is a complete invariant for subshifts as measure preserving transformations, [371. As a consequence, the equivalence of subshifts as measure preserving transformations is a conjugacy invariant.

Let $E=\{1,2, \ldots, n\}$. For a multiindex $\mu=$ (i(1), $\ldots$ , $i(p)$ ) with $i(m) \varepsilon E$, we denote by $I(\mu)$ the length $p$ of $\mu$ and $S_{\mu}=S_{i(1)} \cdots S_{i(p)}$. Then it is easily checked that $S_{\mu}$ is a partial isometry and $S_{\mu} \neq 0$ if and only if $A(i(m), i(m+1))=1$ for $1 \leq m \leq p-1$. Now we begin with an elementary lemma stated in [17], whose proof is an easy exercise for the use of the condition (A).

Lemma 4.3. If $1(\mu)=l(v)=k$ and $S_{\mu}, S_{v} \neq 0$, then

$$
S_{v} * S_{\mu}=\delta_{\mu, \nu} S_{v}(k)^{*} S_{\mu(k)} .
$$

It is known in [17] that the fixed point algebra $\mathrm{F}_{\mathrm{A}}$ of ${ }^{O_{A}}$ by $\alpha$ is an AF-algebra such as

$$
F_{A}=\overline{\bigcup_{m=0}^{\infty} F_{m}}, F_{m}=F_{m}^{1} \oplus \ldots \oplus F_{m}^{n}
$$

and the inclusion is given by $A$. We refer to [4, 201 for $A F-$ algebras. For $\beta \in R$, we put

$$
L_{\beta}=\left\{y=\left(y_{i}\right) \varepsilon R^{n} ; A y=e^{\beta} y, y_{i} \geq 0 \text { and } \varepsilon y_{i}=1\right\}
$$

In the following, we shall construct a trace on $F_{A}$ which is corresponding to each $y_{O} \varepsilon L_{B}$. Then we put $y_{m}=e^{-m B^{3}} y_{O}$, $d_{m}(i)=\operatorname{dim} F_{m}^{i}$ and $w_{m}(i)=d_{m}(i) y_{m}(i)$ for $i \varepsilon E$.

Lemma 4.4. Notation as in above for a fixed $\beta \in R$ and $y_{0}$ $\varepsilon L_{B}$. Then $y_{O}$ induces a trace $\phi$ on $F_{A}$ such as $\phi(e(m, i))$ $=y_{m}(i)$, where $e(m, i)$ is a one-dimensional projection in $F_{m}{ }^{i}$ for $i \in E$.

Proof. We define a trace $\phi_{m}$ on $F_{m}$ by $\phi_{m}(e(m, i))=$ $y_{m}$ (i) for i $\varepsilon$ E. So it suffices to show that $\left(\phi_{m}\right)$ is compatible. We note that $\phi_{0}(1)=\Sigma_{i} y_{0}(i)=1$ and $\phi_{m}\left(\operatorname{Pr} F_{m}{ }^{i}\right)=d_{m}(i) y_{m}(i)=w_{m}(i)$, where $\operatorname{Pr} F_{m}{ }^{i}=0 \oplus \cdots \oplus 0 \oplus \stackrel{i}{i} \oplus 0$ $\ldots \oplus 0 \varepsilon F_{m}$. Since $y_{m}=e^{-m \beta} y_{0}$, it follows that $y_{m}=A y_{m+1}$ for $m \geq 0$. Therefore we have

$$
\begin{aligned}
\Phi_{m}(1) & =\varepsilon_{i} y_{m}(i) d_{m}(i)=\left(y_{m}, d_{m}\right)=\left(y_{m}, t_{A d_{m-1}}\right) \\
& =\left(A y_{m}, d_{m-1}\right)=\left(y_{m-1}, d_{m-1}\right)=\Phi_{m-1}(1),
\end{aligned}
$$

and

$$
w_{m}(i)=\varepsilon_{j} \quad t_{A(j, i)} w_{m+1}(j) d_{m}(i) / d_{m+1}(j) .
$$

Hence $\left(\phi_{m}\right)$ is compatible and so we can define a trace $\phi$ on $F_{A}$ by $\quad \phi \mid F_{m}=\phi_{m}$.

Lemma 4.5. Let $\phi$ be a trace on $F_{A}$ in the preceding lemma and $p=l(\mu), q=l(v), r=l(\xi)$ and $s=l(\eta)$. If $S_{\mu}, S_{v}, S_{\xi}, S_{\eta} \neq 0$ and $p+r=q \div s$, then

$$
\begin{aligned}
\phi\left(S_{\mu} S_{v} * S_{\xi} S_{\eta}^{*}\right) & =\delta_{1}\left(A y_{S}\right)(\eta(s)) & & \text { if } q \leq r \\
& =\delta_{2}\left(a y_{p}\right)(\mu(p)) & & \text { if } q>r
\end{aligned}
$$

Here $\delta_{1}=\delta_{1}(\mu, v, \xi, \eta)=\delta_{\mu, \eta} \delta_{v, \xi} \delta_{\xi}(q+1), \eta(p+1) \cdots \delta_{\xi(r), \eta(s)}$
and $\delta_{2}=\delta_{2}(\mu, v, \xi, \eta)=\delta_{\mu, \eta} \delta_{\nu, \xi} \delta_{\mu}(s+1), \nu(r+1) \cdots \delta_{\mu(p), v(q)}$,
where $\delta_{\nu, \xi}=\delta_{v(1), \xi(1)} \cdots \delta_{\nu(q \wedge r), \xi(q \wedge r)}$
and $\delta_{\mu, \eta}=\delta_{\mu(1), \eta(1)} \cdots \delta_{\mu(p \sim s), \eta(p \wedge s)}$.

Proof. We may assume that $q<r$. Then it follows from Lemma 4.3 that

$$
\begin{aligned}
S_{\mu} S_{\nu}{ }^{*} S_{\xi} S_{\eta}{ }^{*} & =\delta_{\nu, \xi} S_{\mu} S_{\nu}(q){ }^{*} S_{\xi(q)} S_{\xi}(q+1) \cdots S_{\xi}(r) S_{\eta}^{*} \\
& =\delta_{\nu, \xi}{ }^{\Sigma_{n}} A(v(q), h) S_{\mu} S_{h} S_{h}{ }^{*} S_{\xi(q+1)} \cdots \cdots S_{\xi(r)} S_{n}^{*} \\
& =\delta_{\nu, \xi} A(v(q), \xi(q+1)) S_{\mu} S_{\xi(q+1)} \cdots S_{\xi(r)} S_{\eta}{ }^{*} \\
& =\delta_{v, \xi} S_{\mu} S_{\xi(q+1)} \cdots S_{\xi(r)} S_{n}^{*} .
\end{aligned}
$$

Noting that $\phi \mid F_{m}^{i}$ is a usual trace and putting $P_{h}=S_{h} S_{h}{ }^{*}$ for $h \in E$, we have

$$
\begin{aligned}
\phi\left(S_{\mu} S_{v}{ }^{*} S_{\xi} S_{n}{ }^{*}\right) & =\delta_{\nu, \xi} \Sigma_{h} \phi\left(S_{\mu} S_{\xi(q+1)} \cdots S_{\xi(r)} P_{h} S_{n}^{*}\right) \\
& =\delta_{1} A(\mu(p), \xi(q+1)) \Sigma_{h} \phi\left(S_{\eta} P_{h} S_{n}^{*}\right) \\
& =\delta_{1}{ }^{\Sigma}{ }_{h} A(n(s), h) \phi\left(S_{\eta} P_{h} S_{n}^{*}\right) .
\end{aligned}
$$

(In particular, if $q=r$, then we have the above equality directly.) Since $\phi\left(S_{\eta} P_{h} S_{n}^{*}\right)=y_{s}(h)$ if $\phi\left(S_{\eta} P_{h} S_{\eta}{ }^{*}\right) \neq 0$, we have $\Phi\left(S_{\mu} S_{v}{ }^{*} S_{\xi} S_{n}{ }^{*}\right)=\delta_{1} \Sigma_{h} A(n(s), h) y_{S}(h)=\delta_{1}\left(A y_{s}\right)(n(s))$.

Following after [7], we now use the following definition of KMS states:

Definition. Let ( $B, R, \alpha$ ) be a $C^{*}$-dynamical system and $B$ $\varepsilon$ R. Then a state $\phi$ on $B$ is $a(\alpha, \beta)-$ KMS state if $\phi$ satisfies

$$
\phi\left(a \alpha_{i \beta}(b)\right)=\phi(b a)
$$

for all $a, b$ in $a$ norm dense, $a$-invariant *-subalgebra of $B$, where $B$ is the set of entire analytic elements for $\alpha$.

Throughout this note, ( $\alpha, B$ )-KMS states are called 8 -KMS states for brevity. The following shows the existence of KMS states on $O_{A}$.

Corollary 4.6. Let $\phi$ be a trace on $F_{A}$ in Lemma 4.4 and $e$ the expectation of $O_{A}$ onto $F_{A}$. Then $\phi . e$ is a $B-K M S$ state on $O_{A}$, where $B=\log r(A)$.

Proof. It suffices to prove that $\phi\left(S_{\mu} S_{v}{ }^{*} \alpha_{i \beta}\left(S_{\xi} S_{\eta}{ }^{*}\right)\right)=$ $\Phi\left(S_{\xi} S_{\eta} S_{\mu} S_{\nu}{ }^{*}\right)$ if $I(\mu)+I(\xi)=1(\nu)+I(\eta)$ and $I(\xi) \leq 1(v)$. It follows from Lemma 4.5 that

$$
\begin{aligned}
\phi\left(S_{\mu} S_{v}^{* \alpha}{ }_{i \beta}\left(S_{\xi} S_{\eta}^{*}\right)\right) & =e^{(s-r) \beta} \delta_{1}(\mu, v, \xi, \eta)\left(A y_{S}\right)(\eta(s)) \\
& =e^{(s-r) \beta} \delta_{1}(\mu, v, \xi, n)\left(A y_{S}\right)(\xi(r))
\end{aligned}
$$

and

$$
\begin{aligned}
& \phi\left(S_{\xi} S_{n}{ }^{*} S_{\mu} S_{\nu}^{*}\right)=\delta_{2}(\xi, v, \mu, v)\left(A y_{r}\right)(\xi(r)) . \\
& \text { Since } y_{m}=e^{-m \beta} y_{0} \text { and } \delta_{1}(\mu, v, \xi, \eta)=\delta_{2}(\xi, \eta, \mu, v) \text {, we have } \\
& A y_{S}=e^{-S \beta} A y_{O} \text { and so as desired. }
\end{aligned}
$$

## Now we can state a main lemma as follows:

Lemma 4.7. For each $\beta \in R$, let $K_{\beta}$ be the set of all $\beta-$ KMS states for $\alpha$ on $O_{A}$. Then $K_{B}$ is affine-isomorphic to $I_{B}$.

Proof. Define a map $f$ of $K_{\beta}$ to $R^{n}$ by $f(\phi)=\left(\phi\left(P_{i}\right)\right)_{i}$ for $K_{\beta}$, where $P_{i}=S_{i} S_{i}{ }^{*}$ for $i \varepsilon E$. Since $\phi$ is a $\beta-K M S$ state, we have

$$
e^{\beta}\left(P_{i}\right)=\phi\left(S_{i} \alpha_{i \beta}\left(S_{i}^{*}\right)\right)=\phi\left(S_{i}^{*} S_{i}\right)=\varepsilon_{j} A(i, j) \phi\left(P_{j}\right),
$$

so that $A y=e^{B} y$ for $y=\left(\phi\left(P_{i}\right)\right)_{i}$. Obviously f is $W^{*}$ continuous.

Next we shall show that $f$ is a map of $K_{B}$ onto $L_{B}$. By Lemma 4.4, y $\varepsilon L_{B}$ induces a trace $\phi$ on the fixed point algebra $F_{A}$ such that $\phi\left(P_{i}\right)=y(i)$ for $i \varepsilon E$. Let $e$ be the expectation onto $F_{A}$. Then $\psi=e \cdot \phi$ is a $B-K M S$ state on $O_{A}$ by Corollary 4.6 and

$$
f(\psi)(i)=\psi\left(P_{i}\right)=\phi\left(P_{i}\right)=y(i),
$$

so that $\psi \in K_{\beta}$ and $f(\psi)=y$.
Finally we shall prove that $f$ is injective. For a fixed $\phi \varepsilon K_{B}$, let us put $f(\phi)=x \varepsilon R^{n}$. Then $x(m)=\phi\left(P_{m}\right)$ for $m$ $\varepsilon$ E. If $O \neq y=S_{\mu} S_{\mu} * \varepsilon F_{A}$, then $l(\mu)=1(v)=k$ and by Lemma 4.3

$$
\begin{aligned}
e^{k \beta} \phi(y) & =\phi\left(S_{\mu} \alpha_{i B}\left(S_{v}^{*}\right)\right)=\phi\left(S_{\nu}^{*} S_{\mu}\right)=\delta_{\mu, v} \phi\left(S_{\nu(k)}^{*} S_{\mu(k)}\right) \\
& =\delta_{\mu, v} \Sigma_{h} A(\mu(k), h) \phi\left(P_{h}\right)=\delta_{\mu, v} \Sigma_{h} A(\mu(k), h) x(h) \\
& =\delta_{\mu, v}(A x)(\mu(k)) .
\end{aligned}
$$

Since $x \in L_{B}$, it follows that

$$
\begin{aligned}
& e^{k \beta} \phi(y)=\delta_{\mu, v}(A x)(\mu(k))=\delta_{\mu, v} e^{B} x(\mu(k)) \\
& \text { Hence, if } f(\phi)=f(\psi)=x \text { for } \phi, \psi \varepsilon K_{\beta} \text {, then } \phi(a)=\psi(a) \\
& \text { for } a \varepsilon F_{A} \text {. Since } \phi, \psi \varepsilon K_{B} \text {, we have } \phi\left(S_{v}{ }^{*}\right)=0 \text { for } n \geq 1 \\
& \text { and so } \phi(b)=0=\psi(b) \text { for } b \varepsilon F_{A}, \text { so that } f \text { is injective. }
\end{aligned}
$$

Now we reach Theorem 4.2 after above several lemmas :

Proof of Theorem 4.2. The proof is just to apply the Perron-Frobenius theorem to the preceding lemma. Since $A$ is irreducible, $r(A)$ is a unique positive eigenvalue of $A$ with multiplicity 1 [9; (8.7)]. Therefore $L_{B}$ has one element for $\beta=\log r(A)$ only. Hence the statement follows from Lemma 4.7.

Remark. Another refinement based on Theorem 4.1 is given by Bratteli, Elliott and Herman, who constructed, for each closed subset $F$ of $R, a C^{*}$-dynamical system ( $B, R, \tau$ ) admits a $\hat{B}-K M S$ state if and only if $\beta \varepsilon F$. Furthermore the corresponding state for each $\& \varepsilon R$ is unique. Moreover, Bratteli, Elliott and Kishimoto [6] pursued this direction.

Remark. Finally, we can show nonexistence of ground states and ceiling states for $C^{*}$-dynamical system $\left(O_{A}, R, \alpha\right)$ as in [7; Example 5.3.271.

IV-2. III $_{\lambda}$-representation of $O_{A}$.
In the preceding section, we have the unique KMS state for the $C^{*}$-dynamical system $\left(O_{A}, R, \alpha\right)$ under the irreducibility of A. It is known that $O_{n}$ is corresponding to a factor of type $\operatorname{III}_{1 / n}$. We shall determine the type of the factor generated by the GNS representation of $O_{A}$ by the unique KMS state.

Let $A=(A(i, j))$ be an $n x$ matrix whose entries are 0 or 1. For $i, j \varepsilon E, \operatorname{put} E(i, j)=\left\{m \varepsilon N ; A^{m}(i, j)>0\right\}$ and $E(i)=E(i, i)$. We define $d(i)$, the peroid of a state $i \varepsilon E$, by the greatest common devisor of $E(i)$. Suppose that $A$ is irreducible. Then $d(i)=d(j)$ for any $i, j \in E$. Hence we define $d=d(A)$, the period of. $A$, by $d(A)=d(i)$ for any $i \varepsilon E$. The matrix $A$ is said to be periodic of period $d$ if d 22 , and aperiodic if $\mathrm{d}=\mathrm{d}(\mathrm{A})=1$. For $\mathrm{r}=0,1,2, \ldots, \mathrm{~d}-1$, put

$$
D(r)=\{j \varepsilon E ; E(j, 1)=r(\bmod d)\}
$$

Then the following is known, e.g., [19;(8.15)]: If A has period $d \geq 2$, then the state space $E$ can be decomposed into distinct subset $D(0), D(1), \ldots, D(d-1)$, (not necessarily of same size) such that a one step translation from $D(r)$ lead to a state $D(r+1)$, (from $D(d-1)$ to $D(0))$. Each $D(r)$ will be invariant under $A^{d}$, and the restriction of $A^{d}$ to the state of $D(r)$ will be aperiodic. Therefore we have the following decomposition :

$$
A^{d}=B(0) \oplus B(1) \oplus \ldots \oplus B(d-1)
$$

where $B(r)$ is aperiodic for $r=0,1, \ldots, d-1$.
These arguments may come in sight by a graph theoretic
approach. We present here a simple example:

Example. Let $A=\left(\begin{array}{lll}0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0\end{array}\right)$. Then the corresponding digraph of $A$ is expressed by
$1 \rightleftarrows 2 \rightleftarrows 3$,
and $d(A)=2$. Moreover the state space $E=\{1,2,3\}$ is decomposed into the subsets $\{2\}$ and $\{1,3\}$. Incidentally, it follows from [29; Theorem 7] that the fixed point algebra of ${ }^{O_{A}}$ under the gauge action is not simple.

Though $S_{i}{ }^{*} S_{j}=0$ for $i \neq j$ by the condition (A), $S_{i} S_{j}{ }^{*}$ $\neq 0$ in general. So we shall find that such $i$ and $j$ enjoy a relation, which is used in Lemma 4.8 (2). Define a map c of $E$ onto $\{0,1, \ldots, d-1\}$ by

$$
c(i)=r \text { if } i \varepsilon D(r) \text {. }
$$

Sublemma. If $S_{i} S_{j}{ }^{*} \neq 0$, then $c(i)=c(j)$.

Proof. Since $\left(S_{i} * S_{i}\right)\left(S_{j} * S_{j}\right) \neq 0$, there is $h \in E$ with

$$
A(i, h)=1=A(j, h)
$$

by the condition (A). Therefore, if $h \varepsilon D(r)$, then $i, j \varepsilon$ $D(r+1)$ by a one step translation.

Now we define the projections corresponding to the subsets $D(r)$ by

$$
R(r)=\Sigma_{i \varepsilon D(r)} S_{i} S_{i}^{*}=\varepsilon_{i \in D(r)} P_{i}
$$

for $0 \leq r \leq d-1$.

Lemma 4.8. The followings hold:
(1) For $r=0,1, \ldots, d-1, m \quad \varepsilon d Z$, there exist multiindices $\mu$ and $v$ such that $I(\mu)-I(v)=m$ and $R(r) S_{\mu} S_{v}^{* R}(r) \neq 0$.
(2) For $r=0,1, \ldots, d-1$, multiindices $\mu$ and $v$, if $1(\mu)=$ $l(v) \bmod d$, then $R(r) S_{\mu} S_{v}{ }^{*} R(r)=0$.

Proof. (1) For $r$ there exists $k \in E$ such that $k \varepsilon D(r)$. Putting

$$
E^{*}(k)=\{x \varepsilon Z ; x=u-v, u \varepsilon E(k), V \varepsilon E(k)\}
$$

then $E^{*}(k)$ coincides with $d Z$. Therefore there exist multindices $\mu$ and $v$ such that $l(\mu)=u, l(v)=v, u-v$ $=m \varepsilon d Z$, and $\mu(1)=\mu(m)=v(1)=\nu(m)=k$, so that $P_{k} S_{\mu} S_{v}{ }^{*} P_{k}$ $=0$. Since projections $P_{i}(i \varepsilon D(r))$ are mutually orthogonal and $R(r)=\Sigma_{i \varepsilon D(r)} P_{i}$, we have $R(r) S_{\mu} S_{v} * R(r) \neq 0$.
(2) Let $k, m$ be in $D(r)$. We shall show that $P_{k} S_{\mu} S_{v}{ }^{*} P_{m}$ =0. Assume that $P_{k} S_{\mu} S_{v}{ }^{*} P_{m} \neq 0$. Then $k=\mu(1)$ and $m=$
 we have $c(\mu(|\mu|))=c(v(|v|))$ by Sublemma, where $|\xi|=1(\xi)$. Furthermore, since $S_{\mu}, S_{\nu} \neq 0$, it follows that

$$
c(\mu(1))=c(\mu(|\mu|))+|\mu|-1 \quad(\bmod d)
$$

and

$$
c(v(1))=c(v(|v|))+|v|-1 \quad(\bmod d) .
$$

Hence we have $|\mu|=|v|(\bmod d)$, which is a contradiction.

We shall review some notation [10]. Let ( $M, R, \sigma$ ) be a $W^{*}$-system. For $f \varepsilon L^{1}(R)$, let $\sigma_{f}$ be a $\sigma$-weakly continuous linear map of $M$ into $M$ such that

$$
\omega\left(\sigma_{f}(x)\right)=s f(t) \omega\left(\sigma_{t}(x)\right) d t
$$

for $\omega \in M_{*}, x \in M$. The Arveson spectrum of $a$ is defined by

$$
\operatorname{sp}(\sigma)=\cap\left[Z(f) ; f \varepsilon L^{1}(R), \quad \sigma_{f}=0\right\}
$$

Here $Z(f)=\left\{r \varepsilon R^{\wedge} ; f^{\wedge}(r)=0\right\}$, where $R^{\wedge}$ is the dual group of $R$ and $f^{\wedge}$ is the Fourier transform of i. The Connes spectrum of $\sigma$ is defined to be

$$
r(\sigma)=\bigcap_{p} \operatorname{sp}(\sigma \mid p M p)
$$

where $p$ runs all non zero projections in $M^{\sigma} \cap\left(M^{\sigma}\right)^{\prime}$, the center of the fixed point algebra $M^{\sigma}$ of $M$ under $\sigma$.

In the below we assume that $0-1$ matrix $A$ is irreducible, $r(A)$ is the spectral radius of $A, d=d(A)$ is the period of $A, \phi$ is the unique $\log r(A)-\operatorname{KMS}$ state for $\left(O_{A}, R, \alpha\right)$ in Theorem 4.2. Let $\left(\pi_{\phi}, \xi_{\phi}, H_{\phi}\right)$ be the cyclic representation induced by $\phi$.

Theorem 4.9. The von Neumann algebra $M=\pi_{\phi}\left(O_{A}\right)^{-}$generated by $\pi_{\phi}\left(O_{A}\right)$ is a factor of type $I I I_{1 / r(A)} d(A)$.

Proof. Put $\beta=\log r(A)$. Since $\phi$ is the unique $B-K M S$ state, $\phi$ is a factor state by $[7 ; 5.3 .30]$, that is, $M$ is a factor. Let $\sigma$ be an action of $R$ on a $C^{*}$-algebra $O_{A}$ such that $\sigma_{t}\left(S_{j}\right)=e^{-i \beta t} S_{j}(j=1,2, \ldots, n)$, that is, $\sigma_{t}=\alpha_{-\beta t}$. Since $\phi$ is a $B-K M S$ state for $\left(O_{A}, R, \alpha\right), \phi$ is a $(-1)$-KMS state for $\left(O_{A}, R, \sigma\right)$. Since $\phi$ is $\sigma$-invariant, $\quad$ can be extended to the automorphism on the factor $M$, denoted also by $\sigma$. Thus $\left(\sigma_{t}\right)_{t}$ is the modular automorphism group of $M$ associated with $\phi$.

Next we shall consider the fixed point algebra $M^{\sigma}$ of $M$ under $\sigma$. We claim that $M^{\sigma}=\pi_{\phi}\left(O_{A}{ }^{\sigma}\right)^{-}$, the $\sigma$-weak closure of $\pi_{\phi}\left(O_{A}{ }^{\sigma}\right)$. It is trivial that $\pi_{\phi}\left(O_{A}{ }^{\sigma}\right) \subseteq M^{\sigma}$. Conversely, if $x \varepsilon$ $M^{\sigma}$, then we choose $x_{n} \varepsilon \pi_{\phi}\left(O_{A}{ }^{\sigma}\right)$ such that $x_{n}$ converges to $x \quad \sigma$-weakly in M. Putting

$$
y_{n}=s_{T} \sigma_{t}\left(x_{n}\right) d t
$$

then $y_{n} \in \pi_{\phi}\left(O_{A}{ }^{\sigma}\right)$. For $\omega \varepsilon M_{*}$, we have

$$
\begin{aligned}
\omega\left(x-y_{n}\right) & =\omega\left(\delta_{T} \sigma_{t}\left(x-x_{n}\right) d t\right) \\
& =\int_{T}\left(\sigma_{t}(x)-\sigma_{t}\left(x_{n}\right)\right) d t \longrightarrow 0
\end{aligned}
$$

Thus $M^{\sigma} \subseteq \pi_{\phi}\left(O_{A}{ }^{\sigma}\right)^{-}$, so that $M^{\sigma}=\pi_{\phi}\left(O_{A}{ }^{\sigma}\right)^{-}$.
Let $A^{d}=B(0)+B(1)+\ldots+B(d-1)$, where $B(r)$ is
aperiodic for $r=0,1,2, \ldots, d-1$. Then $O_{A}^{\sigma}=F_{A}=F_{B(0)}{ }^{\sigma} \ldots$ $\oplus F_{B(d-1)}$. Since each $B(r)$ is aperiodic, $F_{B(r)}$ is a simple unital $C^{*}$-algebra with a unique trace ${ }^{\tau}{ }_{r}$. Moreover $N_{r}=$ $\pi_{\phi}\left(F_{B(r)}\right)^{-}$is a $I I_{1}$-factor. In fact, let $p$ be a non-zero central projection of $N_{r}$. Since $\phi(x)=\left(x \xi_{\phi}, \xi_{\phi}\right)$ for $x \varepsilon M$ and $\xi_{\phi}$ is separating for $M, \phi$ is a faithful normal state on M. Since $\phi$ is a KMS state for (M, R, $\sigma$ ), $\phi \mid M^{\sigma}$ is a trace. If we put $\tau_{r}^{\prime}(x)=\phi\left(\pi_{\phi}(x) p\right) / \phi(p)$ for $x \varepsilon F_{B(r)}$, then ${ }^{\tau}{ }_{r}^{\prime}$ is a trace on $F_{B(r)}$. By the unicity of traces on $F_{B(r)}$, we have

$$
\tau_{r}^{\prime}(x)=\phi\left(\pi_{\phi}(x) p\right) / \phi(p)=\phi(x)=\tau_{r}(x)
$$

for $x \in F_{B(r)}$. Since $\phi$ is normal, $\phi(a p)=\phi(a) \phi(p)$ for $a$ $\varepsilon N_{r}$, so that $\phi(p)=\phi(p)^{2}$. Since $\phi$ is faithful and $p \neq 0$, it follows that $p=1$. Then $N_{r}$ is a $I I_{1}$-factor with a trace $\phi \mid N_{r}$. For a projection $p$ in $M^{\sigma} \cap\left(M^{\sigma}\right)^{\prime}$, we define an automorphism $\sigma_{t}^{p}$ on pMp by

$$
\sigma_{t}^{p}(p \times p)=p \sigma_{t}(p x p) p \quad \text { for } \quad x \varepsilon M
$$

Then we have

$$
\Gamma(\sigma)=\cap\left\{\operatorname{sp}\left(\sigma^{p}\right) ; 0 \neq p \varepsilon M^{\sigma} \cap\left(M^{\sigma}\right)^{\prime}\right\}=\bigcap_{r=0}^{d-1} \operatorname{sp}\left(\sigma^{R(r)}\right) .
$$

Next we shall show that for $r=0,1, \ldots, d-1$,

$$
\operatorname{sp}\left(\sigma^{R(r)}\right)=\left\{n B d \varepsilon R \cong R^{\lambda} ; n \varepsilon Z\right\}
$$

So we first show that $\operatorname{sp}\left(\sigma^{R(r)} \supseteq\left\{n \beta d \in R \cong R^{\wedge} ; n \in Z\right\}\right.$. For a fixed $n \in Z$, it follows from Lemma 4.8 (1) that there exist multiindices $\mu$ and $v$ such that

$$
1(\mu)-1(v)=n d \quad \text { and } \quad R(r) S_{\mu} S_{v} * R(r) \neq 0
$$

If $f \in \operatorname{Ker} \sigma^{\mathrm{R}(r)}$, then

$$
\sigma_{f}^{R(r)}\left(S_{\mu} S_{v}^{*}\right)=R(r) \sigma_{f}\left(S_{\mu} S_{v}^{*}\right) R(r)=0 .
$$

On the other hand, we have

$$
\begin{aligned}
\sigma_{f} R(r)\left(S_{\mu} S_{v}^{*}\right) & =R(r) \sigma_{f}\left(S_{\mu} S_{v}^{*}\right) R(r) \\
& =R(r)\left(S f(t) \sigma_{t}\left(S_{\mu} S_{v}{ }^{*}\right) d t\right) R(r) \\
& =R(r)\left(f f(t) e^{-i n B d} S_{\mu} S_{v}{ }^{* d t) R(r)}\right. \\
& =f^{\wedge}(n B d) R(r) S_{\mu} S_{v}{ }^{*} R(r)
\end{aligned}
$$

Therefore $f(n \beta d)=0$ and so $n \beta d \varepsilon \operatorname{sp}\left(\sigma^{R(r)}\right.$ ) as desired.
Conversely, let $r \varepsilon R$ and $r \notin \beta d Z \subseteq R$. Then there exists a function $f \varepsilon L^{1}(R)$ such that $f^{\wedge}(r)=1$ and $f^{\wedge} \mid \beta d Z=0$. We shall show that $f$ is in Ker $\sigma^{R(r)}$. Since the *-algebra generated algebraically by $\left\{S_{1}, \ldots, S_{n}\right\}$ is $\sigma$-weakly dense in $M$, it is enough to show that

$$
\sigma_{f}^{R(r)}\left(R(r) S_{\mu} S_{v} * R(r)\right)=0
$$

for multindices $\mu$ and $v$. While we have

$$
\begin{aligned}
& \sigma_{f}^{R(r)}\left(R(r) S_{\mu} S_{v} * R(r)\right)=R(r) \sigma_{f}\left(S_{\mu} S_{v} *\right) R(r) \\
&=R(r)\left(\int f(t) \sigma_{t}\left(S_{\mu} S_{v}^{*}\right) d t\right) R(r) \\
&=R(r)\left(\int f(t) e^{\left.-(1(\mu)-l(v)) B t_{S_{\mu}} S_{v} * d t\right) R(r)}\right. \\
&=f^{\wedge}((1(\mu)-l(v)) B) R(r) S_{\mu} S_{v} * R(r)
\end{aligned}
$$

If $I(\mu)-l(\nu) \varepsilon d Z$, then $f^{\wedge}((I(\mu)-l(\nu)) B)=0$ by the definition of f. If $l(\mu)-1(v) \neq d Z$, then $R(r) S_{\mu} S_{v} * R(r)=0$ by Lemma 4.8 (2). In both cases we have $\sigma_{f} R(r)\left(R(r) S_{\mu} S_{v} * R(r)\right)=$ 0 , so that $f$ is in Ker $\sigma^{P(r)}$. Since $f^{n}(r)=1$, it implies that $r \notin \operatorname{sp}\left(\sigma^{R(r)}\right)$. Therefore we have $\operatorname{sp}\left(\sigma^{R(r)}\right)=\{n \beta d \in R$; $n \in Z\}$ for $r=0,1, \ldots, d-1$.

Hence it follows that

$$
\Gamma(\sigma)=饣_{r=0}^{d-1} \operatorname{sp}\left(\sigma^{R(r)}\right)=\{n \beta d \varepsilon R ; n \in Z\}
$$

Since $B=\log r(A), M$ is a type $I I I_{\lambda}$ factor, where $\lambda=$ $1 / r(A)^{d}$.

IV-3. Eigenvalue problem.
Finally, we shall reformulate the argument in [181, which is based on the discussion of $\S 1$. Precisely, for certain simple $C^{*}-a l g e b r a s$ with periodic dynamics there is a Banach lattice $F$ and a positive operator $R$ on $F$ such that the $C^{*}$-dynamical system has a $\beta-$ KMS state if and only if $e^{\beta}$ is an eigenvalue of $R$. Moreover the set $K$ of all $\beta$-KMS states is affine isomorphic to the set $L_{\beta}$ of all normalized positive eigenvectors corresponding with the eigenvalue $e^{B}$. Thus to find $B-K M S$ states can be formulated as the eigenvalue problem.

Let $A$ be a unital $C^{*}$-algebra and $\left\{\alpha_{t}\right\}_{t \in R}$ a strongly continuous and periodic one-parameter automorphism group on $A$ with period $2 \pi$. The spectral subspace $A(n)$ for $n \varepsilon Z$ is defined by

$$
A(n)=\left\{x \in A ; \alpha_{t}(x)=e^{i n t} x \text { for } t \varepsilon R\right\}
$$

A projection of norm one from $A$ onto the fixed point algebra A(0) is given by

$$
\begin{equation*}
e(x)=\int_{0}^{2 \pi} \alpha_{t}(x) d t / 2 \pi \quad \text { for } \quad x \in A . \tag{1}
\end{equation*}
$$

It is known that the linear span of $\{A(n) ; n \varepsilon Z\}$ is dense in $A$ and $A(n) A(m) \subseteq A(n+m)$ for $n, m \in Z$, e.g., [391.

Let $F$ be the subspace of $A(0)^{*}$ consisting of all selfadjoint and tracial functionals. Then $F$ is a real Banach lattice whose positive cone $F_{+}$is the set of all positive functionals in $F, C f .[2]$. The following lemma is a slight modification of an asymmetric Riesz decomposition theorem [43; Theorem 7.7 in Ch.II and so we omit a proof.

Lemma 4.10. Let $F$ be as in above, and $\left[u_{i}, v_{i}, x_{i}, y_{i}\right.$; $i=1,2, \ldots, m\} \subseteq A(n)$ for a fixed $n \varepsilon Z$. Then $f\left(\Sigma_{i} v_{i}{ }^{*} u_{i}\right)$ $=f\left(\Sigma_{i} y_{i}{ }^{*} x_{i}\right)$ for $f \varepsilon F$ if $\Sigma_{i} u_{i} v_{i}{ }^{*}=\Sigma_{i} x_{i} y_{i}{ }^{*}$.

A bounded linear operator $R$ on $F$ is said to be a reverse operator associated with (A, R, a) if

$$
(R f)\left(x y^{*}\right)=f\left(y^{*} x\right)
$$

for $f \varepsilon F$ and $x, y \in A(1)$. Here we shall discuss on the existence of a reverse operator.

Lemma 4.11. If $A(O)$ is simple, then there exists a unique reverse operator $R$ associated with ( $A, R, a$ ).

Proof. Since $A(0)$ is simple, we have $A(1) A(1) *=A(0)$. For each fixed a $\varepsilon A(0)$, there is a family $\left\{x_{i}, y_{i} ; i=1,2, \ldots\right.$
$\ldots, m\} \in(1)$ such that $a=\Sigma_{i} x_{i} y_{i}{ }^{*}$. Then it follows from Lemma 4.10 that $(R f)(a)=f\left(\Sigma_{i} y_{i}{ }^{*} X_{i}\right)$ is well-defined, and Rf is a tracial linear functional on $A(O)$.

Next we shall show that $R f$ is bounded and $R$ is a bounded linear map on $F$. Note that $I=\Sigma_{i} s_{i} t_{i}{ }^{*}$ for some $s_{i}$, $t_{i} \varepsilon$ $A(1)$. For $b \in A(0)$, we have

$$
\begin{equation*}
(R f)(b)=R f\left(\Sigma_{i} b s_{i} t_{i}^{*}\right)=f\left(\Sigma_{i} t_{i}^{* b s_{i}}\right), \tag{2}
\end{equation*}
$$

so that

$$
|(R f)(b)|=\left|f\left(\Sigma_{i} t_{i}{ }^{*} b s_{i}\right)\right| \leq\|f\| \varepsilon_{i}\left\|t_{i}\right\|\left\|s_{i}\right\|\|b\|
$$

It implies that $R f$ is bounded and moreover. $\|R\| \leq \varepsilon_{i}\left\|t_{i}\right\|\left\|s_{i}\right\|$. Since $R$ is linear by (3), $R$ is a bounded linear operator on F.

Theorem 4.12. Let $L_{B}=\left\{f \varepsilon F_{+} ; R f=e^{\beta} f,\|f\|=1\right\}$ for each $\beta \in R$. Let ( $A, R, \alpha)$ be a $C^{*}$-dynamical system with period $2 \pi$ such that $A$ is unital and the fixed point algebra $A(0)$ is simple. Let $R$ be the reverse operator associated with $(A, R, \alpha)$. Then $K_{\beta}$ is affine isomorphic to $L_{\beta}$ for each $B \in R$.

Proof. Putting $H(g)=g \mid A(0)$ for $g \varepsilon K_{\beta}$, then $H(g)$ is tracial, so that $H(g) \varepsilon F$. Now we shall prove that $H$ is a $W^{*}-$ continuous affine isomorphism of $K_{B}$ onto $L_{B}$. Since $g$ is a $\beta$-KMS state, we have

$$
e^{\beta} g\left(b s t^{*}\right)=g\left(b s \alpha_{i \beta}\left(t^{*}\right)\right)=g\left(t^{*} b s\right)
$$

for $b \in A(0)$ and $s, t \varepsilon A(1)$. It. follows from (2) that $H(g)$ $\varepsilon L_{B}$ because

$$
\begin{aligned}
(\operatorname{RH}(g))(b) & =H(g)\left(\Sigma_{i} t_{i}^{* b s_{i}}\right)=g\left(\Sigma_{i} t_{i}^{* b s_{i}}\right) \\
& =e^{\beta} g\left(\Sigma_{i} b s_{i} t_{i}^{*}\right)=e^{\beta} g(b)=e^{\beta} H(g)(b) .
\end{aligned}
$$

Thus $H$ is a $w^{*}$-continuous affine map of $K_{B}$ into $L_{B}$.
Let $e$ be the norm one projection of $A$ onto $A(0)$
defined by (2) and put $G(f)=f \cdot e$ for $f \varepsilon L_{\beta}$. By similar calculations, $G$ is also a $W^{*}$-continuous affine map of $L_{\beta}$ into $K_{\beta}$, and $H \circ G=i d$ on $I_{B}$. Moreover since $g \mid A(n)=0$ for $n=0$ and $g \varepsilon K_{B}$, we have $G \circ H=i d$ on $K_{B}$. Hence it implies that $H$ is a bijection.

Remark. In the case where there exists a family $\left\{s_{1}, \ldots\right.$, $\left.s_{k}\right\} \subseteq A(1)$ such that $\Sigma_{i} S_{i} s_{i}{ }^{*}=1$, the reverse operator $R$ is positive.

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