ON A CLASS OF C*-ALGEBRAS

GENERATED BY PARTIAL ISOMETRIES

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INTRODUCTION

This thesis is devoted to the study of C*-algebras O_n and O_A which are typical examples of simple C*-algebras.

These C*-algebras were first considered by Cuntz [12], and Cuntz and Krieger [17] respectively. In 1977, Cuntz introduced O_n as a C*-algebra generated by isometries $S_1, \dots, S_n$ acting on a Hilbert space such that $\Sigma_i S_i S_i^* = 1$. He proved that the isomorphism class of O_n does not depend on the choice of generators and O_n is simple. O_n is an example of nuclear C*-algebras which are not strongly amenable. Now one of the great developments in the C*-algebra theory in 1970's is the extension theory due to Brown, Douglas and Fillmore, simply known as the BDF theory. This theory was followed up by the K-theory for C*-algebras. In the BDF theory, the C*-algebra C(T) of all continuous functions on the unit circle T in the plane is adopted as a model. Since C(T) is naturally regarded as O_1, we may use O_n as an available model for the BDF theory and K-theory. As a matter of fact, O_n was the first example of non-commutative C*-algebras taken up in the BDF theory. O_n and O_m are not stably isomorphic if $n \neq m$, because the weak extension group of O_n is isomorphic to $Z^n/(n-1)Z^n$, cf. [38] and [41].

Afterwards, Cuntz and Krieger have generalized the Cuntz algebra O_n. Let $A = (A(i,j))$ be an $n \times n$ matrix whose entries $A(i,j)$ are 0 or 1 and not all zero in any row nor in any column. Then a C*- algebra O_A is generated by partial isometries $S_1, \ldots, S_n$ acting on a Hilbert space satisfying the conditions

(A) $\quad S_i^*S_j = 0 \quad (i \neq j)$, and $S_i^*S_i = \Sigma_j A(i,j)S_jS_j^*$

for $i = 1, \ldots, n$. Under a suitable condition on the matrix A, the isomorphism class of O_A does not depend on the choice of generators. We call O_A the Cuntz-Krieger algebra (associated with A). Note that $O_A = O_n$ if A is the $n \times n$ matrix whose entries are all 1. A C*-algebra O_A is associated with the topological Markov chain (X_A, σ_A).

Now, it is known that a matrix A determining the C*-algebra O_A corresponds to a digraph G as its adjacency matrix. Therefore we can attempt a graph theoretic approach to O_A. This method was initiated by Enomoto and Watatani [29], and it plays one of the central roles in our study of O_A.

This thesis consists of four chapters. We explain briefly the contents of each chapter.

In the first chapter, we will be concerned with automorphisms on O_n. In [1], Archbold considered the 'flip-flop' automorphism θ of $O_2 = C^*(S_1, S_2)$ determined by

$$\theta(S_1) = S_2 \quad \text{and} \quad \theta(S_2) = S_1,$$

which is an analog of the flip-flop automorphism on tensor products. He proved that θ is outer. This was generalized by Enomoto, Takehana and Watatani [26] as a representation to automorphisms on O_n of the symmetric group $S(n)$ with degree n. Furthermore they considered a similar representation of the unitary group $U(n)$ of all $n \times n$ unitary matrices; for $u = (u_{ij}) \in U(n)$

$$\alpha_u(S_k) = \Sigma_j u_{jk} S_j \qquad (k = 1, \ldots, n).$$

By the uniqueness theorem on O_n, α_u can be extended to an automorphism on O_n and they proved that the action α is outer.

Now O_n can be regarded as a semigroup version of the group von Neumann algebra $R(G_n)$ of a free group G_n on n generators. Phillips [40] and Choda [9] showed that $R(G_n)$ is isomorphic to the crossed product of $R(G_{k(n-1)+1})$ by a single automorphism with period k. Choda [9] also determined the fixed point algebra of $R(G_2)$ under an automorphism with period k.

Now we shall determine the fixed point algebras of O_n under certain periodic automorphisms :

Let z be a primitive k-th root of 1 and $z1 \in U(n)$. Then the fixed point algebra of O_n under α_{z1} is generated by a UHF-algebra F_n of type n^∞ and $S_1{}^k$, where S_1 is a generator of O_n. Furthermore, the fixed point algebra is also a Cuntz algebra O_{n^k}.

Since the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ corresponding to the 'flip-flop' automorphism of O_2 is unitarily equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we consider the fixed point algebra of $O_2 = C^*(S_1, S_2)$ under α_u, such that $u = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$ with a primitive k-th root z of 1. We see that it is the subalgebra generated by S_1, $S_2{}^k$ and

$\{S_2^j S_1 S_2^{*j};\ j = 1, \ldots, k-1\}$. In particular, the fixed point algebra under the 'flip-flop' automorphism is isomorphic to the C*-algebra generated by S_1, S_2^2 and $S_2 S_1 S_2^*$.

Here we have a problem; whether α_{-1} and the 'flip-flop' automorphism θ on O_2 are conjugate or not ? Since $S_2 S_1 S_2^*$ is not an isometry, the fixed point algebra O_2^θ seems to be not of type of O_n. As a matter of fact, O_2^θ is a Cuntz-Krieger algebra. It will become clear in Chapter III that the fixed point algebras under α_{-1} and θ are not stably isomorphic, so our problem is solved negatively.

In the last part of this chapter, we shall investigate the relation between the spectrum $\sigma(u)$ of $u \in U(n)$ and $\sigma(\alpha_u)$ of α_u in the Banach algebra of all bounded linear maps on O_A. We prove that $\sigma(\alpha_u)$ is the closed subgroup of the unit circle T in the plane generated by $\sigma(u)$, and for any closed subgroup G of T, there is a $u \in U(n)$ such that $\sigma(\alpha_u) = G$.

In the second chapter, we shall discuss extensions of O_n and O_A by the compacts. In [11], Coburn studied the C*-algebra generated by an isometry acting on a Hilbert space H. He proved that the C*-algebra generated by a simple unilateral shift U_+ on H contains the ideal $K(H)$ of all compact operators on H and

(1) $$0 \longrightarrow K(H) \longrightarrow C^*(U_+) \longrightarrow C(T) \longrightarrow 0$$

is exact, that is, $C^*(U_+)$ is an extension of $C(T)$ by $K(H)$. In the BDF theory, we know that the extension group $\mathrm{Ext}\, C(T)$ coincides with the additive group Z of all integers under the correspondence $n = -\operatorname{ind} U_+^{(n)}$, where $\operatorname{ind} S$ stands for the index of a Fredholm operator S.

Cuntz [12] proved further that if P_n is the C*-algebra generated by isometries $T_1, \ldots, T_n$ on H such that $1 - \Sigma_i T_i T_i^*$ is a non-zero projection, then

(2) $$0 \longrightarrow K(H) \longrightarrow P_n \longrightarrow O_n \longrightarrow 0$$

is exact. $\operatorname{Ext} O_n = Z$ was proved by Paschke and Salinas [38].

In our discussion, we shall first point out that an extension of O_n can be reduced to one of $C(T)$ via a unilateral shift. Then we give a complete set of representatives for extensions of O_n.

Next we shall discuss extensions of O_A. One of our objectives is to find a condition for that α_u defined in Chapter I can be extended to an automorphism on O_A. Now, Evans [30] and Katayama have independently realized a C*-algebra P_n as a 'tensor algebra' on the full Fock space $F(H)$, and constructed a unitary $F(u)$ on $F(H)$ for each $u \in U(n)$. In this realization, the automorphism $\bar{\alpha}_u$ on P_n implemented by $F(u)$ corresponds to the automorphism α_u on O_n. We will here construct a subspace L_A of $F(H)$ associated with an $n \times n$ matrix A, and the C*-algebra P_A generated by the compression to L_A of the creation operators on $F(H)$ is an extension of O_A by $K(H)$. Also we shall consider conditions on $u \in U(n)$ that L_A reduces $F(u)$ and $F(u)|L_A$ implements an automorphism α_u on O_A. As an application, we have a good characterization of O_n; if for all $u \in U(n)$ α_u can be extended to an automorphism on $O_A = C^*(S_1, \ldots, S_n)$, then O_A

$= O_n$. To make these discussions, the graph theoretic approach is very useful. O_A will be sometimes written as O_G if G is the graph with the adjacency matrix A.

Another extension of O_A can be obtained by using the concept of adjoint graphs in the graph theory. We shall prove that $O_G = O_{G*}$ when $G*$ is the adjoint graph of G. As a consequence, we shall see that the reduced C*-algebra generated by the free category of a digraph G is an extension of O_G by $K(H)$.

The main purpose of the third chapter is to classify Cuntz-Krieger algebras O_A for A's with irreducible 3×3 matrices. This will be done in section 3. The irreducibility of A implies the simplicity of O_A. We will make an effective use of the K-theory in our classification problem.

We give attention to the 'position' of the unit 1 of a unital C*-algebra B in the corresponding K_o-group $K_o(B)$. It will be called the marker of B and denoted by mark(B). It is obvious that if unital C*-algebras B and C are isomorphic, then $K_o(B) = K_o(C)$ and $\mathrm{mark}(B) = \mathrm{mark}(C)$, but this fact is very important for the classification. Actually, we shall prove that the following statements are equivalent for 3×3 irreducible matrices A and B ;

(1) O_A is isomorphic to O_B,

(2) $K_o(O_A) = K_o(O_B)$ and $\mathrm{mark}(O_A) = \mathrm{mark}(O_B)$, and

(3) A is primitively equivalent to B.

As a preparatory task for this, we listed up all the strongly connected digraphs with 3 vertices and 3×3 irreducible matrices. We get 29 different matrices. Then we introduced a transformation of matrices by which O_A is left isomorphic. Primitive equivalence also introduced among matrices and this equivalence too makes corresponding algebras isomorphic. The relation (3) $\rightarrow$ (1) $\rightarrow$ (2) follows from these facts, and (2) $\rightarrow$ (3) is then checked one by one.

Next we shall discuss how to change the marker under the tensor product with a matrix algebra M_k. As a corollary, we have another proof of a result on O_n due to Paschke and Salinas [38].

We also define the explosion as a generalization of the adjoint of a digraph. This again leaves isomorphic the corresponding algebras. Using these notions, we can complete to give representatives in the classification. We also discuss the value $\det(1 - A)$ because it is very important in the theory of Markov chains (X_A, σ_A), and show that O_A and O_B are isomorphic for strongly shift equivalent matrices A and B under some additional assumptions.

Concluding this chapter, we shall point out that any finitely generated abelian group can be expressed as the weak extension group and also K_0-group of a simple Cuntz-Krieger algebra. Additionally, we shall discuss the periodicity of the weak extension group of O_A associated with random walks.

In the final chapter, we shall study the existence of KMS states for gauge action on O_A. Here we note that the proof of the uniqueness theorem on O_A is based on the existence of the gauge automorphism α_t $(t \in R)$ on O_A such that

$$\alpha_t(S_j) = e^{it}S_j \qquad j = 1, \ldots , n,$$

where R is the group of real numbers. The action α is called the gauge action. Olesen and Pedersen [33] proved that the C*-dynamical system (O_n, R, α) admits a β-KMS state if and only if $\beta = \log n$, and the corresponding KMS state is unique. Furthermore, the weak closure of the GNS representation of O_n by the unique KMS state is a factor of type $\mathrm{III}_{1/n}$.

On the other hand, there exist matrices A and B such that the spectral radii $r(A)$ and $r(B)$ are different though O_A and O_B are isomorphic. So we want to find out a condition that spectral radii coincide.

We shall generalize the theorem of Olesen and Pedersen on (O_A, R, α). We remark that $r(A) = n$ if A is the $n \times n$ matrix whose entries are all 1, that is, A corresponds to O_n. Now we prove that if A is irreducible, then (O_A, R, α) admits a β-KMS state if and only if $\beta = \log r(A)$, and the corresponding KMS state is unique. It seems to be interesting that the Perron-Frobenius theorem for positive matrices is applied in this proof. Furthermore we obtain that the weak closure of the GNS representation of O_A by the state is a factor of type $\mathrm{III}_{1/r(A)^{d(A)}}$, where $d(A)$ is the period of A. Therefore, since $\log r(A)$ is the topological entropy $h(\sigma_A)$, the pair $(h(\sigma_A), d(A))$ is an invariant for the conjugacy of C*-dynamical system (O_A, R, α). In other words, the equivalence of the subshift σ_A as a measure preserving transformation is an invariant for the conjugacy because $(\log r(A), d(A))$ is a complete invariant for σ_A as a measure preserving transformation. Finally, we discuss a relation between KMS states and eigenvalues of positive maps in a general setting.

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CHAPTER I AUTOMORPHISMS ON O_n

I-1. Action of $U(n)$ on O_n.

A C*-algebra O_n considered by Cuntz is generated by isometries $S_1, \ldots, S_n$ acting on a Hilbert space such that $\Sigma_i S_i S_i^* = 1$. He proved the following uniqueness theorem on O_n:

The uniqueness theorem. The isomorphism class of O_n does not depend on the choice of generators.

That is, if $\{T_1, \ldots, T_n\}$ is another family of isometries such that $\Sigma_i T_i T_i^* = 1$, then there is a canonical isomorphism of $C^*(S_1, \ldots, S_n)$ onto $C^*(T_1, \ldots, T_n)$, where $C^*(S)$ is the C*-algebra generated by S. In other words, if $\{T_1, \ldots, T_n\}$ is as in above, then the map $S_i \longrightarrow T_i$ $(1 \leq i \leq n)$ can be extended to an isomorphism of $C^*(S_1, \ldots, S_n)$ onto $C^*(T_1, \ldots, T_n)$.

Inspired by the flip-flop automorphism of tensor products, Archbold [1] considered the 'flip-flop' automorphism θ of $O_2 = C^*(S_1, S_2)$ determined by

$$\theta(S_1) = S_2 \quad \text{and} \quad \theta(S_2) = S_1.$$

He proved that θ is outer. This might be the first application of the uniqueness theorem.

Following after Archbold, Enomoto, Takehana and Watatani [26] showed that the symmetric group $S(n)$ has a representation as a subgroup of outer automorphisms on O_n for $n \geq 2$. Furthermore they extended it as follows; the group $U(n)$ of $n \times n$ unitary matrices is faithfully represented as a subgroup of outer automorphisms on O_n by

$$\alpha_u(S_k) = \Sigma_j u_{jk}S_j \qquad (k = 1, \ldots , n)$$

for unitary $u = (u_{jk})$. If we take $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then α_u is the 'flip-flop' automorphism on O_2.

Though Archbold considered O_n from the view point of tensor products, it can be regarded as a semigroup version of the group von Neumann algebra $R(G_n)$ of a free group G_n on n generators, cf. [27]. Phillips [40] and Choda [9] showed that $R(G_n)$ is isomorphic to the crossed product of $R(G_{k(n-1)+1})$ by a single automorphism with period k. And Choda determined the fixed point algebra of $R(G_2)$ under a single automorphism with peroid k by $R(G_{k+1})$. Moreover, the fixed point algebra under the gauge automorphism group α_T is determined by Olesen and Pedersen.

In the below, we shall investigate the fixed point algebra of O_n under a periodic automorphism α_u for $u \in U(n)$.

I-2. Fixed point algebras.

First of all, we shall explain notation. Let $W_n{}^k$ ($k = 1,2,\ldots$) be the set of all k-tuples $(j(1), \ldots , j(k))$ with $1 \leq j(i) \leq n$ and let $W_n{}^0 = \{0\}$. Let $\{S_1, \ldots , S_n\}$ be a family of generators of O_n. Then we put

$$S_\mu = S_{j(1)}S_{j(2)}\cdots S_{j(k)}$$

for $\mu = (j(i), \dots, j(k)) \in W_n^k$ and $S_o = 1$. Let F_n^k be the C*-algebra generated by $\{S_\mu S_\upsilon^* ; \mu, \upsilon \in W_n^k\}$ and $F_n^o = \{1\}$. Finally let F_n be the C*-algebra generated by $\{F_n^k ; k = 0,1,2, \dots\}$.

Theorem 1.1. Let z be a primitive k-th root of 1 and $\alpha = \alpha_{z1}$ the automorphism on O_n induced by $z1 \in U(n)$. Then the fixed point algebra B of α is the C*-algebra generated by F_n and S_1^k.

Proof. If $l(\mu) = l(\upsilon)$ where $l(\gamma)$ is the length of γ, then $S_\mu S_\upsilon^* \in B$. Since F_n is generated by $\{S_\mu S_\upsilon^*; l(\mu) = l(\upsilon)\}$ and $S_1^k \in B$, B includes the C*-algebra C generated by F_n and S_1^k. Conversely, let $x \in B$ and $\varepsilon > 0$. Then there is y in the *-algebra Q_n generated by $\{S_1, \dots, S_n\}$ such that $\|x - y\| < \varepsilon$. It is known that y has a unique representation;

$$y = \Sigma_1^m S_1{}^{*i} a_{-i} + a_o + \Sigma_1^m a_i S_1{}^i,$$

where $a_i \in Q_n \cap F_n$. Putting $\beta = (\alpha^{k-1} + \alpha^{k-2} + \dots + \alpha + 1)/k$, every a_i is fixed by β. Since

$$0 = z^{ik} - 1 = (z^i - 1)(z^{i(k-1)} + z^{i(k-2)} + \dots + z^i + 1)$$

for $i = 1,2, \dots$, we have

$$\beta(y) = (1/k)\Sigma_{i=1}^{m} \Sigma_{j=0}^{k-1} z^{*ij} S_1{}^{*i} a_{-i} + a_0 + (1/k)\Sigma_{i=1}^{m} \Sigma_{j=0}^{k-1} z^{ij} a_i S_1{}^i$$

$$= \Sigma_{i=mk} S_1{}^{*i} a_{-i} + a_0 + \Sigma_{i=mk} a_i S_1{}^i$$

so that $\beta(y) \in C$. On the other hand, we have

$$\begin{aligned} \|x - \beta(y)\| &\le (\|x - \alpha^{k-1}(y)\| + \|x - \alpha^{k-2}(y)\| + \dots + \|x - y\|)/k \\ &= (\|\alpha^{k-1}(x - y)\| + \|\alpha^{k-2}(x - y)\| + \dots + \|x - y\|)/k \\ &= \|x - y\| < \varepsilon. \end{aligned}$$

The following theorem shows that the fixed point algebra of an automorphism α in the above is also a Cuntz algebra.

Theorem 1.2. Let α be as in Theorem 1.1. Then the fixed point algebra B is isomorphic to the Cuntz algebra O_{nk}.

Proof. First of all, we shall prove that B coincides with $C = C^*(S_\mu; l(\mu) = k)$. Since $B = C^*(F_n, S_1{}^k)$ by Theorem 1.1 and $B \supseteq C$ clearly, it suffices to show that $F_n \subseteq C$, that is, $S_\mu S_\upsilon{}^* \in C$ if $l(\mu) = l(\upsilon)$. We may assume that $l(\mu) = l(\upsilon) < k$. Then the length of $S_\mu S_\upsilon{}^*$ is enlarged as follows:

$$\begin{aligned} S_\mu S_\upsilon{}^* &= S_\mu(\Sigma_i S_i S_i{}^*)S_\upsilon{}^* \\ &= \Sigma_i (S_\mu S_i)(S_\upsilon S_i)^* \\ &= \Sigma_{i,j} (S_\mu S_i S_j)(S_\upsilon S_i S_j)^*. \end{aligned}$$

Thus $S_\mu S_\upsilon{}^*$ is expressed as a finite sum of $\{S_\gamma S_\delta{}^*; l(\gamma) = l(\delta) = k\}$ by repeating this calculation.

To prove that C is isomorphic to O_{nk}, we shall show that $\{S_\mu; l(\mu) = k\}$ is a family of generators of O_{nk} such that $\Sigma_\mu S_\mu S_\mu{}^* = 1$ by induction. If $k = 2$, then

$$\begin{aligned} \Sigma_\mu S_\mu S_\mu{}^* &= \Sigma_{i,j}(S_i S_j)(S_i S_j)^* \\ &= S_1(\Sigma_i S_i S_i{}^*)S_1{}^* + \dots + S_n(\Sigma_i S_i S_i{}^*)S_n{}^* \\ &= S_1 S_1{}^* + \dots + S_n S_n{}^* \\ &= 1. \end{aligned}$$

Suppose that it is true for $k = p$. Then we have

$$\Sigma_{l(\mu)=p+1} S_\mu S_\mu{}^* = \Sigma_{i=1}^n S_i T S_i{}^*,$$

where $T = \Sigma_{l(\mu)=p} S_\mu S_\mu{}^*$. By the assumption of induction, we have $T = 1$ as desired.

Remark. Let z be a complex number such that $|z| = 1$ and $z^k \neq 1$ for all k. Then the fixed point algebra of α_{z1} is F_n by a result of Olesen and Pedersen [36; lemma 1].

Next we shall discuss the fixed point algebra of automorphism on O_2 induced by $\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$.

Theorem 1.3. Let z be a complex number with period k and α the automorphism on O_2 induced by $\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$. Then the fixed point algebra B is the C*-algebra C generated by S_1, $S_2{}^k$ and $\{S_2{}^j S_1 S_2{}^{*j};\ j = 1, \ldots, k-1\}$.

Proof. For given $x \in B$ and $\varepsilon > 0$, there is $y \in Q = Q_2$ such that $\|x - y\| < \varepsilon$. Putting $w = (y+\alpha(y)+ \ldots + \alpha^{k-1}(y))/k$, we have $w \in B \cap Q$ and $\|x - w\| < \varepsilon$ as in the proof of Theorem 1.1. So it follows that $B \cap Q$ is dense in B. Since $C \subseteq B$, it suffices to show that $B \cap Q \subseteq C$. Every element $y \in Q$ has a unique representation; $y = \Sigma_{i=1}^{m} S_1{}^{*i} a_{-i} + a_0 + \Sigma_{i=1}^{m} a_i S_1{}^i$, where $a_i \in Q \cap F_2$. Since $\alpha(y) = y$ iff $\alpha(a_i) = a_i$ for all i, we may confine ourselves to consider elements in F_2. It is clear that $S_\mu S_\upsilon{}^* \in F_2$ is fixed by α iff $m(\mu) = m(\upsilon) \bmod k$, where $m(\gamma)$ denotes the number of S_2 in S_γ. If $m(\mu) < k$, then we have

$$\begin{aligned} S_\mu &= S_1{}^{i(0)} S_2{}^{j(1)} S_1{}^{i(1)} S_2{}^{j(2)} \ldots S_2{}^{j(r)} S_1{}^{i(r)} \\ &= S_1{}^{i(0)} (S_2{}^{j(1)} S_1{}^{i(1)} S_2{}^{*j(1)}) (S_2{}^{j(1)+j(2)} S_1 S_2{}^{*j(1)+j(2)}) \\ &\quad \ldots (S_2{}^{j(1)+j(2)+\ldots+j(r-1)} S_1{}^{i(r-1)} S_2{}^{*j(1)+\ldots+j(r)}) \\ &\quad S_2{}^{j(1)+\ldots+j(r)} S_1{}^{i(r)}, \end{aligned}$$

so that $S_\mu = XS_2^{m(\mu)}S_1^{\ i}$ for some $X \in C$ and integer i. In particular, if $m(\mu) = k$, then $S_\mu \in C$ since $k = j(1) + \ldots + j(r)$. In general, if $m(\mu) = h$ mod k, then $S_\mu = XS_2^{\ h}S_1^{\ i}$ for some $X \in C$ and some integer i. Therefore, if $m(\mu) = h$ mod k and $h < k$, then

$$S_\mu S_\upsilon^* = XS_2^{\ h}S_1^{\ i}(YS_2^{\ h}S_1^{\ j})^* = X(S_2^{\ h}S_1^{\ i}S_2^{*h})(S_2^{\ h}S_1^{*j}S_2^{*h})Y^*,$$

so that $S_\mu S_\upsilon^* \in C$. Since $B \cap Q$ is generated by F_2 and $S_1^{\ k}$ we have $B \cap Q \subseteq C$. Hence it follows that $B = \mathrm{cl}\ B \cap Q \subseteq C$ where cl means the norm closure, so that $B = C$.

The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ corresponding to the 'flip-flop' automorphism introduced by Archbold is unitarily equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Therefore we have the following.

Corollary 1.4. The fixed point algebra of the 'flip-flop' automorphism on O_2 is isomorphic to the C*-algebra generated by S_1, $S_2^{\ 2}$ and $S_2S_1S_2^*$.

Remark. The fixed point algebra $O_2^{(2)}$ of O_2 under $(2) = Z/2Z$ is isomorphic to O_4 by Theorem 1.2. However, the crossed product $O_2 \times (2)$ of O_2 by (2) is not isomorphic to O_4. In fact, if $O_2 \times (2) \cong O_4$, then O_2 is included in O_4 with the same unit, which is false. Furthermore, $O_4 \times (2)$ is not isomorphic to O_2. Actually, if $O_4 \times (2) \cong O_2$, then $O_4 \otimes M_2 \cong O_2 \times (2)$ by Takai's duality theorem for C*-algebras [42]. Since $O_4 \otimes M_2 \cong O_4$, we have $O_4 \cong O_2 \times (2)$, which is a contradiction.

I-3. Spectra of automorphisms.

Now we shall investigate the relation between the spectrum $\sigma(u)$ of $u \in U(n)$ and the spectrum $\sigma(\alpha_u)$ of α_u in the Banach algebra of all bounded linear maps on O_n.

Theorem 1.5. (i) If $u \in U(n)$, then $\sigma(\alpha_u)$ is the closed subgroup G of the unit circle T in the plane generated by $\sigma(u)$. (ii) For a closed subgoup G of T, there is $u \in U(n)$ such that $\sigma(\alpha_u) = G$.

Proof. (i) We may assume that u is diagonal with eigenvalues $(t_1, \dots , t_n)$. Suppose that u is periodic with period k. Then G is generated by $\{t_1, \dots , t_n\}$. If $z \in G$, then there are integers $s(1), \dots , s(n)$ such that $z = t_1^{s(1)} \dots t_n^{s(n)}$ and $0 \le s(i) \le k-1$ for all i. If we put

$$R = S_1^{s(1)} \dots S_n^{s(n)},$$

then $R \neq 0$ and $\alpha_u(R) = zR$. Since $z \in \sigma(\alpha_u)$, we have $G \subseteq \sigma(\alpha_u)$. Conversely, since $G = \{\exp(2\pi im/k);\ 0 \le m \le k-1\}$ and $(\alpha_u)^k = 1$, we have $\sigma(\alpha_u) \subseteq G$ by the spectral mapping theorem. If u is aperiodic, then $G = T$. Since $\alpha_u(S_1^m) = t_1^m S_1^m$ for all m, we have $\sigma(\alpha_u) = T = G$.

(ii) If $G = \{\exp(2\pi im/k)\ ;\ 0 \le m \le k-1\}$ for some k, then $\sigma(\alpha_u) = G$ for $u = \exp(2\pi i/k)$. If $G = T$, then we take $u = z1$ for some aperiodic z.

Chapter II Extensions of O_A

II-1. Extensions of O_n.

The purpose of this section is to show that extensions of O_n are reduced to the extensions of $C(T)$ via a unilateral shift. In [11], Coburn studied the C*-algebra generated by an isometry acting on a Hilbert space H. He proved that if U_+ is a simple unilateral shift on H, then the C*-algebra $C^*(U_+)$ generated by U_+ contains the ideal $K(H)$ of all compact operators on H and

(1) $$0 \longrightarrow K(H) \longrightarrow C^*(U_+) \longrightarrow C(T) \longrightarrow 0$$

is exact, where $C(T)$ is the C*-algebra of all continuous functions on T. In other words, $C^*(U_+)$ is an extension of $C(T)$ by $K(H)$. In the BDF theory [8], the extension group $\mathrm{Ext}\, C(T)$ coincides with the additive group Z of all integers.

Here we mention a proof of this fact: Let π be the quotient map of $B(H)$ onto $Q(H) = B(H)/K(H)$, where $B(H)$ is the algebra of all bouded linear operators on H. An operator S is essentially normal if $\pi(S)$ is normal. A typical example of an essentially normal operator is a simple unilateral shift U_+. The task to determine $\mathrm{Ext}\, C(T)$ is identified with the classification of essentially normal operators with essential spectrum T. Moreover, it is known in [8; Theorem 3.1] that such an operator S is unitarily equivalent to $U_+^k + K$ (resp. $U_+^{(0)} + K$ and $U_+^{*k} + K$) if $\mathrm{ind} S = - k < 0$ (resp. $\mathrm{ind}\, S$

$= 0$ and $\text{ind}\, S = k > 0$), where $U_+^{(0)}$ is a simple bilateral shift, K is compact and ind S is the index of S. Hence a family of C*-algebras $\{C^*(U_+^{(k)}) + K(H)\}$ is a complete set of representatives for the extensions of $C(T)$, and the identifying map with Ext $C(T)$ and Z is $-\text{ind}\, U_+^{(k)} = k$. We remark that $C^*(U_+) + K(H) = C^*(U_+)$ by (1).

Let P_n be the C*-algebra generated by isometries $T_1, \dots, T_n$ such that $1 - \Sigma_i T_i T_i^*$ is a non-zero projection. Then

$$0 \longrightarrow K(H) \longrightarrow P_n \longrightarrow O_n \longrightarrow 0$$

is exact according to [12; 3.1]. Later, Enomoto, Takehana and Watatani realized P_n as the C*-algebra $C^*_r(G_n^+)$ generated by the left regular representations of a free semigroup G_n^+ on n generators. And they proved that P_n is unique up to isomorphism as well as O_n. Cuntz stated in [12; Remark 1 in § 3] that it seems to be interesting to study more general extensions of O_n by the compacts. Paschke and Salinas proved that Ext $O_n = Z$ by using an index of extensions and showed implicitly that a family $\{P_n\}$ is a complete set of representatives for extensions of O_n corresponding to the negative integers.

The fact that Ext $O_n = Z$ reminds us an analogy with Ext $C(T) = Z$. We shall give attention to the first isometry S_1 among the generators of O_n. A proof of Ext $O_n = Z$ will be obtained by using the C*-algebras P_n^k generated by $\{U_+^{(k)} S_1, S_2, \dots, S_n\}$ and $K(H)$, where U_+ is a simple unilateral shift on ran S_1 and $V^{(k)} = V^k$ $(k \geq 0)$ and V^{*k} $(k < 0)$. As a matter of fact, P_n^k is corresponding to an integer $k =$

$- \operatorname{ind} U_+^{(k)}$, which is the same as the case of $C(T)$.

As usual, an extension of a unital separable C*-algebra B is a *-monomorphism of B into $Q(H)$. Paschke and Salinas used the following index m for extensions of O_n to prove Ext $O_n = Z$: Let τ be an extension of $O_n = C^*(S_1, \ldots, S_n)$ and v_τ the matrix in $Q(H \oplus \ldots \oplus H) = Q(H) \otimes M_n$ with zeros in the second through n-th row and with $\tau(S_1) \ldots \tau(S_n)$ in the first row. Then v_τ is isometric and $v_\tau v_\tau^* = \pi(P_H)$, where P_H is the projection of $H \oplus \ldots \oplus H$ onto $H \oplus 0 \oplus \ldots \oplus 0$. So there is a partial isometry $V = V_\tau$ on $H \oplus \ldots \oplus H$ such that $\pi(V) = v_\tau$ and $VV^* \leq P_H$ [38; Lemma 1.1]. They put $m(\tau) = \dim(1 - V^*V) - \dim(P_H - VV^*)$. Note that $m(\tau) = \operatorname{ind} V$ as an operator of $H \oplus \ldots \oplus H$ into $H \oplus 0 \oplus \ldots \oplus 0$, and so $m(\tau)$ is well-defined. It is known that $m(\tau) = m(\tau')$ if τ and τ' are strongly equivalent, and that $m(\tau) = 0$ iff τ is trivial. Since $m(\tau \oplus \tau') = m(\tau) + m(\tau')$, m is a homomorphism of Ext O_n into Z. The fact to be established is that m is onto. Now we shall give a proof to this fact by using a unilateral shift U_+.

Theorem 2.1. Ext $O_n = Z$.

Proof. It suffices to show that m is onto. For the sake of simplicity, we consider the case of $O_2 = C^*(S_1, S_2)$. Let $V_+ = U_+ \oplus 0$ on $H = \operatorname{ran} S_1 \oplus (\operatorname{ran} S_1)^\perp$. Let us put $P_2^k = C^*(V_+^{(k)} S_1, S_2) + K(H)$. Then it follows from the uniqueness theorem on O_2 that $P_2^k / K(H)$ is isomorphic to O_2 via the quotient map π. We define the extension τ_k for each integer k by $\tau_k(S_1) = \pi(V_+^{(k)}S_1)$ and $\tau_k(S_2) = \pi(S_2)$. Since

$$v_k = \begin{pmatrix} \tau_k(S_1) & \tau_k(S_2) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \pi(V_+^{(k)}S_1) & \pi(S_2) \\ 0 & 0 \end{pmatrix},$$

we have $\pi(V_k) = v_k$ for $V_k = \begin{pmatrix} V_+^{(k)}S_1 & S_2 \\ 0 & 0 \end{pmatrix}$ and

$$\begin{aligned} m(\tau_k) &= \dim(1 - V_k{}^*V_k) - \dim(P_H - V_kV_k{}^*) \\ &= \dim(P_H - (V_+^{(k)}S_1)^*(V_+^{(k)}S_1)) \\ &\quad - \dim(P_H - (V_+^{(k)}S_1)(V_+^{(k)}S_1)^*) \\ &= -k. \end{aligned}$$

Hence this implies that m is onto.

We remark that P_1 is the Coburn algebra and $C(T)$ is regarded as O_1. Putting $P_1{}^k = C^*(U_+^{(k)}U) + K(H)$, we can prove that $\operatorname{Ext} C(T) = Z$. Actually, $P_1{}^k/K(H)$ is isomorphic to $C(T) = C^*(U)$, where U is a simple bilateral shift. If τ_k is the extension defined by $\tau_k(U) = \pi(U_+^{(k)}U)$, then $m(\tau_k) = -k = \operatorname{ind} U_+^{(k)}$. It is easily seen that $\{P_n^k\}_k$ is a complete set of representatives for extensions of O_n.

II-2. Extensions of O_A - tensor representation.

Cuntz and Krieger [17] constructed a new C*-algebra O_A which is associated with a topological Markov chain (X_A, σ_A). Let $A = (A(i,j))$ be an $n \times n$ matrix such that $A(i,j) = 0$ or 1 and every row and column is non-zero. A C*-algebra O_A is generated by non-zero partial isometries $S_1, \dots, S_n$ acting on a Hilbert space satisfying the condition

(A) $S_i^*S_j = 0$ for $i \neq j$, and $S_i^*S_i = \Sigma_j A(i,j)S_jS_j^*$ for all i. They proved:

The uniqueness theorem. The isomorphism class of O_A does not depend on the choice of generators if A satisfies the condition (I), see Lemma 2.1. Furthermore, if A is irreducible, i.e., it is not a permutation and for each i and j there is a k such that $A^k(i,j) > 0$, then O_A is uniquely determined and is simple.

We attempt a graph theoretic approach to Cuntz-Kriegr algebras. A digraph G is an ensemble of a finite set $V(G)$ of vertices $1,2, \dots, n$ and a finite set $E(G)$ of edges which are ordered pairs (i, j) of vertices. It is known that a digraph G is represented by an adjacency matrix A with 0 and 1 as entries: $A(i,j) = 1$ if $(i,j) \in E(G)$ and $A(i,j) = 0$ if not. Thus we identify a digraph with its adjacency matrix.

Now a path from j to i in G is a finite sequence of edges $\{(i_{k-1},i_k)\}$ such that $i_1 = i$ and $i_m = j$. A vertex has an m-cycle if there is a path $\{(i_{k-1},i_k)\}$ from i to i with $i_k \neq i$ for $2 \leq k \leq m-1$. Particularly, a 1-cycle is called a loop. We note that a vertex i has at least two different cycles if and only if $i \in \Sigma_0$, where Σ_0 is refered to [17]. Hence the condition (I) of Cuntz and Krieger is rephrased as follows:

Lemma 2.1 A digraph G satisfies the condition (I) if and only if for each $i \in V(G)$ there is a path from j to i for some $j \in \Sigma_0$.

This reformulation is very useful. For a digraph G satisfying the condition (I), $O_G = O_A$ is unique up to isomorphism. In the below, we always assume that a matrix A and a digraph G satisfy the condition (I).

Now let P_n be as in the above an extension of O_n by the compacts. Evans [30] and Katayama showed independently that P_n is realized as a 'tensor algebra' on the full Fock space $F(H)$, which is analogous to the construction of the CAR algebra on the anti-symmetric Fock space. Furthermore they constructed a unitary $F(u)$ on $F(H)$ for $u \in U(n)$. Then $\bar{\alpha}_u$ on P_n implemented by $F(u)$ corresponds to the automorphism α_u on O_n discussed in the preceding chapter.

In this section, we shall construct a subspace L_A of $F(H)$ associated with a matrix A and consider the C*-algebra P_A on L_A generated by the compressions to L_A of the creation operators on $F(H)$. We shall see that P_A is an extension of O_A by the compacts.

For an n-dimensional Hilbert space H, let $H_m = \otimes^m H$ be the m-fold tensor product and $F(H) = \Sigma_{m=0}^{\infty} \oplus H_m$ the Fock space, where H_0 is the 1-dimensional Hilbert space spanned by the Fock vacuum unit vector Ω. For $f \in H$, there is a bounded operator $o(f)$ on $F(H)$ such that

$$o(f)\Omega = f, \quad o(f)(f_1 \otimes \dots \otimes f_m) = f \otimes f_1 \otimes \dots \otimes f_m,$$

and

$$o(f)^*\Omega = 0, \quad o(f)^*(f_1 \otimes \dots \otimes f_m) = (f_1, f)\, f_2 \otimes \dots \otimes f_m.$$

Then the C*-algebra generated by $\{o(f)\ ;\ f \in H\}$ is isomorphic to P_n which is called the Clifford C*-algebra in [45].

Now we shall consider two subspaces of $F(H)$. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of H. Let L_m be the subspace of H_m spanned by

$$\{e_{i(1)} \otimes \dots \otimes e_{i(m)}\ ;\ A(i(k), i(k+1)) = 1 \quad \text{for} \quad 1 \le k \le m-1\}$$

and $L_A = \Sigma_{m=0}^{\infty} \oplus L_m$, where $L_0 = H_0$ and $L_1 = H_1 = H$. Let M_m be the subspace of H_m spanned by

$$\{e_{i(1)} \otimes \dots \otimes e_{i(m)}\ ;\ \Pi_{k=1}^{m-1} A(i(k), i(k+1)) = 0\}$$

and $M_A = \Sigma_{m=0}^{\infty} \oplus M_m$, where $M_0 = M_1 = \{0\}$. Then $F(H) = L_A \oplus M_A$ and L_A is called the sub-Fock space associated with A. Let us put $S_i = P_{L_A} o(e_i)|L_A$ for $1 \le i \le n$, where P_{L_A} is the projection onto L_A. Then we denote by P_A the C*-algebra generated by $\{S_i\ ;\ 1 \le i \le n\}$.

Theorem 2.2. The C*-algebra P_A acts irreducibly on L_A and contains the compacts $K(L_A)$. Moreover P_A is an extension of O_A by $K(L_A)$, that is,

$$(3) \qquad 0 \longrightarrow K(L_A) \longrightarrow P_A \longrightarrow O_A \longrightarrow 0$$

is exact.

To prove Theorem 2.2, we need following lemmas.

Lemma 2.3. Notation as in the above. Then each S_k is partial isometry such that $S_k{}^*S_k\Omega = \Omega$, $S_kS_k{}^*\Omega = 0$,

$$S_k{}^*S_k(e_{i(1)}\otimes \cdots \otimes e_{i(m)}) = A(k,i(1))e_{i(1)}\otimes \cdots \otimes e_{i(m)}$$

and

$$S_kS_k{}^*(e_{i(1)}\otimes \cdots \otimes e_{i(m)}) = \delta(k,i(1))e_{i(1)} \otimes \cdots \otimes e_{i(m)}$$

for all $g = e_{i(1)}\otimes \cdots \otimes e_{i(m)} \in L_A$, where $\delta(k,i)$ is Kronecker's delta.

Proof. We put $P = P_{L_A}$, $P_k = S_kS_k{}^*$ and $Q_k = S_k{}^*S_k$ for $1 \le k \le n$. Then we have

$$Q_k\Omega = Po(e_k)^*Po(e_k)\Omega = Po(e_k)^*Pe_k = Po(e_k)^*e_k = P\Omega = \Omega.$$

Since $o(e_k)^*\Omega = 0$, it follows that $P_k\Omega = 0$.

Next we have

$$\begin{aligned} Q_ke_i &= Po(e_k)^*Po(e_k)e_i = Po(e_k)^*P(e_k\otimes e_i) \\ &= Po(e_k)^*A(k,i)e_k\otimes e_i \\ &= A(k,i)Pe_i = A(k,i)e_i, \end{aligned}$$

so that $Q_kg = A(k,i)g$ for $g \in L_A$. Since $o(e_k)^*e_i = \delta(k,i)e_i$ we have $P_ke_i = \delta(k,i)e_i$, which implies $P_kg = \delta(k,i(1))g$.

Lemma 2.4. If E is the projection onto $L_0 = H_0$, then

(4) $$S_k{}^*S_k = \Sigma_j\, A(k,j)S_jS_j{}^* + E.$$

Proof. Let P_k and Q_k be as in above. Then we have

$$(\Sigma_jA(k,j)P_j + E)\Omega = \Omega = Q_k\Omega$$

by Lemma 2.3. Next we have

$$(\Sigma_j A(k,j)P_j + E)e_i = \Sigma_j A(k,j)P_j e_i$$
$$= \Sigma_j A(k,j)\delta(j,i)e_i = A(k,i)e_i = Q_k e_i.$$
Hence it implies that $(\Sigma_j A(k,j)P_j + E)g = Q_k g$ for $g \in L_A$.

Remark. It follows from (4) that E is in P_A and

(5) $\qquad \Sigma_j P_j + E = 1 \quad$ on L_A.

Proof of Theorem 2.2. First of all, we shall prove that $P_A x$ is dense in L_A for all $0 \neq x \in L_A$. Since $x \neq 0$, there is m such that the direct summand x_m of x on L_m is non-zero. If $x_m = \Sigma x_m(i(1), \ldots, i(m))e_{i(1)} \otimes \cdots \otimes e_{i(m)}$, where Σ is taken over $(i(1), \ldots, i(m))$ such that $\Pi_{k=1}^{m-1} A(i(k), i(k+1)) = 1$, then there is $\mu = (i(1), \ldots, i(m))$ such that $x_m(\mu) \neq 0$. Since $S_\mu^* x = x_m(\mu)\Omega + y$ for some $y \in \Sigma_{h=1}^{\infty} \oplus L_h$, we have $ES_\mu^* x = x_m(\mu)\Omega \neq 0$. Furthermore, for any $z = e_{j(1)} \otimes \cdots \otimes e_{j(h)} \in L_A$ we have

$$x_m(\mu)^{-1} S_{j(1)} \cdots S_{j(h)} E S_\mu^* x = e_{j(1)} \otimes \cdots \otimes e_{j(h)},$$

which implies that $P_A x$ is dense in L_A. Since E is rank one, P_A contains the compact operators $K(L_A)$.

Let π be the quotient map of $B(L_A)$ onto $Q(L_A)$. Then $\pi(E) = 0$. Noting that the range of P_i is infinite dimensional by the condition (I), we put $T_k = \pi(S_k) \neq 0$ for $1 \leq k \leq n$. Then $\pi(P_A)$ is generated by partial isometries $T_1, \ldots, T_n$, and

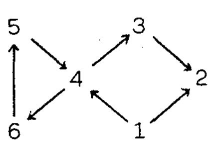
$$T_k^* T_k = \Sigma_j A(k,j)T_j T_j^* \quad \text{and} \quad \Sigma_j T_j T_j^* = 1$$

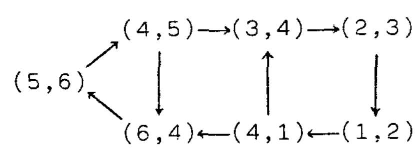
by (4) and (5). Therefore $\pi(P_A)$ is isomorphic to O_A.

II-3. Extensions of O_A - adjoint graphs.

According to [3], the adjoint graph G^* of a digraph G is defined to be a digraph whose vertices $u_1, \dots, u_m$ represent the edges of G and which has an edge $u_i \leftarrow u_j$ if $i_2 = j_1$ where $u_i = (i_1, i_2)$ and $u_j = (j_1, j_2)$. In this section, we shall prove that $O_G = O_{G^*}$. By using this, we shall give another extension of O_A.

First of all, we show an example of an adjoint graph:

G : 

G* : 

Now we shall make sure that the adjoint graph of a digraph with the condition (I) satisfies (I) also.

Lemma 2.5. If a digraph G satisfies the condition (I), then so does the adjoint G^* of G.

Proof. It suffices to show that for each $(i_1, i_2) \in V(G^*)$ there is a vertex $(r,s) \in \Sigma_0$ having a path $P((i_1,i_2),(r,s))$ in G^*. Since G satisfies the condition (I), a vertex i_2 of G has a path $P(i_2, i_0)$ for some $i_0 \in \Sigma_0$, which induces a path $P((i_1,i_2),(i_2,i_3),\dots,(i_{k-1},i_k),(i_k,i_0))$ in G^*. On the other hand, since $i_0 \in \Sigma_0$, there are two different cycles E and F in G such that $i_0 \in V(E) \cap V(F)$, so there is a path $P((i_k,i_0),(r,s))$. Hence there is a path $P((i_k,i_0),(r,s))$ and

$(r,s) \in \Sigma_0$, that is, G^* satisfies (I).

Now we shall realize edges of G as partial isometries $T_{i,j} = S_iP_j$ in O_G, where $P_j = S_jS_j^*$.

Lemma 2.6. Let $O_A = C^*(S_1, \ldots ,S_n)$ and $T_{i,j} = S_iP_j$. then $T_{i,j} = 0$ if and only if $A(i,j) = 0$.

Proof. Note that $T_{i,j} = 0$ iff $P_jS_i^*S_iP_j = 0$, or equivalently $S_i^*S_iP_j = 0$. If $A(i,j) = 1$, then $S_i^*S_i =$ $\Sigma_k A(i,k)P_k \geq P_j$. Therefore we have $S_i^*S_iP_j = P_j$, so that $T_{i,j} = 0$. Conversely, if $A(i,j) = 0$, then

$$S_i^*S_iP_j = \Sigma_k A(i,k)P_kP_j = A(i,j)P_j = 0.$$

Hence it implies $T_{i,j} = 0$.

Theorem 2.7. $O_{G^*} = O_G$.

Proof. Since O_G coincides with the C*-algebra B generated by $\{T_{i,j};\ A(i,j) = 1\}$, we shall show that B is the Cuntz Krieger algebra O_{G^*}, that is, a family $\{T_{i,j};\ A(i,j) = 1\}$ satisfies the condition (A) and $\Sigma_{i,j}T_{i,j}T_{i,j}^* = 1$. By Lemma 2.6 we have

$$\Sigma_{A(i,j)=1}T_{i,j}T_{i,j}^* = \Sigma_{i,j}S_iS_jS_j^*S_i^* = \Sigma_i S_i(\Sigma_j P_j)S_i^* = 1.$$

Next, if $(i,j) \neq (p,q) \in E(G)$, then $i \neq p$ or $j \neq q$. If $i \neq p$, then $T = T_{i,j}T_{i,j}^*T_{p,q}T_{p,q}^* = 0$. On the other hand, if $i = p$ and $j \neq q$, then $T = S_iP_jS_i^*S_iP_qT_{p,q}^* = S_iP_jP_qT_{p,q}^* = 0$ because $P_jP_q = 0$. Finally, by the definition of the adjoint graph and Lemma 2.5 we have

$$\Sigma_{(i,j)} A^*((p,q),(i,j)) T_{i,j} T_{i,j}^*$$
$$= \Sigma_{A(q,j)=1} A^*((p,q),(q,j)) S_q P_j S_q^*$$
$$= \Sigma_j T_{q,j} S_q^* = S_q(\Sigma_j P_j) S_q^* = P_q.$$

Since $T_{p,q}^* T_{p,q} = P_q S_p^* S_p P_q = P_q$, it follows that $\{T_{i,j};\ A(i,j) = 1\}$ satisfies the condition (A).

After [32], we say that a category $D(G)$ is the free category of a digraph G if $D(G)$ is a category whose morphisms consist of all paths in G and whose objects consist of $V(G)$. Let $s(g)$ be the source of $g \in D(G)$ and $t(g)$ the target of g. Let $l^2(D(G))$ be the Hilbert space of all square summable sequences on $D(G)$ with the orthonormal basis $\{e_d ;\ d \in D(G)\}$, where $e_d(g) = \delta_{d,g}$ for $g \in D(G)$. For each $i \in V(G)$ let H_i be the subspace of $l^2(D(G))$ spanned by $\{e_d ;\ d \in D(G),\ s(d) = i\}$. Now we shall define the left regular representation u of $D(G)$ on $l^2(D(G))$. For each $g \in D(G)$, a partial isometry u_g on $l^2(D(G))$ is defined by $u_g e_h = e_{gh}$ if $s(g) = t(h)$ and $u_g e_h = 0$ if not. Let $C^*_r(G)$ denote the C*-algebra generated by $\{u_g;\ g \in D(G)\}$. Since $u_g^* u_h = e_k$ if $h = gk$ for some k and $u_g^* e_h = 0$ if not, every H_i is invariant under $C^*_r(G)$. So, putting $\rho_i(a) = a|H_i$ for $a \in C^*_r(G)$ and $i \in V(G)$, then ρ_i is a representation of $C^*_r(G)$ on H_i and $\oplus_{i \in V(G)} \rho_i$ is the identity representation of $C^*_r(G)$ on $l^2(D(g))$.

Theorem 2.8. The representation ρ_i is irreducible for

$C^*_r(G)$ and $\rho_i(C^*_r(G))$ contains the compacts $K(H_i)$. Furthermore, if G satisfies the condition (I), then

$$0 \longrightarrow K(H_i) \longrightarrow \rho_i(C^*_r(G)) \longrightarrow O_G \longrightarrow 0$$

is exact.

Proof. By the definition of u_g, we have $u_g{}^*u_g e_h = e_h$ if $s(g) = t(h)$ and $u_g{}^*u_g e_h = 0$ if not, and $u_g u_g{}^* e_b = e_b$ if $b = gh$ for some h and $u_g u_g{}^* e_b = 0$ if not. Therefore $u_g{}^*u_g$ (resp. $u_g u_g{}^*$) is the projection on $[e_h;\ s(g) = t(h)]$ (resp. $[e_{gh};\ h \in D(G)]$), where $[M]$ denotes the subspace spanned by M. Since $P = 1 - \Sigma_{t \in E(G)} u_t u_t{}^*$ is the projection on $[e_j;\ j \in V(G)]$, it follows that $\rho_i(P)$ is the projection $[e_i]$ for every $i \in V(G)$.

To show the irreducibility of ρ_i, we shall prove that $\rho_i(C^*_r(G))x$ is dense in H for all non-zero $x \in H_i$. Let $x = \Sigma_{s(b)=i} x(b)e_b \in H_i$. Then there is $g \in D(G)$ such that $s(g) = i$ and $x(g) \neq 0$. Since $u_g{}^*x = \Sigma\, x(gh)e_h$ where Σ is taken over h such that $s(h) = i$ and $s(g) = t(h)$, we have $Pu_g{}^*x = x(gi)e_i = x(g)e_i \neq 0$. Moreover, if $k \in D(G)$ and $s(k) = i$, then

$$\rho_i(u_k)\rho_i(P)\rho_i(u_k)^*x = u_k P u_k{}^*x = x(g)u_k e_i = x(g)e_k.$$

Hence it follows that ρ_i is irreducible on H_i. Since $\rho_i(P)$ is rank one, $\rho_i(C^*_r(G))$ contains $K(H_i)$.

Let π be the quotient map of $B(H_i)$ onto $Q(H_i)$. It is clear that $\pi\rho_i(C^*_r(G))$ is generated by partial isometries $\{T_g;\ g \in E(G)\}$, where $T_g = \pi\rho_i(u_g)$. So we shall show that $C^*(T_g;\ g \in E(G))$ is the C^*-algebra O_{G^*}. Since $\rho_i(P)$ is rank one, we have $\Sigma\, T_g T_g^* = 1$. Furthermore $\rho_i(u_g)^*\rho_i(u_g)$ (resp. $\rho_i(u_g)\rho_i(u_g)^*$) is the projection on $[e_h;\ s(g) = t(h), s(h) = i]$ (resp. $[e\ ;\ k = gh$ for some $h \in D(G)$ with $s(h) = i]$). It follows that $T_g^*T_g = \Sigma_{h\in E(G)} A^*(g,h) T_h T_h^*$, where A^* is the adjacency matrix of G^*. Hence it implies that $C^*(T_g;\ g \in E(G)) = O_{G^*}$, so that $\rho_i(C^*_r(G))/K(H_i) = O_G$ by the preceding theorem.

II-4. Applications to automorphisms on O_A.

In the first chapter, we have discussed a representation of the unitary group $U(n)$ into the outer automorphisms on O_n. Unfortunately, for general $O_A = C^*(T_1, \ldots, T_n)$ there are unitaries u such that $\alpha_u(T_i) = \Sigma_k u_{ki} T_k$ cannot be extended to automorphisms on O_A. For example, if $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then α_u can be extended to an automorphism on O_A if and only if u is diagonal. As applications of extensions in II-2, we shall characterize unitary matrices such that α_u can be extended to automorphisms on O_A.

Let H be an n-dimensional Hilbert space with an orthonormal basis $\{e_i\}$. For each $u \in U(n)$, let us put $U_o = 1$ on $H_o = H$, $U_m = x^m u$ on $H_m = x^m H$ for $m \geq 1$, and $F(u) = \Sigma_m + U_m$ on the full Fock space $F(H)$. Then $F(u)$ is unitary. Evans and Katayama showed that $F(u)$ implements an automorphism $\bar{\alpha}_u$ on $P_A = C^*(o(e_i);\ 1 \leq i \leq n)$ such that

$$\bar{\alpha}_u(o(e_i)) = F(u)o(e_i)F(u)^* = \Sigma_k\, u_{ki}\, o(e_k).$$

First of all, we shall consider a condition on $u \in U(n)$ such that the sub-Fock space L_A associated with A reduces $F(u)$.

Lemma 2.9. Let $A = (A(i,j))$ be an $n \times n$ matrix and $u = (u_{ij}) \in U(n)$ such that $A(i,j) = 0$ and $A(k,m) = 1$ imply $u_{ki}u_{mj} = 0$ for all i,j,k,m. Then the sub-Fock space L_A associated with A is reducing for $F(u)$.

Proof. It is obvious that $U_oM_o = U_o\{0\} = \{0\}$ and $U_1M_1 = U_1\{0\} = \{0\}$. If $e_i \otimes e_j \in M_2$, i.e., $A(i,j) = 0$, then

$$U_2(e_i \otimes e_j) = (u \otimes u)(e_i \otimes e_j) = (\Sigma_k u_{ki}e_k) \otimes (\Sigma_m u_{mj}e_m)$$
$$= \Sigma_{A(k,m)=0}\, u_{ki}u_{mj}e_k \otimes e_m + \Sigma_{A(k,m)=1} u_{ki}u_{mj}e_k \otimes e_m$$
$$= \Sigma_{A(k,m)=0}\, u_{ki}u_{mj}e_k \otimes e_m \in M_2$$

by the assumption. Similarly we have $U_mM_m \subseteq M_m$ for $m \geq 3$.

Theorem 2.10. Let $O_A = C^*(T_1, \ldots , T_n)$. Then the following statements are equivalent for $u \in U(n)$;

(1) $\alpha_u(T_i) = \Sigma_k u_{ki}T_i$ can be extended to an automorphism on O_A,

(2) $(1 - A(i,j))A(k,m)u_{ki}u_{mj} = 0$ for all i,j,k,m, and

(3) $A(i,j) = 0$ and $A(k,m) = 1$ imply $u_{ki}u_{mj} = 0$ for all i,j,k,m.

Proof. It is clear that (2) and (3) are equivalent by noting the case that $A(i,j) = 0$ and $A(k,m) = 1$. Suppose that (1) is hold. If $A(i,j) = 0$, then $T_iT_j = 0$, so that

$$0 = \alpha_u(T_iT_j) = \alpha_u(T_i)\alpha_u(T_j) = (\Sigma_k u_{ki}T_k)(\Sigma_m u_{mj}T_m)$$
$$= \Sigma_{k,m}u_{ki}u_{mj}T_kT_m = \Sigma_{A(k,m)=1}u_{ki}u_{mj}T_kT_m.$$

Since $\{T_kT_m;\ A(k,m)=1\}$ is linearly independent, we have $u_{ki}u_{mj}=0$ if $A(k,m)=1$, which implies (3).

Conversely, suppose that u satisfies (3). By Lemma 2.9, a unitary $F(u)|L_A$ implements an automorphism γ_u on P_A. For generators $S_i = Po(e_i)|L_A$ of P_A where $P = P_{L_A}$, we have

$$\begin{aligned}\gamma_u(S_i) &= F(u)S_iF(u)^*|L_A = F(u)Po(e_i)PF(u)^*|L_A\\ &= PF(u)o(e_i)F(u)^*|L_A = P\Sigma_k u_{ki}o(e_k)|L_A\\ &= \Sigma_k u_{ki}S_k.\end{aligned}$$

Since $\gamma_u(K(L_A)) = K(L_A)$, γ_u induces an automorphism α_u on O_A such that $\alpha_u(\pi(X)) = \pi(\gamma_u(X))$ for $X \in P_A$ by Theorem 2.8, where π is the quotient map of P_A onto O_A and $T_i = \pi(S_i)$. Moreover we have

$$\alpha_u(T_i) = \alpha_u(\pi(S_i)) = \pi(\gamma_u(S_i)) = \pi(\Sigma_k u_{ki}S_k) = \Sigma_k u_{ki}T_k.$$

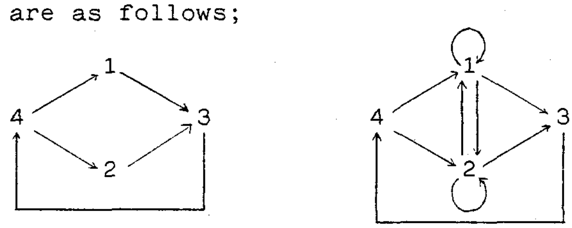
Corollary 2.11. Let $O_A = C^*(T_i;\ 1 \le i \le n)$. Then $O_A = O_n$ if and only if α_u can be extended to an automorphism on O_A for all $u \in U(n)$.

Proof. Suppose that α_u can be extended to an automorphism on O_A for all $u \in U(n)$. Then (2) in Theorem 2.10 holds true for all $u \in U(n)$. For $n \ge 3$, let q be the matrix whose entries are $1/n$, and $r = 2q - 1$. Then we have

$$(1 - A(i,j))A(k,m) = 0$$

for $1 \le i,j,k,m \le n$. Since $A(k,m) = 1$ for some k and m, it follows that $1 - A(i,j) = 0$ for all i and j, so that $O_A = O_n$. If $n = 2$, then we consider $r = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}/\sqrt{2}$.

Incidentally, we shall discuss outerness of automorphisms on O_A. So we introduce a notion on vertices of digraphs. Vertices i and j of a digraph G are equivalent if $A(i,k) = A(j,k)$ and $A(k,i) = A(k,j)$ for all $k \in V(G)$. Typical examples are as follows;



Then 1 and 2 are equivalent.

Corollary 2.12. Let G be a digraph with n vertices such that $1, \ldots, m$ are equivalent. If $u = (u_{ij})$ is a unitary matrix such that $u_{ij} = \delta_{ij}$ for $m+1 \leq i,j \leq n$, then α_u can be extended to an automorphism on O_A. Furthermore, if G is strongly connected , then α_u is outer except $u = 1$.

Proof. It suffices to show that $A(i,j) = 1$ or $A(k,p) = 0$ if $u_{ki}u_{pj} = 0$. Note that $k = i$ or $1 \leq k,i \leq m$ if $u_{ki} \neq 0$. So we must consider the following four cases; (i) $k = i$ and $p = j$, (ii) $k = i$ and $1 \leq p, j \leq m$, (iii) $p = j$ and $1 \leq k,i \leq m$, and (iv) $1 \leq k,i \leq m$ and $1 \leq p,j \leq m$.

(i) implies that $A(i,j) = A(k,p)$. (ii) implies that $A(k,p) = A(i,p)$ and $A(i,p) = A(i,j)$ by the equivalence of p and j. Similarly (iii) implies that $A(i,j) = A(k,p)$. Finally (iv) implies that $A(k,p) = A(k,j) = A(i,j)$. Hence we have $A(i,j) = A(k,p)$ for all cases, so that $A(i,j) = 1$ or $A(k,p) = 0$.

Suppose that G is strongly connected. To prove that α_u is outer, we may assume that u is diagonal and $u_{11} \neq 1$. It follows from [29; Remark in § 3] that α_u is outer if 1 has a loop. If 1 has no loop, then 1 has a q-cycle $\{(i_{k-1}, i_k)\}$ such that $i_k \neq i_p$ for $k \neq p$ and $m+1 \leq i_k \leq n$ for $2 \leq k \leq q-1$. (See the above examples.) Since $u_{11} \neq 1$ and $u_{jj} = 1$ for $m+1 \leq j \leq n$, it follows that α_u is outer.

Chapter III K-theory for O_A

III-1. Prologue.

Cuntz and Krieger proved that the weak extension group $\mathrm{Ext}^w O_A$ is isomorphic to $Z^n/(1 - A)Z^n$, the Bowen-Franks invariant for a subshift σ_A. And Cuntz [14] showed that $K_o(O_A)$ is isomorphic to $Z^n/(1 - {}^tA)Z^n$. In addition, it is known that $K_o(B)$ is realized as $B^p/\approx$ for any unital purely infinite simple C*-algebra B, where $\approx$ is the von Neumann equivalence among the non-zero projections B^p in B, so that we identify the corresponding class in K_o-group with the von Neumann equivalence class $[\![P]\!]_\approx$ of $P \in B^p$. Moreover O_A is unital, purely infinite and simple for irreducible A. We here remark that A is irreducible if and only if the corresponding digraph G of A is strongly connected, i.e., for any vertices $i \neq j$ of G there are paths $P(i,j)$ and $P(j,i)$.

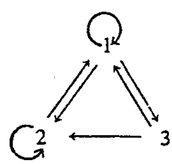
Now we shall introduce a new invariant for unital C*-algebras: Let B be a C*-algebra with unit 1. Then $[\![1]\!]$ stands for the corresponding class in $K_o(B)$ for 1. For $g,h \in K_o(B)$ we write $g \sim h$ if $g = \alpha(h)$ for some automorphism α of $K_o(B)$. Putting $K_o(B)^- = K_o(B)/\sim$, the marker of B is the equivalence class $[\![1]\!]^-$ of $[\![1]\!]$. In particular, since $K_o(B)$ is identified with $B^p/\approx$ for a unital purely infinite simple C*-algebra B, we have $\mathrm{mark}(B) = [\![1]\!]_\approx^-$.

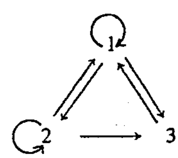
The following theorem is evident but very important:

Theorem 3.1. If B and C are unital C*-algebras which are isomorphic, then $K_o(B) = K_o(C)$ and $\mathrm{mark}(B) = \mathrm{mark}(C)$.

Here we show simple examples to apply Theorem 1.1.

Example 3.1.1. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = {}^tA$, whose corresponding digraphs are as follows;

G; 

H; 

It is easily seen that G and H are strongly connected and $K_o(O_G) = K_o(O_H) = Z_2 = (Z/2Z)$. Therefore $K_o(O_G)^- = K_o(O_H)^- = Z_2^- = \{\bar{0}, \bar{1}\}$. Since G and H are strongly connected, these C*-algebras are unital purely infinite and simple. If $O_G = C^*(S_1, S_2, S_3)$ and $P_i = S_i S_i^*$ for $i = 1,2,3$, then

$$\begin{pmatrix} \llbracket P_1 \rrbracket \\ \llbracket P_2 \rrbracket \\ \llbracket P_3 \rrbracket \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \llbracket P_1 \rrbracket \\ \llbracket P_2 \rrbracket \\ \llbracket P_3 \rrbracket \end{pmatrix} = \begin{pmatrix} \llbracket P_1 \rrbracket + \llbracket P_2 \rrbracket + \llbracket P_3 \rrbracket \\ \llbracket P_1 \rrbracket + \llbracket P_2 \rrbracket + \llbracket P_3 \rrbracket \\ \llbracket P_1 \rrbracket \end{pmatrix}$$

so that $\llbracket 1 \rrbracket = \llbracket P_1 \rrbracket = \llbracket P_2 \rrbracket$ and $\llbracket P_3 \rrbracket = \llbracket P_1 \rrbracket$. Hence $\llbracket 1 \rrbracket$ must be a generator of Z_2, that is, $\mathrm{mark}(O_G) = \bar{1}$. On the other hand, we have $\mathrm{mark}(O_H) = \bar{0}$. Actually $\llbracket 1 \rrbracket = \llbracket P_1 \rrbracket$ and $\llbracket P_1 \rrbracket + \llbracket P_2 \rrbracket = \llbracket P_2 \rrbracket$, so that $\llbracket 1 \rrbracket = \llbracket P_1 \rrbracket$ is neutral in $K_o(O_H)$. By Theorem 3.1, O_G and O_H are non-isomorphic.

Next we shall consider the case that $K_o(B) = Z$.

Example 3.1.2. For each non-negative integer $n \in Z^-$, there is a Cuntz-Krieger algebra O_G such that $K_o(O_G) = Z$ and $\mathrm{mark}(O_G) = n$.

For $n \geq 1$, let $A(n)$ be a matrix with degree $n+4$;

$$A(n) = \begin{pmatrix} 1 & 1 & 1 & 1 & & & \\ 1 & 1 & 0 & 0 & & & \\ 1 & 0 & 1 & 1 & & & \\ 0 & 1 & 0 & 1 & \ddots & & \\ & \vdots & & & \ddots & \ddots & \\ & 1 & & & & \ddots & 1 \\ 1 & 0 & & & & & 1 \end{pmatrix} \quad \text{and} \quad A(0) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then $K_o(O_{G(n)}) = Z$ and $\mathrm{mark}(O_{G(n)}) = n$ for $n \geq 0$.

III-2. Transfered graphs.

In order to classify simple Cuntz-Krieger algebras O_A (for 3×3 matrices A), we shall introduce transfered graphs of digraphs. First of all, we begin with the following simplest example:

Example 3.2.1. Let $O_2 = C^*(S_1,S_2)$, $T_1 = S_1$ and $T_2 = S_2S_1{}^*$. Then $C^*(T_1,T_2)$ is isomorphic to O_B, where $B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $C^*(T_1,T_2) = C^*(S_1,S_2)$ as a set, that is, $O_2 = O_B$.

The above example inspires us the following definition. Let G be a digraph and $\Gamma^-(i) = \{j \in V(G);\ j \longrightarrow i\}$, where $i \longrightarrow j$ stands for the edge (i,j).

Definition 3.2. Suppose that $\Gamma^-(k) = \Gamma^-(m)$ for some $k \neq m \in V(G)$. Then the transfered graph $H = G(k \to m)$ from k to m is defined by $V(H) = V(G)$ and $E(H) = (E(G) \setminus \{(m,i) \in E(G);\ i \in V(G)\}) \cup \{(m,k)\}$, that is, take away all edges whose targets are m and add the edge $k \to m$.

The adjacency matrix B of $G(k \to m)$ is determined as follows: Let A_i be the i-th row vector of A. Then $\Gamma^-(k) = \Gamma^-(m)$ means $A_k = A_m$. We then put

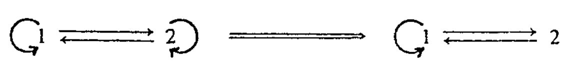
$$B(i,j) = \begin{cases} A(i,j) & \text{for } i \neq m, \\ \delta_{k,j} & \text{for } i = m. \end{cases}$$

For the sake of convenience, we denote it by

$$A \xRightarrow[A_k \longrightarrow A_m]{} B.$$

So Example 3.2.1 is changed in the following form;

Example 3.2.2.



$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad A_1 \longrightarrow A_2 \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

In general, we obtain that the transfered graph preserves isomorphisms between Cuntz-Krieger algebras.

Theorem 3.3. Let $H = G(k \to m)$ be the transfered graph of a digraph G from k to m. Then O_H is isomorphic to O_G.

Proof. Let A and B be the adjacency matrices of G and H respectively, and $O_A = C^*(S_1, \ldots, S_n)$. Since $k \neq m$

by definition, we assume that $k = 2$ and $m = 1$. Now we put $T_1 = S_1S_2{}^*$ and $T_i = S_i$ for $2 \le i \le n$. Since $A_1 = A_2$ where A_i is the i-th row vector of A, we have $S_1{}^*S_1 = S_2{}^*S_2$ and so

$$T_1T_1{}^* = S_1S_2{}^*S_2S_1{}^* = S_1(S_1{}^*S_1)S_1{}^* = S_1S_1{}^*,$$

so that $T_iT_i{}^* = S_iS_i{}^*$ for $1 \le i \le n$. Furthermore, since $B(i,j) = A(i,j)$ for $i \neq 1$ and $B(1,j) = \delta_{2,j}$, we have

$$\begin{aligned} T_1{}^*T_1 &= S_2S_1{}^*S_1S_2{}^* = S_2S_2{}^*S_2S_2{}^* = S_2S_2{}^* = \Sigma_j\ \delta_{2,j}S_jS_j{}^* \\ &= \Sigma_j\ B(1,j)S_jS_j{}^* = \Sigma_j\ B(1,j)T_jT_j{}^*, \end{aligned}$$

and for $2 \le i \le n$

$$T_i{}^*T_i = S_i{}^*\ S_i = \Sigma_j\ A(i,j)S_jS_j{}^* = \Sigma_j\ B(i,j)T_jT_j{}^*.$$

Hence $C^*(T_1, \ldots, T_n)$ is isomorphic to the Cuntz-Krieger algebra O_B. It is clear that $C^*(T_1, \ldots, T_n) \subseteq C^*(S_1, \ldots, S_n)$. On the other hand, since

$$T_1T_2 = S_1S_2{}^*S_2 = S_1S_1{}^*S_1 = S_1,$$

we have $C^*(S_1, \ldots, S_n) = C^*(T_1, \ldots, T_n)$ as a set. Since $C^*(S_1, \ldots, S_n)$ does not depend on the choice of generators, $C^*(T_1, \ldots, T_n)$ is isomorphic to O_B.

Next we shall generalize the above transfered graph of a digraph.

Definition 3.4. Let A be an $n \times n$ matrix, and $E_i = (0,\ldots,0,\overset{i}{1},0,\ldots,0)$ for $1 \le i \le n$. Suppose that

$$A_p = E_{k(1)} + \cdots + E_{k(r)} + A_{m(1)} + \cdots + A_{m(s)}$$

for some $k(1), \ldots, k(r), m(1), \ldots, m(s)$ which are mutually different and $p \notin \{m(1), \ldots, m(s)\}$. Then an $n \times n$ matrix B is defined by

$$B(i,j) = \begin{cases} A(i,j) & \text{for } i \neq p \\ 1 & \text{for } i = p \text{ and } j \in \{k(1),..,k(r),m(1),..,m(s)\} \\ 0 & \text{otherwise,} \end{cases}$$

and B is called to be primitively transfered from A, in symbol, $A \underset{\text{prim}}{\Longrightarrow} B$, or more precisely,

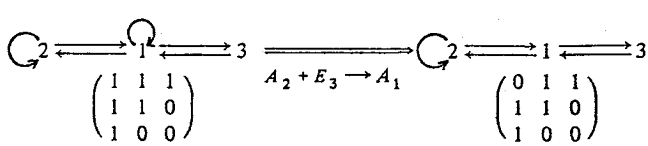
$$(*) \quad A \xrightarrow[E_{k(1)} + \cdots + E_{k(r)} + A_{m(1)} + \cdots + A_{m(s)} \longrightarrow A_p]{} B.$$

The primitive transformation '$\underset{\text{prim}}{\Longrightarrow}$' generates the following equivalence relation which is called the primitive equivalence; $A \underset{\text{prim}}{\sim} B$ if and only if there are matrices C_1, ... , C_q such that

$$A \underset{\text{prim}}{\Longleftrightarrow} C \underset{\text{prim}}{\Longleftrightarrow} \cdots \underset{\text{prim}}{\Longleftrightarrow} C \underset{\text{prim}}{\Longleftrightarrow} B,$$

where $C \underset{\text{prim}}{\Longleftrightarrow} D$ means that $C \underset{\text{prim}}{\Longrightarrow} D$ or $D \underset{\text{prim}}{\Longrightarrow} C$.

Example 3.2.3.



Here we have a generarization of Theorem 3.3.

Theorem 3.5. If A is primitively equivalent to B, then O_A is isomorphic to O_B.

Proof. We assume $(*)$ and $p = 1$. Let $O_A = C^*(S_1,\ldots,S_n)$, $P_i = S_iS_i{}^*$ and $Q_i = S_i{}^*S_i$. Then we put

$$T_1 = S_1(P_{k(1)} + \ldots + P_{k(r)} + S_{m(1)}{}^* + \ldots + S_{m(s)}{}^*)$$

and $T_i = S_i$ for $i \neq 1$. Since

$$A_1 = E_{k(1)} + \dots + E_{k(r)} + A_{m(1)} + \dots + A_{m(s)},$$

it follows that

$$Q_1 = P_{k(1)} + \dots + P_{k(r)} + Q_{m(1)} + \dots + Q_{m(s)}.$$

Then $\{P_{k(i)}, Q_{m(j)}; 1 \leq i \leq r, 1 \leq j \leq s\}$ is a family of orthogonal projections. Furthermore, since $k(1), \dots, k(r), m(1), \dots, m(s)$ are mutually different, a family $\{P_{k(i)}, P_{m(j)}; 1 \leq i \leq r, 1 \leq j \leq s\}$ is orthogonal. Hence we have

$$T_1 T_1^* = S_1(S_1^* S_1)S_1,$$

so that $T_i T_i^* = P_i$ for $1 \leq i \leq n$. On the other hand,

$$T_i^* T_i = \Sigma_j B(1,j)P_j = \Sigma_j B(1,j)T_j T_j^*$$

and for $i \neq 1$

$$T_i^* T_i = Q_i = \Sigma_j A(i,j)P_j = \Sigma_j B(i,j)T_j T_j^*.$$

Therefore $C^*(T_1, \dots, T_n)$ is isomorphic to O_B. It is clear that $C^*(T_1, \dots, T_n) \subseteq C^*(S_1, \dots, S_n)$. Since $1 \neq m(j)$ for $1 \leq j \leq s$ by definition, we have

$$T_1(T_{k(1)}T_{k(1)}^* + \dots + T_{k(r)}T_{k(r)}^* + T_{m(1)} + \dots + T_{m(s)})$$
$$= S_1(P_{k(1)} + \dots + P_{k(r)} + Q_{m(1)} + \dots + Q_{m(s)})$$
$$= S_1(S_1^* S_1)$$
$$= S_1,$$

so that $C^*(T_1, \dots, T_n) = C^*(S_1, \dots, S_n)$ as a set. Hence it implies that O_A is isomorphic to O_B. Since O_A does not depend on the choice of generators, O_B does not depend on them either.

III-3. Classifications of O_A.

Now we shall classify Cuntz-Krieger algebras O_A for 3 x 3 irreducible matrices A and pose a classification table expressed by the corresponding digraphs.

Theorem 3.6. Let A and B be 3 x 3 irreducible matrices. Then the followings are equivalent:

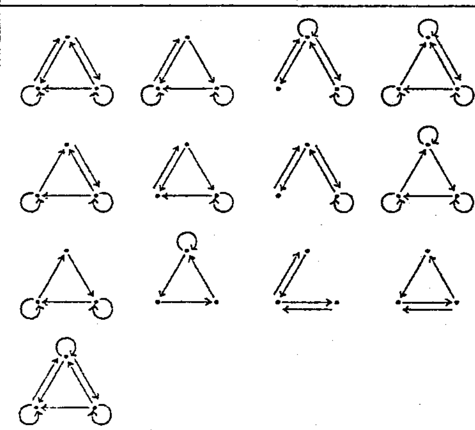
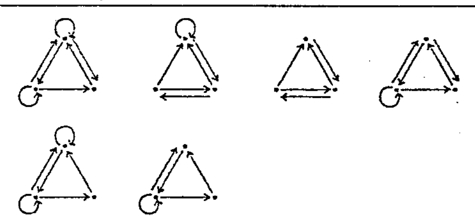
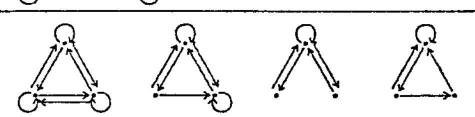
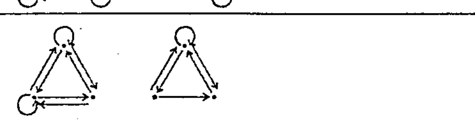
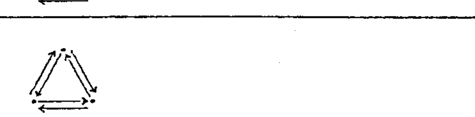
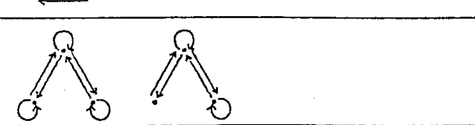
(1) O_A is isomorphic to O_B,

(2) $K_o(O_A) = K(O_B)$ and $\mathrm{mark}(O_A) = \mathrm{mark}(O_B)$, and

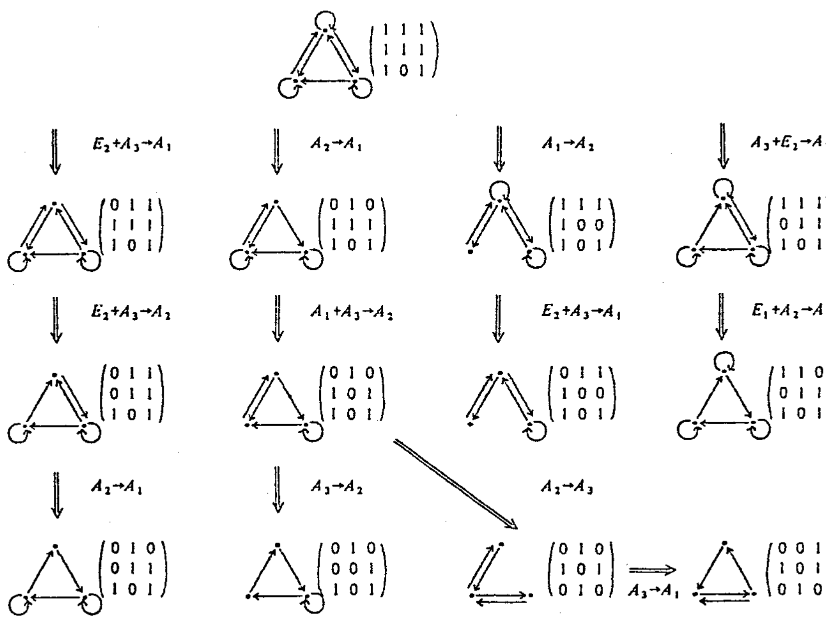
(3) A is primitively equivalent to B.

Proof. By theorems 3.1 and 3.3, it suffices to show that (2) implies (3). By using a computer, we listed up all strongly connected digraphs with 3 vertices satisfying the condition (I). (Note that A is irreducible if and only if the corresponding digraph G of A is strongly connected.) Then these digraphs are classified by K_o and marker of O_A, which are shown in the following classification table. The final step of the proof is to show that digraphs with the same K_o and marker in the table are primitively equivalent. This can be checked one by one.

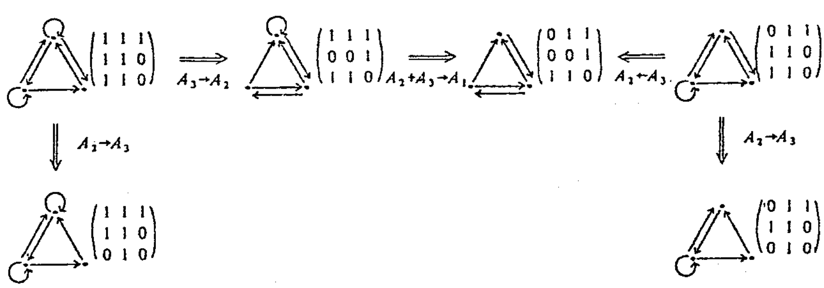
The classification table of $\mathcal{O}_A$ for 3×3 irreducible matrices.

$K_0(\mathcal{O}_A)$	marker	digraph	representative
0	$\bar{0}$		$\mathcal{O}_2$
$\mathbb{Z}_2$	$\bar{0}$		$\mathcal{O}_3 \otimes M_2$
	$\bar{1}$		$\mathcal{O}_3$
$\mathbb{Z}_3$	$\bar{1}$		$\mathcal{O}_4$
$\mathbb{Z}_4$	$\bar{2}$		$\mathcal{O}_5 \otimes M_2$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\bar{0}$		
$\mathbb{Z}$	$\bar{0}$		

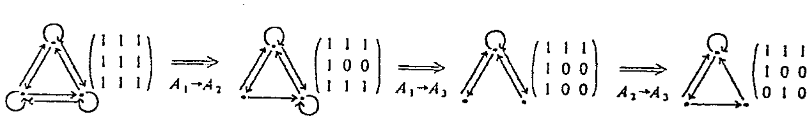
The case that $K_o(O_G) = 0$.



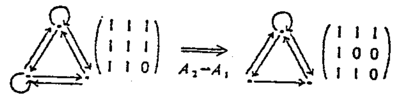
The case that $K_o(O_G) = Z_2$ and $\text{mark}(O_G) = \bar{0}$.



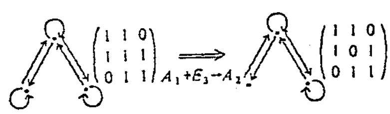
The case that $K_o(O_G) = Z_2$ and $\text{mark}(O_G) = \bar{1}$.



The case that $K_o(O_G) = Z_3$ and $\mathrm{mark}(O_G) = \bar{1}$.



The case that $K_o(O_G) = Z$ and $\mathrm{mark}(O_G) = \bar{0}$.



This completes the proof except to determine the representatives, which will be done in the following sections 4 and 5.

Remark. In the first chapter, we have discussed fixed point algebras of periodic automorphisms on O_n and determined the one of the 'flip-flop' automorphism θ on O_2 considered by Archbold. Now the fixed point algebra $C^*(S_1, S_2{}^2, S_2S_1S_2{}^*)$ of θ is the Cuntz-Krieger algebra O_A such that $A(3,1) = 0$ and $A(i,j) = 1$ for otherwise i,j. Therefore it follows from the classification table that $K_o(O_A) = 0$. On the other hand, it is known that the fixed point algebra of O_2 under α_{-1} is isomorphic to O_4, so that Z_3 is its K_o-group. Hence they are not conjugate.

III-4. Tensor products of O_A by matrix algebras.

Paschke and Salinas [38] studies the tensor product of O_n by the matrix algebra M_k, and proved that O_n and $O_n \otimes M_k$ are non-isomorphic if k and $n - 1$ are relatively prime. In this section, we shall investigate transferences of markers under the tensor product by M_k. Let us define $k \cdot x^-$ by $(kx)^-$ for $x^- \in K_o(B)^-$ and an integer k. (Since $kx = x + \ldots + x$ (k times), $k \cdot x^-$ does not depend on representatives of x^-, that is, $k \cdot x^-$ is well-defined.)

Theorem 3.7. For a unital C*-algebra B, $\mathrm{mark}(B \otimes M_k) = k \cdot \mathrm{mark}(B)$.

Proof. Note that $M_k \otimes K(H) = K(\Sigma_1^k \oplus H_i)$ is spatially isomorphic to $K(H)$ by an isomorphism ϕ^{-1}, where $H_i = H$. Let e be a one-dimensional projection in $K(H)$. Then $\phi(e)$ is one-dimensional, so that we may assume that $\phi(e) = e \oplus 0 \oplus \ldots \oplus 0$. Since

$$\begin{aligned} [\![1 \otimes (1 \oplus \ldots \oplus 1) \otimes e]\!]_\approx &= [\![1 \otimes (e \oplus \ldots \oplus e)]\!]_\approx \\ &= [\![1 \otimes (e \oplus 0 \oplus \ldots \oplus 0)]\!]_\approx + \ldots + [\![1 \otimes (0 \oplus \ldots \oplus 0 \oplus e]\!]_\approx \\ &= k \cdot [\![1 \otimes \phi(e)]\!]_\approx = k \cdot [\![1 \otimes e]\!]_\approx, \end{aligned}$$

we have $\mathrm{mark}(B \otimes M_k) = k \cdot \mathrm{mark}(B)$.

Corollary 3.8. If $K_o(B) = Z_n$ and $\mathrm{mark}(B) = \bar{1}$, then $\mathrm{mark}(B \otimes M_k) = \bar{k}$ for $2 \le n \le \infty$, where $Z_\infty = Z$.

Corollary 3.9. (Paschke-Salinas) If k and $n-1$ are not relatively prime, then $O_n \otimes M_k$ and O_n are non-isomorphic.

Proof. It is known that $K_o(O_n) = Z_{n-1}$ and the equivalence class of 1 is a generator of $K_o(O_n)$, [15; 3.7]. Note that k is a generator of Z_m if and only if $\alpha(1) = k$ for some $\alpha \in \mathrm{Aut}\, Z_m$, i.e., $\bar{1} = \bar{k}$. Hence we have $\mathrm{mark}(O_n) = \bar{1}$. Suppose that $O_n \otimes M_k$ is isomorphic to O_n. Since

$$\bar{k} = \mathrm{mark}(O_n \otimes M_k) = \mathrm{mark}(O_n) = \bar{1}\ ,$$

k is a generator of Z_{n-1}. Therefore there is an integer j such that $jk = 1 \mod n-1$. Furthermore, since $jk + a(n-1) = 1$ for some $a \in Z$, k and $n - 1$ are relatively prime.

Remark. We point out that $O_A \otimes M_k$ is also a Cuntz-Krieger algebra. Actually, since $O_A \otimes M_k$ is generated by

$$\left\{ \begin{pmatrix} & & S_i \\ & & \\ & & \end{pmatrix}, \begin{pmatrix} 0 & & \\ P_i & & \\ & & \end{pmatrix}, \ldots, \begin{pmatrix} & & \\ & & \\ & P_i & 0 \end{pmatrix} ; 1 \le i \le n \right\},$$

we have

$$B = \begin{pmatrix} 0 & & & A \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

As another application, we consider the inclusion among Cuntz algebras O_n.

Theorem 3.10. O_m *is included in* O_n *containing the unit if and only if* $m = n + (n - 1)k$ *for some integer* $k \geq 0$.

Proof. Assume that O_n includes O_m. Then it follows from [13; Remark 7] that $m \leq n$. Furthermore, we have $n[\![1]\!] = [\![1]\!] = m[\![1]\!]$ in $K_o(O_n)$,so that $(m - n)[\![1]\!]$ is the neutral element in $K_o(O_n) = Z_{n-1}$. Then $m - n = k(n - 1)$ for some k. We prove the converse by induction. Let $O_n = C^*(S_1, \ldots, S_n)$. The case $k = 0$ is trivial. If $m = n + (n - 1)$, then we put $T_j = S_1S_j$ for $j = 1, \ldots, n$ and so $C^*(T_j, S_k; 1 \leq j \leq n, 2 \leq k \leq n)$ is isomorphic to O_m. Next, if $m = n + 2(n-1)$ then we put $U_i = T_1S_i$ for $i = 1, \ldots, n$ and also $C^*(U_i, T_j, S_k; 1 \leq i \leq n, 2 \leq j,k \leq n)$ is isomorphic to O_m. We can construct C^*-algebras isomorphic to O_m in such a way.

III-5. Explosions of digraphs.

The adjoint graph G^* of a digraph G is defined to be a digraph whose vertices $u_1, \ldots, u_m$ represent the edges of G and which has an edge $u_i \longleftarrow u_j$ if $i_2 = j_1$, where $u_i = (i_1, i_2)$ and $u_j = (j_1, j_2)$. We shall generalize the adjoint of a digraph in order to determine completely the repesentatives in the preceding classification of O_A. This process will be called explosion.

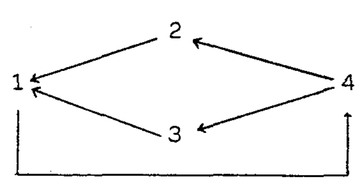
Definition 3.11. Let G be a digraph. Suppose that the number of $\Gamma^{-}(i)$ is greater than 2 for some $i \in V(G)$. (For simplicity, assume that $i = 1$.) Decompose $\Gamma^{-}(1) = V \cup W$ such that $1 \in V$ if $1 \in \Gamma^{-}(1)$. Then the explosion H of G at 1 (with respect to V and W) is defined as follows;

$V(H) = (V(G) \setminus \{1\}) \cup \{v_o, w_o\}$, and

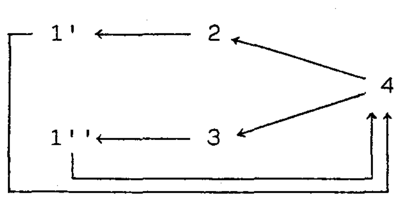
$$E(H) = (E(G) \setminus \{(1,j),(k,1);\ j,k \in V(G)\})$$
$$\cup\{(v_o,v),(w_o,w);\ v \in V \setminus \{1\},\ w \in W\}$$
$$\cup\{(i,v_o),(i,w_o);\ (i,1) \in E(G)\},$$

and if $1 \in \Gamma^{-}(1)$, $\{(v_o,v_o),(v_o,w_o)\}$ is added to the set on the right hand side. This operation is called as explosion, and every digraph obtained by repeating explosions is called an explosion of G.

Example 3.5.1. Let G be a digraph;



Then the explosion H of G at 1 is the following;



Moreover, it is easily seen that the explosion of H at 3 is the adjoint G^* of G.

More generally, it is obvious that the adjoint of a digraph G is an explosion of G, and the adjoint operation multiplies the number of vertices.

Now, by using explosions, we can increase the number of vertices by one. As an application, the classification problem of O_A for $n \times n$ matrices A is included in one of O_B for $(n+1) \times (n+1)$ matrices B by the following theorem, whose idea is the same as Theorem 2.7.

Theorem 3.12. If a digraph H is an explosion of a digraph G, then O_H is isomorphic to O_G.

Proof. We may assume that H is the explosion of G at 1 and $\Gamma^-(1) = V \cup W$ such as $1 \in V$ if $1 \in \Gamma^-(1)$. Let $O_G = C^*(S_1, \dots . S_n)$, $P_i = S_iS_i^*$ and $P_Y = \Sigma_{i\in Y} P_i$. Then, if we put $T_V = S_1P_V$, $T_W = S_1P_W$ and $T_k = S_k$ for $2 \leq k \leq n$, then we have

$$T_VT_V^* + T_WT_W^* = S_1(P_V + P_W)S_1^* = S_1(S_1^*S_1)S_1^* = P_1,$$

so that

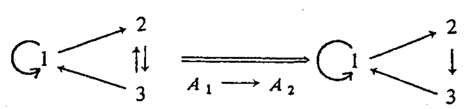
$$T_VT_V^* + T_WT_W^* + \Sigma_{k=2}^n T_kT_k^* = \Sigma_k P_k.$$

Furthermore, since $T_V^*T_V = P_V$ and $T_W^*T_W = P_W$, the family of partial isometries $T_V, T_W, T_2, \dots, T_n$ satisfies the condition (A). Hence the C^*-algebra generated by them is the Cuntz-Krieger algebra O_H and is included in O_G. Since $S_1 = T_V + T_W$, it coincides with O_G, so that O_H is isomorphic to O_G.

The following corollary shows that there are many Cuntz-Krieger algebras isomorphic to O_2.

Corollary 3.13. If $C(n) = \begin{pmatrix} 1 & & & & 1 \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}$ is of degree n, then $O_{C(n)}$ is isomorphic to O_2.

Proof. The case of $n = 2$ is Example 3.1.1. Consider the adjoint graph of $C(n)$ and its transfered graph inductively;



C(2)* C(3)

Theorems 3.12 and 3.3 implies that $O_{C(n)}$ is isomorphic to O_2.

Concluding this section, we shall complete the representatives in the table by applying Theorems 3.3 and 3.11. The above corollary proves the case that $K_0(O_A) = 0$. Next we shall prove the case that $K_0(O_A) = Z_2$ and $\text{mark}(O_A) = \bar{0}$. Let $O_3 = C^*(S_1, S_2, S_3)$. Then $O_3 \otimes M_k$ is generated by

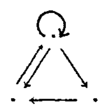
$$\begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & S_3 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

so that $O_3 \otimes M_2$ is isomorphic to O_B, where $B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

We have

$$B \xRightarrow[B_1 \longrightarrow B_4]{} C \xRightarrow[C_1 \longrightarrow C_2]{} D = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

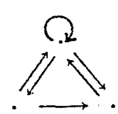
On the other hand, D is the explosion of G at 2, where

G;  $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$

In the case that $K_o(O_A) = Z_3$ and $\text{mark}(O_A) = \bar{1}$, if A is the 4×4 matrix whose entries are 1, i.e., $O_A = O_4$, then

$$A \xRightarrow[A_1 \longrightarrow A_2]{} B \xRightarrow[B_1 \longrightarrow B_3]{} C \xRightarrow[C_1 \longrightarrow C_4]{} D \xRightarrow[D_4 \longrightarrow D_3]{} E,$$

and moreover E is the explosion of H at 3, where

H;  $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$

Finally, the case that $K_o(O_A) = Z_4$ and $\text{mark}(O_A) = \bar{2}$ is stated in [17].

III-6. Shift equivalence and determinant.

A matrix A is strongly shift equivalent to a matrix B if there are matrices R and S such that $A = RS$ and $B = SR$, cf. [37]. If A and B are strongly shift equivalent, then O_A and O_B are stably isomorphic [17]. While we have the following example by the classification table: There are strongly shift equivalent matrices A and B such that O_A

is not isomorphic to O_B. As a matter of fact, let $A = RS$ and $B = SR$ where

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then $K_o(O_A) = K_o(O_B) = Z_2$. On the other hand, since $\mathrm{mark}(O_A) = \bar{1}$ and $\mathrm{mark}(O_B) = \bar{0}$, O_A is not isomorphic to O_B.

The following theorem shows that, under an additional assumption, O_A and O_B are isomorphic for strongly shift equivalent matrices A and B.

Theorem 3.14. Let R and S be matrices such that $\Sigma_i R(i,j) = 1$ for all j, and RS and SR satisfy the condition (I). Then O_{RS} and O_{SR} are isomorphic.

Proof. Let R (resp. S) be an $n \times m$ (resp. $m \times n$) matrix and put $A = RS$ and $B = SR$. Let $H_j (j = 1, \ldots, m)$ and K_i $(i = 1, \ldots, n)$ be infinite dimensional Hilbert spaces, and P_j (resp. Q_i) the projection of $H = \Sigma \oplus H_j$ (resp. $K = \Sigma \oplus K_i$) onto H_j (resp. K_i). Take partial isometries U_i and V_j of K into H such that

$(*)$ $\qquad U_iU_i^* = Q_i, \quad U_i^* U_i = \Sigma_j R(i,j)P_j,$ and

$(**)$ $\qquad V_jV_j^* = P_j, \quad V_j^* V_j = \Sigma_k S(j,k)Q_k.$

And let C (resp. D) be the C^*-algebra generated by $\{U_iV_j;\ 1 \le i \le n,\ 1 \le j \le m\}$ (resp. $\{V_jU_i;\ 1 \le i \le n,\ 1 \le j \le m\}$). Then we shall prove that C (resp. D) is isomorphic to O_A (resp. O_B) and C is isomorhic to D.

If we put $T_i = U_i(\Sigma_j V_j)$, then $T_iT_i^* = Q_i$ and

$$T_i^*T_i = \Sigma_j R(i,j)(\Sigma_j V_j)^*P_j(\Sigma_j V_j) = \Sigma_j R(i,j)V_j^*V_j$$
$$= \Sigma_k(\Sigma_j R(i,j)S(j,k))Q_k = \Sigma_k A(i,k)Q_k$$

for all i. Hence it implies that the C*-algebra $C^*(T_i;\ 1 \le i \le n)$ is isomorphic to O_A. On the other hand, since $V_j^*V_j = \Sigma_k S(j,k)Q_k = \Sigma_k S(j,k)T_kT_k^*$ and $T_iV_j^*V_j = R(i,j)U_iV_j$, we have $C = C^*(T_i;\ 1 \le i \le n)$ as a set, so that C is isomorphic to O_A. Similarly D is isomorphic to O_B.

By the assumption of R and (*), $W = \Sigma_i U_i$ is an isometry from H onto K. Since

$$(W^*U_iV_jW)(W^*U_iV_jW)^* = R(i,j)P_j \quad \text{and} \quad W^*U_iV_jW(U_i^*U_i) = V_jU_i$$

by (*) and (**), we have

$$C^*(W^*U_iV_jW;\ 1 \le i \le n,\ 1 \le j \le m) = C^*(V_jU_i;\ 1 \le i \le n,\ 1 \le j \le m)$$

as a set. Therefore C and D are isomorphic.

Next we shall discuss an topological invatiant $\det(1 - A)$. It is known that, identifying a digraph with its adjacency matrix as usual, for a digraph G,

$$\det(x - G^*) = x^k \det(x - G),$$

where $k = V(G^*)^= - V(G)^=$ and $M^=$ is the cardinal number of M. A key of a proof is to find matrices A and B such that $G^* = AB$ and $G = BA$. Inspired by this, we shall reformulate explosions of digraphs. Here a matrix A is represented by (a_{ij}).

Definition 3.15. Let $G = (a_{ij})$ be an $n \times n$ matrix (digraph) with $\Gamma^-(1)^= \ge 2$. Then a digraph H is the explosion of G at 1 (with respect to V_1 and V_2) if $H = G_o E_o$ such that G_o(resp. E_o) is an $(n+1) \times n$ (resp. $n \times (n+1)$) matrix expressed by

$$E_o = \begin{array}{c|c} & \begin{matrix} p & q & 2 & \cdots & n \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ \vdots \\ n \end{matrix} & \begin{pmatrix} 1 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \end{array} \quad \text{and} \quad G_o = \begin{array}{c|c} & \begin{matrix} 1 & 2 & \cdots & n \end{matrix} \\ \begin{matrix} p \\ q \\ 2 \\ \vdots \\ n \end{matrix} & \begin{pmatrix} a_{11} & a_{p2} & \cdots & a_{pn} \\ 0 & a_{q2} & \cdots & a_{qn} \\ a_{21} & \cdots & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} \end{array},$$

where $a_{pj} = \begin{cases} 1 & \text{if } j \in V_1, \\ 0 & \text{if not,} \end{cases}$ and $a_{qj} = \begin{cases} 1 & \text{if } j \in V_2, \\ 0 & \text{if not.} \end{cases}$

Lemma 3.16. Definitions 3.11 and 3.15 are identical. Moreover, if notation is as in above, then $E = E_o G_o$.

Proof. We represent the original explosion of G as its adjacency matrix;

$$H = \begin{array}{c|c} & \begin{matrix} p & q & 2 & \cdots & n \end{matrix} \\ \begin{matrix} p \\ q \\ 2 \\ \vdots \\ n \end{matrix} & \begin{pmatrix} a_{11} & a_{11} & a_{p2} & \cdots & a_{pn} \\ 0 & 0 & a_{q2} & \cdots & a_{qn} \\ a_{21} & \cdots & \cdots & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{pmatrix} \end{array}.$$

Thus elementary calculations lead us the conclusion.

The lemma gives us another proof of Theorem 3.11 by joining Theorem 3.14.

By the way, it is proved that $|\det(1 - A)|$ is a stable invariant for O_A. And Cuntz conjectures that $\det(1 - A)$ is a stable invariant for O_A. Now we have the following results on $\det(1 - A)$.

Theorem 3.17. If H is an explosion of a digraph G, then

$$\det(x - H) = x \det(x - G),$$

and so $\det(1 - H) = \det(1 - G)$.

Proof. Since $V(H)^{=} = V(G)^{=} + 1$, the statement follows from the preceding lemma.

Theorem 3.18. If H is a transfered graph of G, then $\det(1 - H) = \det(1 - G)$.

Proof. Suppose that $G \xrightarrow[E_K+A_M \longrightarrow A_1]{} H$, where $E_K = \Sigma_{i=1}^{r} E_{k(i)}$ and $A_M = \Sigma_{j=1}^{s} A_{m(j)}$. Since $A_1 = E_K + A_M$, it follows from the definition of transfered graphs that

$$\det(1 - G) = \det\left(\begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{pmatrix} - \begin{pmatrix} E_K+A_M \\ A_2 \\ \vdots \\ A_n \end{pmatrix}\right) = \det\begin{pmatrix} E_1-E_K-E_M \\ E_2-A_2 \\ \vdots \\ E_n-A_n \end{pmatrix}$$

$$= \det\left(\begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{pmatrix} - \begin{pmatrix} E_K+E_M \\ A_2 \\ \vdots \\ A_n \end{pmatrix}\right) = \det(1 - H).$$

III-7. Weak extension groups of O_A.

Let $Q(H)$ be the Calkin algebra on an infinite dimensional separable Hilbert space H and π the quotient map of $B(H)$ onto $Q(H)$. For a separable unital C*-algebra B, let ext(B) be the set of all unital *-monomorphisms (extensions) of B into $Q(H)$. Extensions τ and σ are weakly equivalent if there is a unitary $u \in Q(H)$ such that $\tau(x) = u\,\sigma(x)u^*$ for all $x \in B$. Let Ext (B) denote the set of all weak equivalence classes in ext(B), which is called the weak extension group of B. Cuntz and Krieger determined the weak extension group of O_A by the Bowen-Franks invariant $Z^n/(1 - A)Z^n$.

In this section, we shall prove that any finitely generated abelian group is represented by the weak extension group of a simple Cuntz-Krieger algebra.

Theorem 3.19. Let H be a finitely generated abelian group. Then there is a simple Cuntz-Kreiger algebra O_A such that $\mathrm{Ext}^w O_A = H$.

Now it is known that every finitely generated abelian group H is represented;

$$H = Z \oplus \cdots \oplus Z \oplus Z_{n(1)} \oplus \cdots \oplus Z_{n(m)},$$

where $Z_n = Z/nZ$. So we shall devide into several cases. In the beginning, we shall consider the simple case $H = Z$, which is a key in the proof. It is known that $\mathrm{Ext}^w O_{n+1} = Z_n$. However, we shall pose another Cuntz-Krieger algebras O_A with the same property. We omit often 0 entries of matrices in the below.

Lemma 3.20. Let $G(n)$ be the digraph whose (adjacency) matrix is of degree $n+1$ and expressed by

$$\begin{pmatrix} 0 & & & 0 & 1 \\ 1 & \ddots & & \vdots & \vdots \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & 0 & \vdots \\ & & & 1 & 1 \end{pmatrix}.$$

Then $\mathrm{Ext}^w O_{G(n)} = Z_n$ for $n \geq 1$. Particularly, $\mathrm{Ext}^w O_{G(1)}$ is trivial.

Proof. Since $G(n)$ has an $(n+1)$-cycle and the vertex $n+1$ has a loop, $G(n)$ satisfies the condition (I) and is strongly connected. It implies that $O_{G(n)}$ is simple. We have also

$$\begin{pmatrix} 1 & & & & \\ \vdots & \ddots & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ 1 & \cdots & \cdots & \cdots & 1 \end{pmatrix}(1 - G(n)) \begin{pmatrix} 1 & & & & 1 \\ & \ddots & & & 2 \\ & & \ddots & & \vdots \\ & & & \ddots & n \\ & & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -n \end{pmatrix}.$$

By the Elementarteilersatz [42; § 118], it follows that

$$Z^{n+1}/(1 - G(n))Z^{n+1} = Z_n,$$

so that $\mathrm{Ext}^w O_{G(n)} = Z_n$.

Next we shall consider the case $H = Z \oplus \dots \oplus Z \oplus Z_n$.

Lemma 3.21. Let $G(k|n)$ be the digraph whose matrix is of degree $k+n+1$ and expressed by

$$\begin{pmatrix} 1 & & & & 1 & 0 \\ & \ddots & & & \vdots & \vdots \\ & & 1 & & 1 & 0 \\ 1 & \cdots & 1 & & & \\ \vdots & & \vdots & G(n) & & \\ 1 & \cdots & 1 & & & \\ 0 & \cdots & 0 & & & \end{pmatrix}.$$

Then $\mathrm{Ext}^{w}O_{G(k|m)} = Z \oplus \ldots \oplus Z \oplus Z_n$, where $Z \oplus \ldots \oplus Z$ is k-copies of Z. Particularly, $\mathrm{Ext}^{w}O_{G(k|1)} = Z \oplus \ldots \oplus Z$.

Proof. Since $A(i,k+n) = 1$ for $1 \le i \le k$, $A(k+n,j) = 1$ for $1 \le j \le k$ and the strongly connected digraph $G(n)$ satisfies the condition (I), $G(k|n)$ satisfies the condition (I) and is strongly connected. So $O_{G(k|n)}$ is simple. Moreover we have

$$\begin{pmatrix} 1 & & & -1 \\ & \ddots & & \vdots \\ & & 1 & -1 \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} (G(k|n)-1) \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ -1 & \cdots & \underset{\widehat{k}}{-1} & & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ & \\ 0 & G(n)-1 \end{pmatrix}.$$

By Lemma 3.20, it implies that $\mathrm{Ext}^{w}O_{G(k|n)} = Z \oplus \ldots \oplus Z \oplus Z_n$.

For the case $H = Z_m \oplus Z_n$, we shall apply Lemma 3.20 again.

Lemma 3.22. Let $G(m,n)$ be the digraph which is expressed as follows. Then $\mathrm{Ext}^{w}O_{G(m,n)} = Z_m \oplus Z_n$.

$$\begin{pmatrix} & & 1 & \\ G(m) & & \vdots & \\ & & 1 & \\ & & 0 & \\ & & 1 & \\ & 1 & & \\ & \vdots & & G(n) \\ & 1 & & \\ & 0 & & \end{pmatrix}.$$

Proof. Since $A(m+1,m+2) = 1 = A(m+2,m+1)$, $G(m,n)$ satisfies (I) and is strongly connected, so that $O_{G(m,n)}$ is simple by similar calculations, we have

$$(G(m,n)-1)\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & -1 & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & & & \\ & \ddots & & -1 & & \\ & & 1 & -1 & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} = \begin{pmatrix} G(m)-1 & \\ & G(n)-1 \end{pmatrix}.$$

It follows from Lemma 3.20 that $\mathrm{Ext}^{w} O_{G(m,n)} = Z_m \oplus Z_n$.

Here we shall remark that $\mathrm{Ext}^{w} O_{G(p,m,n)} = Z_p \oplus Z_m \oplus Z_n$ if we define $G(p,m,n)$ anologously. Now let us join Lemmas 3.21 and 3.22. Let $G(k|m,n)$ be the digraph expressed by

$$\begin{pmatrix} 1 & & 1 & 0 & 1 & \\ & \ddots & \vdots & \vdots & \vdots & \\ & 1 & 1 & 0 & 1 & \\ 1 \cdots & 1 & & & 1 & \\ \vdots & \vdots & & & \vdots & \\ \vdots & \vdots & G(m) & & 1 & \\ 1 \cdots & 1 & & & 0 & \\ 0 \cdots & 0 & & & 1 & \\ & & & 1 & & \\ & & & \vdots & & G(n) \\ & & & 1 & & \\ & & & 0 & & \end{pmatrix}.$$

Then we have

$$
(G(k|m,n)-1)
\begin{pmatrix}
1 & & & & & \\
 & \ddots & & & & \\
 & & 1 & & & \\
 & & & 1 & & & & \\
 & & & & \ddots & & & \\
 & & & & & 1 & & \\
 & & & & & & 1 & \\
 & & & & & & & \ddots \\
 & & & & & -1 & & & 1
\end{pmatrix}
\begin{pmatrix}
1 & & & & & & & & \\
 & \ddots & & & & & & & \\
 & & 1 & & & & & & \\
 & & & 1 & & & & & \\
 & & & & \ddots & & -1 & & \\
 & & & & & 1 & -1 & & \\
 & & & & & & 1 & & \\
 & & & & & & & \ddots & \\
 & & & & & & & & 1
\end{pmatrix}
$$

$$
= \begin{pmatrix}
 & \begin{matrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{matrix} & \\
\begin{matrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \\ 0 & & 0 \end{matrix} & G(m)-1 & \\
 & & G(n)-1
\end{pmatrix}.
$$

On the other hand, $G(k|m,n)$ satisfies (I) and is strongly connected by $A(k,k+m) = 1 = A(k+m,k)$. So it follows from Lemma 3.21 that $\mathrm{Ext}^w O_{G(k|m,n)} = Z \oplus \ldots \oplus Z \oplus Z_m \oplus Z_n$. Thus it is easily seen that there is a strongly connected digraph $G = G(k|n(1),n(2), \ldots ,n(m))$ such that $\mathrm{Ext}^w O_G = Z \oplus \ldots \oplus Z \oplus Z_{n(1)} \oplus \ldots \oplus Z_{n(m)}$, which completes the proof of Theorem 3.19.

Finally we shall discuss the periodicity of weak extension groups of Cuntz-Krieger algebras associated with random walks. We consider the following example associated with a random walk PPRW reflecting at both boundaries, cf. [35].

Example 3.7.1. Let $A(n)$ $(n \geq 2)$ be the digraph;

$$\circlearrowleft 1 \rightleftarrows 2 \rightleftarrows \cdots \rightleftarrows n-1 \rightleftarrows n \circlearrowright .$$

Then $\mathrm{Ext}^{W}O_{A(n)} = Z$ for $n = 3m$ and 0 for otherwise.

In fact, it is proved that $\mathrm{Ext}^{W}O_{B(n+1)} = \mathrm{Ext}^{W}O_{A(n)} = \mathrm{Ext}^{W}O_{B(n-2)}$, where $B(n)$ is the digraph;

$$1 \rightleftarrows 2 \rightleftarrows \cdots \rightleftarrows n-1 \rightleftarrows n \circlearrowright .$$

Since $A(n)$ (resp. $B(n)$) is expressed by

$$\begin{pmatrix} 1 & 1 & & & \\ 1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & 1 & 1 \end{pmatrix} \quad \left(\text{resp. } \begin{pmatrix} 0 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & 1 & 1 \end{pmatrix}\right),$$

we have

$$\begin{pmatrix} 1 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{pmatrix} (B(n+1)-1) \begin{pmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix} = \begin{pmatrix} -1 & \\ & A(n)-1 \end{pmatrix}.$$

Therefore $\mathrm{Ext}^{W}O_{B(n+1)} = \mathrm{Ext}^{W}O_{A(n)}$.

Next, if we put

$$I_n = \begin{pmatrix} 1 & & & & \\ 0 & \ddots & & & \\ -1 & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \quad \text{and} \quad J_n = \begin{pmatrix} 1 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 1 \end{pmatrix},$$

then we have

$$I_n(A(n)-1)J_n = \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & -1 & 1 & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & -1 & 1 \\ & & & & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & B(n-2)-1 \end{pmatrix},$$

so that $\mathrm{Ext}^w O_{A(n)} = \mathrm{Ext}^w O_{B(n-2)}$.

Next we shall replace the edges of a polygon by directed edges $\rightleftarrows$, which is associated with a random walk CRW on a circle, cf. [33].

Example 3.7.2. Let $C(n)$ $(n \geq 3)$ be the digraph;

$$\begin{array}{ccccccc} 1 & \rightleftarrows & 2 & \rightleftarrows & \cdots & \rightleftarrows & k \\ \uparrow\downarrow & & & & & & \uparrow\downarrow \\ n & \rightleftarrows & n-1 & \rightleftarrows & \cdots & \rightleftarrows & k+1 \end{array}$$

It is somewhat surprising that $\mathrm{Ext}^w O_{C(n)}$ is periodic with period 6: The weak extension group of $O_{C(3)}$ (resp. $O_{C(4)}, \ldots, O_{C(8)}$) is $Z_2 \oplus Z_2$ (resp. Z_3, 0, $Z \oplus Z$, 0, Z_3).

Note that $C(n) = \begin{pmatrix} 0 & 1 & & 1 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & & 1 & 0 \end{pmatrix}$.

Let us put

$$B_0 = \begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} (A - 1), \quad J_i = \begin{pmatrix} 1 & & \overset{n-i}{-1} & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{array}{l} \begin{pmatrix} 1 & & & & \\ 1 & \cdots & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \end{array} (n-i$$

and $B_{i+1} = B_i J_i$ for $1 \leq i \leq n-3$. If we put $b_i^{(k)} = B_i(k,i)$ and $c_i^{(k)} = B_i(k,n-i)$ for $k = 1,2$ and $1 \leq i \leq n-3$, then $B_{i+1}(k,n-(i+1)) = c_i^{(k)} - b_i^{(k)} + 1$ and $b_{i+1}^{(k)} = c_i^{(k)}$. Hence we have

(B) $\qquad b_{i+2}^{(k)} = b_{i+1}^{(k)} - b_i^{(k)} + 1$

for $k = 1,2$ and $1 \leq i \leq n-4$. Furthermore, since we have

$$B_{n-3} = \begin{pmatrix} b_{n-3}{}^{(1)} & 1 & b_{n-2}{}^{(1)} & & \\ b_{n-3}{}^{(2)} & 0 & b_{n-2}{}^{(1)} & * & \\ 1 & 1 & 0 & & \\ & & & -1 \ddots & \\ & & & & -1 \end{pmatrix}$$

it follows that

$$B_{n-2} = B_{n-3}J_{n-3} = \begin{pmatrix} b_{n-2}{}^{(1)} & b_{n-2}{}^{(1)} - b_{n-3}{}^{(1)} + 1 & & * \\ b_{n-2}{}^{(2)} & b_{n-2}{}^{(2)} - b_{n-3}{}^{(2)} & & \\ & & -1 & \\ & & & -1 \end{pmatrix},$$

so that $\mathrm{Ext}^W O_{C(n)}$ depends only on the matrix

$$\begin{pmatrix} b_{n-2}{}^{(1)} & b_{n-2}{}^{(1)} - b_{n-3}{}^{(1)} + 1 \\ b_{n-2}{}^{(2)} & b_{n-2}{}^{(2)} - b_{n-3}{}^{(2)} \end{pmatrix}.$$

On the other hand, if $d_{i+1} = d_i - d_{i-1}$, then

$$d_i = d_{i-1} - d_{i-2} = (d_{i-2} - d_{i-3}) - d_{i-2} = -d_{i-3}$$
$$= -d_{i-4} + d_{i-5} = -(d_{i-5} - d_{i-6}) + d_{i-5} = d_{i-6},$$

so that d_n is of period 6. Since $d_i = b_i - 1$ satisfies that $d_{i+1} = d_i - d_{i-1}$ by (B), $\{b_n\}$ is of period 6 and so is $\mathrm{Ext}^W O_{C(n)}$. In addition, since $b_1{}^{(1)} = b_2{}^{(1)} = b_1{}^{(2)} = 1$ and $b_2{}^{(2)} = 2$, the equation (B) implies the conclusion.

Finally we shall give an example of a sequence of digraphs $S(n)$ such that $\mathrm{Ext}^W O_{S(n)}$ is not periodic. Replacing the edges of an n-simplex $\Delta(n)$ $(n \geq 3)$ by directed edges $\rightleftarrows$, we obtain the digraph $S(n)$.

Example 3.7.3. Let $S(n)$ be the digraph whose matrix A is given by $A(i,j) = 1 - \delta_{i,j}$. Then $\mathrm{Ext}^{w} O_{s(n)} = Z_2{}^{n-2} \oplus Z_{2n-4}$.

As a matter of fact, we have

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ 0 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ \vdots & \ddots & & \\ \vdots & & \ddots & \\ 1 & & & 1 \end{pmatrix} (1-A) \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ 0 & -1 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & 1 \\ & & \vdots \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 2 & \\ & & & & 4-2n \end{pmatrix}.$$

Chapter IV KMS states on O_A

IV-1. KMS states.

A proof of the uniqueness theorem on O_A is based on the existence of the gauge automorphism $\alpha_t (t \in R)$ on O_A by

$$\alpha_t(S_j) = e^{it}S_j \quad \text{for } 1 \le j \le n,$$

where R is the group of real numbers. The action α is called the gauge action on O_A. Olesen and Pedersen [36] proved the following theorem on the C*-dynamical system (O_A, R, α), cf. also [30] :

Theorem 4.1. The C*-dynamical system (O_A, R, α) admits a β-KMS state if and only if $\beta = \log n$, and the corresponding KMS state is unique.

Now we remark that if $A(i,j) = 1$ for $1 \le i,j \le n$, then $O_A = O_n$ and the spectral radius $r(A)$ of A is just n.

Under these situation, we shall give a natural generalization of Theorem 4.1. As a matter of fact, we shall point out that the Perron-Frobenius theorem for positive matrices is applicable to the existence of KMS states on the C*-dynamical system (O_A, R, α). More precisely,

Theorem 4.2. If A is irreducible, then (O_A, R, α) admits a β-KMS state if and only if $\beta = \log r(A)$, and the corresponding KMS state is unique.

The topological entropy of a subshift σ_A is defined by $h(\sigma_A) = \log r(A)$. Therefore, the above theorem shows that the topological entropy of a subshift is the value β which gives a unique β-KMS state for (O_A, R, α), and consequently, if C*-dynamical systems (O_A, R, α) and (O_B, R, α) are conjugate, and A and B are irreducible, then their topological entropies coincide.

Incidentally, the period of A will be concerned with a factor representation of type III_λ in the following section, in which it will be proved that the period is also a conjugacy invariant for (O_A, R, α). It is known that a pair of the topological entropy and the period is a complete invariant for subshifts as measure preserving transformations, [37]. As a consequence, the equivalence of subshifts as measure preserving transformations is a conjugacy invariant.

Let $E = \{1,2, \ldots, n\}$. For a multiindex $\mu = (i(1), \ldots, i(p))$ with $i(m) \in E$, we denote by $l(\mu)$ the length p of μ and $S_\mu = S_{i(1)} \cdots S_{i(p)}$. Then it is easily checked that S_μ is a partial isometry and $S_\mu \neq 0$ if and only if $A(i(m), i(m+1)) = 1$ for $1 \leq m \leq p-1$. Now we begin with an elementary lemma stated in [17], whose proof is an easy exercise for the use of the condition (A).

Lemma 4.3. If $l(\mu) = l(\nu) = k$ and $S_\mu, S_\nu \neq 0$, then

$$S_\nu{}^*S_\mu = \delta_{\mu,\nu}S_{\nu(k)}{}^*S_{\mu(k)}.$$

It is known in [17] that the fixed point algebra F_A of O_A by α is an AF-algebra such as

$$F_A = \overline{\bigcup_{m=0}^{\infty} F_m},\ F_m = F_m{}^1 \oplus \dots \oplus F_m{}^n$$

and the inclusion is given by A. We refer to [4, 20] for AF-algebras. For $\beta \in R$, we put

$$L_\beta = \{y = (y_i) \in R^n;\ Ay = e^\beta y,\ y_i \geq 0 \text{ and } \Sigma\, y_i = 1\}.$$

In the following, we shall construct a trace on F_A which is corresponding to each $y_0 \in L_\beta$. Then we put $y_m = e^{-m\beta}y_0$, $d_m(i) = \dim F_m{}^i$ and $w_m(i) = d_m(i)y_m(i)$ for $i \in E$.

Lemma 4.4. Notation as in above for a fixed $\beta \in R$ and $y_0 \in L_\beta$. Then y_0 induces a trace ϕ on F_A such as $\phi(e(m,i)) = y_m(i)$, where $e(m,i)$ is a one-dimensional projection in $F_m{}^i$ for $i \in E$.

Proof. We define a trace ϕ_m on F_m by $\phi_m(e(m,i)) = y_m(i)$ for $i \in E$. So it suffices to show that (ϕ_m) is compatible. We note that $\phi_0(1) = \Sigma_i\, y_0(i) = 1$ and $\phi_m(\Pr F_m{}^i) = d_m(i)y_m(i) = w_m(i)$, where $\Pr F_m{}^i = 0\oplus \cdots \oplus 0 \oplus \overset{i}{\breve{1}} \oplus 0 \dots \oplus 0 \in F_m$. Since $y_m = e^{-m\beta}y_0$, it follows that $y_m = Ay_{m+1}$ for $m \geq 0$. Therefore we have

$$\begin{aligned}\phi_m(1) &= \Sigma_i\, y_m(i)d_m(i) = (y_m,\ d_m) = (y_m,\ {}^tAd_{m-1})\\ &= (Ay_m,\ d_{m-1}) = (y_{m-1},\ d_{m-1}) = \phi_{m-1}(1),\end{aligned}$$

and

$$w_m(i) = \Sigma_j\ {}^tA(j,i)\ w_{m+1}(j)\ d_m(i)/d_{m+1}(j).$$

Hence (ϕ_m) is compatible and so we can define a trace ϕ on F_A by $\phi | F_m = \phi_m$.

Lemma 4.5. Let ϕ be a trace on F_A in the preceding lemma and $p = l(\mu)$, $q = l(\nu)$, $r = l(\xi)$ and $s = l(\eta)$. If $S_\mu, S_\nu, S_\xi, S_\eta \neq 0$ and $p + r = q + s$, then

$$\begin{aligned} \phi(S_\mu S_\nu{}^* S_\xi S_\eta{}^*) &= \delta_1(Ay_s)(\eta(s)) \qquad \text{if } q \leq r \\ &= \delta_2(ay_p)(\mu(p)) \qquad \text{if } q > r. \end{aligned}$$

Here $\delta_1 = \delta_1(\mu,\nu,\xi,\eta) = \delta_{\mu,\eta}\ \delta_{\nu,\xi}\ \delta_{\xi(q+1),\eta(p+1)} \cdots \delta_{\xi(r),\eta(s)}$
and $\delta_2 = \delta_2(\mu,\nu,\xi,\eta) = \delta_{\mu,\eta}\ \delta_{\nu,\xi}\ \delta_{\mu(s+1),\nu(r+1)} \cdots \delta_{\mu(p),\nu(q)}$,
where $\delta_{\nu,\xi} = \delta_{\nu(1),\xi(1)} \cdots \delta_{\nu(q\wedge r),\xi(q\wedge r)}$
and $\delta_{\mu,\eta} = \delta_{\mu(1),\eta(1)} \cdots \delta_{\mu(p\wedge s),\eta(p\wedge s)}$.

Proof. We may assume that $q < r$. Then it follows from Lemma 4.3 that

$$\begin{aligned} S_\mu S_\nu{}^* S_\xi S_\eta{}^* &= \delta_{\nu,\xi} S_\mu S_{\nu(q)}{}^* S_{\xi(q)} S_{\xi(q+1)} \cdots S_{\xi(r)} S_\eta{}^* \\ &= \delta_{\nu,\xi} \Sigma_h A(\nu(q),h) S_\mu S_h S_h{}^* S_{\xi(q+1)} \cdots S_{\xi(r)} S_\eta{}^* \\ &= \delta_{\nu,\xi} A(\nu(q),\xi(q+1)) S_\mu S_{\xi(q+1)} \cdots S_{\xi(r)} S_\eta{}^* \\ &= \delta_{\nu,\xi} S_\mu S_{\xi(q+1)} \cdots S_{\xi(r)} S_\eta{}^*. \end{aligned}$$

Noting that $\phi | F_m{}^i$ is a usual trace and putting $P_h = S_h S_h{}^*$ for $h \in E$, we have

$$\begin{aligned} \phi(S_\mu S_\nu{}^* S_\xi S_\eta{}^*) &= \delta_{\nu,\xi} \Sigma_h \phi(S_\mu S_{\xi(q+1)} \cdots S_{\xi(r)} P_h S_\eta{}^*) \\ &= \delta_1 A(\mu(p),\xi(q+1)) \Sigma_h \phi(S_\eta P_h S_\eta{}^*) \\ &= \delta_1 \Sigma_h A(\eta(s),h) \phi(S_\eta P_h S_\eta{}^*). \end{aligned}$$

(In particular, if $q = r$, then we have the above equality directly.) Since $\phi(S_\eta P_h S_\eta{}^*) = y_s(h)$ if $\phi(S_\eta P_h S_\eta{}^*) \neq 0$, we have $\phi(S_\mu S_\nu{}^* S_\xi S_\eta{}^*) = \delta_1 \Sigma_h A(\eta(s),h) y_s(h) = \delta_1(Ay_s)(\eta(s))$.

Following after [7], we now use the following definition of KMS states:

Definition. Let (B, R, α) be a C*-dynamical system and $\beta \in R$. Then a state ϕ on B is a (α, β)-KMS state if ϕ satisfies

$$\phi(a\,\alpha_{i\beta}(b)) = \phi(ba)$$

for all a,b in a norm dense, α-invariant *-subalgebra of B, where B is the set of entire analytic elements for α.

Throughout this note, (α, β)-KMS states are called β-KMS states for brevity. The following shows the existence of KMS states on O_A.

Corollary 4.6. Let ϕ be a trace on F_A in Lemma 4.4 and e the expectation of O_A onto F_A. Then $\phi \circ e$ is a β-KMS state on O_A, where $\beta = \log r(A)$.

Proof. It suffices to prove that $\phi(S_\mu S_\nu{}^*\alpha_{i\beta}(S_\xi S_\eta{}^*)) = \phi(S_\xi S_\eta{}^* S_\mu S_\nu{}^*)$ if $l(\mu) + l(\xi) = l(\nu) + l(\eta)$ and $l(\xi) \leq l(\nu)$. It follows from Lemma 4.5 that

$$\begin{aligned}\phi(S_\mu S_\nu{}^*\alpha_{i\beta}(S_\xi S_\eta{}^*)) &= e^{(s-r)\beta}\,\delta_1(\mu,\nu,\xi,\eta)(Ay_s)(\eta(s)) \\ &= e^{(s-r)\beta}\,\delta_1(\mu,\nu,\xi,\eta)(Ay_s)(\xi(r))\end{aligned}$$

and

$$\phi(S_\xi S_\eta{}^* S_\mu S_\nu{}^*) = \delta_2(\xi,\nu,\mu,\nu)(Ay_r)(\xi(r)).$$

Since $y_m = e^{-m\beta}y_0$ and $\delta_1(\mu,\nu,\xi,\eta) = \delta_2(\xi,\eta,\mu,\nu)$, we have $Ay_s = e^{-s\beta}Ay_0$ and so as desired.

Now we can state a main lemma as follows:

Lemma 4.7. For each $\beta \in R$, let K_β be the set of all β-KMS states for α on O_A. Then K_β is affine-isomorphic to L_β.

Proof. Define a map f of K_β to R^n by $f(\phi) = (\phi(P_i))_i$ for K_β, where $P_i = S_iS_i^*$ for $i \in E$. Since ϕ is a β-KMS state, we have

$$e^\beta (P_i) = \phi(S_i\alpha_{i\beta}(S_i^*)) = \phi(S_i^*S_i) = \Sigma_j A(i,j)\phi(P_j),$$

so that $Ay = e^\beta y$ for $y = (\phi(P_i))_i$. Obviously f is w^*-continuous.

Next we shall show that f is a map of K_β onto L_β. By Lemma 4.4, $y \in L_\beta$ induces a trace ϕ on the fixed point algebra F_A such that $\phi(P_i) = y(i)$ for $i \in E$. Let e be the expectation onto F_A. Then $\psi = e\circ\phi$ is a β-KMS state on O_A by Corollary 4.6 and

$$f(\psi)(i) = \psi(P_i) = \phi(P_i) = y(i),$$

so that $\psi \in K_\beta$ and $f(\psi) = y$.

Finally we shall prove that f is injective. For a fixed $\phi \in K_\beta$, let us put $f(\phi) = x \in R^n$. Then $x(m) = \phi(P_m)$ for $m \in E$. If $0 \neq y = S_\mu S_\mu^* \in F_A$, then $l(\mu) = l(\nu) = k$ and by Lemma 4.3

$$\begin{aligned} e^{k\beta}\phi(y) &= \phi(S_\mu\alpha_{i\beta}(S_\nu^*)) = \phi(S_\nu^*S_\mu) = \delta_{\mu,\nu}\phi(S_{\nu(k)}^*S_{\mu(k)}) \\ &= \delta_{\mu,\nu}\Sigma_h A(\mu(k),h)\phi(P_h) = \delta_{\mu,\nu}\Sigma_h A(\mu(k),h)\, x(h) \\ &= \delta_{\mu,\nu}(Ax)(\mu(k)). \end{aligned}$$

Since $x \in L_\beta$, it follows that

$$e^{k\beta}\phi(y) = \delta_{\mu,\nu}(Ax)(\mu(k)) = \delta_{\mu,\nu}e^{\beta}x(\mu(k)).$$

Hence, if $f(\phi) = f(\psi) = x$ for $\phi,\psi \in K_\beta$, then $\phi(a) = \psi(a)$ for $a \in F_A$. Since $\phi,\psi \in K_\beta$, we have $\phi(S_\nu{*}^n) = 0$ for $n \geq 1$ and so $\phi(b) = 0 = \psi(b)$ for $b \in F_A$, so that f is injective.

Now we reach Theorem 4.2 after above several lemmas :

Proof of Theorem 4.2. The proof is just to apply the Perron-Frobenius theorem to the preceding lemma. Since A is irreducible, r(A) is a unique positive eigenvalue of A with multiplicity 1 [9; (8.7)]. Therefore L_β has one element for $\beta = \log r(A)$ only. Hence the statement follows from Lemma 4.7.

Remark. Another refinement based on Theorem 4.1 is given by Bratteli, Elliott and Herman, who constructed, for each closed subset F of R, a C*-dynamical system (B,R,τ) admits a β-KMS state if and only if $\beta \in F$. Furthermore the corresponding state for each $\beta \in R$ is unique. Moreover, Bratteli, Elliott and Kishimoto [6] pursued this direction.

Remark. Finally, we can show nonexistence of ground states and ceiling states for C*-dynamical system (O_A, R, α) as in [7; Example 5.3.27].

IV-2. III_λ-representation of O_A.

In the preceding section, we have the unique KMS state for the C*-dynamical system (O_A, R, α) under the irreducibility of A. It is known that O_n is corresponding to a factor of type $III_{1/n}$. We shall determine the type of the factor generated by the GNS representation of O_A by the unique KMS state.

Let $A = (A(i,j))$ be an $n \times n$ matrix whose entries are 0 or 1. For $i,j \in E$, put $E(i,j) = \{m \in N ; A^m(i,j) > 0\}$ and $E(i) = E(i,i)$. We define $d(i)$, the peroid of a state $i \in E$, by the greatest common devisor of $E(i)$. Suppose that A is irreducible. Then $d(i) = d(j)$ for any $i,j \in E$. Hence we define $d = d(A)$, the period of A, by $d(A) = d(i)$ for any $i \in E$. The matrix A is said to be periodic of period d if $d \geq 2$, and aperiodic if $d = d(A) = 1$. For $r = 0,1,2,\dots,d-1$, put

$$D(r) = \{j \in E ; E(j,1) = r \pmod d\}.$$

Then the following is known, e.g., [19;(8.15)] : If A has period $d \geq 2$, then the state space E can be decomposed into distinct subset $D(0), D(1),\dots, D(d-1)$, (not necessarily of same size) such that a one step translation from $D(r)$ lead to a state $D(r+1)$, (from $D(d-1)$ to $D(0)$). Each $D(r)$ will be invariant under A^d, and the restriction of A^d to the state of $D(r)$ will be aperiodic. Therefore we have the following decomposition :

$$A^d = B(0) \oplus B(1) \oplus \dots \oplus B(d-1),$$

where $B(r)$ is aperiodic for $r = 0,1,\dots,d-1$.

These arguments may come in sight by a graph theoretic approach. We present here a simple example:

Example. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Then the corresponding digraph of A is expressed by

$$1 \rightleftarrows 2 \rightleftarrows 3 ,$$

and $d(A) = 2$. Moreover the state space $E = \{1, 2, 3\}$ is decomposed into the subsets $\{2\}$ and $\{1, 3\}$. Incidentally, it follows from [29; Theorem 7] that the fixed point algebra of O_A under the gauge action is not simple.

Though $S_i{}^*S_j = 0$ for $i \neq j$ by the condition (A), $S_iS_j{}^*$ $\neq 0$ in general. So we shall find that such i and j enjoy a relation, which is used in Lemma 4.8 (2). Define a map c of E onto $\{0,1, \ldots ,d-1\}$ by

$$c(i) = r \quad \text{if} \quad i \in D(r).$$

Sublemma. If $S_iS_j{}^* \neq 0$, then $c(i) = c(j)$.

Proof. Since $(S_i{}^*S_i)(S_j{}^*S_j) \neq 0$, there is $h \in E$ with

$$A(i, h) = 1 = A(j, h)$$

by the condition (A). Therefore, if $h \in D(r)$, then $i, j \in D(r+1)$ by a one step translation.

Now we define the projections corresponding to the subsets $D(r)$ by

$$R(r) = \Sigma_{i \in D(r)} S_iS_i{}^* = \Sigma_{i \in D(r)} P_i$$

for $0 \leq r \leq d-1$.

Lemma 4.8. The followings hold:

(1) For $r = 0,1,\ldots,d-1$, $m \in dZ$, there exist multiindices μ and ν such that $l(\mu) - l(\nu) = m$ and $R(r)S_\mu S_\nu{}^*R(r) \neq 0$.

(2) For $r = 0,1,\ldots,d-1$, multiindices μ and ν, if $l(\mu) = l(\nu)$ mod d, then $R(r)S_\mu S_\nu{}^*R(r) = 0$.

Proof. (1) For r there exists $k \in E$ such that $k \in D(r)$. Putting

$$E^*(k) = \{x \in Z \,;\, x = u - v,\ u \in E(k),\ v \in E(k)\},$$

then $E^*(k)$ coincides with dZ. Therefore there exist multiindices μ and ν such that $l(\mu) = u$, $l(\nu) = v$, $u - v = m \in dZ$, and $\mu(1) = \mu(m) = \nu(1) = \nu(m) = k$, so that $P_kS_\mu S_\nu{}^*P_k = 0$. Since projections $P_i (i \in D(r))$ are mutually orthogonal and $R(r) = \Sigma_{i\in D(r)}P_i$, we have $R(r)S_\mu S_\nu{}^*R(r) \neq 0$.

(2) Let k,m be in $D(r)$. We shall show that $P_kS_\mu S_\nu{}^*P_m = 0$. Assume that $P_kS_\mu S_\nu{}^*P_m \neq 0$. Then $k = \mu(1)$ and $m = \nu(1)$, so that $c(\mu(1)) = c(\nu(1))$. Since $S_{\mu(|\mu|)}S_{\nu(|\nu|)}{}^* \neq 0$, we have $c(\mu(|\mu|)) = c(\nu(|\nu|))$ by Sublemma, where $|\xi| = l(\xi)$. Furthermore, since S_μ, $S_\nu \neq 0$, it follows that

$$c(\mu(1)) = c(\mu(|\mu|)) + |\mu| - 1 \quad (\text{mod } d)$$

and

$$c(\nu(1)) = c(\nu(|\nu|)) + |\nu| - 1 \quad (\text{mod } d).$$

Hence we have $|\mu| = |\nu|$ (mod d), which is a contradiction.

We shall review some notation [10]. Let (M, R, σ) be a W*-system. For $f \in L^1(R)$, let σ_f be a σ-weakly continuous linear map of M into M such that

$$\omega(\sigma_f(x)) = \int f(t)\omega(\sigma_t(x))\, dt$$

for $\omega \in M_*$, $x \in M$. The Arveson spectrum of σ is defined by

$$sp(\sigma) = \bigcap \{Z(f)\ ;\ f \in L^1(R),\quad \sigma_f = 0\}.$$

Here $Z(f) = \{r \in R^\wedge;\ f^\wedge(r) = 0\}$, where $R^\wedge$ is the dual group of R and $f^\wedge$ is the Fourier transform of f. The Connes spectrum of σ is defined to be

$$\Gamma(\sigma) = \bigcap_p sp(\sigma\,|pMp),$$

where p runs all non zero projections in $M^\sigma \cap (M^\sigma)'$, the center of the fixed point algebra M^σ of M under σ.

In the below we assume that 0-1 matrix A is irreducible, r(A) is the spectral radius of A, $d = d(A)$ is the period of A, ϕ is the unique log r(A) - KMS state for (O_A, R, α) in Theorem 4.2. Let $(\pi_\phi, \xi_\phi, H_\phi)$ be the cyclic representation induced by ϕ.

Theorem 4.9. The von Neumann algebra $M = \pi_\phi(O_A)^-$ generated by $\pi_\phi(O_A)$ is a factor of type $III_{1/r(A)^{d(A)}}$.

Proof. Put $\beta = \log r(A)$. Since ϕ is the unique β-KMS state, ϕ is a factor state by [7;5.3.30], that is, M is a factor. Let σ be an action of R on a C*-algebra O_A such that $\sigma_t(S_j) = e^{-i\beta t}S_j$ $(j = 1, 2, \ldots, n)$, that is, $\sigma_t = \alpha_{-\beta t}$. Since ϕ is a β-KMS state for (O_A, R, α), ϕ is a (-1)-KMS state for (O_A, R, σ). Since ϕ is σ-invariant, σ can be extended to the automorphism on the factor M, denoted also by σ. Thus $(\sigma_t)_t$ is the modular automorphism group of M associated with ϕ.

Next we shall consider the fixed point algebra M^{σ} of M under σ. We claim that $M^{\sigma} = \pi_{\phi}(O_A{}^{\sigma})^{-}$, the σ-weak closure of $\pi_{\phi}(O_A{}^{\sigma})$. It is trivial that $\pi_{\phi}(O_A{}^{\sigma})^{-} \subseteq M^{\sigma}$. Conversely, if $x \in M^{\sigma}$, then we choose $x_n \in \pi_{\phi}(O_A{}^{\sigma})$ such that x_n converges to x σ-weakly in M. Putting

$$y_n = \int_T \sigma_t(x_n)dt,$$

then $y_n \in \pi_{\phi}(O_A{}^{\sigma})$. For $\omega \in M_*$, we have

$$\omega(x - y_n) = \omega(\int_T \sigma_t(x - x_n)dt)$$
$$= \int_T (\sigma_t(x) - \sigma_t(x_n))dt \longrightarrow 0.$$

Thus $M^{\sigma} \subseteq \pi_{\phi}(O_A{}^{\sigma})^{-}$, so that $M^{\sigma} = \pi_{\phi}(O_A{}^{\sigma})^{-}$.

Let $A^d = B(0) + B(1) + \dots + B(d-1)$, where $B(r)$ is aperiodic for $r = 0,1,2,\dots,d-1$. Then $O_A{}^{\sigma} = F_A = F_{B(0)} \oplus \dots \oplus F_{B(d-1)}$. Since each $B(r)$ is aperiodic, $F_{B(r)}$ is a simple unital C*-algebra with a unique trace τ_r. Moreover $N_r = \pi_{\phi}(F_{B(r)})^{-}$ is a II_1-factor. In fact, let p be a non-zero central projection of N_r. Since $\phi(x) = (x\xi_{\phi}, \xi_{\phi})$ for $x \in M$ and ξ_{ϕ} is separating for M, ϕ is a faithful normal state on M. Since ϕ is a KMS state for (M, R, σ), $\phi|M^{\sigma}$ is a trace. If we put $\tau_r'(x) = \phi(\pi_{\phi}(x)p)/\phi(p)$ for $x \in F_{B(r)}$, then τ_r' is a trace on $F_{B(r)}$. By the unicity of traces on $F_{B(r)}$, we have

$$\tau_r'(x) = \phi(\pi_{\phi}(x)p)/\phi(p) = \phi(x) = \tau_r(x)$$

for $x \in F_{B(r)}$. Since ϕ is normal, $\phi(ap) = \phi(a)\phi(p)$ for $a \in N_r$, so that $\phi(p) = \phi(p)^2$. Since ϕ is faithful and $p \neq 0$, it follows that $p = 1$. Then N_r is a II_1-factor with a trace $\phi|N_r$. For a projection p in $M^{\sigma} \cap (M^{\sigma})'$, we define an automorphism $\sigma_t{}^p$ on pMp by

$$\sigma_t{}^p(pxp) = p\sigma_t(pxp)p \quad \text{for } x \in M.$$

Then we have

$$\Gamma(\sigma) = \bigcap\{\operatorname{sp}(\sigma^{p})\ ;\ 0 \neq p \in M^{\sigma} \cap (M^{\sigma})'\} = \bigcap_{r=0}^{d-1} \operatorname{sp}(\sigma^{R(r)}).$$

Next we shall show that for $r = 0,1,\ldots,d-1$,

$$\operatorname{sp}(\sigma^{R(r)}) = \{n\beta d \in R \cong R\hat{}\ ;\ n \in Z\} \ .$$

So we first show that $\operatorname{sp}(\sigma^{R(r)}) \supseteq \{n\beta d \in R \cong R\hat{}\ ;\ n \in Z\}$. For a fixed $n \in Z$, it follows from Lemma 4.8 (1) that there exist multiindices μ and ν such that

$$l(\mu) - l(\nu) = nd \quad \text{and} \quad R(r)S_{\mu}S_{\nu}^{*}R(r) \neq 0.$$

If $f \in \operatorname{Ker} \sigma^{R(r)}$, then

$$\sigma_f^{R(r)}(S_{\mu}S_{\nu}^{*}) = R(r)\sigma_f(S_{\mu}S_{\nu}^{*})R(r) = 0.$$

On the other hand, we have

$$\begin{aligned}\sigma_f^{R(r)}(S_{\mu}S_{\nu}^{*}) &= R(r)\sigma_f(S_{\mu}S_{\nu}^{*})R(r)\\ &= R(r)(\textstyle\int f(t)\sigma_t(S_{\mu}S_{\nu}^{*})dt)R(r)\\ &= R(r)(\textstyle\int f(t)e^{-in\beta d}S_{\mu}S_{\nu}^{*}dt)R(r)\\ &= f\hat{}(n\beta d)R(r)S_{\mu}S_{\nu}^{*}R(r).\end{aligned}$$

Therefore $\hat{f}(n\beta d) = 0$ and so $n\beta d \in \operatorname{sp}(\sigma^{R(r)})$ as desired.

Conversely, let $r \in R$ and $r \notin \beta dZ \subseteq R$. Then there exists a function $f \in L^{1}(R)$ such that $f\hat{}(r) = 1$ and $f\hat{}\,|\,\beta dZ = 0$. We shall show that f is in $\operatorname{Ker} \sigma^{R(r)}$. Since the $*$-algebra generated algebraically by $\{S_1,\ldots,S_n\}$ is σ-weakly dense in M, it is enough to show that

$$\sigma_f^{R(r)}(R(r)S_{\mu}S_{\nu}^{*}R(r)) = 0$$

for multiindices μ and ν. While we have

$$\begin{aligned}\sigma_f^{R(r)}(R(r)S_{\mu}S_{\nu}^{*}R(r)) &= R(r)\sigma_f(S_{\mu}S_{\nu}^{*})R(r)\\ &= R(r)(\textstyle\int f(t)\sigma_t(S_{\mu}S_{\nu}^{*})dt)R(r)\\ &= R(r)(\textstyle\int f(t)e^{-(l(\mu)-l(\nu))\beta t}S_{\mu}S_{\nu}^{*}dt)R(r)\\ &= f\hat{}((l(\mu)-l(\nu))\beta)R(r)S_{\mu}S_{\nu}^{*}R(r).\end{aligned}$$

If $l(\mu)-l(\nu) \in dZ$, then $f^{\wedge}((l(\mu)-l(\nu))\beta) = 0$ by the definition of f. If $l(\mu)-l(\nu) \notin dZ$, then $R(r)S_{\mu}S_{\nu}{}^{*}R(r) = 0$ by Lemma 4.8 (2). In both cases we have $\sigma_f{}^{R(r)}(R(r)S_{\mu}S_{\nu}{}^{*}R(r)) = 0$, so that f is in $\mathrm{Ker}\ \sigma^{R(r)}$. Since $f^{\wedge}(r) = 1$, it implies that $r \notin sp(\sigma^{R(r)})$. Therefore we have $sp(\sigma^{R(r)}) = \{n\beta d \in R\ ;\ n \in Z\}$ for $r = 0,1,\ldots,d-1$.

Hence it follows that

$$\Gamma(\sigma) = \bigcap_{r=0}^{d-1} sp(\sigma^{R(r)}) = \{n\beta d \in R\ ;\ n \in Z\}.$$

Since $\beta = \log r(A)$, M is a type III_{λ} factor, where $\lambda = 1/r(A)^{d}$.

IV-3. Eigenvalue problem.

Finally, we shall reformulate the argument in [18], which is based on the discussion of §1. Precisely, for certain simple C*-algebras with periodic dynamics there is a Banach lattice F and a positive operator R on F such that the C*-dynamical system has a β-KMS state if and only if e^{β} is an eigenvalue of R. Moreover the set K of all β-KMS states is affine isomorphic to the set L_{β} of all normalized positive eigenvectors corresponding with the eigenvalue e^{β}. Thus to find β-KMS states can be formulated as the eigenvalue problem.

Let A be a unital C*-algebra and $\{\alpha_t\}_{t\in R}$ a strongly continuous and periodic one-parameter automorphism group on A with period 2π. The spectral subspace $A(n)$ for $n \in Z$ is defined by

$$A(n) = \{x \in A;\ \alpha_t(x) = e^{int}x \ \text{ for } \ t \in R\}.$$

A projection of norm one from A onto the fixed point algebra $A(0)$ is given by

(1) $\quad e(x) = \int_0^{2\pi} \alpha_t(x)dt/2\pi \quad$ for $x \in A$.

It is known that the linear span of $\{A(n);\ n \in Z\}$ is dense in A and $A(n)A(m) \subseteq A(n+m)$ for $n,m \in Z$, e.g., [39].

Let F be the subspace of $A(0)^*$ consisting of all self-adjoint and tracial functionals. Then F is a real Banach lattice whose positive cone F_+ is the set of all positive functionals in F, cf. [2]. The following lemma is a slight modification of an asymmetric Riesz decomposition theorem [43; Theorem 7.7 in Ch.I] and so we omit a proof.

Lemma 4.10. Let F be as in above, and $\{u_i, v_i, x_i, y_i;\ i=1,2,\dots, m\} \subseteq A(n)$ for a fixed $n \in Z$. Then $f(\Sigma_i v_i^* u_i) = f(\Sigma_i y_i^* x_i)$ for $f \in F$ if $\Sigma_i u_i v_i^* = \Sigma_i x_i y_i^*$.

A bounded linear operator R on F is said to be a reverse operator associated with (A, R, α) if

$$(Rf)(xy^*) = f(y^*x)$$

for $f \in F$ and $x,y \in A(1)$. Here we shall discuss on the existence of a reverse operator.

Lemma 4.11. If $A(0)$ is simple, then there exists a unique reverse operator R associated with (A, R, α).

Proof. Since $A(0)$ is simple, we have $A(1)A(1)^* = A(0)$. For each fixed $a \in A(0)$, there is a family $\{x_i, y_i\ ;\ i=1,2,\dots$

$\ldots, m\} \subseteq A(1)$ such that $a = \Sigma_i x_i y_i{}^*$. Then it follows from Lemma 4.10 that $(Rf)(a) = f(\Sigma_i y_i{}^* x_i)$ is well-defined, and Rf is a tracial linear functional on $A(0)$.

Next we shall show that Rf is bounded and R is a bounded linear map on F. Note that $1 = \Sigma_i s_i t_i{}^*$ for some s_i, $t_i \in A(1)$. For $b \in A(0)$, we have

(2) $$(Rf)(b) = Rf(\Sigma_i bs_i t_i{}^*) = f(\Sigma_i t_i{}^* bs_i),$$

so that

$$|(Rf)(b)| = |f(\Sigma_i t_i{}^* bs_i)| \le \|f\| \; \Sigma_i \|t_i\| \; \|s_i\| \, \|b\|.$$

It implies that Rf is bounded and moreover $\|R\| \le \Sigma_i \|t_i\| \, \|s_i\|$. Since R is linear by (3), R is a bounded linear operator on F.

Theorem 4.12. Let $L_\beta = \{f \in F_+;\ Rf = e^\beta f,\ \|f\| = 1\}$ for each $\beta \in R$. Let (A, R, α) be a C*-dynamical system with period 2π such that A is unital and the fixed point algebra $A(0)$ is simple. Let R be the reverse operator associated with (A, R, α). Then K_β is affine isomorphic to L_β for each $\beta \in R$.

Proof. Putting $H(g) = g|A(0)$ for $g \in K_\beta$, then $H(g)$ is tracial, so that $H(g) \in F$. Now we shall prove that H is a w*-continuous affine isomorphism of K_β onto L_β. Since g is a β-KMS state, we have

$$e^\beta g(bst^*) = g(bs\,\alpha_{i\beta}(t^*)) = g(t^*bs)$$

for $b \in A(0)$ and $s,t \in A(1)$. It follows from (2) that $H(g) \in L_\beta$ because

$$(RH(g))(b) = H(g)(\Sigma_i t_i{}^*bs_i) = g(\Sigma_i t_i{}^*bs_i)$$
$$= e^{\beta}g(\Sigma_i bs_it_i{}^*) = e^{\beta}g(b) = e^{\beta}H(g)(b).$$

Thus H is a w*-continuous affine map of K_β into L_β.

Let e be the norm one projection of A onto $A(0)$ defined by (2) and put $G(f) = f\circ e$ for $f \in L_\beta$. By similar calculations, G is also a w*-continuous affine map of L_β into K_β, and $H\circ G = \mathrm{id}$ on L_β. Moreover since $g|A(n) = 0$ for $n = 0$ and $g \in K_\beta$, we have $G\circ H = \mathrm{id}$ on K_β. Hence it implies that H is a bijection.

Remark. In the case where there exists a family $\{s_1, \ldots, s_k\} \subseteq A(1)$ such that $\Sigma_i s_is_i{}^* = 1$, the reverse operator R is positive.

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