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PSEUDO-ORBIT TRACING PROPERTY AND STRONG TRANSVERSALITY OF DIFFEOMORPHISMS ON CLOSED MANIFOLDS

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1. Introduction

We are interested in the dynamical property of a diffeomorphism $f$ having the pseudo-orbit tracing property of a closed manifold $M$. Let $d$ be a metric for $M$. A sequence of points $\{x_i\}_{i \in \mathbb{Z}}$ of $M$ is called a $\delta$-pseudo-orbit of $f$ if $d(f(x_i), x_{i+1}) < \delta$ for $i \in \mathbb{Z}$. A sequence $\{x_i\}_{i \in \mathbb{Z}}$ is said to be $f$-$\varepsilon$-traced by $y \in M$ if $d(f^i(y), x_i) < \varepsilon$ for $i \in \mathbb{Z}$.

We say that $f$ has the pseudo-orbit tracing property (abbrev. POTP) if for every $\varepsilon > 0$ there is $\delta > 0$ such that every $\delta$-pseudo-orbit of $f$ can be $f$-$\varepsilon$-traced by some point.

In [5] Robinson proved that every Axiom A diffeomorphism satisfying strong transversality has POTP. Thus it will be natural to ask whether POTP implies Axiom A and strong transversality. For this problem we have partial results that are answered in [4] for dim $M=2$ and in [7] for dim $M=3$. However we have no answer for higher dimensions.

Our aim is to prove the following

**Theorem.** The $C^1$ interior of all diffeomorphisms having POTP of a closed manifold $M$, $\mathcal{P}(M)$, coincides with the set of all Axiom A diffeomorphisms satisfying strong transversality.

We say that $f$ has the $C^1$ uniform pseudo-orbit tracing property (abbrev. $C^1$-UPOTP) if there is a $C^1$ neighborhood $\mathcal{U}(f)$ of $f$ with the property that for $\varepsilon > 0$ there is $\delta > 0$ such that every $\delta$-pseudo-orbit of $g \in \mathcal{U}(f)$ is $g$-$\varepsilon$-traced by some point. Since every Axiom A diffeomorphism satisfying strong transversality has $C^1$-UPOTP (see [6, Theorem]), if we establish our theorem, then the following corollary is obtained.

**Corollary.** The set of all diffeomorphism having $C^1$-UPOTP is characterized as the set of all Axiom A diffeomorphisms satisfying strong transversality.
It was proved in [4] that all periodic points of \( f \in \mathcal{P}(M) \) are hyperbolic. From this we can prove that each \( f \) belonging to \( \mathcal{P}(M) \) satisfies Axiom A with no-cycle. Recently it was shown in general by Aoki [1]. Therefore, to conclude our theorem it remains only to prove the following proposition.

**Proposition.** Every \( f \in \mathcal{P}(M) \) satisfies strong transversality.

Unfortunately this can not be proved by the techniques mentioned in [4] and [7]. Thus we need a new technique for the proof of the proposition.

2. Proof of Proposition

Let \( \text{Diff}(M) \) denote the set of all diffeomorphisms of \( M \) endowed with \( C^1 \) topology, and let \( p = f^n(p) \) \((n > 0)\) be a hyperbolic periodic point of \( f \in \text{Diff}(M) \). Even if \( p \) is hyperbolic, when \( \dim M \geq 3 \), it is not easy to construct an \( f^n \)-invariant foliation in a neighborhood of \( p \) that is compatible with the local stable manifold (i.e. the leaf passing through \( p \) is the local stable manifold of \( p \)). In this paper, by using Franks's lemma we make a new diffeomorphism \( g \) \((g^n(p) = p)\), arbitrarily near to \( f \) in \( C^1 \) topology, which has a \( g^n \)-invariant compatible foliation in a neighborhood of \( p \) (see lemmas 1 and 2). This foliation will play an essential role in the proof of the proposition.

Let \( f \in \text{Diff}(M) \) satisfy Axiom A with no-cycle. The non-wandering set \( \Omega(f) \) of \( f \) is expressed as a finite disjoint union of basic sets \( \{ \Lambda_i(f) \} \), and for a sufficiently small \( \varepsilon_0 > 0 \) and \( x \in \Omega(f) \) there are a local stable manifold \( W^s_{\varepsilon_0}(x, f) \) and a local unstable manifold \( W^u_{\varepsilon_0}(x, f) \). Let \( \Lambda(f) \) be a basic set of \( f \). Since \( \dim W^s_{\varepsilon_0}(x, f) = \dim W^u_{\varepsilon_0}(y, f) \) \((x, y \in \Lambda(f))\), we denote by \( \text{Ind} \Lambda(f) \) the dimension of \( W^s_{\varepsilon_0}(x, f) \) for \( x \in \Lambda(f) \). If \( g \in \text{Diff}(M) \) is \( C^1 \) close to \( f \), then the number of basic sets \( \{ \Lambda_i(g) \} \) of \( g \) coincides with that of basic sets \( \{ \Lambda_i(f) \} \) since \( f \) is \( \Omega \)-stable.

Put \( B_\varepsilon(x) = \{ y \in M | d(x, y) \leq \varepsilon \} \) for \( \varepsilon > 0 \) and let \( \rho \) be a usual \( C^1 \) metric of \( \text{Diff}(M) \). Then we have the following

**Lemma 1.** Let \( \varepsilon_0 > 0 \) be as above and let \( \Lambda(f) \) be a basic set such that \( 1 \leq \text{Ind} \Lambda(f) \leq \dim M - 1 \). Then, for a periodic point \( p \in \Lambda(f) \) \((f^n(p) = p, n > 0)\), a neighborhood \( \mathcal{U}(f) \subseteq \text{Diff}(M) \) and a number \( \gamma > 0 \) there are \( 0 < \varepsilon_1 < \varepsilon_0/2 \), \( g \in \mathcal{U}(f) \) and a basic set \( \Lambda(g) \) for \( g \) such that

(i) \( B_{4\varepsilon_1}(f^i(p)) \cap B_{4\varepsilon_1}(f^j(p)) = \phi \) for \( 0 \leq i \neq j \leq n - 1 \),
(ii) \( g(x) = \begin{cases} 
\exp_{f^i(p)} \circ D_{f^i(p)} f \circ \exp_{f^i(p)}^{-1}(x) & \text{if } x \in B_{4\varepsilon_1}(f^i(p)) \text{ for } 0 \leq i \leq n-1, \\
f(x) & \text{if } x \notin \bigcup_{i=0}^{n-1} B_{4\varepsilon_1}(f^i(p)), \end{cases} \)

(iii) \( g^n(p) = p \in \Lambda(g) \) and \( \rho(W^\sigma_{\varepsilon_0}(p, f), W^\sigma_{\varepsilon_0}(p, g)) < \gamma \) for \( \sigma = s, u \) (i.e. there is a \( C^1 \) diffeomorphism \( \xi^\sigma : W^\sigma_{\varepsilon_0}(p, f) \to W^\sigma_{\varepsilon_0}(p, g) \) such that \( \rho(\xi^\sigma, \text{id}) < \gamma \) (\( \sigma = s, u \)).

Proof. Since \( \Lambda(f) \) is hyperbolic, there is \( \varepsilon > 0 \) such that \( d(f^n(x), f^n(y)) < \varepsilon \) \((x, y \in \Lambda(f) \) and \( n \in \mathbb{Z}\)) implies \( x = y \) (see [5]). By \( \Omega \)-stability theorem, there exists a neighborhood \( \mathcal{U}_0(f) \subset \mathcal{U}(f) \) of \( f \) such that for every \( g \in \mathcal{U}_0(f) \) there is a homeomorphism \( h_g \), which maps \( \Omega(f) \) onto the non-wandering set \( \Omega(g) \) of \( g \), satisfying

\[
\begin{align*}
&g \circ h_g = h_g \circ f, \\
d(h_g, \text{id}|_{\Omega(f)}) < \varepsilon, \\
&\rho(W^\sigma_{\varepsilon_0}(p, f), W^\sigma_{\varepsilon_0}(h_g(p), g)) < \gamma \text{ for } \sigma = s, u.
\end{align*}
\]

By Franks's lemma [2, lemma 1.1], we can find \( g \in \mathcal{U}_0(f) \) and \( 0 < \varepsilon_1 < \varepsilon_0 / 2 \) such that

\[
B_{4\varepsilon_1}(f^i(p)) \cap B_{4\varepsilon_1}(f^j(p)) = \emptyset \text{ (0 \leq i \neq j \leq n-1)} \text{ and}
\]

\[
g(x) = \begin{cases} 
\exp_{f^i(p)} \circ D_{f^i(p)} f \circ \exp_{f^i(p)}^{-1}(x) & \text{if } x \in B_{4\varepsilon_1}(f^i(p)) \text{ for } 0 \leq i \leq n-1, \\
f(x) & \text{if } x \notin \bigcup_{i=0}^{n-1} B_{4\varepsilon_1}(f^i(p)), \end{cases}
\]

We write \( \Lambda(g) = h_g(\Lambda(f)) \) for simplicity. Then \( h_g(p) \in \Lambda(g) \) and \( \text{Ind } \Lambda(f) = \text{Ind } \Lambda(g) \). Clearly \( g(f^i(p)) = \exp_{f^i(p)} \circ D_{f^i(p)} f \circ \exp_{f^i(p)}^{-1}(f^i(p)) = f^{i+1}(p) \) for \( 0 \leq i \leq n-1 \) and so \( g(p) = f(p), g^2(p) = f^2(p), \ldots, g^n(p) = f^n(p) = p \). Since

\[
d(f^i(h_g^{-1}(p)), f^i(p)) = d(h_g^{-1}(g^i(p)), f^i(p)) = d(h_g^{-1}(f^i(p)), f^i(p)) < \varepsilon \text{ (i \in \mathbb{Z})},
\]

we have \( h_g(p) = p \). Therefore \( \rho(W^\sigma_{\varepsilon_0}(p, f), W^\sigma_{\varepsilon_0}(p, g)) < \gamma \) (\( \sigma = s, u \)) and \( p \in \Lambda(g) \).

Since \( f \) satisfies Axiom A, by definition there is a \( Df \)-invariant continuous splitting \( T_{\Omega(f)} M = E^s \oplus E^u \) and a constant \( 0 < \lambda < 1 \) such that
\[ \|Df\|_{E^m} \leq \lambda^m \text{ and } \|Df\|_{E^m} \leq \lambda^m \text{ for } m > 0. \] We denote by \( E^s_\varepsilon \) a fiber of \( E^s \) at \( x \in \Omega(f) (\sigma = s, u) \), and put \( E^s_\varepsilon = \{ v \in E^s_\varepsilon \| v \| \leq \varepsilon \} \) for \( \varepsilon > 0. \)

Let \( g \in \text{Diff}(M), p = g^n(p) \in \Lambda(g) (n > 0) \) and \( \varepsilon_1 > 0 \) be as in lemma 1. Then it is easily checked that for \( 0 < \varepsilon \leq \varepsilon_1 \), we have \( \exp_p(E^s_\varepsilon) = W^s_\varepsilon(p, g) \) and \( \dim \exp_p(E^s_\varepsilon) = \dim W^s_\varepsilon(p, g) (\sigma = s, u) \).

Fix \( \varepsilon_2 \) with \( 0 < \varepsilon_2 = \varepsilon_2(g, n) < \varepsilon_1 \) such that \( x \in B_{\varepsilon_2}(p) \) implies \( g^n(x) \in B_{\varepsilon_2}(g^n(p)) \) for \( 0 \leq i \leq n-1 \), and define

\[ \tilde{W}^s_{\varepsilon_2}(x, g) = \exp_p \left( E^s_\varepsilon + \exp_p^{-1}(x) \right) \]

for \( x \in \exp_p(E^s_\varepsilon) \). Then, since \( \bigcup_{\varepsilon \in E^s_\varepsilon} (E^s_\varepsilon + v) \) is a foliation defined in a neighborhood of \( O_p \in T_pM \) and since \( \exp_p \) is a local diffeomorphism, we have that \( \{ \tilde{W}^s_{\varepsilon_2}(x, g) : x \in \exp_p(E^s_\varepsilon) \} \) is a foliation defined in a neighborhood of \( p \) in \( M \) such that \( \tilde{W}^s_{\varepsilon_2}(p, g) = W^s_{\varepsilon_2}(p, g) \).

**Lemma 2.**

(i) \( \tilde{W}^s_{\varepsilon_2}(x, g) \) is a \( C^1 \) manifold and \( \dim \tilde{W}^s_{\varepsilon_2}(x, g) = \dim \tilde{W}^s_{\varepsilon_2}(p, g) \).

(ii) \( g^n(\tilde{W}^s_{\varepsilon_2}(x, g)) \subset \tilde{W}^s_{\varepsilon_2}(g^n(x), g) \) for \( x \in \exp_p(E^s_\varepsilon) \cap g^{-n}(\exp_p(E^s_\varepsilon)) \).

(iii) there exists \( C > 0 \) such that if \( \{ x, g^n(x), \cdots, g^{nk}(x) \} \subset \exp_p(E^s_\varepsilon) \) for some \( k > 0 \), then \( d(g^{nk}(x), g^{nk}(y)) \leq C \lambda^k d(x, y) \) for \( y \in \tilde{W}^s_{\varepsilon_2}(x, g) \).

Proof. Assertion (i) is clear, and (ii) is easily obtained. To show (iii) put \( T_p(\varepsilon_2) = \{ v \in T_pM \| v \| \leq \varepsilon_2 \} \). Since \( \exp_p : T_pM \to M \) and \( \exp_p^{-1} : B_{\varepsilon_2}(p) \to T_pM \) are into diffeomorphisms there is \( K > 0 \) such that

\[ d(\exp_p(v), \exp_p(w)) \leq K\|v - w\| \quad (v, w \in T_p(\varepsilon_2)), \]

\[ \|\exp_p^{-1}(x) - \exp_p^{-1}(y)\| \leq Kd(x, y) \quad (x, y \in B_{\varepsilon_2}(p)). \]

If \( \{ x, g^n(x), \cdots, g^{nk}(x) \} \subset \exp_p(E^s_\varepsilon) \) for some \( k > 0 \), then for \( y \in \tilde{W}^s_{\varepsilon_2}(x, g) \) there is \( v_y \in E^s_\varepsilon(\varepsilon_2) \) such that \( y = \exp_p(v_y + \exp_p^{-1}(x)) \). Thus we have

\[ g^n(y) = \exp_p \left( D_pf^n(v_y) + \exp_p^{-1}(g^n(x)) \right) \]

(since \( D_pf^n(\exp_p^{-1}(x)) = \exp_p^{-1}(g^n(x)) \)), and so

\[ \left( D_pf^n \circ \exp_p^{-1} \circ g^n \right)(y) = D_pf^n(v_y) + D_pf^n(\exp_p^{-1}(g^n(x))). \]
from which
\[ g^{2n}(x) = \exp_p(D_pf^{2n}(v_x) + D_pf(\exp_p^{-1}(g^{2n}(x)))]. \]
Since \( g^n(x) \in B_{\epsilon_1}(p) \), we have \( (\exp_p \circ D_pf^n \circ \exp_p^{-1})(g^n(x)) = g^{2n}(x) \); i.e. \( D_pf^n(\exp_p^{-1}(g^n(x))) = \exp_p^{-1}(g^{2n}(x)) \). Thus \( g^{2n}(y) = \exp_p(D_pf^{2n}(v_y) + \exp_p^{-1}(g^{2n}(x))) \). By repetition we have
\[ g^{nk}(y) = \exp_p(D_pf^{nk}(v_y) + \exp_p^{-1}(g^{nk}(x))). \]
from which
\[ d(g^{nk}(x), g^{nk}(y)) \leq \frac{K}{\exp_p^{-1}(g^{nk}(x)) - \exp_p^{-1}(g^{nk}(y))} \]
\[ = \frac{K}{\exp_p^{-1}(g^{nk}(v_y))} \]
\[ \leq K\lambda^{nk}||v_y||. \]
Clearly, \( ||v_y|| = ||\exp_p^{-1}(x) - \exp_p^{-1}(y)|| \leq Kd(x, y) \) since \( \exp_p^{-1}(y) = v_y + \exp_p^{-1}(x) \). Therefore, \( d(g^{nk}(x), g^{nk}(y)) \leq K^2\lambda^{nk}d(x, y) \). Assertion (iii) was proved.

Let \( f \) be as before, and denote by \( W^s(x, f) \) the stable manifold and by \( W^u(x, f) \) the unstable manifold for \( x \in \Omega(f) \) respectively.

\textbf{Lemma 3.} Let \( \Lambda_1(f) \) and \( \Lambda_2(f) \) be two distinct basic sets for \( f \). Suppose that there are \( p = f^n(p) \in \Lambda_1(f) \) \((n > 0)\), \( q \in \Lambda_2(f) \) and \( x \in M \setminus \Omega(f) \) such that \( x \in W^s(p, f) \cap W^u(q, f) \). Then, for neighborhood \( \mathcal{U}(f) \subset \text{Diff} (M) \) there are \( 0 < \epsilon_1 < \epsilon_0/2, g \in \mathcal{U}(f) \) and two distinct basic sets \( \Lambda_1(g) \) and \( \Lambda_2(g) \) for \( g \) such that
\[
\begin{align*}
\text{(I)} & \quad B_{4\epsilon_2}(f^j(p)) \cap B_{4\epsilon_2}(f^j(p)) = \phi \quad \text{for} \quad 0 \leq i \neq j \leq n-1, \\
\text{(II)} & \quad g(z) = \begin{cases} 
\exp_{f^i(p)} \circ D_{f^i(p)} \circ \exp_{f^i(p)}^{-1}(z) & \text{if} \quad z \in B_{\epsilon_1}(f^i(p)) \quad \text{for} \quad 0 \leq i \leq n-1, \\
\emptyset & \text{if} \quad z \notin \bigcup_{i=0}^{n-1} B_{4\epsilon_2}(f^i(p)), \\
f(z) & \text{if} \quad z \notin \bigcup_{i=0}^{n-1} B_{4\epsilon_2}(f^i(p)), \end{cases} \\
\text{(III)} & \quad p = g^n(p) \in \Lambda_1(g) \quad q \in \Lambda_2(g), \\
x \in W^s(p, g) \cap W^u(q, g), \\
T_x W^s(p, g) = T_x W^s(p, f) \quad \text{and} \quad T_x W^u(q, g) = T_x W^u(q, f). 
\end{align*}
\]
Proof. Fix \( \mathcal{U}(f) \subset \text{Diff} (M) \). By lemma 1, for any \( \gamma > 0 \) there are
0 < \varepsilon_1 < \varepsilon_0 / 2, \ g \in \mathcal{U}(f) \) and a basic set \( \Lambda_1(g) \) satisfying properties (i), (ii) and (iii) of lemma 1. Put \( \Lambda_2(g) = \Lambda_2(f) \). Then \( q \in \Lambda_2(g) \). Since \( \gamma \) is arbitrarily small, by (iii) there are a new diffeomorphism \( \tilde{g} \in \mathcal{U}(f) \) and a small neighborhood \( U(x) \) of \( x \) such that \( \tilde{g}(y) = g(y) \) for all \( y \notin U(x) \) and such that

\[
\begin{cases}
  x \in W^u(p, \tilde{g}) \cap W^u(q, \tilde{g}), \\
  T_x W^u(p, \tilde{g}) = T_x W^u(p, f), \\
  T_x W^u(q, \tilde{g}) = T_x W^u(q, f),
\end{cases}
\]

For simplicity we identify \( \tilde{g} \) with \( g \). Thus (I), (II) and (III) are concluded.

Let \( g \in \mathcal{U}(f), \ p = g^n(p) \in \Lambda_1(g) \) and \( \varepsilon_1 > 0 \) be as in lemma 3 and suppose that \( \dim M - \text{Ind} \Lambda_1(f) > 2 \). Take \( 0 < \varepsilon_2 \leq \varepsilon_1 \) be as in lemma 2, and fix \( \alpha > 0 \) such that \( D_p f^2_p(E^u_p(\alpha)) \subset E^u_p(\varepsilon_2) \). Put \( D^u(p) = \exp_p(E^u_p(\alpha)) \). Then we have

\[
d(\tilde{W}_{\varepsilon_2}(g^{2n}(F^u(p, g)), g), \tilde{W}_{\varepsilon_2}(F^u(p, g), g)) > 0,
\]

\[
d(\tilde{W}_{\varepsilon_2}(F^u(p, g), g), \tilde{W}_{\varepsilon_2}(g^{-2n}(D^u(p)), g)) > 0
\]

where

\[
F^u(p, g) = D^u(p) \setminus g^{-n}(D^u(p))
\]

is a fundamental domain of \( W^u_{\varepsilon_2}(p, g) \) (recall that \( \exp_p(E^u_p(\varepsilon)) = W^u_{\varepsilon}(p, g) \) for \( 0 < \varepsilon \leq \varepsilon_2 \)).
Let $G$ be a linear subspace of $E_p$ such that $1 \leq \dim G < \dim E_p$ and write $B_r^G(E) = B_r(E) \cap \exp_p(E_p(\varepsilon_2))$ for a subset $E$ of $M$. Then we can find $0 < r_0 < \varepsilon_2$ such that

$$F^w(p, g) \setminus B_{r_0}(\exp_p(G \cap E_p(\varepsilon_2)) \cap F^w(p, g)) \neq \emptyset$$

for every $G$. Since

$$r_0' = d(\bar{W}_{\varepsilon_2}(g^{2n}(F^w(p, g)), g), \bar{W}_{\varepsilon_2}(F^w(p, g), g)) > 0,$$
$$r_0'' = d(\bar{W}_{\varepsilon_2}(F^w(p, g), g), \bar{W}_{\varepsilon_2}(g^{-2n}(D^w(p)), g)) > 0,$$

we define a positive number $r_1 = \frac{1}{4} \min\{r_0, r_0', r_0''\}$.

Put

$$\Gamma(p) = \bigcup_{y \in \exp_p(E_p(\varepsilon_2))} \bar{W}_{\varepsilon_2}(y, g).$$

Then, for any $z \in \Gamma(p)$, we can find only one point $y \in \exp_p(E_p(\varepsilon_2))$ such that $z \in \bar{W}_{\varepsilon_2}(y, g)$, and so we write

$$\pi(z) = y.$$

Then $\pi: \Gamma(p) \to \exp_p(E_p(\varepsilon_2))$ is differentiable and which plays an essential role in the proof of the proposition. For $z \in \Gamma(p) \setminus \bar{W}_{\varepsilon_2}(p, g)$, there is an integer $N_z > 0$ such that $g^{ni}(\pi(z)) \in D^w(p)$ for $0 \leq i \leq N_z$ (especially $g^{nN_z}(\pi(z)) \in F^w(p, g)$) and $g^{n(N_z+1)}(\pi(z)) \notin D^w(p)$.

**Lemma 4.** Under the above notations, there is $0 < \varepsilon_3 < r_1$ such that $\text{diam } \pi(B_\varepsilon(g^{nN_z}(z))) < r_1$ for every $z \in \left( \bigcup_{y \in \bar{W}_{\varepsilon_2}(y, g)} \bar{W}_{\varepsilon_2}(y, g) \right) \setminus \bar{W}_{\varepsilon_2}(p, g)$.

**Proof.** If this is false, for $k > 0$ there are

$$z_k \in \left( \bigcup_{y \in \bar{W}_{\varepsilon_2}(y, g)} \bar{W}_{\varepsilon_2}(y, g) \right) \setminus \bar{W}_{\varepsilon_2}(p, g)$$

and $N_k = N_{z_k} > 0$ such that $\text{diam } \pi(B_{\varepsilon_k}(g^{nN_k}(z_k))) \geq r_1$. Since $z_k \in \bar{W}_{\varepsilon_2}(\pi(z_k), g)$, we have $N_k \to \infty$ as $k \to \infty$ (because of $\pi(z_k) \in \bar{W}_{\varepsilon_2}(p, g)$). From $g^{ni}(\pi(z_k)) \in D^w(p) \cap \exp_p(E_p(\varepsilon_2))$ for $0 \leq i \leq N_k$, we have

$$d(g^{nN_k}(\pi(z_k)), g^{nN_k}(z_k)) \leq C \lambda^{nN_k}d(\pi(z_k), z_k) \to 0 \text{ as } k \to \infty.$$
by lemma 2 (iii).

For \( k > 0 \) there are \( w_k, w'_k \in \exp_p(E^u_p(\varepsilon_2)) \), \( v_k \in \tilde{W}^s_{\varepsilon_2}(w_k, g) \cap B_{\frac{1}{k}}(g^{nNk}(z_k)) \) and \( v'_k \in \tilde{W}^s_{\varepsilon_2}(w'_k, g) \cap B_{\frac{1}{k}}(g^{nNk}(z_k)) \) such that \( dv_k, dv'_k \geq r_1 \). If \( w_k \to w \) and \( w'_k \to w' \) \((k \to \infty)\), then \( w, w' \in \exp_p(E^u_p(\varepsilon_2)) \) and \( d(w, w') \geq r_1 \). When \( v_k \to v \) and \( v'_k \to v' \) as \( k \to \infty \), we have \( v = v' \in \exp_p(E^u_p(\varepsilon_2)) \) by the properties

\[
\begin{align*}
g^{nNk}(z_k) &\in \exp_p(E^u_p(\varepsilon_2)), \\
d(g^{nNk}(z_k), g^{nNk}(z_k)) &\to 0 \text{ as } k \to \infty, \\
d(v_k, g^{nNk}(z_k)) &< \frac{1}{k} \quad \text{and} \quad d(v'_k, g^{nNk}(z_k)) < \frac{1}{k}.
\end{align*}
\]

Since \( \tilde{W}^s_{\varepsilon_2}(y, g) \) \((y \in \exp_p(E^u_p(\varepsilon_2))\) is continuous with respect to \( y \), we have \( v \in \tilde{W}^s_{\varepsilon_2}(w, g) \). Thus \( v = w \) since \( \tilde{W}^s_{\varepsilon_2}(w, g) \cap \exp_p(E^u_p(\varepsilon_2)) \) is a single point and \( v, w \in \exp_p(E^u_p(\varepsilon_2)) \). In this way we get \( w = v = v' = w' \), thus contradicting.

We are in a position to prove the proposition. Hereafter let \( \dim M \geq 4 \) and \( f \in \mathcal{P}(M) \). Notice that \( f \) satisfies Axiom A with no-cycle.

Fix \( x \in M \setminus \Omega(f) \). Then \( f \) has distinct basic sets \( \Lambda_i(f) \) \((i=1, 2)\) such that \( x \in W^s(\Lambda_1(f), f) \cap W^u(\Lambda_2(f), f) \). If \( \text{Ind} \Lambda_1(f) = \dim M \) or \( \dim M - 1 \), then by the proof of [4, Theorem 2] we have \( T_xM = T_xW^u(x, f) + T_xW^s(x, f) \). Thus it is enough to prove the above equality for the case when \( 1 \leq \text{Ind} \Lambda_1(f) \leq \dim M - 2 \).

Since \( \Omega(f) = \overline{\mathcal{P}(f)} \), there is \( f' \in \mathcal{P}(M) \) arbitrarily near to \( f \) in a \( C^1 \) topology satisfying

(a) \( f(y) = f'(y) \) for all \( y \) outside of a small neighborhood of \( x \),

(b) there are \( p = f^n(p) \in \Lambda_1(f) \) for some \( n > 0 \) and \( q \in \Lambda_2(f) \) such that \( x \in W^s(p, f') \cap W^u(q, f') \), \( T_xW^s(p, f') = T_xW^s(x, f) \) and \( T_xW^u(q, f') = T_xW^u(x, f) \).

By (a) there are basic sets \( \Lambda_i(f') \) \((i=1, 2)\) for \( f' \) such that \( \Lambda_i(f') = \Lambda_i(f) \) \((i=1, 2)\) since \( f \) is \( \Omega \)-stable. We shall prove that \( T_xM = T_xW^s(p, f') + T_xW^u(q, f') \) for the case when \( 1 \leq \text{Ind} \Lambda_1(f) \leq \dim M - 2 \). For simplicity we identify \( f' \) with \( f \).

Let \( \mathcal{U}(f) \) be a small neighborhood of \( f \) such that \( \mathcal{U}(f) \subset \mathcal{P}(M) \). Then, by lemma 3 there are \( g \in \mathcal{U}(f) \) and basic sets \( \Lambda_i(g) \) \((i=1, 2)\) satisfying lemma 3 (I), (II) and (III). Thus \( T_xW^s(p, g) = T_xW^s(x, f) \) and \( T_xW^u(q, g) = T_xW^u(x, f) \). Let \( \varepsilon_3 > 0 \) be as in lemma 4 and define

\[
V_k(p) = \bigcup_{y \in g^{-n_k(E^u_p(p, g))}} \tilde{W}^s_{\varepsilon_3}(y, g) \text{ for } k \geq 0
\]
where \( F^w(p, g) \) is the fundamental domain of \( W_{v_3}^w(p, g) \) (see (1)). Notice that \( V_k(p) \subset \Gamma(p) \) for \( k \geq 0 \) and that \( V_k(p) \rightarrow W_{v_3}^\omega(p, g) = W_{v_3}^\omega(p, g) \) as \( k \rightarrow \infty \) since \( g^{-nk}(F^w(p, g)) \rightarrow \{p\} \) as \( k \rightarrow \infty \). Thus there is \( k_0 > 0 \) such that

\[
V_{k_0}(p) \subset \bigcup_{y \in W_{v_3}^\omega(p, g)} W_{v_3}^\omega(y, g).
\]

Obviously \( \bigcup_{k \geq k_0} V_k(p) \) is a neighborhood of \( p \) in \( M \).

Pick \( l > 0 \) such that \( g^l(x) \in \text{int} \left( \bigcup_{k \geq k_0} V_k(p) \right) \) and \( g^{-l}(x) \in W_{v_0/2}^\omega(g^{-l}(q), g) \), and denote by \( C^w(g^l(x)) \) the connected component of \( g^l(x) \) in \( W^\omega(g^l(q), g) \)

\[
\cap \left( \bigcup_{k \geq k_0} V_k(p) \right).
\]

Clearly, \( \exp_p^{-1}(C^w(g^l(x))) \subset T_p M \).

For a linear subspace \( E \) of \( T_p M \) and \( v > 0 \) we write

\[
E_v(g^l(x)) = \{ v + \exp_p^{-1}(g^l(x)) \mid v \in E \text{ with } \|v\| \leq v \}.
\]

Then there are a linear subspace \( E' \) of \( T_p M \) and a number \( 0 < v_0 \leq v_3 \) such that

\[
T_{g^l(x)} \exp_p(E_v(g^l(x))) = T_{g^l(x)} C^w(g^l(x))
\]
and \( \exp_p(E'_v(g'(x))) \subseteq \bigcup_{k \geq k_0} V_k(p) \) for \( 0 < v \leq v_0 \).

Since \( g'(x) \notin \Omega(g) \), there exists \( 0 < v_1 \leq v_0 \) such that \( B_{v_1}(g'(x)) \cap g'(B_{v_1}(g'(x))) = \emptyset \) for \( i \in \mathbb{Z} \setminus \{0\} \). Let \( \mathcal{U}(g) \) be a neighborhood of \( g \) such that \( \mathcal{U}(g) \subset \mathcal{U}(f) \). By (4) there are \( 0 < v_2 < v_1 \) and \( \varphi \in \text{Diff}(M) \) such that

\[
\begin{align*}
\varphi|_{B_{v_2}(g'(x))} &= \text{id}, \\
\varphi(g'(x)) &= g'(x), \\
\varphi(\exp_p(E'_{v_2}(g'(x)))) &\subset C^\infty(g'(x)), \\
\dim \varphi(\exp_p(E'_{v_2}(g'(x)))) &= \dim C^\infty(g'(x)), \\
g' \in \mathcal{U}(g) \text{ where } g' = \varphi^{-1} \circ g.
\end{align*}
\]

We denote \( \exp_p(E'_{v_2}(g'(x))) \) by \( \exp_p(E'_{v_2}(g''(x))) \) because of \( g''(x) = g'(x) \).

It is clear that there are two distinct basic sets \( \Lambda_i(g') \) \((i = 1, 2)\) such that \( \Lambda_i(g') = \Lambda_i(g) \) \((i = 1, 2)\) since \( g \) is \( \Omega \)-stable, and that

\[
\begin{align*}
W^s_{\varepsilon_0}(p, g') &= W^s_{\varepsilon_0}(p, g), \\
W^s_{\varepsilon_0}(q, g') &= W^s_{\varepsilon_0}(q, g), \\
T_x W^s(x, g') &= T_x W^s(x, g) \quad (\sigma = s, u), \\
\exp_p(E'_{v_2}(g''(x))) &\subset W^u(g''(q), g') \cap \Gamma(p),
\end{align*}
\]

\[
\dim \exp_p(E'_{v_2}(g''(x))) = \dim W^u(q, g') = \dim C^\infty(g'(x)).
\]

**Lemma 5.** Under the above notations, \( \exp_p(E'_{v_2}(g''(x))) \) meets transversely \( W^s_{\varepsilon_3}(p, g') \) at \( g''(x) \).

Proof. Let \( \varepsilon_2 > 0 \) be as in lemma 2. Since \( W^s_{\varepsilon_3}(p, g') \subset \exp_p(E'_{\varepsilon_2}(g''(x))) \) and \( W^s_{\varepsilon_3}(p, g') \subset \exp_p(E'_{\varepsilon_2}(g''(x))) \), to get the conclusion it is enough to prove

\[
\dim \pi(\exp_p(E'_{v_2}(g''(x)))) \geq \dim W^s_{\varepsilon_3}(p, g').
\]

Here \( \pi: \Gamma(p) \to \exp_p(E'_{v_2}(g''(x))) \) is the map defined as in (3).

Assume that \( \dim \pi(\exp_p(E'_{v_2}(g''(x)))) < \dim W^u_{\varepsilon_3}(p, g') \) and put \( C^s_{\varepsilon}(g''(x)) = B_\varepsilon(g''(x)) \cap g''^2(W^u_{\varepsilon_0}(g''^{-1}(q), g')) \) for \( \varepsilon > 0 \). Take \( 0 < \varepsilon < v_2 \) such that \( C^s_{\varepsilon}(g''(x)) \) is the connected component of \( g''(x) \) in \( B_\varepsilon(g''(x)) \cap g''^2(W^u_{\varepsilon_0}(g''^{-1}(q), g')) \) for \( 0 < \varepsilon \leq \varepsilon \), and such that \( B_\varepsilon(g''(x)) \cap g''^2(W^u_{\varepsilon_0}(g''^{-1}(q), g')) \subset \exp(E'_{v_2}(g''(x))). \)
Claim 1. Let $0 < \varepsilon \leq \tilde{\varepsilon}$. If $d(g_i^{-i}(g^{il}(x)), g_i^{-i}(w)) < \varepsilon$ for $i \geq 0$, then $w \in C_{\varepsilon}(g^{il}(x))$. 

It is clear that $d(g_i^{-1}(x), g_i^{-1}(q)) < \varepsilon_0/2$ for all $i \geq 0$. On the other hand, since $d(g_i^{-1}(x), g_i^{-1}(q)) < \varepsilon_0/2$ ($i \geq 0$),

$$d(g_i^{-2i}(w), g_i^{-1}(q)) \leq d(g_i^{-2i}(w), g_i^{-1}(x)) + d(g_i^{-1}(x), g_i^{-1}(q)) < \varepsilon_0$$

for all $i \geq 0$, and so $g_i^{-2i}(w) \in W_{\varepsilon_0}(g_i^{-1}(q), g_i')$. Thus $w \in C_{\varepsilon}(g^{il}(x)) = B_{\varepsilon}(g^{il}(x)) \cap g^{2i}(W_{\varepsilon_0}(g_i^{-1}(q), g_i'))$ since $d(g^{il}(x), w) < \varepsilon$.

We divide the proof of this lemma into two cases:

Case 1. $C_{\varepsilon}(g^{il}(x)) \subset W_{\varepsilon_3}(p, g')$,

Case 2. $C_{\varepsilon}(g^{il}(x)) \not\subset W_{\varepsilon_3}(p, g')$.

For case 1, put $\varepsilon = \tilde{\varepsilon}/2$ and let $0 < \delta = \delta(\varepsilon, g') < \varepsilon$ be the number in the definition of POTP of $g'$. Recall that $F^u(p, g') = F^u(p, g)$ and fix $y \in \bigcup_{y \in B_{\varepsilon}(g^{il}(x)) \cap W_{\varepsilon_3}(p, g')}$ such that $W_{\varepsilon_3}(y, g') \cap B_{\delta}(g^{il}(x)) \neq \emptyset$. For $z \in W_{\varepsilon_3}(y, g') \cap B_{\delta}(g^{il}(x))$,

$$\{\cdots, g_i^{-1}(x), x, g^{i}(x), \cdots, g^{i-1}(x), z, g^{i}(z), g^{i+1}(z), \cdots\}$$

is a $\delta$-pseudo-orbit of $g'$. Thus there exists $w \in M$ such that $d(g_i^{il}(w), g^{il}(z)) < \varepsilon$ ($i \geq 0$) and $d(g_i^{-i}(w), g_i^{-i}(g^{il}(x))) < \varepsilon$ ($i \geq 1$). Since $d(w, z) < \varepsilon$ and $d(z, g^{il}(x)) < \delta < \varepsilon / 2$, we have $d(g^{il}(x), w) < \tilde{\varepsilon}$. Therefore $d(g_i^{-i}(w), g_i^{-i}(g^{il}(x))) < \varepsilon$ for all $i \geq 0$, and so $w \in C_{\varepsilon}(g^{il}(x))$ by claim 1.

Obviously, there is $\tilde{k} = \tilde{k}(z) > 0$ such that $g^{nk}(z) \in V_{\varepsilon}(p) = \bigcup_{y \in F^u(p, g')} W_{\varepsilon_3}(y, g')$. By the choice of $\varepsilon$ and by the definition of $F^u(p, g)$ we have $B_{\varepsilon}(g^{nk}(z)) \cap W_{\varepsilon_3}(p, g') = \emptyset$. However, $w \in C_{\varepsilon}(g^{il}(x)) \subset W_{\varepsilon_3}(p, g')$ implies $(g^{nk}(w)) \in W_{\varepsilon_3}(p, g')$. Thus $g^{nk}(w) \in B_{\varepsilon}(g^{nk}(z)) \cap W_{\varepsilon_3}(p, g') \neq \emptyset$ (since $d(g^{nk}(w), g^{nk}(z)) < \varepsilon$). This is a contradiction and so the lemma is proved for case 1.

For case 2, take $k_1 k_0$ such that $k_1 \geq k_1$ implies $C_{\varepsilon}(g^{il}(x)) \cap V_{\varepsilon}(p) \neq \emptyset$. By the choice of $\varepsilon$,
for all $k \geq k_1$ since $\dim \pi(C^u_{\varepsilon}(g^l(x))) < \dim W^u_{\varepsilon_3}(p, g')$ (see (2)). To simplify we write

$$W_k(p) = \bigcup_{y \in g' - nk(\pi(B_\varepsilon(g^n((C^u_{\varepsilon}(g^l(x)) \cap V_k(p))))))} \mathcal{W}^u_{\varepsilon_3}(y, g'),$$

$$W(p) = \bigcup_{k \geq k_1} W_k(p) \cup W^u_{\varepsilon_3}(p, g').$$

Then $W(p) \subset \Gamma(p)$ and

$$\pi(W(p)) = \bigcup_{k \geq k_1} \pi(W_k(p)) \cup \{p\}$$

$$= \bigcup_{k \geq k_1} g'^{-nk}(\pi(B_\varepsilon(g^{nk}(C^u_{\varepsilon}(g^l(x)) \cap V_k(p)))))) \cup \{p\}$$

is not a neighborhood of $p$ in $W^u_{\varepsilon_3}(p, g')$.

**Claim 2.** Put $\varepsilon = \varepsilon/2$ and let $\delta = \delta(\varepsilon, g') < \varepsilon$ be the number in the definition of POTP of $g'$. Then we have $B_\delta(g^l(x)) \subset W(p)$. 
For every \( z \in B_\delta(g'^i(x)) \setminus W^s_w(p, g) \), there exists \( w \in M \) such that \( d(g'^i(w), g'^i(z)) < \epsilon \) and \( d(g'^{-i-1}(w), g'^{-i-1}(g'^i(x))) < \epsilon \) for all \( i \geq 0 \) since

\[
\{\cdots, g'^{-1}(x), x, g'(x), \cdots, \\
g'^{-1}(x), z, g'(z), g'^2(z), \cdots\}
\]
is a \( \delta \)-pseudo-orbit of \( g' \). Thus \( d(g'^{-i}(w), g'^{-i}(g'^i(x))) < \epsilon \) for all \( i \geq 0 \) (since \( d(g'^i(x), w) \leq d(g'^i(x), z) + d(z, w) < \epsilon + \delta < \epsilon \)), and so \( w \in C_\epsilon^\delta(g'^i(x)) \) by claim 1. Fix \( \overline{k} = \overline{k}(w) \geq k_1 \) such that \( w \in V_{\overline{k}}(p) \cap C_\epsilon^\delta(g'^i(x)) \). Then \( g'^{nk}(z) \in B_\epsilon(V_0(p) \cap g'^nk(C_\epsilon^\delta(g'^i(x)))) \) since \( d(g'^{nk}(w), g'^{nk}(z)) < \epsilon \). Thus we have \( z \in W_{\overline{k}}(p) \subset W(p) \).

By claim 2 we have \( \pi(B_\delta(g'^i(x))) \subset \pi(W(p)) \). If we establish that \( \pi(B_\delta(g'^i(x))) \) is a neighborhood of \( p \) in \( W^u_\epsilon(p, g') \), then we get a contradiction and therefore the proof of this lemma is completed.

If \( \pi(B_\delta(g'^i(x))) \) is not a neighborhood of \( p \) in \( W^u_\epsilon(p, g') \), then for every \( i > 0 \) there is \( y_i \in W^u_\epsilon(p, g') \) such that \( y_i \notin \pi(B_\delta(g'^i(x))) \) and \( d(y_i, p) < \frac{1}{i} \). Since \( \overline{W}^s_\epsilon(y_i, g') \to W^s_\epsilon(p, g') \) as \( i \to \infty \),

\[
\overline{W}^s_\epsilon(y_i, g') \cap B_\delta(g'^i(x)) \neq \emptyset
\]
for sufficiently large \( i \) and thus \( y_i \in \pi(B_\delta(g'^i(x))) \). This is a contradiction and so \( \pi(W(p)) \) is a neighborhood of \( p \) in \( W^u_\epsilon(p, g') \). For any case lemma 5 was proved.

The proof of the transversality at \( x \) for case \( 1 \leq \text{Ind } \Lambda_1(f) \leq \dim M - 2 \) follows from lemma 5. Indeed, since \( \exp_p(E'_{v_2}(g'^i(x))) \) meets transversely \( W^s_\epsilon(p, g') \) at \( g'^i(x) \), we have

\[
T_{g'^i(x)}M = T_{g'^i(x)}\exp_p(E'_{v_2}(g'^i(x))) + T_{g'^i(x)}W^s_\epsilon(p, g')
= T_{g'^i(x)}W^s(g'^i(q), g') + T_{g'^i(x)}W^s_\epsilon(p, g')
\]
by (4). Thus

\[
T_xM = T_xW^s(p, g') + T_xW^s(q, g')
= T_xW^s(x, g) + T_xW^s(x, g)
= T_xW^s(x, f) + T_xW^s(x, f).
\]
Therefore the proof of the proposition is completed.
References


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