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Von Wright-Anderson's Decision Procedures for Lewis's Systems S2 and S3

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In [1] A. R. Anderson described decision procedures for Lewis's system *S4* and for von Wright's system *M*. In this note similar procedures for Lewis's systems *S2* and *S3*[1)2)] will be developed.

By virtue of the application of the results of my previous paper [4] not only the proof which shows the adequacy of our decision procedures will be considerably simplified comparing with [1], but also intrinsic interrelations between Gentzen's and von Wright-Anderson's methods will be made clear.

Familiarity with [1] and [4] will be presupposed.

§ 1. Preliminaries.

1.1. DEFINITION of *constituent* of a (modal) formula α is as follows:

(1) A propositional variable contained in α is a constituent of α.

(2) A subformula (of α) of the form $\Box\beta$ is a constituent of α.

1.2. Construction of a truth-table for α, denoted by $\mathfrak{T}(\alpha)$, the notion of *T-rows* and of *F-rows* of it, and the *value* of a subformula of α in *Row* (i) etc. are just the same as in [1].

§ 2. A decision procedure for *S2*.

2.1. DEFINITION. The number of the logical symbols $\Box$ contained in a formula α is called the *order* of α.

2.2. DEFINITION. α is an *E2-tautology* if and only if every F-row of $\mathfrak{T}(\alpha)$, denoted by *Row* (i), satisfies at least one of the following two conditions:

I. Some constituent of the form $\Box\beta$ has the value T in *Row* (i), where β has the (assigned or calculated) value F in *Row* (i).

II. Some constituents of the form $\Box\gamma_1, \Box\gamma_2, \cdots, \Box\gamma_n$ $(n\geqq1)$, all have

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1) Lewis and Langford [3].

2) Anderson reported in [1] (without detail) that he also solved the decision problem of *S3* in a similar way as [2]. But checking his unpublished solution the author is of opinion that it is incorrect.

the value T in *Row* (i), and some constituent of the form $\Box\beta$ has the value F in *Row* (i), where the formula $(\gamma_1 \& \gamma_1 \& \cdots \& \gamma_n) \supset \beta$ is an *E2*-tautology.

2.3. DEFINITION. α is an *S2-tautology* if and only if every F-row of $\mathfrak{T}(\alpha)$, *Row* (i), satisfies at least one of the following three conditions:

I. and II. are just the same as in Definition 2.2.

III. Some constituent of the form $\Box\beta$ has the value F in *Row* (i), where β is an *E2*-tautology.

2.4. REMARK. Both the formula $(\gamma_1 \& \gamma_2 \& \cdots \& \gamma_n) \supset \beta$ appearing in the above condition II and the formula β in the condition III, are clearly of less order than α, hence by induction on the order we can effectively determine whether or not a formula α is an *E2*-tautology as well as an *S2*-tautology.

2.5. **Theorem.** *If α is an S2-tautology, then α is provable in S2.*

Proof. It is sufficient to show that under the assumption of the theorem the sequent $\rightarrow \alpha$ is provable in *S2**[3]. When α is of order zero, *S2*-tautology clearly coincides with *LK*-tautology[4], therefore $\rightarrow \alpha$ is provable with *LK*-rules only, hence provable a fortiori in *S2**. When α is of positive order we may assume that for any formula of less order than α *S2*-tautologyhood entails *S2*-provability. Now we define a formula λ_i for every F-row of $\mathfrak{T}(\alpha)$, *Row* (i), for $i = 1, 2, \cdots, r$, such that the sequent $\rightarrow \lambda_i$ is provable in *S2**.

In case *Row* (i) satisfies condition I, let λ_i be the formula $\Box\beta \supset \beta$. In case *Row* (i) satisfies II, let λ_i be $(\Box\gamma_1 \& \cdots \& \Box\gamma_n) \supset \Box\beta$; because of the hypothesis of induction the sequent $\rightarrow (\gamma_1 \& \cdots \& \gamma_n) \supset \beta$, or what is the same, the sequent $\gamma_1, \cdots, \gamma_n \rightarrow \beta$ is provable in *E2**, hence the sequent $\rightarrow \lambda_i$ is surely provable in *S2** by $(\rightarrow\Box)$-rule. In case *Row* (i) satisfies III, let λ_i be $\Box\beta$; $\rightarrow \beta$ being provable in *E2**, $\rightarrow \Box\beta$ is certainly provable in *S2** by (RT).

Now the formula $(\lambda_1 \& \cdots \& \lambda_r) \supset \alpha$ is clearly an *LK*-tautology, and so the sequent $\lambda_1, \cdots, \lambda_r \rightarrow \alpha$ is provable in *S2**. On the other hand every sequent $\rightarrow \lambda_i$ $(i = 1, 2, \cdots, r)$ is provable in *S2**, therefor $\rightarrow \alpha$ is provable in *S2**, what was to be shown.

2.6. **Theorem.** *If α is provable in S2, then α is an S2-tautology.*

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3) See [4].

4) A formula is an *LK*-tautology if and only if there exists no F-row at all in its truth-table.

Proof. We shall prove more generally that if a sequent is provable in *S2** its interpretation[3] is an *S2*-tautology. To prove this we must show that for every rule of inference of *S2** tautologyhood(s) of the upper sequent(s) entails tautologyhood of the lower sequent. But as to the rule ($\Box\rightarrow$), (RM) and (RT), the conditions I, II and III guarantees the fact respectively; as to *LK*-rules with the only exception of cut-rule we find no difficulty at all. The cut-elimination theorem for *S2**, however, enables us to do without the cut-rule. Thus we have proved our Theorem 2. 6.[5]

By 2. 4., 2. 5. and 2. 6. we can get a decision procedure for *S2*.

§ 3. A decision pocedure for *S3*.

3. 1. Definition. α is an *E3-tautology* if and only if every F-row of $\mathfrak{T}(\alpha)$, *Row* (i), satisfies at least one of the following two conditions:

I. Some constituent of the form $\Box\beta$ has the value T in *Row* (i), where β has the value F in *Row* (i).

II. Some constituents of the form $\Box\gamma_1, \Box\gamma_2, \cdots, \Box\gamma_n$ ($n \geqq 1$), all have the value T in *Row* (i), and some constituent of the form $\Box\beta$ has the the value F in *Row* (i), where 1° for some constituents $\Box\theta_1, \Box\theta_2, \cdots, \Box\theta_m$ ($m \geqq 0$) the formula $(\gamma_1 \,\&\, \gamma_2 \,\&\, \cdots \,\&\, \gamma_n) \supset \beta \vee (\Box\theta_1 \vee \Box\theta_2 \vee \cdots \vee \Box\theta_m)$ is an *LK*-tautology; 2° the formula $(\Box\gamma_1 \,\&\, \cdots \,\&\, \Box\gamma_n) \supset \beta$ is of less order than α[6] and is an *E3*-tautology.

3. 2. Definition. α is an *S3-tautology* if and only if every F-row of $\mathfrak{T}(\alpha)$, *Row* (i), satisfies at least one of the following three conditions:

I. and II. are just the same as in Definition 3. 1.

III. Some constituent of the form $\Box\beta$ has the value F in *Row* (i), where 1° for some constituents $\Box\theta_1, \cdots, \Box\theta_m$ ($m \geqq 0$) the formula $\beta \vee (\Box\theta_1 \vee \cdots \vee \Box\theta_m)$ is an *LK*-tautology; 2° β itself is an *E3*-tautology.

3. 3. **Theorem.** *If and only if α is an S3-tautology, α is provable in S3.*

Proof. By methods analogous to the proofs of Theorems 2. 5. and 2. 6.

Thus we have got a decision procedure for *S3*.

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5) Anderson's proof for the fact that the rule of detachment preserves tautologyhood corresponds, we might say, to the proof for the cut-elimination theorem. See [4].

6) This condition is necessary. Surely the order of $(\Box\gamma_1 \,\&\cdots\&\, \Box\gamma_n) \supset \beta$ is less than that of $(\Box\gamma_1 \,\&\cdots\&\, \Box\gamma_n) \supset \Box\beta$, but it may be greater than that of α. For instance, let α be $\Box\Box\Box p \supset \Box\Box p$, where p is a propositional variable. The order of α is 4. But the formula $(\Box\Box p \,\&\, \Box\Box\Box p) \supset p$, which might appear in the condition II as the above formula, is of order 5.

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