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ZHU'S ALGEBRA OF RANK ONE LATTICE VERTEX OPERATOR SUPERALGEBRAS

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1. Introduction

In this paper, we explicitly determine the algebraic structure of Zhu's algebra for a lattice vertex operator superalgebra which is constructed from a rank one odd lattice, and give a proof of rationality.

Y. Zhu introduced an associative algebra associated to a vertex operator algebra in order to study the structure of its modules [11]. Such an associative algebra is now called Zhu's algebra. Roughly speaking, the structure of such a module is determined from the action of the weight 0 component operator on the lowest weight space, which is described through the action of Zhu's algebra and the structure of the vertex operator algebra. Then, for instance, we have one to one correspondence between the classes of inequivalent admissible modules of a vertex operator algebra and the classes of inequivalent modules of Zhu's algebra [11].

The notion of Zhu's algebra has been vastly used to classify all simple modules for vertex operator algebras (affine vertex operator algebras [7], Virasoro vertex operator algebras [10] and lattice vertex operator algebras associated to even lattices of rank one [5], etc.). In this paper, we will study Zhu's algebra $A(V_L)$ associated to a vertex operator superalgebra V_L for a rank one *odd* lattice L.

Zhu's algebra $A(V_L)$ of V_L for an even lattice is studied in [5] and the classification of the simple modules for V_L is given, which provides another proof of the classification results known in [2]. In our odd lattice case, V_L is a vertex operator *superalgebra*, and by virtue of super symmetries in some sense, the structure of its Zhu's algebra is much simpler comparing to the even lattice case. It is a quotient algebra of the polynomial ring with one variable. By using the explicit structure of Zhu's algebra we can easily classify all simple modules for V_L. Though the classification of the simple modules for V_L for an even lattice L has been known in [2] and the method given in [2] can be applied to noneven case including our case, it is worthy to study the reason of such simpleness appeared in super case.

For an even lattice L, the rationality, more precisely, the regularity of V_L is proved in [4] by the method deeply depending on the one in [2]. In this paper, we will give a rough sketch of the proof of the regularity of V_L for a rank one odd lattice L emphasizing the differences between the super and nonsuper case.

Now we state the precise structure of Zhu's algebra $A(V_L)$ which is constructed from a rank one odd lattice L. Let $L=\mathbb{Z}\alpha$ be an integral lattice of rank one generated by α such that $(\alpha|\alpha)=k$, where k is a positive odd integer, and let V_L be the vertex operator superalgebra associated to L. Zhu's algebra $A(V_L)$ in our case is isomorphic to the following quotient algebra of the polynomial ring $\mathbb{C}[x]$:

$$A(V_L)\cong\mathbb{C}[x]/(F_k(x)),$$

where $F_k(x)=\prod_{n\in I_k}(x-n)$ and $I_k=\{0,\pm1,\dots,\pm(k-1)/2\}$. This enables us to obtain a complete list of the simple V_L-modules.

We note that V_L is isomorphic to the charged free fermions for $k=1$, and to the $N=2$ superconformal vertex algebra with central charge $C=1$ for $k=3$ [8].

This paper is organized as follows. In Section 2.1 we define vertex operator superalgebras, their modules, and the notions of rationality and regularity. The definition of Zhu's algebra corresponding to vertex operator superalgebras is given in Section 2.2. We construct vertex operator superalgebras V_L in Section 3.1 and determine Zhu's algebra $A(V_L)$ in Section 3.2. The proof of regularity of V_L is given in Section 3.3. In Applications we consider vertex operator superalgebras V_L in special cases $k=1$ and $k=3$.

2. Vertex operator superalgebras and their Zhu's algebra

2.1. Vertex operator superalgebras and modules A vertex operator superalgebra is a $(1/2)\mathbb{Z}$-graded vector space $V=\oplus_{n\in(1/2)\mathbb{Z}}V_n=V_{\bar{0}}\oplus V_{\bar{1}}$, $(V_{\bar{0}}=\oplus_{n\in\mathbb{Z}}V_n$, $V_{\bar{1}}=\oplus_{n\in\mathbb{Z}+(1/2)}V_n)$ such that dim $V_n<\infty$ for $n\in(1/2)\mathbb{Z}$ and $V_n=0$ for sufficiently small $n\in(1/2)\mathbb{Z}$, equipped with a linear map

$$\begin{aligned} V &\to (\mathrm{End}\,V)[[z,z^{-1}]]\\ a &\mapsto Y(a,z)=\sum_{n\in\mathbb{Z}}a_nz^{-n-1}\end{aligned}$$

and with two distinguished homogeneous elements $\mathbf{1}\in V_0$ (called the vacuum vector), $\omega\in V_2$ (called the Virasoro element) satisfying the following conditions (V1) $\sim$ (V6): for $a,b\in V$,

(V1) $a_nb=0$ for sufficiently large $n\in\mathbb{Z}$,

(V2) $$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(a,z_1)Y(b,z_2)-(-1)^{\tilde{a}\tilde{b}}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(b,z_2)Y(a,z_1)$$
$$=z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(a,z_0)b,z_2),$$

where $\tilde{a}=0$ (resp. $\tilde{a}=1$) according to $a\in V_{\bar{0}}$ (resp. $a\in V_{\bar{1}}$), and the formal δ function $\delta(z)$ is defined to be $\delta(z)=\sum_{n\in\mathbb{Z}}z^n$ and any binomial expressions are expanded into non-negative powers of the second variable, (V3) $Y(\mathbf{1}, z) = \mathrm{id}_V$, $Y(a, z)\mathbf{1} \in V[[z]]$, $\lim_{z\to 0} Y(a, z)\mathbf{1} = a$, and

(V4) set $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, then

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12} c_V \delta_{m+n,0} \quad (c_V \in \mathbb{C},\ m,\ n \in \mathbb{Z}),$$

(V5) $L_0 a = na$ for $a \in V_n$ $\left(n \in \frac{1}{2}\mathbb{Z}\right)$,

(V6) $\frac{d}{dz} Y(a, z) = Y(L_{-1}a,\ z)$.

The scalar c_V is called central charge. We say an element $a \in V_n$ homogeneous with weight n, denoted $n = \mathrm{wt}(a)$.

The notion of a module for a vertex operator superalgebra is defined in the following way. A weak V-module M is a vector space equipped with a linear map

$$\begin{aligned} V &\to (\mathrm{End}\, M)[[z, z^{-1}]] \\ a &\mapsto Y_M(a, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \end{aligned}$$

satisfying the following conditions (M1) $\sim$ (M3): for any $a, b \in V$ and $u \in M$,

(M1) $a_n u = 0$ for sufficiently large $n \in \mathbb{Z}$,

(M2) $Y_M(\mathbf{1}, z) = \mathrm{id}_M$,

(M3) $z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(a, z_1)Y_M(b, z_2) - (-1)^{\tilde{a}\tilde{b}} z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(b, z_2)Y_M(a, z_1)$
$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(a, z_0)b, z_2)$.

An admissible V-module M is a weak V-module M which carries a $(1/2)\mathbb{Z}_{\geq 0}$-grading $M = \oplus_{n\in(1/2)\mathbb{Z}_{\geq 0}} M_n$ subject to the conditions: for $m \in \mathbb{Z}, n \in (1/2)\mathbb{Z}_{\geq 0}$ and homogeneous $a \in V$,

$$a_m M_n \subseteq M_{\mathrm{wt}(a)+n-m-1}.$$

Let M be a weak V-module such that L_0 is semisimple on M and $M = \oplus_{\lambda\in\mathbb{C}} M_\lambda$ the eigenspace decomposition with respect to L_0. If $\dim M_\lambda < \infty$ for all $\lambda \in \mathbb{C}$ and for fixed $\lambda \in \mathbb{C}$, $M_{\lambda+n} = 0$ for sufficiently small $n \in (1/2)\mathbb{Z}$, M is called an (ordinary) V-module. An admissible V-module M is called simple if 0 and M are the only $(1/2)\mathbb{Z}_{\geq 0}$-graded admissible V-submodules.

DEFINITION 2.1 ([4]). A vertex operator superalgebra V is called rational if any admissible V-module is a direct sum of simple admissible V-modules. A vertex operator superalgebra V is called regular if any weak V-module is a direct sum of simple ordinary V-modules.

2.2. Zhu's algebra We review the definition of Zhu's algebra for a vertex operator superalgebra [9].

Let $V = \oplus_{n\in(1/2)\mathbb{Z}} V_n$ be a vertex operator superalgebra. Let us define binary operations $*, \circ : V \times V \to V$ as follows: for homogeneous $a \in V$ and $b \in V$,

$$a * b = \begin{cases} \operatorname{Res}_z Y(a,z)\dfrac{(1+z)^{\mathrm{wt}(a)}}{z} b = \displaystyle\sum_{i\geq 0}\binom{\mathrm{wt}(a)}{i} a_{i-1}b & (a,\ b \in V_{\bar{0}}), \\ 0 & (a \text{ or } b \in V_{\bar{1}}), \end{cases}$$

$$a \circ b = \begin{cases} \operatorname{Res}_z Y(a,z)\dfrac{(1+z)^{\mathrm{wt}(a)}}{z^2} b = \displaystyle\sum_{i\geq 0}\binom{\mathrm{wt}(a)}{i} a_{i-2}b & (a \in V_{\bar{0}},\ b \in V), \\ \operatorname{Res}_z Y(a,z)\dfrac{(1+z)^{\mathrm{wt}(a)-1/2}}{z} b = \displaystyle\sum_{i\geq 0}\binom{\mathrm{wt}(a)-1/2}{i} a_{i-1}b & (a \in V_{\bar{1}},\ b \in V). \end{cases}$$

We extend both operations $*, \circ$ to V by linearity. Let $O(V)$ be the linear span of elements of the form $a \circ b$ in V. The space $A(V)$ is defined by the quotient space $V/O(V)$. In the following, we set $[a] = a + O(V) \in A(V)$ for $a \in V$.

For any homogeneous $a \in V_{\bar{0}}$ and $b \in V$, we have (cf. [9])

$$\operatorname{Res}_z\left(Y(a,z)\frac{(1+z)^{\mathrm{wt}(a)}}{z^{n+2}} b\right) \in O(V) \quad (n \geq 0). \tag{2.1}$$

If $a \in V_{\bar{1}}$, then

$$a \circ \mathbf{1} = \operatorname{Res}_z Y(a,z)\frac{(z+1)^{\mathrm{wt}(a)-1/2}}{z}\mathbf{1} = a_{-1}\mathbf{1} = a \in O(V).$$

Since $O(V)$ is a $\mathbb{Z}_2$-graded subspace, we see $O(V) = O_{\bar{0}}(V) + V_{\bar{1}}$ where $O_{\bar{0}}(V) = O(V) \cap V_{\bar{0}}$. Thus we have $A(V) = V_{\bar{0}}/O_{\bar{0}}(V)$.

It follows from [9] that $O(V)$ is a two-sided ideal of V with respect to $*$ and the operation $*$ induces an associative algebra structure on $A(V)$. Moreover the image $[\mathbf{1}]$ of the vacuum in $A(V)$ becomes the identity element of $A(V)$. We call the associative algebra $A(V)$ Zhu's algebra of V.

For any homogeneous $a \in V_{\bar{0}}$, we denote $o(a)$ by the weight 0 component operator $a_{\mathrm{wt}(a)-1}$. Clearly, for any admissible V-module $M = \oplus_{n\in(1/2)\mathbb{Z}_{\geq 0}} M_n$, $o(a)$ preserves each homogeneous space M_n. The action of operators $o(a)$ on the lowest weight spaces of admissible V-modules leads to the following fundamental theorem (see Theorem 1.2, Theorem 1.3 of [9]):

Theorem 2.1. (1) *If $M = \oplus_{n\in(1/2)\mathbb{Z}_{\geq 0}} M_n$ is an admissible V-module, then M_0 is an $A(V)$-module under the action $[a] \mapsto o(a)$ for $a \in V_{\bar{0}}$.*
(2) *If W is an $A(V)$-module, then there exists an admissible V-module $M = \oplus_{n\in(1/2)\mathbb{Z}_{\geq 0}} M_n$ such that $M_0 \cong W$ as an $A(V)$-module.*
(3) *The map $M \mapsto M_0$ gives a bijection between the set of inequivalent simple admissible V-modules and the set of inequivalent simple $A(V)$-modules.*

3. Zhu's algebra $A(V_L)$ for a rank one odd lattice L

3.1. The structure of V_L Let $L = \mathbb{Z}\alpha$ be a rank one integral lattice with the symmetric nondegenerate bilinear form $(\cdot|\cdot)$ given by $(\alpha|\alpha) = k$, where k is a positive odd integer.

Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend the bilinear form $(\cdot|\cdot)$ on L to $\mathfrak{h}$ by $\mathbb{C}$-linearity. Let $\hat{\mathfrak{h}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h} \oplus \mathbb{C}K$ be the affinization of $\mathfrak{h}$ regarding $\mathfrak{h}$ as an abelian Lie algebra. Lie bracket on $\hat{\mathfrak{h}}$ is given by

$$[t^m \otimes h, t^n \otimes h'] = m(h|h')\delta_{m+n,0}K, \quad [t^m \otimes h, K] = 0 \quad (h,\ h' \in \mathfrak{h},\ m,\ n \in \mathbb{Z}).$$

Let $\hat{\mathfrak{h}}^- = t^{-1}\mathbb{C}[t^{-1}] \otimes \mathfrak{h}$, $\mathfrak{b} = \mathbb{C}[t] \otimes \mathfrak{h} \oplus \mathbb{C}K$, which are commutative subalgebras of $\hat{\mathfrak{h}}$. The relations $K \cdot 1 = 1$, $(\mathbb{C}[t] \otimes \mathfrak{h}) \cdot 1 = 0$ define the one dimensional module $\mathbb{C}$ of $\mathfrak{b}$. We set $M(1) = \mathrm{Ind}_{\mathfrak{b}}^{\hat{\mathfrak{h}}} \mathbb{C}$ and denote this $\hat{\mathfrak{h}}$-module by π_1. Remark that

$$M(1) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{b})} \mathbb{C} \cong S(\hat{\mathfrak{h}}^-) \quad \text{as a linear space,}$$

where $U(\cdot)$ denotes the corresponding universal enveloping algebra and $S(\cdot)$ the corresponding symmetric algebra.

Let $\mathbb{C}[L]$ the group algebra of the additive group L. Thus $\mathbb{C}[L]$ has a basis $\{e^\beta\}_{\beta\in L}$. The space $\mathbb{C}[L]$ has a natural $\hat{\mathfrak{h}}$-module structure π_2 by letting

$$\pi_2(K) = 0, \quad \pi_2(t^n \otimes h)e^\beta = \delta_{n,0}(h|\beta)e^\beta \quad (h \in \mathfrak{h},\ n \in \mathbb{Z},\ \beta \in L).$$

Let us define the $\hat{\mathfrak{h}}$-module structure on $V_L = M(1) \otimes_{\mathbb{C}} \mathbb{C}[L]$ by $\pi = \pi_1 \otimes 1 + 1 \otimes \pi_2$.

Here we give the definition of the vertex operator $Y(a, z)$ for $a \in V_L$. For $h \in \mathfrak{h}$, we set $h_n = t^n \otimes h$ ($n \in \mathbb{Z}$) and $h(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n-1}$. Let $a = \alpha_{-n_1} \cdots \alpha_{-n_r} \otimes e^\beta \in V_L$ ($n_1, \ldots, n_r \in \mathbb{Z}_{>0}$, $\beta \in L$). The vertex operator $Y(a, z)$ is defined as

$$Y(a, z) = {}^{\circ}_{\circ}\, \partial^{(n_1-1)}\alpha(z) \cdots \partial^{(n_k-1)}\alpha(z)\Gamma_\beta(z)\, {}^{\circ}_{\circ},$$

where

$$\partial^{(n)} = \frac{1}{n!}\frac{d^n}{dz^n}, \quad \Gamma_\beta(z) = e^\beta z^{\beta_0} \exp\left(\sum_{n>0} \frac{\beta_{-n} z^n}{n}\right) \exp\left(\sum_{n>0} \frac{\beta_n z^{-n}}{-n}\right),$$

and e^β is the operator of left multiplication by $1 \otimes e^\beta$. The normal ordering procedure ${}^{\circ}_{\circ} \cdot {}^{\circ}_{\circ}$ follows the definition in [8].

Theorem 3.1 ([6]). *V_L is a simple vertex operator superalgebra with the vacuum vector $\mathbf{1} = 1 \otimes 1$ and the Virasoro element $\omega = (1/(2k))\, \alpha_{-1}^2 \mathbf{1}$.*

Note that for $a = \alpha_{-n_1} \cdots \alpha_{-n_r} \otimes e^{\beta}$ $(= \alpha_{-n_1} \cdots \alpha_{-n_r} e^{\beta}\mathbf{1}) \in V_L$ $(n_1, \dots, n_r \in \mathbb{Z}_{>0}, \beta \in L)$, its weight is

$$\mathrm{wt}(a) = \sum_{i=1}^{r} n_i + \frac{1}{2}(\beta|\beta).$$

We next discuss modules for the vertex operator superalgebra V_L. Let $L^{\circ} \supset L$ be the dual lattice of L. Then one has the coset decomposition

$$L^{\circ} = \cup_{i \in I_k} \left(L + \frac{i}{k}\alpha \right),$$

where $I_k = \{0, \pm 1, \dots, (k-1)/2\}$. Let $\mathbb{C}[\mathfrak{h}]$ be the group algebra and set $\mathbb{C}[S] = \oplus_{\beta \in S} \mathbb{C} e^{\beta}$ for any subset S of $\mathfrak{h}$. We define the vector space

$$V(i) = M(1) \otimes_{\mathbb{C}} \mathbb{C}\left[L + \frac{i}{k}\alpha \right]$$

for $i \in I_k$.

Theorem 3.2 ([6]). *$V(i)$ $(i \in I_k)$ are inequivalent simple V_L-modules.*

3.2. Zhu's algebra $A(V_L)$ Now we state one of the main results in this paper.

Theorem 3.3. *Zhu's algebra $A(V_L)$ of the vertex operator superalgebra V_L is isomorphic to the following quotient algebra of the polynomial ring $\mathbb{C}[x]$:*

$$A(V_L) \cong \mathbb{C}[x]/(F_k(x)),$$

where $F_k(x) = \prod_{n \in I_k}(x - n)$.

As a corollary, we have

Corollary 3.1. *The set of the simple modules $\{V(i)\}_{i \in I_k}$ gives the complete list of the simple V_L-modules.*

The proof of Theorem 3.3 is given after we establish several lemmas.

Let $n \geq 0$, $a \in V_L$. Note that

$$\mathrm{Res}_z \left(Y(\alpha_{-1}\mathbf{1}, z) \frac{1+z}{z^{n+2}} a \right) \in O(V_L),$$

by (2.1). Thus we have $\alpha_{-n-1}a + \alpha_{-n-2}a \equiv 0 \pmod{O(V_L)}$, and then

$$\alpha_{-n}a \equiv (-1)^{n-1}\alpha_{-1}a \pmod{O(V_L)} \quad (n \geq 1,\ a \in V_L). \tag{3.2}$$

Let $p_n(x_1, x_2, \dots)$ be the elementary Schur polynomials

$$\exp\left(\sum_{n=1}^{\infty} \frac{x_n}{n} y^n\right) = \sum_{n=0}^{\infty} p_n(x_1, x_2, \dots) y^n,$$

and $p_n(x_1, x_2, \dots) = 0$ for $n \in \mathbb{Z}_{<0}$. For any operator x, we define

$$\binom{x}{n} = \begin{cases} \frac{1}{n!} x(x-1)\cdots(x-n+1) & (n \in \mathbb{Z}_{>0}), \\ 1 & (n = 0), \\ 0 & (n \in \mathbb{Z}_{<0}). \end{cases}$$

Then one can easily see that

$$p_n(x, -x, x, -x, \dots) = \binom{x}{n}. \tag{3.3}$$

Let $\beta, \gamma \in L, i \in \mathbb{Z}$ and $\Gamma_\beta(z) = \sum_{i\in\mathbb{Z}} e_i^\beta z^{-i-1}$. Since

$$\begin{aligned} \Gamma_\beta(z)e^\gamma\mathbf{1} &= z^{(\beta|\gamma)} \exp\left(\sum_{n=1}^{\infty} \frac{\beta_{-n}}{n} z^n\right) e^{\beta+\gamma}\mathbf{1} \\ &= \sum_{n=0}^{\infty} p_n(\beta_{-1}, \beta_{-2}, \dots)e^{\beta+\gamma}\mathbf{1}\ z^{n+(\beta|\gamma)}, \end{aligned}$$

(3.2) and (3.3) show that

$$\begin{aligned} e_i^\beta e^\gamma \mathbf{1} &= p_{-(\beta|\gamma)-i-1}(\beta_{-1}, \beta_{-2}, \dots)e^{\beta+\gamma}\mathbf{1} \\ &\equiv p_{-(\beta|\gamma)-i-1}(\beta_{-1}, -\beta_{-1}, \dots)e^{\beta+\gamma}\mathbf{1} \\ &\equiv \binom{\beta_{-1}}{-(\beta|\gamma)-i-1} e^{\beta+\gamma}\mathbf{1} \pmod{O(V_L)}. \end{aligned} \tag{3.4}$$

From the definition of the operation $*$, we have $\alpha_{-1}\mathbf{1} * a = \alpha_{-1}a + \alpha_0 a$ ($a \in V_L$) and then

$$\alpha_{-1}^n\mathbf{1} = \underbrace{\alpha_{-1}\mathbf{1} * \cdots * \alpha_{-1}\mathbf{1}}_{n-th} = (\alpha_{-1}\mathbf{1})^{*n} \quad (n \geq 1). \tag{3.5}$$

Lemma 3.1. (1) $\binom{\alpha_{-1}+(k-1)/2}{k}\mathbf{1} \equiv 0 \pmod{O(V_L)}$.
(2) *$A(V_L)$ is spanned by vectors $[\alpha_{-1}^n\mathbf{1}]$ where $n \geq 0$.*

Proof. (1) Since $e^{\alpha}\mathbf{1}$ is an odd element, we have

$$0 \equiv e^{\alpha}\mathbf{1} \circ e^{-\alpha}\mathbf{1} \equiv \sum_{i\geq 0}\binom{(k-1)/2}{i}\binom{\alpha_{-1}}{k-i}\mathbf{1}$$
$$\equiv \binom{\alpha_{-1}+(k-1)/2}{k}\mathbf{1} \pmod{O(V_L)},$$

by (3.4) and the formula

$$\sum_{i=0}^{n}\binom{n}{i}\binom{x}{m-i} = \binom{x+n}{m} \quad (m \geq n \geq 0,\ m,\ n \in \mathbb{Z}_{\geq 0}, \text{ and } x\text{: an operator}).$$

(2) Let $2L = \{2n\alpha | n \in \mathbb{Z}\}$. Then $2L$ is a sublattice of L. Since the vertex operator algebra V_{2L} for the lattice $2L$ is a vertex operator subalgebra of V_L, we have a homomorphism $\nu : A(V_{2L}) \to A(V_L)$. The map ν is surjective as $[e^{n\alpha}\mathbf{1}] = 0$ in $A(V_L)$ for any odd integer n.

Therefore Theorem 3.2 of [5] shows $A(V_L)$ is generated by $[e^{2\alpha}\mathbf{1}], [e^{-2\alpha}\mathbf{1}]$ and $[\alpha_{-1}\mathbf{1}]$.

Let $m \neq 0$. We note that

$$\alpha_{-1}\mathbf{1} * e^{2m\alpha}\mathbf{1} = (\alpha_{-1}+\alpha_0)e^{2m\alpha}\mathbf{1} \equiv kme^{2m\alpha}\mathbf{1} \pmod{O(V_L)}$$

as $\alpha_{-1}e^{2m\alpha}\mathbf{1} \equiv -kme^{2m\alpha}\mathbf{1} \pmod{O(V_L)}$. Then, by (1), we have

$$0 \equiv \binom{\alpha_{-1}+(k-1)/2}{k}\mathbf{1} * e^{2m\alpha}\mathbf{1} \equiv \binom{[\alpha_{-1}\mathbf{1}]+(k-1)/2}{k} * e^{2m\alpha}\mathbf{1}$$
$$\equiv \binom{km+(k-1)/2}{k} e^{2m\alpha}\mathbf{1} \pmod{O(V_L)}.$$

Thus we see that $e^{2m\alpha}\mathbf{1} \equiv 0 \pmod{O(V_L)}$.

Since $e^{2\alpha}\mathbf{1} \equiv 0$ and $e^{-2\alpha}\mathbf{1} \equiv 0 \pmod{O(V_L)}$, $A(V_L)$ is generated by $[\alpha_{-1}\mathbf{1}]$. Therefore it implies this lemma. □

It follows from Lemma 3.1 (1) that we have a relation

$$0 = n!\binom{[\alpha_{-1}\mathbf{1}]+(k-1)/2}{k} = F_k([\alpha_{-1}\mathbf{1}]) \tag{3.6}$$

in $A(V_L)$. Now we can prove Theorem 3.3. Let ϕ be the $\mathbb{C}$-linear map defined by

$$\phi : \quad \mathbb{C}[x] \to A(V_L)$$

$$x^n \mapsto [\alpha_{-1}\mathbf{1}]^{*n}.$$

Then the map ϕ is a homomorphism of an associative algebra by (3.5) and it is surjective by Lemma 3.1 (2). From (3.6), it is enough to show that $\ker\phi$ is generated by $F_k([\alpha_{-1}\mathbf{1}])$. Suppose $\ker\phi \neq (F_k([\alpha_{-1}\mathbf{1}]))$, then $\dim_{\mathbb{C}} \mathbb{C}[x]/(F_k(x)) < k$. Therefore, the number of the simple modules of the associative algebra $A(V_L)$ must be strictly less than k, which gives a contradiction as $A(V_L)$ has the k inequivalent simple modules which correspond to the simple modules $V(i)$ ($i \in I_k$) of V_L by Theorem 2.1. □

3.3. Regularity of V_L The regularity of V_L can be shown in the same way given in [4], in which the regularity of lattice vertex operator algebras associated to even lattices is proved. However, in order to describe the difference between odd and even cases, we will give the outline of the proof of the regularity of V_L.

The following lemma is fundamental in our proof, whose proof is suggested by C. Dong (one can see the same statement in the proof of Lemma 3.15 of [4]. Also see [3], Proposition 11.9):

Lemma 3.2. *Let V be a vertex operator superalgebra and M be a nonzero weak V_L-module. If V is simple, then for any nonzero vectors $a \in V$ and $u \in M$, $Y(a, z)u \neq 0$.*

Proof. Set $I = \{b \in V | Y(b, z)u = 0\}$. Suppose that $Y(a, z)u = 0$, then $I \neq \{0\}$. First of all, we prove that I is an ideal of V. By the associativity, which is a result of the Jacobi identity, we have for any $b \in V, c \in I$ and some $m \in \mathbb{Z}_{>0}$,

$$(z_0 + z_2)^m Y(Y(b, z_0)c, z_2)u = (z_0 + z_2)^m Y(b, z_0 + z_2)Y(c, z_2)u = 0$$

as $Y(c, z)u = 0$. Thus we have $Y(Y(b, z_0)c, z_2)u = 0$, which implies $b_n c \in I$ for all $n \in \mathbb{Z}$. Since V is simple, we see $I = V$. Therefore, $Y(b, z)u = 0$ for all $b \in V$. This gives us a contradiction because $Y(\mathbf{1}, z)u = u \neq 0$. □

Now we return to the case of $V = V_L$. For any nonzero weak V_L-module M, let us define the vacuum space of M by

$$\Omega_M = \{u \in M | \alpha_n u = 0 \quad \text{for } n > 0\}.$$

Using Lemma 3.2, we can prove that $\Omega_M \neq 0$ by argument similar to Lemma 3.15 of [4]. For $\beta \in L$, the following operator on M is called the Z-operator (cf. [2]):

$$Z(\beta, z) = \exp\left(\sum_{n>0} \frac{\beta_{-n} z^n}{-n}\right) \Gamma_\beta(z) \exp\left(\sum_{n>0} \frac{\beta_n z^{-n}}{n}\right) = \sum_{n\in\mathbb{Z}} Z(\beta, n) z^{-n-1}.$$

For the odd lattice L, the following identities are proved in the same way as in [2]:

$$[\beta_m, Z(\gamma, n)] = \delta_{m,0}(\beta|\gamma)Z(\gamma, n), \tag{3.7}$$

$$Z(\beta, n)\beta_0 = (-n-1)Z(\beta, n) \tag{3.8}$$

for $\beta, \gamma \in L, m, n \in \mathbb{Z}$. Lemma 3.2, (3.7) and (3.8) show:

Lemma 3.3. *Let M be a nonzero weak V_L-module. Then there exist a nonzero vector $w \in \Omega_M$ and $\lambda \in L^\circ$ such that*

$$\beta_0 w = \lambda(\beta)w \quad \textit{for any } \beta \in \mathfrak{h}.$$

Proof. Let u be a nonzero vector of Ω_M. By Lemma 3.2, we have $\Gamma_\alpha(z)u \neq 0$. Since

$$\Gamma_\alpha(z) = \exp\left(\sum_{n>0} \frac{\alpha_{-n}z^n}{n}\right) Z(\alpha, z) \exp\left(\sum_{n>0} \frac{\alpha_n z^{-n}}{-n}\right),$$

$Z(\alpha, z)u \neq 0$, i.e., $Z(\alpha, n)u \neq 0$ for some $n \in \mathbb{Z}$. Then, from (3.7) and (3.8), we obtain $Z(\alpha, n)u \in \Omega_M$ and

$$\begin{aligned} \alpha_0 Z(\alpha, n)u &= ([\alpha_0, Z(\alpha, n)] + Z(\alpha, n)\alpha_0)u \\ &= ((\alpha|\alpha) - n - 1)Z(\alpha, n)u \\ &= (k - n - 1)Z(\alpha, n)u. \end{aligned}$$

Put $w = Z(\alpha, n)u$. Let $\lambda \in \mathfrak{h}^*$ such that $\alpha_0 w = \lambda(\alpha)w$. By the nondegenerate form $(\cdot|\cdot)$ on $\mathfrak{h}$, we have

$$\lambda = \frac{k-n-1}{k}\alpha$$

under the identification $\mathfrak{h}^*$ with $\mathfrak{h}$. We see that $(\lambda|\beta) \in \mathbb{Z}$ for any $\beta \in L$, and then $\lambda \in L^\circ$. □

Let w be a nonzero vector of Ω_M subject to the condition in Lemma 3.3. It is not difficult to prove that the V_L-submodule generated by w is simple and isomorphic to $V(i)$ for some $i \in I_k$. In particular, if M is a simple weak V_L-module, then M is isomorphic to $V(i)$ for some $i \in I_k$.

REMARK 3.1. The proof of the fact that $\lambda \in L^\circ$ given in [2] does not work for our odd lattice (see the proof of Lemma 3.5 of [2]). However, the proof of Lemma 3.4 of [2] implicitly shows this as we have seen in Lemma 3.3.

We prove the following theorem in the same way as in Theorem 3.16 of [4].

Theorem 3.4. *The vertex operator superalgebra V_L is regular. In particular, V_L is rational.*

Proof. It suffices to prove that any weak V_L-module is completely reducible, since any simple V_L-module is an ordinary V_L-module. Let M be a nonzero weak V_L-module. Let W be the sum of all simple ordinary V_L-submodules in M. Suppose $M' = M/W \neq 0$. Let $u\ (\neq 0) \in \Omega_M \backslash \Omega_W$. It follows from Lemma 3.2 that there exists $n \in \mathbb{Z}$ such that $Z(\alpha, n)(u + W) \neq 0$ in M', and then $Z(\alpha, n)u \notin W$. Taking $w = Z(\alpha, n)u$, we see that $w\ (\neq 0) \in \Omega_M \backslash \Omega_W$,

$$\beta_0 w = \lambda(\beta) w \quad \text{for any } \beta \in \mathfrak{h},$$

and $\lambda \in L^\circ$ from Lemma 3.3. Then the V_L-submodule generated by w is simple and is not contained in W. This gives us a contradiction. □

3.4. Applications: Rationality of vertex operator superalgebras associated to the charged free fermions and the $N = 2$ superconformal algebra It is known that, if $(\alpha|\alpha) = 1$, then V_L is isomorphic to the charged free fermions F [8], and if $(\alpha|\alpha) = 3$, then V_L is isomorphic to the $N = 2$ superconformal vertex algebra with central charge $C = 1$ [8]. The rationality of the charged free fermions F is shown in Theorem 4.1 of [9].

Let us consider the $N = 2$ superconformal algebra. It is a graded superalgebla spanned by the basis $L_n, T_n, G_r^{\pm}, C$ $\{n \in \mathbb{Z}, r \in \mathbb{Z}+(1/2)\}$, and has (anti)-commutation relations given by

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{m^3-m}{12} C\delta_{m+n,0}, \\
[L_m, G_r^{\pm}] &= \left(\frac{1}{2}m - r\right) G_{m+r}^{\pm}, \\
[L_m, T_n] &= -nT_{m+n}, \\
[T_m, T_n] &= \frac{m}{3} C\delta_{m+n,0}, \\
[T_m, G_r^{\pm}] &= \pm G_{m+r}^{\pm}, \\
\{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)T_{r+s} + \frac{C}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}, \\
[L_m, C] &= [T_n, C] = [G_r^{\pm}, C] = \{G_r^+, G_s^+\} = \{G_r^-, G_s^-\} = 0
\end{aligned}$$

for all $m, n \in \mathbb{Z}$ and $r, s \in \mathbb{Z}+(1/2)$. Given complex numbers h, q and c, we denote the Verma module generated by the highest weight vector $|h, q, c\rangle$ with L_0 eigenvalue h, T_0 eigenvalue q and central charge c by $\mathcal{M}(h, q, c)$. Note that the highest weight vector $|h, q, c\rangle$ is annihilated by L_n, T_n, and $G_r^{\pm}$ for $n \in \mathbb{Z}_{>0}, r \in \mathbb{Z}_{\geq 0} + (1/2)$. It follows from [8] that the vertex algebra $\mathcal{M}(0,0,c)$ has a unique simple quotient $L(0,0,c)$ and if $(\alpha|\alpha) = 3$, then the lattice vertex operator superalgebra V_L is isomorphic to the $N = 2$ superconformal vertex algebra $L(0,0,1)$. The classification results of its simple modules are given by the Kac determinant of the $N = 2$ superconformal algebra in [1].

As a consequence of a particular case $k = 3$ of Corollary 3.1 and Theorem 3.4, we have

Theorem 3.5. *The $N = 2$ superconformal vertex algebra $L(0,0,1)$ is rational, and its simple modules are only $L(0,0,1)$ and $L(1/6, \pm 1/3, 1)$.*

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