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DEFORMATIONS OF WEAK FANO 3-FOLDS WITH ONLY TERMINAL SINGULARITIES

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0. Introduction.

DEFINITION 0.1. Let X be a normal Gorenstein projective variety of dimension 3 over $\mathbb{C}$ which has only terminal singularities.
(1) If $-K_X$ is ample, we call X a Fano 3-fold.
(2) If $-K_X$ is nef and big, we call X a weak Fano 3-fold.

DEFINITION 0.2. Let X be a normal Gorenstein projective variety of dimension 3 with only terminal singularities. Let $(\Delta, 0)$ be a germ of the 1-parameter unit disk. Let $\mathfrak{f} : \mathfrak{X} \to (\Delta, 0)$ be a small deformation of X over $(\Delta, 0)$. We call $\mathfrak{f}$ a smoothing of X when the fiber $\mathfrak{X}_s = \mathfrak{f}^{-1}(s)$ is smooth for each $s \in (\Delta, 0) \setminus \{0\}$.

We treat the following problem in this paper:

Problem. Let X be a weak Fano 3-fold with only terminal singularities.
When does X have a smoothing ?

For the case of Fano 3-fold X, X has a smoothing by the result of Namikawa and Mukai ([12], [9]). Moreover by the method of Namikawa, we can show the following theorem:

Theorem 0.3 (Namikawa, Takagi) (cf. [12], [25]). *Let X be a weak Fano 3-fold with only terminal singularities. Assume that there exists a birational projective morphism $\pi : X \to \overline{X}$ from X to a Fano 3-fold with only canonical singularities $\overline{X}$ such that* $\dim(\pi^{-1}(x)) \leq 1$ *for any $x \in \overline{X}$. Then X has a smoothing.*

In this paper, we will show the following theorem:

Main Theorem. *Let X be a weak Fano 3-fold with only terminal singularities.*
(1) *The Kuranishi space $Def(X)$ of X is smooth.*
(2) *There exists $\mathfrak{f} : \mathfrak{X} \to (\Delta, 0)$ a small deformation of X over $(\Delta, 0)$ such that the*

fiber $\mathfrak{X}_s = \mathfrak{f}^{-1}(s)$ *has only ordinary double points for any* $s \in (\Delta, 0) \setminus \{0\}$.
(3) *If* X *is* $\mathbb{Q}$*-factorial, then* X *has a smoothing.*

We remark that if the condition of (3) "$\mathbb{Q}$-factorial" is dropped, then there is an example that X remains singular under any small deformation (see Example 3.7).

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NOTATION. $\mathbb{C}$: the complex number field.
$\sim$: linear equivalence.
K_X : canonical divisor of X.
Let G be a group acting on a set S. We set

$$S^G := \{s \in S \mid gs = s \text{ for any } g \in G\}.$$

In this paper, $(\Delta, 0)$ means a germ of a 1-parameter unit disk.

Let X be a compact complex space or a good representative for a germ, and $\mathfrak{g} : \mathfrak{X} \to (\Delta, 0)$ a 1-parameter small deformation of X. We denote the fiber $\mathfrak{g}^{-1}(s)$ for $s \in (\Delta, 0)$ by $\mathfrak{X}_s$.

(Ens): the category of sets.

Let k be a field. We set (Art_k): the category of Artin local k-algebras with residue field k.

Let V be a $\mathbb{Z}$-module. The symbol $V_{\mathbb{C}}$ means $V \otimes_{\mathbb{Z}} \mathbb{C}$.

1. Proof of (1) of Main theorem.

We use the following theorem of Takagi:

Theorem 1.1 (cf. [25]). *Let* X *be a weak Fano* 3*-fold with only terminal singularities, then there exists a divisor* $D \in |-2K_X|$ *such that* D *is smooth.* (*In this paper, we call such* D *a smooth member of* $|-2K_X|$.)

Lemma 1.2. *Let* X *be a weak Fano* 3*-fold with only terminal singularities,* D *a smooth member of* $|-2K_X|$, $B \to A$ *a surjection in* $(Art_{\mathbb{C}})$, X_B *an infinitesimal deformation of* X *over* B, *and* $X_A := X_B \times_{\mathrm{Spec}(B)} \mathrm{Spec}(A)$. *Set* $D_A \in |-2K_{X_A/\mathrm{Spec}(A)}$ *such that* $D_A|_X = D$. *Then there exists* $D_B \in |-2K_{X_B/\mathrm{Spec}(B)}|$ *such that* $D_B|_{X_A} = D_A$.

Proof. Since by the Kawamata-Viehweg vanishing theorem we have $H^i(X, -2K_X) = 0$ for all $i > 0$, we can show above lemma as in [16, page 63, proof of (iii)]. □

Proof of (1) of Main theorem. Set $A_n = \mathbb{C}[t]/(t^{n+1})$, $\alpha_n : A_{n+1} \to A_n$, $S_n = \mathrm{Spec}(A_n)$. Let D be a smooth member of $|-2K_X|$, and X_{n+1} an infinitesimal deformation of X over S_{n+1}, and $X_n = X_{n+1} \times_{S_{n+1}} S_n$. By Lemma 1.2, there exists $D_{n+1} \in |-2K_{X_{n+1}/S_{n+1}}|$ such that $D_{n+1}|_X = D$. Set $D_n = D_{n+1}|_{X_n}$, $\pi_{n+1} : Y_{n+1} = \mathrm{Spec}(\mathcal{O}_{X_{n+1}} \oplus \mathcal{O}_{X_{n+1}}(K_{X_{n+1}/S_{n+1}})) \to X_{n+1}$ a double cover ramified along D_{n+1}, and $\pi_n : Y_n = \mathrm{Spec}(\mathcal{O}_{X_n} \oplus \mathcal{O}_{X_n}(K_{X_n/S_n})) \to X_n$ a double cover ramified along D_n. We remark that $Y = Y_0$ is a Calabi-Yau 3-fold with only terminal singularities. Let $G = \mathbb{Z}/2\mathbb{Z}$. We have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Ext}^1_{\mathcal{O}_{Y_{n+1}}}(\Omega^1_{Y_{n+1}/S_{n+1}}, \mathcal{O}_{Y_{n+1}})^G & \xrightarrow{T^1_Y(\alpha_n)^G} & \mathrm{Ext}^1_{\mathcal{O}_{Y_n}}(\Omega^1_{Y_n/S_n}, \mathcal{O}_{Y_n})^G \\
\beta_{n+1}\downarrow & & \downarrow\beta_n \\
\mathrm{Ext}^1_{\mathcal{O}_{X_{n+1}}}(\Omega^1_{X_{n+1}/S_{n+1}}, \mathcal{O}_{X_{n+1}}) & \xrightarrow[T^1_X(\alpha_n)]{} & \mathrm{Ext}^1_{\mathcal{O}_{X_n}}(\Omega^1_{X_n/S_n}, \mathcal{O}_{X_n}).
\end{array}$$

We remark that β_{n+1} and β_n are defined because π_{n+1} and π_n are finite morphisms. (cf. [10, Proposition 4.1]). By [10, Proof of Theorem 1, page 431], $T^1_Y(\alpha_n)$ is surjective. Thus $T^1_Y(\alpha_n)^G$ is a surjection because G is finite. β_n is a surjection by Lemma 1.2 and we have that $T^1_X(\alpha_n)$ is also surjective. By $\mathbb{T}^1$-lifting criterion (cf. [4], [5]), we proved (1) of Main theorem. □

2. Proof of (2) of Main theorem.

We use the result of Namikawa and Steenbrink on deformations of Calabi-Yau 3-folds to prove (2) of Main theorem. Let Y be a Calabi-Yau 3-fold with only terminal singularities, $\{q_1, q_2, \ldots, q_n\} = \mathrm{Sing}(Y)$, $\nu : \tilde{Y} \to Y$ be a good resolution of Y, and $E_i = \nu^{-1}(q_i)$. ("good" means the restriction of $\nu : \nu^{-1}(V) \to V$ is an isomorphism and its exceptional divisor E_i is simple normal crossings for each i.)

Proposition 2.1 (cf. [15]). *If (Y, q_i) is not the ordinary double point, then the homomorphism $\iota_i : H^2_{E_i}(\tilde{Y}, \Omega^2_{\tilde{Y}}) \to H^2(\tilde{Y}, \Omega^2_{\tilde{Y}})$ is not injective.*

Proof of (2) of Main theorem. By Theorem 1.1, there exists a smooth member D of $|-2K_X|$. We remark that $D \cap \mathrm{Sing}(X) = \emptyset$. Let $\{p_1, p_2, \ldots, p_n\} = \mathrm{Sing}(X)$, and $\pi : Y = \mathrm{Spec}(\mathcal{O}_X \oplus \mathcal{O}_X(K_X)) \to X$ be a double cover ramified along D. Then Y is a Calabi-Yau 3-fold with only terminal singularities. Let $G = \mathbb{Z}/2\mathbb{Z} = \{\mathrm{id}_X, \sigma\}$, $\pi^{-1}(p_i) = \{q_{i1}, q_{i2}\}$. Then we have that $\mathrm{Sing}(Y) = \{q_{ij} \mid i = 1, 2, \ldots, n, j = 1, 2\}$ because D is smooth. Let Y_{ij} be a sufficiently small open neighborhood of q_{ij}, $V_{ij} = Y_{ij} \setminus \{q_{ij}\}$, $U = X \setminus \mathrm{Sing}(X)$, and $V = Y \setminus \mathrm{Sing}(Y)$. Let $\nu : \tilde{Y} \to Y$ be a G-equivariant good resolution of Y, and $E_{ij} = \pi^{-1}(q_{ij})$. Let $\omega \in H^0(\omega_Y)$ be a nowhere vanishing section. We remark that $\sigma(\omega) = -\omega$. We consider the following commutative diagram:

$$\begin{array}{ccccc} [H^1(V, \Omega^2_V)]^{[-1]} & \xrightarrow{\alpha'} & [\oplus_{i,j} H^2_{E_{ij}}(\tilde{Y}, \Omega^2_{\tilde{Y}})]^{[-1]} & \xrightarrow{\iota} & [H^2(\tilde{Y}, \Omega^2_{\tilde{Y}})]^{[-1]} \\ \uparrow \wr \tau & & \uparrow \oplus_{i,j}\tau_{ij} & & \\ H^1(V, \Theta_V)^G & \xrightarrow{\alpha} & \oplus_{i,j} H^1(V_{ij}, \Theta_{V_{ij}})^G & & \end{array}$$

where $F^{[-1]} = \{x \in F \mid \sigma(x) = -x\}$ for a $\mathbb{C}$-vector space F with a G-action.

By Proposition 2.1, ι_{ij} is not injective if $(X, p_i) \simeq (Y, q_{ij})$ is not the ordinary double point. So there exists an element $\eta' \in [H^1(V, \Theta_V)]^{[-1]}$ such that $\alpha'(\eta)_{ij} \not\equiv 0$ for any i, j where $(X, p_i) \simeq (Y, q_{ij})$ is not the ordinary double point. Let $\eta \in H^1(V, \Theta_V)^G$ such that $\tau(\eta) = \eta'$. Let $\beta : [H^1(V, \Theta_V)]^G = \mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y)^G \to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = H^1(U, \Theta_U)$ be the homomorphism defined in the proof of (1) of Main theorem. By (1) of Main theorem, there exists a small deformation of X over $(\Delta, 0)\mathfrak{f} : \mathfrak{X} \to (\Delta, 0)$ which is a realization of $\beta(\eta)$. Using the method of Namikawa (cf. [11, Theorem 5], [15, Theorem (2.4)]), we can reach a smooth 3-fold by small deformations by continuing the process above. □

DEFINITION 2.2. Let X be a normal $\mathbb{Q}$-Gorenstein projective variety of dimension 3 over $\mathbb{C}$ which has only terminal singularities.

(1) The index i_p of a singular point $p \in X$ is defined by

$$i_p := \min\{m \in \mathbb{N} \mid mK_X \text{ is a Cartier divisor near } p\}.$$

(2) The sigular index $i(X)$ of X is defined by

$$i(X) := \min\{m \in \mathbb{N} \mid mK_X \text{ is a Cartier divisor}\}.$$

(3) If $-K_X$ is ample, we call X a $\mathbb{Q}$-Fano 3-fold.

(4) If $-K_X$ is nef and big, we call X a weak $\mathbb{Q}$-Fano 3-fold.

We considered deformations of $\mathbb{Q}$-Fano 3-folds in [7]. The method of (2) of Main theorem is also useful for weak $\mathbb{Q}$-Fano 3-folds of singular index 2.

DEFINITION 2.3. Let (X, p) be a germ of a 3-dimensional terminal singularity and $G = \mathbb{Z}/2\mathbb{Z}$.

(1) We call (X, p) a quotient singularity of type $(1/2)(1, 1, 1)$ if (X, p) is isomorphic to the singularity of the following type: Let x_1, x_2, x_3, be coordinates of the germ $(\mathbb{C}^3, 0)$. We define a G-action on $(\mathbb{C}^3, 0)$ by $x_1 \mapsto -x_1$, $x_2 \mapsto -x_2$, $x_3 \mapsto -x_3$. $(X, p) \simeq (\mathbb{C}^3/G, 0)$.

(2) We call (X, p) a quotient of the ordinary double point if (X, p) is isomorphic to the singularity of the following type: Let x_1, x_2, x_3, x_4 be coordinates of the germ

$(\mathbb{C}^4, 0)$. We define a G-action on $(\mathbb{C}^4, 0)$ by $x_1 \mapsto -x_1$, $x_2 \mapsto -x_2$, $x_3 \mapsto x_3$, $x_4 \mapsto -x_4$. $(X, p) \simeq \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \mid (\mathbb{C}^4, 0)\}/G$.

Theorem 2.4. *Let X be a weak $\mathbb{Q}$-Fano 3-fold with only terminal singularities of singular index $i(X) = 2$, and assume that there exists a smooth member of $|-2K_X|$. Then there exists a small deformation of X over $(\Delta, 0)$ $\mathfrak{f} : \mathfrak{X} \to (\Delta, 0)$ such that the fiber $\mathfrak{X}_s = \mathfrak{f}^{-1}(s)$ has only quotient singularities of type $(1/2)(1, 1, 1)$, ordinary double points or quotients of ordinary double points for any $s \in (\Delta, 0) \setminus \{0\}$.*

To prove this theorem, we use an analogous proposition of Proposition 2.1. Let X be a weak $\mathbb{Q}$-Fano 3-fold with only terminal singularities of singular index $i(X) = 2$, and assume that there exists a smooth member of $| - 2K_X|$. Let D be a smooth member of $|-2K_X|$. Let $\pi : Y = \mathrm{Spec}(\mathcal{O}_X \oplus \mathcal{O}_X(K_X)) \to X$ be a double cover ramified along D. Then Y is a Calabi-Yau 3-fold with only terminal singularities. Let $p \in X$ be a singularity of index $i_p = 2$, and $\pi^{-1}(p) = q$. We remark that $\pi|_{(Y,q)} : (Y, q) \to (X, p)$ is a canonical cover of (X, p). Let $G = \mathbb{Z}/2\mathbb{Z}$. $\nu : \tilde{Y} \to Y$ be a G-equivariant good resolution of Y, and $E = \nu^{-1}(q)$. We know the following proposition which is analogous to Proposition 2.1 and is a result of Namikawa.

Proposition 2.5 (cf. [13]). *If (X, p) is a singular point of index $i_p = 2$, and if (Y, q) is not the ordinary double point, then the homomorphism $\iota^{[-1]} : H^2_E(\tilde{Y}, \Omega^2_{\tilde{Y}})^{[-1]} \to H^2(\tilde{Y}, \Omega^2_{\tilde{Y}})^{[-1]}$ is not injective.*

This proposition leads us to Theorem 2.4 by the same method of the proof of (2) of Main theorem.

3. Proof of (3) of Main theorem.

We first prove the following theorem to prove (3) of Main theorem.

Theorem 3.1. *Let X be a weak Fano 3-fold with only terminal singularities. Assume that X is $\mathbb{Q}$-factorial, then there exists a divisor $S \in | - K_X|$ such that S is smooth.*

To prove Theorem 3.1, we use some known results as follows.

DEFINITION 3.2. Let X be a weak Fano 3-fold with only terminal singularities. Fano index of X is defined by

$$F(X) = \max\{r \in \mathbb{N} \mid \text{there exists a Cartier divisor } H \text{ such that } -K_X \sim rH\}.$$

Theorem 3.3 (Reid, Shin) (cf. [20], [24]). *Let X be a Fano 3-fold with only canonical singularities. Then we have,*
(1) $\dim Bs|-K_X| \leq 1$,
(2) *if* $F(X) > 1$ *then* $Bs|-K_X| = \emptyset$,
(3) *if* $\dim Bs|-K_X| = 1$ *then a general member of* $|-K_X|$ *is smooth at base points of* $|-K_X|$, *and*
(4) *if* $\dim Bs|-K_X| = 0$ *then* $Bs|-K_X| = \{p\}$ *one point, a general member of* $|-K_X|$ *has the ordinary double point at* p, *and* $p \in \mathrm{Sing}(X)$.

Theorem 3.4 (Mella) (cf. [6, Theorem (2.4)]). *In the case of* (4) *of theorem* (3.3), *if* $p \in \mathrm{Sing}(X)$ *is a terminal singularity, then* $X \cong X_{2,6} \subset \mathbb{P}(1,1,1,1,2,3)$. *Moreover for any Zariski open set* U *containing* p, U *is not* $\mathbb{Q}$*-factorial.*

Theorem 3.5 (Reid, Ambro) (cf. [20], [1]). *Let X be a weak Fano 3-fold with only canonical singularities, then a general member of* $|-K_X|$ *has only canonical singularities.*

Proof of Theorem 3.1. Let $\pi : X \to \bar{X}$ be a multi-anti-canonical morphism, then $\bar{X}$ is a Fano 3-fold with only canonical singularities, and π is crepant ($K_X = \pi^*(K_{\bar{X}})$).

In the case of $Bs|-K_{\bar{X}}| = \emptyset$ or $\dim Bs|-K_{\bar{X}}| = 1$, then a general member of $|-K_{\bar{X}}|$ is smooth at its base point by Theorem 3.3, and there exists a divisor $S \in |-K_X|$ such that S is smooth by Theorem 3.5.

In the case of $\dim Bs|-K_{\bar{X}}| = 0$ (in this case $Bs|-K_X| = \{p\}$ by Theorem (3.3.4)), there exists a divisor $\bar{S} \in |-K_{\bar{X}}|$ which has the ordinary double point at p such that $S = \pi^*(\bar{S})$ has only canonical singularities. If we can not take a smooth S, then $\pi|_S : S \to \bar{S}$ is an isomorphism near p because p is the ordinary double point. Then there exists a Zariski open set U containing $\pi^{-1}(p)$ such that $\pi|_U : U \to \bar{X}$ is an open immersion. So $p \in \bar{X}$ is terminal. By Theorem 3.4, $\pi(U)$ is not $\mathbb{Q}$-factorial. Thus U is not $\mathbb{Q}$-factorial and X is not $\mathbb{Q}$-factorial which is a contradiction. $\square$

By (2) of Main theorem, the following theorem is enough to prove (3) of Main theorem.

Theorem 3.6. *Let X be a weak Fano 3-fold with only ordinary double points. Assume that X is $\mathbb{Q}$-factorial. Then X has a smoothing.*

Proof. Let $\nu : \tilde{X} \to X$ be a small resolution of X, $\{p_1, p_2, \ldots, p_n\} = \mathrm{Sing}(X)$, $U = X - \mathrm{Sing}(X)$, X_i a sufficiently small open neighborhood of p_i, $U_i = X_i \setminus \{p_i\}$, and $C_i = \nu^{-1}(p_i)$. Since $H^1(X, \Omega^1_X) \simeq H^1(X, \nu_*\Omega^1_{\tilde{X}})$ (cf. [10, Lemma 2.2]), We have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1(X,\Omega^1_X) & \xrightarrow{\alpha_1} & H^1(\tilde{X},\Omega^1_{\tilde{X}}) & \xrightarrow{\alpha_2} & H^0(X,R^1\nu_*\Omega^1_{\tilde{X}}) \\
 & & \uparrow\lambda_1 & & \uparrow\lambda_2 & & \uparrow\lambda_3 \\
0 & \longrightarrow & H^1(X,\mathcal{O}^*_X)_{\mathbb{C}} & \xrightarrow{\beta_1} & H^1(\tilde{X},\mathcal{O}^*_{\tilde{X}})_{\mathbb{C}} & \xrightarrow{\beta_2} & H^0(X,R^1\nu_*\mathcal{O}^*_{\tilde{X}})_{\mathbb{C}}
\end{array}$$

λ_2 is surjective because $h^2(\tilde{X},\mathcal{O}_{\tilde{X}})=0$, and β_1 is also surjective because X is $\mathbb{Q}$-factorial and ν is small. Thus we have that α_2 is the zero map, and its dual $\oplus_{i=1}^n H^2_{C_i}(\tilde{X},\Omega^2_{\tilde{X}}) \to H^2(\tilde{X},\Omega^2_{\tilde{X}})$ is also the zero map. By Theorem 3.1, there exists $D\in|-K_X|$ a smooth member of $|-K_X|$. Then $D\cap \mathrm{Sing}(X)=\phi$. We consider the following commutative diagram defined by ν^*D:

$$\begin{array}{ccc}
\oplus_{i=1}^n H^2_{C_i}(\tilde{X},\Omega^2_{\tilde{X}}) & \xrightarrow{\oplus_i\delta_i} & \oplus_{i=1}^n H^2_{C_i}(\tilde{X},\Theta_{\tilde{X}}) \\
\downarrow & & \downarrow\oplus_i\iota_i \\
H^2(\tilde{X},\Omega^2_{\tilde{X}}) & \longrightarrow & H^2(\tilde{X},\Theta_{\tilde{X}}).
\end{array}$$

δ_i is an isomorphism for any i, and we have that ι_i is the zero map for any i. We consider the following exact commutative diagram:

$$\begin{array}{ccccc}
H^1(U,\Theta_U) & \xrightarrow{\gamma'} & \oplus_{i=1}^n H^2_{C_i}(\tilde{X},\Theta_{\tilde{X}}) & \xrightarrow{\oplus_i\iota_i} & H^2(\tilde{X},\Theta_{\tilde{X}}) \\
\| & & \uparrow & & \\
H^1(U,\Theta_U) & \xrightarrow[\gamma]{} & \oplus_{i=0}^n H^1(U_i,\Theta_{U_i}). & &
\end{array}$$

Then there exists an element $\eta\in H^1(U,\Theta_U)$ such that $\gamma'(\eta)_i \neq 0$ for any $i=1,2,\dots,n$. Thus $\gamma(\eta)_i\neq 0$ for any $i=1,2,\dots,n$. By (1) of Main theorem, there exists a small deformation of X over $(\varDelta,0)\mathfrak{f}:\mathfrak{X}\to(\varDelta,0)$ which is a realization of η. Then $\mathfrak{f}$ is a smoothing of X. □

EXAMPLE 3.7. Let $\overline{X}$ be the projective cone over the smooth del Pezzo surface S of degree 8. Then $\overline{X}$ is a Gorenstein Fano 3-fold with $\rho=1$ which has only one Gorenstein rational singularity $\overline{p}$ at its vertex. Let $f:Z\to\overline{X}$ be the blowing-up at $\overline{p}$, then f is a crepant resolution of $\overline{X}$ and $Z\simeq \mathrm{Proj}(\mathcal{O}_S\oplus\omega_S^{-1})$. Let E be an exceptional divisor of f which is isomorphic to $\mathbb{F}_1$, and C be the (-1)-curve on E. Then Z is a weak Fano 3-fold with $(-1,-1)$-curve C. Let $\nu:Z\to X$ be a birational contraction which contracts C. Then X is a weak Fano 3-fold which has only one ordinary double point $\nu(C)=p$. Let $F=\nu(E)$, then $F\simeq\mathbb{P}^2$ passing through p. So X is not $\mathbb{Q}$-factorial. We have that X is not smoothable, in fact there exists a sufficiently small open neighborhood U of F $(F\subset U)$ which is not smoothable by [14, Proposition 1.3].

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