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<thead>
<tr>
<th><strong>Title</strong></th>
<th>The inverse scattering problem for the Dirac operator and the modified Korteweg-de Vries equation</th>
</tr>
</thead>
<tbody>
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<td><strong>Author(s)</strong></td>
<td>Ohmiya, Mayumi</td>
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Osaka University
THE INVERSE SCATTERING PROBLEM FOR THE DIRAC OPERATOR AND THE MODIFIED KORTEWEG-DE VRIES EQUATION

MAYUMI OHMIYA

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The main purpose of the present paper is to construct the solution of the initial value problem for the modified Korteweg-de Vries (KdV) equation

\[ v_t - 6v^2v_x + v_{xxx} = 0, \quad -\infty < x, t < \infty. \]

The subscripts \( x, t \) denote partial differentiations. We study smooth real valued solutions which tend to \( \pm m \) as \( x \to \pm \infty \) for a positive constant \( m \).

As an analogue of the method of Gardner, Greene, Kruskal and Miura (GGKM) [3], we construct these solutions in terms of the scattering data of the one dimensional Dirac operator

\[ L_{iv} = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} D + i \begin{bmatrix} 0 & -v \\ v & 0 \end{bmatrix}, \quad D = d/dx. \]

In [9], Zakharov and Shabat have studied the initial value problem for the non-linear Schrödinger equation

\[ iu_x + u_{xx} - |u|^2u = 0 \]

with the step type initial data as above. They have developed the inverse scattering theory of

\[ L_u = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} D + \begin{bmatrix} 0 & u^* \\ u & 0 \end{bmatrix} \]

on formal basis, where \( u^* \) is the complex conjugate of \( u \). They have constructed the exact solutions of (0.2) in terms of the scattering data of \( L_u \), assuming that the reflection coefficient identically vanishes.

Now, \( L_{iv} \) can be obtained from \( L_u \) by putting \( u = iv \), where \( v \) is a real valued function. By virtue of this restriction, the argument can be considerably simplified and, in the sequel, we can complete the inverse scattering theory of \( L_{iv} \). This result enables us to construct the solutions with general step type initial data.
In § 1, we describe preliminary materials which concern the Jost solutions and the scattering data of $L_u$. In § 2, we derive the fundamental integral equation. In § 3, the solvability of the fundamental integral equation is established. In § 4, the inverse scattering problem for $L_{iv}$ are discussed. Finally, in § 5, the solutions of the initial value problem for the modified KdV equation (0.1) are constructed.

Throughout the paper, $c^*$ denotes the complex conjugate of $c$.

The author wishes to express his hearty thanks to Professor Shunichi Tanaka for his invaluable suggestion.

1. Scattering data

In this section, we expose the generality of the scattering data of $L_u$ without the assumption $u-iv$. In deriving the following results, methods developed for the Schrödinger operator and other operators have been used in modified form. For these results, we refer to [1], [2] [4], [6], [8] and [9].

Let $m$ be a positive real number. Put

$$m_\pm = m \exp (i\alpha_\pm), \quad -\pi \leq \alpha_\pm \leq \pi.$$ 

For a complex valued measurable function $u(x)$ which tends to $m_\pm$ as $x \to \pm \infty$, consider the eigenvalue problem

$$L_u y = \lambda y, \quad y = i(y_1, y_2), \quad \lambda = \xi + i\kappa,$$

on the real axis $(-\infty, \infty)$.

Let $\zeta = \zeta(\lambda)$ be the two-valued algebraic function defined by

$$\zeta^2 = \lambda^2 - m^2$$

and $R$ be the upper leaf of the two-sheeted Riemann surface associated with $\zeta$. We assume $\text{Im} \zeta > 0$ for $\lambda \in R$. For $\xi \in R = \mathbb{R} \setminus [-m, m]$, put

$$\sigma = \sigma(\xi) = (\text{sgn} \xi)(\xi^2 - m^2)^{1/2}.$$ 

For a two-dimensional vector $y = (y_1, y_2)$ and a matrix $A = (a_{ij})$ of order 2, put

$$y^* = \begin{pmatrix} y_1^* & y_2^* \end{pmatrix}, \quad y^* = \begin{pmatrix} y_2 & y_1 \end{pmatrix},$$

$$A^* = \begin{bmatrix} a_{22}^* & a_{21}^* \\ a_{12}^* & a_{11}^* \end{bmatrix}, \quad A^* = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}.$$ 

If $y = y(x)$ is a solution of (1.1), then $y^*$ is a solution of (1.1), $\lambda$ being replaced by $\lambda^*$.

For solutions $y(x)$ and $z(x)$ of (1.1), the Wronskian

$$[y; z] = y_1 z_2 - y_2 z_1$$
is constant.

Put
\[
f_+^0(x, \lambda) = i(m^{-1}(\lambda - \xi)^{-1}) \exp (i\xi x),
\]
\[
f_-^0(x, \lambda) = (1, m^{-1}(\lambda - \xi)^{-1}) \exp (-i\xi x).
\]

They are solutions of (1.1) for \(u(x) = m_\pm\) respectively.

Gasymov [4; Theorem 1.2.1] has shown the following.

**Theorem 1.1** (Gasymov [4]). If we assume
\[
\sigma_\pm(x) = \pm \int_{x}^{\pm \infty} \left(1 + |y|\right)|u(y) - m_\pm|\,dy + \sup_{y > x} |u(y) - m_\pm| < \infty,
\]
then there exist unique solutions \(f_\pm(x, \lambda)\) of (1.1) such that
\[
f_\pm(x, \lambda) = f_\pm^0(x, \lambda) + o(1)
\]
as \(x \to \pm \infty\). \(f_\pm(x, \lambda)\) are analytic in \(\lambda \in \mathbb{R}\). Moreover there exist matrix functions
\[
A_\pm(x, y) = (A_{\pm ij}(x, y))_{i, j = 1, 2}
\]
such that
\[
(1.2) \quad f_\pm(x, \lambda) = f_\pm^0(x, \lambda) \pm \int_x^{\pm \infty} A_\pm(x, y)f_\pm^0(y, \lambda)\,dy.
\]

Furthermore
\[
|A_{\pm ij}(x, y)| \leq C_\pm \sigma_\pm(x + y)
\]
and
\[
A_\pm^*(x, y) = A_\pm(x, y)
\]
are valid. We have
\[
u(x) = -2iA_+^*(x, x) + m_+.
\]

**Proof.** Put
\[
u(x) = (f_0^+, (x, \lambda), f_0^-(x, \lambda))
\]
then we have
\[
E(x, \lambda) = (f_0^+, (x, \lambda), f_0^-(x, \lambda))
\]
This integral equation can be solved by successive approximation which leads to the existence of the solution and its analyticity.

We refer to [4; pp. 53–63] for the existence of kernels \(A_\pm\). Q.E.D.

The functions \(f_\pm(x, \lambda)\) are called the Jost solutions.

If we assume that \(u = iv\) and \(\alpha_\pm = \pm 2^{-1}\pi\), where \(v\) is real, then the proof of this theorem can be considerably simplified as follows. Put
\[
(1.3) \quad E(x, \lambda) = \begin{bmatrix}
0 & u^*(y) - m_+^{-1} \\
-u(y) + m_+ & 0
\end{bmatrix}f_+(y, \lambda)\,dy.
\]

This integral equation can be solved by successive approximation which leads to the existence of the solution and its analyticity.

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The functions \(f_\pm(x, \lambda)\) are called the Jost solutions.

If we assume that \(u = iv\) and \(\alpha_\pm = \pm 2^{-1}\pi\), where \(v\) is real, then the proof of this theorem can be considerably simplified as follows. Put
\[
(1.4) \quad E(\lambda) = \begin{bmatrix}
im^{-1}(\xi - \lambda) & 1 \\
im^{-1}(\xi - \lambda) & 1
\end{bmatrix}
\]
If we set
\[ h_{\pm}(x, \zeta) = E(\lambda)^{-1} f_{\pm}(x, \lambda) \exp \left( \mp i\zeta x \right) \quad (\lambda \neq 0), \]
then \( h_{\pm}(x, \zeta) \) are analytic in \( \zeta \), \( \text{Im} \zeta > 0 \). Assuming
\[(1.5) \quad h_{+}(x, \zeta) = \theta(0, 1) + \int_0^\infty K_{+}(x, y) \exp (2i\zeta y) dy, \quad K_{+} = \theta(K_{+1}, K_{+2}), \]
put (1.5) into (1.3). And we have
\[(1.6) \quad K_{+1}(x, y) + \int_x^{x+y} (v(z) - m) K_{+2}(z, x+y-z) dz = -v(x+y) + m \]
\[(1.7) \quad K_{+2}(x, y) + \int_x^{x+y} (v(z) + m) K_{+1}(z, y) dz = 0. \]
These integral equations can be solved by successive approximation. From this, \( K_{\pm} \) are real vectors. We have
\[ v(x) = -K_{+1}(x, 0) + m = K_{-2}(x, 0) - m. \]
The matrix
\[ 2^{-1} \begin{bmatrix} K_{+2}(x, 2^{-1}(y-x)) & K_{+1}(x, 2^{-1}(y-x)) \\ K_{+1}(x, 2^{-1}(y-x)) & K_{+2}(x, 2^{-1}(y-x)) \end{bmatrix} \]
coincides with the kernels \( A_{\pm}(x, y) \) in Theorem 1.1.
Returning to the case of general complex potential, put
\[ f_{\pm}(x, \xi) = f_{\pm}(x, \xi + i0), \quad \xi \in \mathbb{R}. \]
We have
\[ [f_{\pm}(x, \xi); f_{\pm}^*(x, \xi)] = 2\sigma(\sigma - \xi)/m^2. \]
Since \( \sigma(\sigma - \xi) \) does not vanish for \( \xi \in \mathbb{R}, \) \( f_{+}(x, \xi) \) and \( f_{-}^*(x, \xi) \) are linearly independent solutions of (1.1). Therefore one can express
\[(1.8) \quad f_-(x, \xi) = a_+(\xi) f_+^*(x, \xi) + b_+(\xi) f_+(x, \xi). \]
Similarly, we have
\[ f_+(x, \xi) = a_-(\xi) f_+^*(x, \xi) + b_-(\xi) f_-(x, \xi). \]
We have
\[ a_+(\xi) = a_-(\xi) = a(\xi) = m^2[f_+; f_-]/2\sigma(\sigma - \xi) \]
and
\[(1.9) \quad b_+(\xi) = -b_-(\xi) = m^2[f_+^*; f_-^*]/2\sigma(\sigma - \xi). \]
We have
\[(1.10) \quad |a(\xi)|^2 = 1 + |b_+(\xi)|^2. \]
This implies that $a(\xi)$ does not vanish for $\xi \in R^m$.

The coefficient $a(\xi)$ can be extended to the analytic function

$$a(\lambda) = m^2[f_+(x, \lambda); f_-(x, \lambda)]/2\xi(\xi - \lambda), \quad \lambda \in R.$$  

Put (1.2) into (1.9) and (1.11) and calculate the Wronskians, and we can obtain the integral representations of $a(\lambda)$ and $b_\pm(\xi)$. For instance, we have

$$a(\lambda) = \frac{(\xi - \lambda)^2 - m^2}{2\xi(\xi - \lambda)} \exp \left\{ i (\alpha_+-\alpha_-) \right\}$$

$$+ \frac{1}{2\xi(\xi - \lambda)} \int_0^\infty \left\{ \alpha_1(x) + (\xi - \lambda)\alpha_2(x) + (\xi - \lambda)^2\alpha_3(x) \right\} \exp (2i\xi x) dx,$$

where $\alpha_j(x)$ ($j=1, 2, 3$) which are integrable can be expressed explicitly in terms of the kernels $A_\pm$.

Because $f_\pm$ are linearly dependent at the zero of $a(\lambda)$, they are square integrable by their asymptotic property. By virtue of formal selfadjointness of $L_u$, zeros $a(\lambda)$ belong to $(-m, m)$. Let $\lambda^0$ be one of zeros of $a(\lambda)$. Then

$$f_-(x, \lambda^0) = d^\theta f_+(x, \lambda^0)$$

is valid for some constant $d^\theta$. We have

$$a'(\lambda^0) = -i(2\eta^0)^{-1}m_+ d^\theta \int_{-\infty}^{\infty} |f_+(x, \lambda^0)|^2 dx,$$

where $\eta^0 = (m^2 - \lambda^0)^{1/2}$. Hence $\lambda^0$ is a simple zero of $a(\lambda)$.

Similarly to [6; pp133–134], we can show that $a(\lambda)$ has only finite number of zeros. We denote them by $\lambda_1, \lambda_2, \cdots, \lambda_N$. Put

$$r_\pm(\xi) = b_\pm(\xi)/a(\xi), \quad \xi \in R^m,$$

which are called reflection coefficients. We have

$$r_\pm(\xi) = O(\xi^{-1}), \quad |\xi| \to \infty,$$

and

$$|r_\pm(\xi)| < 1, \quad \xi \in R^m.$$  

Put

$$n_{\pm j} = \left\{ \int_{-\infty}^{\infty} f^*_\pm(x, \lambda_j)f_\pm(x, \lambda_j) dx \right\}^{-1}, \quad j = 1, 2, \cdots, N.$$

We call the collection

$$\{r_\pm(\xi), n_{\pm j}, \lambda_j, j = 1, 2, \cdots, N\}$$

the scattering data of $L_u$. 
In the following, we assume that \( u = iv \) and \( \alpha_\pm = \pm 2^{-1} \pi \), where \( v \) is real. Putting (1.5) into (1.9) and (1.1), we have

\[
(1.15) \quad a(\lambda) = \lambda \left( 1 + \int_0^\infty \alpha(x) \exp(2i\xi x) dx \right) \xi
\]

and

\[
(1.16) \quad b_+ (\xi) = (2i\sigma)^{-1} \int_0^\infty \beta(x) \exp(-2i\sigma x) dx ,
\]

where \( \alpha(x) \) and \( \beta(x) \) are real valued integrable functions which can be expressed explicitly in terms of kernels \( K_\pm \). By (1.14) and (1.15), we have

\[
a(-\lambda) = -a(\lambda)
\]

and

\[
b_+ (\xi) = O(\xi^{-1}).
\]

Hence, if \( \lambda^0 \) is a zero of \( a(\lambda) \), then \( a(\lambda) \) vanishes also at \( \lambda = -\lambda^0 \). Therefore zeros of \( a(\lambda) \) consist of \( \pm \kappa_j \), where

\[
0 = \kappa_0 < \kappa_1 < \cdots < \kappa_n < m .
\]

The linear dependence of \( f_\pm \) implies that of \( h_\pm \exp(\pm i\xi x) \). Therefore we have

\[
h_-(\eta, \eta_j) \exp (\eta_j x) = d\ h_+(\eta, \eta_j) \exp (-\eta_j x) , \quad j = 0, 1, \ldots, n ,
\]

for some real number \( d_j \), where \( \eta_j = (m^2 - \kappa_j^2)^{1/2} \). Put

\[
c_{\pm 0} = \left\{ \int_{-\infty}^\infty | f_+(\eta, 0) |^2 d\eta \right\}^{-1} = \text{id}_0/2a'(0) ,
\]

\[
c_{\pm j} = 2 \left\{ \int_{-\infty}^\infty | f_+(\eta, \pm \kappa_j) |^2 d\eta \right\}^{-1} = \text{id}_j/\eta_j a'(\pm \kappa_j) , \quad j = 1, 2, \ldots, n .
\]

Define \( c_{-j} \) by

\[
(1.17) \quad c_{-j} = -(2a'(0))^{-2} ,
\]

\[
c_{+j} = -m^2(\eta_j a'(\kappa_j))^{-2} , \quad j = 1, 2, \ldots, n .
\]

By (1.12), \( c_{\pm j} \) are positive numbers.

In place of (1.14), we call the collection

\[
\{ r_{\pm}(\xi), c_{\pm j}, \kappa_j , \quad j = 0, 1, 2, \ldots, n \}
\]

the scattering data of \( L_{iv} \).

By the similar arguments as in [2, p 149], we can show that the condition

\[
(1.18) \quad r(\xi) \rightarrow \mp i \quad (\xi \rightarrow \pm m)
\]

are valid, if and only if
Moreover the condition
\[(1.19) \quad r(\xi) < \delta < 1, \quad \xi \in \mathbb{R}_+, \]
is valid, if and only if
\[1 + \int_0^\infty \alpha(y)dy = 0.\]

Put
\[B_1(\lambda) = \lambda^{-1} \Pi J^{-1}(\xi - i\eta_1)^{-1}(\xi + i\eta_1)\]
and
\[B_2(\lambda) = \lambda^{-1}(\xi + im) \Pi J^{-1}(\xi - i\eta_1)^{-1}(\xi + i\eta_1).\]

If the condition (1.18) holds, then \(B_1(\lambda)a(\lambda)\) is analytic in \(\zeta, \text{Im}\zeta > 0\), and has no zero. If we set
\[a_0(\zeta) = B_1(\lambda)a(\lambda)\]
and
\[g(x) = \pi^{-1} \int_{-\infty}^\infty \log a_0(\sigma) \exp (-2i\sigma x)d\sigma,\]
where integration is taken in \(L^2\)-sense, then, by (1.15) and the Payley-Wiener's theorem, \(g(x)\) is a real valued function which vanishes for \(x<0\). Hence we have
\[(1.20) \quad g(x) + g(-x) = \pi^{-1} \int_{-\infty}^\infty \log |a_0(\sigma)|^2 \exp (-2i\sigma x)d\sigma\]
and
\[(1.21) \quad \log a_0(\zeta) = 2^{-1} \left\{ \int_0^\infty g(x) \exp (2i\zeta x)dx - \int_{-\infty}^0 g(-x) \exp (-2i\zeta x)dx \right\} .\]

Eliminating \(g(x)\) in (1.21) by (1.20), we have
\[\log a_0(\zeta) = (2\pi i)^{-1} \int_{-\infty}^\infty (\sigma - \zeta)^{-1} \log |a_0(\sigma)|^2d\sigma.\]

Hence, by (1.10), we obtain
\[(1.22) \quad a(\lambda) = B_1(\lambda)^{-1} \exp \left\{ (2\pi i)^{-1} \int_{-\infty}^\infty (\sigma - \zeta)^{-1} \log [\xi^{-2}\sigma^2(1 - |r(\xi)|^2)]^{-1}d\sigma \right\} .\]

Similarly to above, we have
\[(1.23) \quad a(\lambda) = B_2(\lambda)^{-1} \exp \left\{ (2\pi i)^{-1} \int_{-\infty}^\infty (\sigma - \zeta)^{-1} \log (1 - |r(\xi)|^2)^{-1}d\sigma \right\},\]
if (1.18) holds. Thus we can reconstruct \(a(\lambda)\) from the reflection coefficient \(r(\xi)\).
2. The fundamental integral equation

In this and subsequent sections, we assume that \( u = iv \) and \( \alpha_\pm = \pm 2^{-1} \pi \), where \( v \) is real.

In [8], Zakharov and Shabat have derived integral equations which connect kernels \( A_\pm \) with the scattering data of \( L_u \). In this section we derive similar integral equations which connect kernels \( K_\pm \) with the scattering data of \( L_v \).

By (1.8) we have

\[
\begin{align*}
\Lambda(\xi) - \Lambda(\sigma)^* & = \Lambda(\xi)^\ast \Lambda(\sigma) - \Lambda(1, 0) = \{h_+(x, \sigma)^\ast(0, 1)\}^t, \\
& + r_+(\xi)J(\xi) \exp(2iv\sigma)h_+(x, \sigma),
\end{align*}
\]

where

\[
J(\xi) = E(\xi + i0)^t E(\xi + i0) = \xi^{-1} \begin{bmatrix} \sigma & -im \\ -im & \sigma \end{bmatrix}
\]

Now, multiply \( \pi^{-1} \exp(2i\sigma y) \) on the above identity and integrate over \((-\infty, \infty)\) with respect to \( \sigma \), where integrations are taken in \( L^2 \)-sense. We have

\[
\pi^{-1} \int_{-\infty}^{\infty} \{a(\xi)^{-1} J(\xi)h_-(x, \sigma)^{-t}(1, 0)\} \exp(2i\sigma y) d\sigma = 2i \sum_{j=0}^{n} R_j,
\]

where \( R_j \) is the residue at \( \xi = i\eta_j \) of

\[
a(\lambda)^{-1} J(\lambda)h_-(x, \xi) \exp(2i\xi y) \]

which is a meromorphic function in \( \xi \), \( \text{Im} \xi > 0 \), with simple poles \( i\eta_j \). We have

\[
R_j = i\xi_j \exp(-2\eta_j(x+y)) \begin{bmatrix} -\eta_j/m & 1 \\ 1 & -\eta_j/m \end{bmatrix} h_+(x, i\eta_j).
\]

Hence we have

\[
(2.1+) \quad K_+(x, y) + F_+(x+y)^t(0, 1) + \int_{0}^{\infty} F_+(x+y+z)K_+(x, z)dz = 0 \quad (y > 0),
\]

where

\[
(2.2+) \quad F_+(x) = 2 \sum_{j=0}^{n} e^{-i\eta_j/m} \begin{bmatrix} -\eta_j/m & 1 \\ 1 & -\eta_j/m \end{bmatrix} \exp(-2\eta_j x)
\]

\[
+ \pi^{-1} \int_{-\infty}^{\infty} r(\xi)J(\xi) \exp(2i\sigma x) d\sigma.
\]

Similarly we have

\[
(2.1-) \quad K_-(x, y) + F_-(x+y)^t(1, 0) + \int_{-\infty}^{0} F_-(x+y+z)K_-(x, z)dz = 0 \quad (y < 0),
\]

where
\[
F_{-}(x) = 2 \sum_{j=0}^{\infty} e^{-j} \begin{bmatrix} -\eta_j/m & 1 \\ 1 & -\eta_j/m \end{bmatrix} \exp(2\eta_jx) + \pi^{-1} \int_{-\infty}^{\infty} r(\xi)J(\xi) \exp(-2i\sigma x) d\sigma.
\]

By (1.15) and (1.16), we have
\[
r(\xi)^* = r(-\xi).
\]

This shows that \(F_{\pm}(x)\) are real matrices.

We call (2.1\(\pm\)) the fundamental integral equations.

3. Solvability of the fundamental equation

In this section we discuss the solvability of the fundamental equation (2.1) as an integral equation for \(K\).

Assuming that \(G\) is bounded integrable in \((a, \infty)\) for any \(a\), put
\[
(T_{G,x}f)(y) = \int_{0}^{\infty} G(x+y+z)f(z)dz
\]
for \(f \in L^1(0, \infty)\). Then \(T_{G,x}\) is a completely continuous operator as an operator on \(L^1(0, \infty)\).

We have

**Theorem 3.1.** If \(F(x)\) defined by (2.2) is bounded integrable in \((a, \infty)\) for any \(a\), then \(I+T_{F^*x}\) has the bounded inverse for any \(x\), where \(I\) is the identity.

**Proof.** Suppose \(\phi\) is a solution of
\[
(I+T_{F^*x})\phi = 0
\]
in \(L^1(0, \infty)\). By the boundedness of \(F\), that of \(\phi\) follows. So \(\phi\) belongs to \(L^2(0, \infty)\). Put
\[
h(\xi) = \langle h_1(\xi), h_2(\xi) \rangle = \int_{0}^{\infty} \phi(x) \exp(2i\xi x)dx, \quad \text{Im} \xi > 0,
\]
\[
X(\xi) = \langle h_1(\xi), h_2(\xi), h_2^\ast(\xi), h_1^\ast(\xi) \rangle,
\]
\[
R(x, \sigma) = r(\xi)J(\xi)^\ast \exp(2i\sigma x),
\]
\[
H(x, \sigma) = \begin{bmatrix} E & R(x, \sigma)^* \\ R(x, \sigma) & E \end{bmatrix}
\]
and
\[
H_j(x) = 2c_j \exp(-2\eta_jx) \begin{bmatrix} 1 & -\eta_j/m \\ -\eta_j/m & 1 \end{bmatrix},
\]
where \(E\) is the unit matrix of order 2. Then we have
\[ (3.1) \quad 0 = \int_0^\infty \phi(y)^* (I + T_{\sigma^*, x}) \phi(y) dy = 2^{-1} \int_{-\infty}^\infty X(\sigma)^* H(x, \sigma) X(\sigma) d\sigma + \sum_{j=0}^n h(i\eta_j)^* H_j(x) h(i\eta_j). \]

\( H_j \) are nonnegative definite real symmetric matrices. On the other hand, the Hermitian matrix \( H \) is unitarily equivalent to the diagonal matrix

\[
\begin{pmatrix}
1 + |r(\xi)| & 0 & 0 & 0 \\
0 & 1 - |r(\xi)| & 0 & 0 \\
0 & 0 & 1 - |r(\xi)| & 0 \\
0 & 0 & 0 & 1 - |r(\xi)|
\end{pmatrix}.
\]

Hence, by (1.14), the right hand side of (3.1) contains only positive terms. Therefore we have

\[ X(\sigma)^* H(x, \sigma) X(\sigma) = 0 \]

for any \( x, \sigma \). Therefore \( h(\sigma) = 0 \) follows. This shows \( \phi(x) = 0 \). Q.E.D.

By Theorem 3.1, the operator equation

\[ (I + T_{\sigma^*, x}) \phi = \psi_x \]

is uniquely solvable for a continuous \( L^1 \)-valued function \( \psi_x \). We denote the unique solution by \( \phi_x \). Then, by Theorem 3.1, \( \phi_x \) is a continuous \( L^1 \)-valued function. Moreover we have

**Lemma 3.2.** Suppose that \( F \) is absolutely continuous and \( F, F' \) are in \( L^1(a, \infty) \) for any \( a \). Let \( \psi_x \) be continuously differentiable in \( x \) as a \( L^1 \)-valued function, then the solution \( \phi_x \) is differentiable in \( x \) and

\[ (I + T_{\sigma^*, x}) \phi' = \psi_x - T_{\sigma^*, x} \phi_x \]

holds.

A proof for this Lemma is completely parallel to [7; Lemma 4.3, pp 342-343].

Put \( \psi_x = -F(x+y)^\prime(0, 1) \) and the equation (3.2) coincides with the fundamental equation (2.1). By Theorem 3.1 and Lemma 3.2, \( K(x, y) \) is differentiable in the ordinary sense. Put

\[ (3.3) \quad \nu(x) = -K_1(x, 0) + m \]

and

\[ (3.4) \quad f(x, \lambda) = \exp (i\xi x) E(\lambda) \left\{ t(0, 1) + \int_0^\infty K(x, y) \exp (2i\xi y) dy \right\}, \]

where \( E(\lambda) \) is the matrix defined by (1.4). Then we have
Theorem 3.3. If \( F \) is absolutely continuous and \( F, F' \) are in \( L^1(a, \infty) \) for any \( a \), then \( f \) defined by (3.4) is differentiable in \( x \) and satisfies

\[
L_{ii}f = \lambda f
\]

for \( v = v(x) \) defined by (3.3).

Proof. Put

\[
J(x, y) = (K_2(x, y) - (v(x) + m)K_1(x, y)) - (v(x) - m)K_2(x, y)).
\]

Then, (3.5) holds if and only if \( J(x, y) = 0 \). We have

\[
F_1(x) = 2mF_4(x),
\]

where

\[
F(x) = \begin{bmatrix} F_1(x) & F_2(x) \\ F_3(x) & F_4(x) \end{bmatrix}.
\]

By this relation, we have

\[
J(x, y)^* + \int_0^\infty F(x + y + z)J(x, z)dx = 0.
\]

Hence, by Theorem 3.1, \( J(x, y) = 0 \) follows. Q.E.D.

4. The inverse problem

Let \( n \) be a nonnegative integer, \( \kappa_j (j = 0, 1, \cdots, n) \) be nonnegative numbers such that

\[
0 = \kappa_0 < \kappa_1 < \cdots < \kappa_n < m
\]

and \( c_j (j = 0, 1, \cdots, n) \) be positive numbers. Suppose \( r(\xi) (\xi \in R_m) \) be a function which satisfies the conditions

\[
r(-\xi) = r(\xi)^* , \quad |r(\xi)| < 1, \xi \in R_m ,
\]

\[
r(\xi) = O(\xi^{-1}) \quad (\xi \to \pm \infty).
\]

Moreover we assume that either

\[
r(\xi) \to \mp i \quad (\xi \to \pm m),
\]

or

\[
|r(\xi)| < \delta < 1, \quad \xi \in R_m .
\]

Determine \( a(\xi) \) from \( r(\xi) \) by (1.22) and (1.23) respectively. Put

\[
a(\xi) = a(\xi + i0)
\]

\[
r_+(\xi) = r(\xi), \quad r_-(\xi) = -a(\xi)^{-1}a(-\xi)r_+(\xi -)
\]
and define $c_j$ from $c_j = c_j$ according to (1.16).

Put

$$F_{\pm}(x) = 2 \sum_{j \in \mathbb{Z}} c\left[\begin{array}{ll} -\eta_j/m \\ 1 \\ -\eta_j/m \end{array}\right] \exp\left(\mp 2\eta_j x\right) + \pi^{-1} \int_{-\infty}^{\infty} r_{\pm}(\xi) J(\xi) \exp\left(\pm 2i\sigma x\right) d\sigma.$$  

We assume that $F_{\pm}(x)$ are absolutely continuous and $F_{\pm}(\pm x), F'_{\pm}(\pm x)$ belong to $L^1(a, \infty)$ for any $a$.

Let $K_{\pm}(x, y)$ be the unique solutions of the fundamental equations (2.1±) whose kernels $F_{\pm}$ are defined above.

Put

$$v_{+}(x) = -K_{+}(x, 0) + m$$

and

$$v_{-}(x) = K_{-}(x, 0) - m.$$  

By Theorem 3.3,

$$f_{+}(x, \lambda) = \exp\left(i\xi x\right) E(\lambda) \left\{ t(0, 1) + \int_0^\infty K_{+}(x, y) \exp\left(2i\sigma y\right) dy \right\}$$

and

$$f_{-}(x, \lambda) = \exp\left(-i\xi x\right) E(\lambda) \left\{ t(1, 0) + \int_{-\infty}^0 K_{-}(x, y) \exp\left(-2i\sigma y\right) dy \right\}$$

satisfy (1.1) for $v = v_{\pm}$ respectively.

Next we show that $v_{\pm}(x)$ coincide. This follows immediately, once the equality

$$a(\xi)^{-1} f_{-}(x, \xi) = f_{-}(x, \xi) + r_{-}(\xi) f_{+}(x, \xi), \quad \xi \in \mathbb{R},$$

is established, where

$$f_{\pm}(x, \xi) = f_{\pm}(x, \xi+i0), \quad \xi \in \mathbb{R}_m.$$  

Put

$$g(x, \sigma) = h_{\pm}(x, \sigma) + \exp(2i\sigma x) r_{\pm}(\xi) J(\xi) h_{\pm}(x, \sigma)$$

and

$$G(x, y) = \pi^{-1} \int_{-\infty}^{\infty} \{ g(x, \sigma) - t(1, 0) \} \exp\left(2i\sigma y\right) d\sigma ,$$

where

$$h_{\pm}(x, \sigma) = t(1, 0) + \int_0^\infty K_{\pm}(x, y) \exp(2i\sigma y) dy.$$  

Then we have

$$G(x, y) = K_{+}(x, y) + F_{+}^0(x+y) t(0, 1) + \int_0^\infty F_{+}^0(x+y+z) K_{+}(x, z) dz ,$$
where

\[ F^\alpha_0(x) = \pi^{-1} \int_{-\infty}^{\infty} r_\alpha(\xi) J(\xi) \exp(2i\sigma x) d\sigma. \]

**Lemma 4.1.** The function \( g(x, \sigma) \) can be extended to the domain, \( \text{Im} \xi > 0 \), as a meromorphic function \( g(x, \zeta) \) whose poles are simple and exhausted by \( i\eta_j \) (\( j = 0, 1, 2, \ldots, n \)).

**Proof.** Putting

\[ q_j(x, \zeta) = -ic_j(\zeta - i\eta_j)^{-1} \left[ \frac{\zeta}{\text{im} \sigma} \right] \left[ \frac{-1}{\eta_j/\sigma} \right] \exp(2i\sigma x) \left\{ \frac{\eta_i(m)}{\eta_j} \right\} i(0, 1) \]

\[ + \int_0^\infty K_+(x, \sigma) \exp(2i\sigma x) d\sigma. \]

and

\[ g_j(x, \sigma) = g(x, \sigma) - i(0, 1) - \sum_{j=0}^n q_j(x, \sigma), \quad \sigma \in \mathbb{R}. \]

We have

\[ \pi^{-1} \int_{-\infty}^{\infty} q_j(x, \sigma) \exp(2i\sigma y) d\sigma \]

\[ = 2c_j \exp(-2\eta_j(x+y)) \left[ \frac{\eta_i/m}{\eta_j} \right] \left[ \frac{-1}{\eta_j/\sigma} \right] i(0, 1) + \int_0^\infty K_+(x, \sigma) \exp(2i\sigma \sigma) d\sigma. \]

By the fundamental equation,

\[ G(x, y) = \pi^{-1} \sum_j \int_{-\infty}^{\infty} q_j(x, \sigma) \exp(2i\sigma y) d\sigma, \quad (x+y, y > 0), \]

follows. Therefore, we have

\[ (4.2) \quad \int_{-\infty}^{\infty} g_j(x, \sigma) \exp(2i\sigma y) d\sigma = 0, \quad (x+y, y > 0). \]

So, \( g_j(x, \sigma) \) can be extended to the analytic function \( g_j(x, \zeta), \text{Im} \zeta > 0 \). Q.E.D.

Put

\[ J(\lambda) = \lambda^{-1} \left[ \begin{array}{cc} \zeta & -im \\ -im & \zeta \end{array} \right], \quad \lambda \in \mathbb{R}, \]

\[ h(x, \zeta) = a(\lambda) J(\lambda)^{-1} g(x, \zeta) \]

and

\[ f(x, \lambda) = \exp(-i\zeta x) J(\lambda) h(x, \zeta). \]

By Lemma 4.1, \( f(x, \lambda) \) is holomorphic in \( \lambda \in \mathbb{R} \).

We have

**Theorem 4.2.** The function \( h(x, \zeta) \) defined by (4.3) is represented as
(4.4) \[ h(x, \zeta) = \mathcal{I}(0, 1) + \int_{-\infty}^{0} K(x, y) \exp(-2i\zeta y) dy, \]

where \( K(x, y) \) is the unique solution of the fundamental equation (2.1-).

Proof. By the absolute continuity of \( F \) and the integrability of \( F' \), the existence and integrability of \( K_{+, y}(x, y) \) follows. Hence \( \sigma g_+(x, \sigma) \) is bounded as a function of \( \sigma \). By (4.2), we can apply the Phragmén-Lindelöf type argument (see [6;pl68, problem 32]) and conclude that \( \zeta g_{\pm}(x, \zeta) \) is bounded in the domain \( \text{Im} \, \zeta > 0 \) for \( x > 0 \). This implies that as \(|\zeta| \to \infty (\text{Im} \, \zeta \geq 0)\)

\[ h(x, \zeta) - \mathcal{I}(1, 0) \to 0, \]

where convergence is uniform. Hence we have

\[ \int_{-\infty}^{\infty} \{h(x, \sigma) - \mathcal{I}(1, 0)\} \exp(2i\sigma y) d\sigma = 0, \quad (y > 0). \]

Therefore, the representation (4.4) holds.

By direct calculation, we have

\[ a^{-1}(\xi) J(\xi) h_{+, y}(x, \sigma) = h_{+(x, \sigma)} + \exp(-2i\sigma x) r_{-(\xi)} J(\xi) h(x, \sigma), \]

Hence the kernel \( K(x, y) \) satisfies the fundamental equation (2.1-). Q.E.D.

By this Theorem, the equality \( K(x, y) = K_{+, y}(x, y) \)

follows. This shows that

\[ f(x, \lambda) = f_{-}(x, \lambda), \quad x > 0. \]

So we have shown the fulfillment of the equality (4.1). Therefore \( v_{\pm}(x) \) coincide for \( x > 0 \).

From the fundamental equation, the estimates

\[ |K_{\pm}(x, y)| < C_{\pm} \sup_{\pm2\pi \pm(x+y)} |F_{\pm}(x)| \]

follows. Hence, we have finally

**Theorem 4.3.** Let \( r(\xi) \) satisfy the conditions formulated at the beginning of this section and also we assume that \( m_{\pm}(\pm x) \) belong to \( L^1(a, \infty) \) for any \( a \), where

\[ m_{\pm}(x) = \sup_{\pm2\pi \pm x} |F_{\pm}(x)|. \]

Then

\[ \{r_{\pm}(\xi), c_{\pm}, \kappa, j = 0, 1, \ldots, n\} \]

are the scattering data of \( L_{iv} \).
For the application of this result to the construction of the solution of the modified KdV equation (0.1), we need the relation between the smoothness of the potential \( v \) and that of the reflection coefficient \( r(\xi) \).

Let \( S \) be the space of \( C^\infty \)-functions which are rapidly decreasing together with all their derivatives and \( D_m \) be the set of \( C^m \)-functions which tend to \( \pm m \) as \( x \to \pm \infty \) and whose derivatives belong to \( S \).

We have

**Lemma 4.4.** Suppose that the potential \( v \) is \( n \)-times continuously differentiable function with integrable derivatives. Then \( K_{\pm 1}^{(j,k)}(x, y) = (\partial/\partial x)^j (\partial/\partial y)^k K_+(x, y) \) exist for \( j, k; 1 \leq j+k \leq n \) and the estimates

\[
|K_{\pm 1}^{(j,k)}(x, y) + v^{(j+k)}(x+y)| + |K_{\pm 2}^{(j,k)}(x, y)| \leq C_+ \sigma_+(x+y)
\]

hold.

The proof of this Lemma is completely parallel to that of [7; Lemma 1.3, p 334].

Next we have

**Theorem 4.6.** The potential \( v \) belongs to \( D_m \) if and only if \( \xi^{-1} r(\xi) \) belongs to \( S \) as the function of a variable \( \sigma \).

**Proof.** If we express \( \alpha(x) \) and \( \beta(x) \) defined by (1.15) and (1.16) in terms of \( K_{\pm 1} \), by calculating the Wronskians in (1.8) and (1.9), then, by Lemma 4.4, \( \alpha(x) \) and \( \beta(x) \) are infinitely differentiable except at \( x=0 \) and rapidly decreasing together with all derivatives.

By (2.1), we have

\[
h_+(x, \sigma) = a(\xi) J(\xi) h_+(x, \sigma) + b(\xi) h_+(x, \sigma) \exp (2i\sigma x).
\]

Multiply \( \pi^{-1} \exp (2i\sigma y)(-|x| < y < 0) \) on the second component of the above relation, integrate over \((-\infty, \infty)\) with respect to \( \sigma \), differentiate with respect to \( y \) and let \( y \uparrow 0 \). Then we have an explicit representation for \( \beta(x) \)

\[
\beta(x) = \nu'(x) - (\nu(x) - m) \int_{-\infty}^{x} (\nu^2(z) - m^2) dz + 2m \int_{x}^{\infty} (\nu^2(z) - m^2) dz
\]

\[
+ \int_{0}^{x} \alpha'(z) K_{\pm 1}(x, z) + (2m \alpha(z) - \beta(x+z)) K_{\pm 2}(x, z) dz.
\]

Hence \( \beta(x) \) is infinitely differentiable even at \( x=0 \), i.e, \( \beta(x) \) belongs to \( S \).

Next we assume

\[
1 + \int_{0}^{\infty} \alpha(x) dx \neq 0.
\]

Then, by Lemma 4.4, \((2i\sigma \xi{\infty}(\xi))^{-1}\) is a \( C^\infty \)-function of \( \sigma \). As mentioned
above, \(2i\sigma b(\xi)\) belongs to \(S\). Hence
\[
\xi^{-r}(\xi) = 2i\sigma b(\xi)/2i\sigma \xi a(\xi)
\]
belongs to \(S\).

On the other hand if we assume
\[
1 + \int_0^\infty \alpha(x)dx = 0,
\]
then we have
\[
\int_{-\infty}^\infty \beta(x)dx = 0.
\]
This implies that there exists \(\gamma(x) \in S\) such that
\[
\gamma'(x) = \beta(x).
\]
This shows
\[
b(\xi) = \int_{-\infty}^\infty \gamma(x) \exp (-2i\sigma x) dx.
\]
The condition (4.5) implies that \((\xi a(\xi))^{-1}\) is a \(C^\infty\)-function with bounded derivatives. Therefore \(\xi^{-r}(\xi)\) belongs to \(S\).

The proof for the converse statement can be obtained by induction based on Lemma 3.2. Q.E.D.

5. Construction of the solution of the modified KdV equation

Put
\[
B_{s(t)} = -4D^3 + 3 \begin{bmatrix} v^2 & v_s \\ v_s & v_x \end{bmatrix} D + 3D \begin{bmatrix} v^2 & v_x \\ v_x & v_x \end{bmatrix}.
\]
Then, by direct calculation, the modified KdV equation (0.1) is equivalent to
\[
dL_{s(t)}/dt = [B_s(t), L_{s(t)}] = B_{s(t)}L_{s(t)} - L_{s(t)}B_{s(t)}.
\]
Let \(v = v(t) = v(x, t)\) be a smooth solution of (0.1). Suppose
\[
L_{s(t)} f_\pm = \lambda f_\pm.
\]
Differentiate this with respect to \(t\), then, by (5.1),
\[
df_\pm/\left.dt - B_{s(t)} f_\pm
\]
satisfy the differential equation (5.2). Hence if \(v\) belongs to \(D_m\) for each \(t\), then, by the asymptotic property and the uniqueness of the Jost solution, we have
\[
df_\pm /\left.dt - B_{s(t)} f_\pm = (\mp 4i\xi^2 \mp 6i\xi m^2) f_\pm.
\]
Differentiating (1.8) with respect to $t$ and eliminating $df_{\pm}/dt$ by (5.3), we have
\[
d a/dt f_{\pm} + \{db_{\pm}/dt + (8i\sigma^3 + 12m^2i\sigma) b_{\pm}\} f_{\pm} = 0.
\]
So we have
\[
a(\xi, t) = a(\xi, 0)
\]and
\[
b_{\pm}(\xi, t) = b_{\pm}(\xi, 0) \exp \{\pm (8i\sigma^3 + 12m^2i\sigma)t\}.
\]
Hence $a(\lambda, t)$ is independent of $t$ and so are its zeros $\pm \kappa_j (j = 0, 1, \ldots, n)$. Similarly we have
\[
c_{\pm}(t) = c_{\pm}(0) \exp \{\pm (8\eta^3 - 12m^2\eta)t\}.
\]
Conversely, suppose that
\[
\{r_{\pm}(\xi), c_{\pm}, \kappa_j, j = 0, 1, \ldots, n\}
\]are the scattering data of the operator $L_{\nu^7}, \nu \in D_m$. Define $r_{\pm}(\xi, t) = b_{\pm}(\xi, t)/a(\xi)$ and $c_{\pm}(t)$ by (5.4) and (5.5). Put
\[
F_{\pm}(x, t) = 2 \sum_{j=0}^{n} c_{\pm}(t) \begin{bmatrix} -\eta_j/m & 1 \\ 1 & -\eta_j/m \end{bmatrix} \exp (\mp 2\eta_j x) + \pi^{-1} \int_{-\infty}^{\infty} r_{\pm}(\xi, t) J(\xi) \exp (\pm 2i\sigma x) d\sigma.
\]
Then, by Theorem 3.1, the fundamental equations (2.1±) with the kernels $F_{\pm}(x, t)$ are uniquely solvable. We denote the solutions by $K_{\pm}(x, y, t)$. Put
\[
v_{\pm}(x, t) = -K_{\pm}(x, 0, t) + m
\]
As $r(\pm m, t) = r(\pm m)$, the condition required to show $v_{\pm}(x, t) = v_{\pm}(x, t)$ is clearly satisfied. Thus, by Theorem 4.3 and 4.5, we have

**Theorem 5.1.** If $\nu(x)$ belongs to $D_m$, then there exists the unique potential $\nu(x, t) \in D_m$ whose scattering data is
\[
\{r_{\pm}(\xi, t), c_{\pm}(t), \kappa_j, j = 0, 1, \ldots, n\}
\]for each $t$.

We have finally

**Theorem 5.2.** The potential $\nu(x, t)$ defined by (5.6) satisfies the modified KdV equation (0.1).
Proof. It is sufficient to show that the relation (5.3) holds. In fact, differentiate (5.2) with respect to $t$ and eliminate $df_{j}/dt$ by (5.3). Then we have

$$(dL_{ij}(t)/dt - [B_{j}(t), L_{ij}(t)])f = 0.$$ 

By direct calculation, the relation (5.3) is equivalent to

$$(5.7) dh_{j}/dt = g_{\pm},$$

where

$$h_{+}(x, \zeta, t) = \int_{0}^{\infty} K_{+}(x, y, t) \exp(2i\zeta y) dy,$$

$$h_{-}(x, \zeta, t) = \int_{0}^{\infty} K_{-}(x, y, t) \exp(-2i\zeta y) dy$$

and

$$g_{\pm}(x, \zeta, t) = 12\zeta^{2}h_{\pm 2} + 12i\zeta h_{\pm 3} - 4h_{\pm 4} + 6 \left[ \begin{array}{cc} v_{x}^{2} & v_{x} \\ v_{x} & v_{x}^{2} \end{array} \right] (\pm i\zeta h_{\pm 0} + h_{\pm 1}) + 3 \left[ \begin{array}{cc} v_{xx} & v_{x} \\ v_{xx} & 2v_{xx} \end{array} \right] h_{\pm 1} \mp 6i\zeta m h_{\pm 0}.$$ 

Substitute (5.8) into this and integrate by part. Then we have

$$g_{+}(x, \zeta, t) = \int_{0}^{\infty} J(x, y, t) \exp(2i\zeta y) dy,$$

where

$$J(x, y, t) = -K_{+xxx} + 3 \left[ \begin{array}{cc} v_{x}^{2} + m^{2} & v_{x} \\ v_{x} & v_{x}^{2} + m^{2} \end{array} \right] K_{+x}.$$ 

As $F(x, y)$ is differentiable with respect to $t$, so is $K_{+}$. The relation

$$F_{t} + F_{xxx} - 6m^{2}F_{x} = 0$$

is valid. Hence we have

$$(5.9) \quad K_{+}(x, y, t) + \int_{0}^{\infty} F(x+y+z, t)K_{+}(x, z, t) dz = D(x, y, t),$$

where

$$D(x, y, t) = \int_{0}^{\infty} (F_{xx}(x+y+z, t) - 6m^{2}F_{x}(x+y+z, t)K_{+}(x, z, t)) dz + (F_{xx}(x+y+z, t) - 6m^{2}F_{x}(x+y, t))' \int(0, 1).$$

By direct calculation, we can show that $J(x, y, t)$ satisfies (5.9). Therefore, by Theorem 3.1, $K_{+} = J$ follows. Q.E.D.

Next we discuss the reflectionless solution which can be obtained under the assumption $r(\xi) \equiv 0$. This implies
This shows that we can express the unique solution \( K(x, y) \) of the fundamental equation as

\[
K(x, y) = 2 \sum_{j=0}^{n} c_j \begin{bmatrix} -\eta_j/m & 1 \\ 1 & -\eta_j/m \end{bmatrix} f_j(x) \exp (-2\eta_j (x+y)) ,
\]

where \( f_j(x) = (f_1(x), f_2(x)) \). Substitute this into the fundamental equation (2.1), and we have the system of the \( 2(n+1) \) linear algebraic equations

\[(5.10) \quad f_j(x) + \sum_{j=0}^{n} c_j \begin{bmatrix} -\eta_j/m & 1 \\ 1 & -\eta_j/m \end{bmatrix} (\eta_j + \eta_j)^{-1} \exp (-2\eta_j f_j(x)) = -\epsilon'(1, 0), \quad (i = 0, 1, \ldots, n),
\]

whose coefficient matrix is easily seen to be nondegenerate. Let \( f_j(x) \) \((i = 1, 2 \text{ and } j=0, 1, \ldots, n)\) be the unique solutions of (5.10). Then we have the reflectionless potential

\[(5.11) \quad \psi_0(x) = 2 \sum_{j=1}^{n} c_j (m^{-1} \eta_j f_1(x) - f_2(x)) \exp (-2\eta_j x + m).
\]

Put

\[
h_{\pm j}(x) = c_j (1 \mp m^{-1} \eta_j) \exp (-\eta_j x (f_1(x) \pm f_2(x))),
\]

where \( j=1, 2, \ldots, n \) for + and \( j=0, 1, \ldots, n \) for -. Then we can rewrite the formula (5.11) as

\[(5.12) \quad \psi_0(x) = \sum_{j=1}^{n} h_{+ j}(x) \exp (-\eta_j x) - \sum_{j=0}^{n} h_{- j}(x) \exp (-\eta_j x + m).
\]

The functions \( h_{\pm j} \) satisfy the linear algebraic equations

\[
h_{\pm j}(x) + a_{\pm j} \exp (-\eta_j x) \sum_{j=0}^{n} h_{\pm j}(x) \exp (-\eta_j x) = -a_{\pm j} \exp (-\eta_j x),
\]

where \( a_{\pm j} = c_j (1 \mp m^{-1} \eta_j) \). Put

\[
A_+ = \left( a_{\pm j} \exp (-\eta_j x) (\eta_j + \eta_j)^{-1} \right)_{i, j=1, 2, \ldots, n}
\]

and

\[
A_- = \left( a_{\pm j} \exp (-\eta_j x) (\eta_j + \eta_j)^{-1} \right)_{i, j=0, 1, \ldots, n}
\]

Then \( E_k + A_+ \) and \( E_{k+1} + A_- \) are positive definite, where \( E_k \) is the unit matrix of order \( k \). (See [5; Lemma 1].)

We have

**Proposition 5.3.** The equality
holds.

Proof. By the Cramer's formula, we have
\[ h_i(x) = D_i/\det(E_n + A_+), \]
where \( D_i \) is the determinant obtained by replacing the \( i \)-th column of \( \det(E_n + A_+) \) by \((-a_{i+1} \exp(-\eta_1 x), -a_{i+2} \exp(-\eta_2 x), \ldots, -a_{i+n} \exp(-\eta_n x)) \). On the other we have

\[ d \{\log \det(E_{n+1} + A)\}/dx = \sum_{i=0}^{n} \Delta_i/\det(E_{n+1} + A), \]
where \( \Delta_i \) is the determinant obtained by replacing the \( i \)-th column of \( \det(E_{n+1} + A) \) by \((-a_{i+1} \exp(-(\eta_1 + \eta_i) x), -a_{i+2} \exp(-(\eta_2 + \eta_i) x), \ldots, -a_{i+n} \exp(-(\eta_n + \eta_i) x)) \). Hence we have

\[ \Delta_i = \exp(-\eta_i x)D_i. \]

Therefore we have

\[ d \{\log \det(E_{n+1} + A)\}/dx = \sum_{i=0}^{n} h_i(x) \exp(-\eta_i x). \]

Completely parallel to above, we have

\[ d \{\log \det(E_{n+1} + A_-)\}/dx = \sum_{i=0}^{n} h_{-i}(x) \exp(-\eta_i x). \]

Q.E.D.

If the reflectionless scattering data \( S_0 = \{0, c_j(t), \kappa_j, j=0, 1, \ldots, n\} \) depend on \( t \) as (5.5), we denote the unique solutions of (5.10) which correspond to \( S_0 \) by \( f_{ij}(x, t) \) \((i=1, 2 \text{ and } j=0, 1, \ldots, n)\). Then we have the explicit formula of the reflectionless solutions

\[ v(x, t) = 2 \sum_{j=0}^{n} c_j(m^{-1} \eta_j f_{1j}(x, t) - f_{2j}(x, t)) \exp(-2\eta_j z_j) + m, \]
where \( z_j = x - (4\eta_j^2 - 6m^2)t \).

Now suppose \( n=0 \) in (5.13), and we have

\[ v_0(x, t) = m \tanh(m(x + 2m^2 t + \delta)), \]
where \( \delta = (2m)^{-1} \log(c^{-1}m) \). Thus the reflectionless solutions (5.13) contain the traveling wave solution \( v_0(x, t) \).
References


