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On the Metric Temporal Logic for Continuous Stochastic Processes

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On the Metric Temporal Logic for Continuous Stochastic Processes

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Abstract

In this thesis, we prove the measurability of an event for which a general continuous–time stochastic process satisfies a continuous–time Metric Temporal Logic (MTL) formula. Continuous–time MTL can define temporal constraints for physical systems in natural way. Then several researches deal with the probability of continuous MTL semantics for stochastic processes. However, proving measurability for such events is by no means an obvious task, even though it is essential. The difficulty comes from the semantics of "until operator", which is defined by the logical sum of uncountably many propositions. Given the difficulty involved in proving the measurability of such an event using classical measure-theoretic methods, we employ a theorem from stochastic analysis. This theorem is utilized to prove the measurability of hitting times for stochastic processes, and it stands as a profound result within the theory of capacity. Next, we provide an example that illustrates the failure of probability approximation when discretizing the continuous semantics of MTL–formulas with respect to time. Additionally, we prove that the probability of the discretized semantics converges to that of the continuous semantics when we impose restrictions on diamond operators to prevent nesting. Furthermore, we propose a new discretization of one–dimensional stochastic differential equation using time–change method and estimate its discretization error. Then we apply the scheme to the approximation of the probability of MTL events for time–inhomogeneous SDE.

Contents

Chapter 1

Introduction

In this thesis, we prove the measurability of the event that the sample path satisfies the conditions defined by metric temporal logic (MTL) and consider an approximation of the probability of such events. Specifically, we prove the measurability of any event defined by the MTL–formula. Additionally, we investigate methods for approximating the probabilities of these events and demonstrated, through counterexamples, that the approach proposed in previous studies generally does not converge to the true probability. Conversely, by imposing stronger constraints than previous studies, we provide proof that similar approximations converge to the true probability.

MTL is a fragment of temporal logic. Temporal logic is a set of rules for combining conditions on sample paths. Defining rules for combining conditions allows for the inductive definition of a set of propositions following those rules. A distinctive feature of temporal logic is its ability to create conditions that depend on the progression of time, and it can generate arbitrarily complex conditions. While Boolean algebra makes similar rules, Boolean algebra permits only logical negation and disjunction, and conditions depending on the progression of time cannot be formulated. Temporal logic can be seen as an extension of Boolean algebra in a sense, as it allows the combination of conditions depending on the progression of time with Boolean algebraic operations. Another characteristic of temporal logic is that these complex propositions can be expressed using simple symbols.

Such specification of time–dependent properties are considered for *model checking* [BK08]. Model checking is a technique to check automatically whether an automated system works correctly. It was first proposed in the 1980s by European [QS82] and American [CE82] researchers independently, to attempt to assess the correctness of computer programs automatically without a large amount of human ingenuity. Nowadays model checking is being extended to real–time probabilistic system validation for unmanned aerial vehicle (UAV), biology, and motion planning for robotics, etc (see [PHLS00, JP14, WRW$^+$16, JL12, FT15]). Typical questions treated by model checking are:

- Safety: Can a system avoid an unsafe event permanently?

- Liveness: Will a system eventually reach a required event?
- Fairness: Will a repetitive attempt to carry out a task be eventually granted?

All of these properties can be represented by temporal logic.

In particular, Metric Temporal Logic (MTL) is applied in fields such as robotics and biochemistry, utilizing the ”until operator” symbol. For instance, in a scenario where multiple drones are autonomously delivering goods, a proposition like ”Complete delivery to point A within 2 hours of dispatch, followed by delivery to point B within 1 hour, without colliding with other drones in the meantime” can be defined using MTL symbols. In financial engineering, MTL can interpret parts of derivative payoffs. For knock-in options, the payoff occurs if the asset price reaches a certain level by the expiration. The proposition ”The asset price reaches a specified level at some time before expiration” can be expressed using MTL symbols.

In this thesis, we investigate events where the sample path of a continuous–time stochastic process satisfies continuous–time Metric Temporal Logic (MTL, see [GR21]). MTL has variations such as continuous–time MTL and discrete–time MTL depending on how it is defined. When defining conditions for sample paths, using the same MTL symbols can have different meanings depending on whether it is defined in continuous or discrete time. Discrete–time MTL considers only states at pre–defined discrete time points and defines conditions based on them. On the other hand, continuous-time MTL has the advantage of considering states continuously over real–time [FR10].

In the sample path of a continuous–time stochastic process, states corresponding to all real–time points are determined. Therefore, even if conditions are defined at discrete time points using discrete-time MTL, it is generally impossible to consider states between those time points, potentially missing important events occurring at some continuous time. Applying continuous–time MTL to sample paths allows for considering important events that may occur between discrete time points.

The problem of applying continuous–time MTL to continuous-time stochastic processes and calculating probabilities is studied in the field of cyber–physical systems [CL07]. Cyber–physical systems involve close interactions between computers and physical elements, with examples including autonomous vehicle systems, medical monitoring, industrial control systems, robotics systems, recycling, and autopilot aircraft. These systems often experience influences from turbulence, molecular–level noise, or human uncertainty. Previous research has attempted to define complex conditions using continuous–time MTL for such systems and calculate the probabilities of satisfying those conditions.

Now, a nontrivial problem that arises is the measurability of events satisfying conditions defined by MTL. When conditions are defined using MTL for sample paths, ensuring the measurability of events satisfying those conditions is essential for discussing the probability of satisfying the conditions. Probability is defined only for events with measurability, and the analysis of probability variables and processes begins by ensuring measurability. However, applying continuous–time MTL to sample paths does not immediately lead to measurability from conventional results in probability theory. The reason is that the ”until operator” in continuous–time MTL combines an uncountable number of conditions to define new conditions. Generally, the measurability of events is guaranteed only for countable combinations, making it challenging to immediately deduce the measurability of events expressed by MTL.

In this thesis, we apply a theory called Capacity Theory [Del72] to prove the measurability of any event defined by Metric Temporal Logic (MTL). Capacity Theory ensures measurability under projections. Initially, we show that the until operator in MTL can be represented by a projection. Then, by applying the results of Capacity Theory to the representation of the until operator, we inductively prove the measurability of any event defined by MTL. While obtaining the results of measurability, we impose certain assumptions on the probability process, which are commonly applicable in the theory of stochastic analysis. Therefore, our results resolve the foundational non–triviality in discussions, including previous research.

With this result, it is ensured that probabilities can be defined within the conventional framework. However, the problem of actually calculating these probabilities remained.

Therefore, we tackle the problem of approximating the probabilities of continuous-time MTL using a method similar to previous research. Previous studies addressed the problem of approximating the probabilities of continuous–time MTL by defining discrete–time MTL corresponding to continuous-time MTL. Since the probability of a sample path satisfying conditions defined by MTL cannot be analytically determined without a priori knowledge of the sample path itself, previous research defined similar conditions in discrete–time MTL for each condition in continuous–time MTL. Adopting this definition allows for using the same symbols in continuous–time MTL and discrete–time MTL to define conditions with similar meanings. Thus, in discrete–time MTL, it was expected that as the time intervals became smaller, approaching zero, the conditions would converge to those expressed by continuous-time MTL. In other words, it was speculated that by approaching zero in the discretization of time, discrete-time MTL would converge to continuous–time MTL, and attempts were made to approximate the probabilities defined by continuous–time MTL.

However, in this thesis, we show that this approximation generally does not converge to the true probabilities. In previous research, the approach involved narrowing the time interval of discrete–time MTL to zero while simultaneously approximating the continuous–time probability process. Previous research cited the fact of distribution convergence as the basis for the approximation, but it is not immediately evident that probability convergence follows from distribution convergence. Using specific counterexamples, we show that the probability defined by discrete–time MTL does not converge to the corresponding probability defined by continuous–time MTL.

On the other hand, we prove that when a continuous-time stochastic process is modeled by a stochastic differential equation (SDE), restricting the symbols representing Metric Temporal Logic (MTL) converges to the true probability. Without restrictions on the symbols representing MTL, the until operator can be nested in multiple layers, leading to complex conditions. The counterexamples we provide in this thesis involve defining intricate conditions by nesting the until operator in 2 or 3 layers. We explore the approximation of probabilities when the until operator is not nested in more than 2 layers and specifically apply MTL to sample paths of continuous–time stochastic processes described by SDEs.

In Chapter 2, we resolve foundational discussions in previous studies and demonstrate cases where conventional methods are not applicable. Furthermore, we show that under certain limitations, conventional methods can be applied. This result indicates the need for a more rigorous investigation of conventional findings and discussions about how far the conditions can be extended. More precisely, we obtain the following results:

(i) We identify a counterexample of an MTL–formula such that the probability derived from the discrete semantics, referenced for the exact solution of the SDE, does not converge to the probability derived from the continuous semantics for the exact solution.

(ii) When we restrict the syntax of the MTL formula such that the until operator does not nest (referred to as $\flat$MTL–formula), the probability for the exact solution of the SDE, based on discrete semantics, converges to the probability based on continuous semantics for the exact solution of the SDE.

(iii) Under appropriate conditions on the SDE, we show the convergence of the probability that a locally uniform approximation of the solution satisfies the discrete semantics to the probability of continuous semantics for the exact solution.

In addition to these results, we propose a new discretization of stochastic differential equations in Chapter 3. Specifically, we present a new discretization method for stochastic differential equations without drift terms and prove its strong convergence to the true weak solution. To calculate the probabilities satisfying MTL, it is necessary not only to discretize MTL but also to discretize the solutions of stochastic differential equations. This is because the solutions of stochastic differential equations are generally not explicitly obtainable and need to be approximated through discretization. As a result, we obtain a strong convergence rate of the discretization which is finer than Euler's method when the diffusion coefficient is β–Hölder continuous with $\beta < 1/2$. In addition, we show that the new scheme can be applied to the approximation of the probability of $\flat$MTL in continuous semantics for the exact solution.

Chapter 2

On the metric temporal logic for continuous stochastic processes

2.1 Introduction

Stochastic processes have emerged as a valuable tool for analyzing real–time dynamics characterized by uncertainties. They consist of a family of random variables indexed in real–time and find applications in diverse domains such as molecular behavior, mathematical finance, and turbulence modeling. To formally analyze the temporal properties of real–time systems, *Metric Temporal Logic (MTL)* has been introduced as a logical framework, specifying constraints that real-time systems must satisfy (see Chapter VI in [Pri67]). The increasing demand for MTL specifications in industrial applications [Koy90, CZ11, KF08] has sparked interest in investigating the probability that a stochastic process satisfies the semantics of MTL–formulas. In contrast to discrete–time stochastic systems, which are limited to describing events within a discretized time domain, continuous–time MTL allows for the precise representation of constraints on events occurring between discrete times.

This chapter focuses on MTL–specifications on stochastic systems interpreted by the continuous–time domain. In particular, we are interested in the probability in which an MTL–formula is satisfied by a continuous–time stochastic system. However, before discussing the probability of event occurrences, it is crucial to make sure of the measurability of these events. Although previous research [FT15, JP14, MLD16] considered the probability of events in which an MTL–formula is satisfied, they did not prove the measurability of such events. The subtle problem arises in the definition of probability because temporal operators in MTL are defined by unions of uncountably many sets, while measurability is guaranteed in the case of a union of countably many sets in general.

In this chapter, we prove the measurability of events where sample paths of stochastic processes satisfy the propositions defined by MTL, under mild assumptions. We assume that the stochastic process is measurable as a mapping from the product space of the time domain and the sample space — a common assumption in stochastic analysis. Although establishing measurability for events represented by MTL–formulas with temporal operators is made challenging by the presence of the union or intersection of uncountably many sets in the definition, we overcome this difficulty by introducing the concept of reaching time for sub-formulas of the given MTL–formula. By leveraging the reaching time, we prove the measurability of the sub-formulas inductively. In the proof of measurability of MTL–formulas, we utilize the measurability of the corresponding reaching times. The measurability of such reaching times itself is non–trivial and proven using the theory of capacity (see [Del72]).

While establishing measurability shows that the probability of continuous-time MTL semantics for stochastic processes is well-defined, the problem of calculating such probabilities remains a challenge. Although the previous researches [FT15, MLD16] proposed an approximation by discretizing the semantics of MTL–formulas with respect to time, we show that probability based on discretization does not converge the probability based on continuous semantics in general. We give an example that involves multi–level nested temporal operators. This motivates the need for a more comprehensive and precise discussion of approximations, which has been generally overlooked in previous studies. As a part of such an effort, we show that if a formula only has simplified temporal operators, and these operators never appear in nested positions, the discretization converges the continuous semantics.

In summary, this chapter contributes to the understanding of the probability foundation and approximation for stochastic processes satisfying continuous-time MTL semantics. We investigate the measurability of events, provide proofs under mild assumptions, and explore the possibility of approximation by discretization with respect to time. Our findings shed light on the challenges involved and emphasize the importance of refining approximation techniques in future studies.

This chapter is organized as follows. In Section 2.2, we refer to some related works and the novelty of our results compared to previous studies. In Section 2.3, we provide a comprehensive exposition of the fundamental concepts that will be utilized extensively in this chapter. These include the definitions of measurability, probability space, stochastic process, Brownian motion, stochastic differential equation, and metric temporal logic. In Section 2.7, we prove the measurability of the path of a stochastic process satisfies the semantics of the MTL–formula in both continuous and discrete sense. In Section 2.8, we provide a counterexample that the probability that a stochastic process satisfies the discrete semantics of an MTL–formula does not converge to the probability of path satisfying the semantics of the same MTL–formula in a continuous sense. On the other hand, in Section 2.11, we prove the convergence of probability of discrete semantics for general stochastic differential equations (SDEs) under some restriction on the syntax of propositional formulas. We set the restriction so that temporal operators do not nest.

As a result, we show that the convergence result relies on the depth of nests of temporal operators in an MTL–formula.

2.2 Related Works

Temporal reasoning has been extensively studied (for an overview, see [GR21]), and it has gained increasing attention due to the growing demands in various industrial applications for real-time systems.

Pnueli [Ami77] introduced linear temporal logic (LTL) as a means to express qualitative timing properties of real-time systems using temporal operators. Koyman [Koy90] extended this logic to include quantitative specifications by indexing the temporal operators with intervals of time, leading to the development of metric temporal logic (MTL). Unlike other extensions of LTL with timing constraints, such as timed propositional temporal logic (TPTL) [AH89], MTL does not allow explicit reference to clocks, making it practical for implementation. A more detailed survey of temporal logic for real-time systems can be found in [Kon13].

In this chapter, we focus on MTL for a continuous–time stochastic process with a continuous state space. Such processes are commonly used as probabilistic models to describe various phenomena with continuous or intermittent effects caused by environmental noise. In particular, the process represented by the *stochastic differential equation (SDE)* is widely used of model statistical dynamics, asset prices in mathematical finance [Shr05, Shr04], computer network [ARA00] and future position of aircraft [JPS03], to named a few [KF08, JP14, PAM17].

Considering the wide range of applications, it is natural to consider the probability in which the given stochastic system satisfies properties defined by MTL–formulas. The previous studies [FT15, MLD16] already considered the probabilities in which stochastic systems satisfy MTL properties and gave an approximation based on the discretization of time and state spaces.

However, to talk probabilities consistently, we need to show the measurability of events under consideration. The subtle problem arises in the definition of probability because temporal operators in MTL are defined by unions of uncountably many sets, while measurability is guaranteed for the unions of countably many sets in general. The previous studies did not prove but simply assumed the measurability of events in which stochastic processes satisfy the MTL–formula. Further, their approximation by discretization assumed that the timed behavior of stochastic processes satisfies Non–Zenoness, which means that the behavior does not change its value infinitely in finite time. However, stochastic processes, such as solutions of SDEs, generally do not satisfy Non–Zenoness assumption, because of the inherent "rough" properties of stochastic processes, as they are neither smooth nor differentiable everywhere (see, for example, Chapter 2 in [KS91]).

In this chapter, we prove the measurability of events in which stochastic systems satisfy MTL–formulas interpreted by the continuous–time domain, under mild assumptions, with reference to the fundamental theory of stochastic analysis which is developed to study the approximation of probability measures by describing the structure of classes of sets [Del72, Kec95]. Our result guarantees the existence of probability in which stochastic systems satisfy MTL–formulas.

Further, we give examples in which probabilities defined by discretization do not converge to probabilities defined in the continuous–time domain, even if the time interval used for discretization goes to 0. Our examples show that the approximation by discretization, proposed by previous studies, does not work in general. Our examples involve either the until operator, or triple nesting "always" and "possibly".

On the positive side, we show that if MTL–formulas only have "always" and "possibly" operators and they do not nest, discretization of time converges to probability defined in the continuous time domain.

2.3 Preliminaries

In this section, we introduce several fundamental concepts that are discussed throughout this chapter. To begin, let us start with the definition of measurability and probability space. When defining an event, it is crucial to ensure that the event is measurable to give meaning to its probability. Once the probability space is defined, we proceed to define the product space of two probability spaces. Next, we define a general stochastic process and its path. Following that, we introduce the definition of Brownian motion and stochastic differential equation. These two concepts are fundamental to stochastic analysis and form its core. Lastly, we introduce the syntax and semantics of MTL–formulas, which are defined for every path of a stochastic process.

2.4 Measurability and Probability

In this subsection, we introduce the basic definitions used in the measure theory and probability. Readers who are familiar with these theories may skip this subsection. More details are available in [Rud66].

Definition 1 (σ–algebra and Measurable space)**.** *Let Ω be a set and $\mathcal{F}$ be a family of subsets of Ω, i.e., $\mathcal{F} \subset 2^{\Omega}$. $\mathcal{F}$ is called σ–algebra if it satisfies the following three conditions:*

(i) $\Omega \in \mathcal{F}$ *and* $\emptyset \in \mathcal{F}$.

(ii) *If* $A \in \mathcal{F}$, *then* $\Omega \setminus A \in \Omega$.

(iii) *If* $A_n \in \mathcal{F}$ *for* $i = 1, 2, 3, \cdots$, *then* $\bigcup_{i=1}^{\infty} A_n \in \mathcal{F}$ *and* $\bigcap_{i=1}^{\infty} A_n \in \mathcal{F}$

If $\mathcal{F}$ is σ–algebra, $(\Omega, \mathcal{F})$ is called measurable space*. If $(\Omega, \mathcal{F})$ is a measurable space and $A \in F$, we say that A is $\mathcal{F}$–*measurable *or merely* measurable*.*

Definition 2 (Borel space)**.** *Let E be a topological space. A measurable space is called* Borel space *on E, denoted by $(E, \mathcal{B}(E))$, if $\mathcal{B}(E)$ is the smallest σ–algebra which contains every open set. Every set belonging to $\mathcal{B}(E)$ is called* Borel set*.*

Definition 3 (Measure space and probability space)**.** *Let* $(\Omega, \mathcal{F})$ *be a measurable space. A function* $\mu : \mathcal{F} \to [0, \infty]$ *is called a* measure *on* $(\Omega, \mathcal{F})$ *if* μ *satisfies the following two conditions:*

(i) $\mu(\emptyset) = 0$.

(ii) If $A_i \in \mathcal{F}$ *for* $i = 1, 2, 3, \cdots$ *and* $A_i \cap A_j = \emptyset$ *for* $j \neq i$*, then*

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

We call the triplets $(\Omega, \mathcal{F}, \mathbb{P})$ measure space*.*

Especially, if $\mathbb{P}(\Omega) = 1$*, we refer to a measure* $\mathbb{P}$ *on* $(\Omega, \mathcal{F})$ *as a* probability*, and call* $(\Omega, \mathcal{F}, \mathbb{P})$ *a* probability space*.*

Definition 4 (Lebesgue measure)**.** *Let* $E = [0, \infty)$ *or* $\mathbb{R}^n$ *with the natural topology and* $(E, \mathcal{B}(E))$ *be Borel space on* E*. It is well known that there exists a unique measure* μ *on* E *such that*

$$\mu(\prod_{i=1}^{n} \langle a_i, b_i \rangle) = \prod_{i=1}^{n} (b_i - a_i), \tag{2.1}$$

for every rectangle $\prod_{i=1}^{n} \langle a_i, b_i \rangle$ *on* E*. We call such a measure* Lebesgue measure*.*

Definition 5 (Complete probability space)**.** *A probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is said to be* complete *if every subset* G *of a measurable set* F *such that* $\mathbb{P}(F) = 0$ *is also measurable.*

Definition 6 (Almost)**.** *(i) Let* $(X, \mathcal{A}, \mu)$ *be a measure space, and let* $P(x)$ *be a proposition defined on* $x \in X$*. We say that* $P(x)$ *holds for* almost all $x \in X$ *if there exists a measurable set* $N \in \mathcal{A}$ *such that* $\mu(N) = 0$ *and* $N^C \subset \{x \in X; P(x)\}$*.*

(ii) Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, and let* $P(\omega)$ *be a proposition defined on* $\omega \in \Omega$*. We say that* $P(\omega)$ *holds* almost surely *if there exists a measurable set* $N \in \mathcal{F}$ *such that* $\mathbb{P}(N) = 0$ *and* $N^C \subset \{\omega \in \Omega; P(\omega)\}$*.*

In the context of probability theory, the phrase "almost surely" is often denoted as "a.s.". Hence, we frequently use the notation $P(\omega)$*, a.s. to indicate that* $P(\omega)$ *holds almost surely.*

Remark 1. *Whether* $P(\omega)$ *holds almost surely depends on the probability measure* $\mathbb{P}$*. When we specially focus on the probability measure* $\mathbb{P}$*, we express that "*$P(\omega)$ holds almost surely $\mathbb{P}$*" or "*$P(\omega)$ holds a.s. $\mathbb{P}$*".*

Remark 2. *Let* $P_1(\omega)$ *and* $P_2(\omega)$ *be two proposition defined on* $\omega \in \Omega$*. If* $P_1(\omega)$ *holds almost surely and* $P_2(\omega)$ *holds almost surely, then both* $P_1(\omega)$ *and* $P_2(\omega)$ *hold almost surely. To see this, let* $\Omega_1 \in \mathcal{F}$ *and* $\Omega_2 \in \mathcal{F}$ *and suppose that* $\Omega_1 \subset \{\omega \in \Omega; P_1(\omega)\}$, $\Omega_2 \subset$

$\{\omega \in \Omega; P_2(\omega)\}$, and $\mathbb{P}(\Omega_1) = \mathbb{P}(\Omega_2) = 1$. Then $\Omega_1 \cap \Omega_2 \subset \{\omega \in \Omega; P_1(\omega)$ and $P_2(\omega)\}$ and

$$\mathbb{P}((\Omega_1 \cap \Omega_2)^C) = \mathbb{P}(\Omega_1^C \cup \Omega_2^C) = \mathbb{P}(\Omega_1^C) + \mathbb{P}(\Omega_2^C) - \mathbb{P}(\Omega_1^C \cap \Omega_2^C) \leq \mathbb{P}(\Omega_1^C) + \mathbb{P}(\Omega_2^C) = 0.$$

Therefore, $\mathbb{P}(\Omega_1 \cap \Omega_2) = 1$.

By repeating similar arguments, we can see the following statement: if $P_i(\omega)$ holds almost surely for $i = 1, \cdots, n$, then $P_1(\omega)\wedge, \cdots, \wedge P_n(\omega)$ holds almost surely.

Definition 7 (Product measurable space)**.** *Let $(G, \mathcal{G})$ and $(H, \mathcal{H})$ be two measurable spaces. The product σ–algebra of $\mathcal{G}$ and $\mathcal{H}$, denoted $\mathcal{G} \otimes \mathcal{H}$, is the smallest σ–algebra on $G \times H$ which contains all set of the form $A \times B$, where $A \in \mathcal{G}$ and $B \in \mathcal{H}$. The resulting measurable space $(G \times H, \mathcal{G} \otimes \mathcal{H})$ is called the product measurable space of $(G, \mathcal{G})$ and $(H, \mathcal{H})$.*

Fact 1. *Let $(G, \mathcal{G})$ and $(H, \mathcal{H})$ me two measurable space, $x \in G$, $y \in H$, and $E \in \mathcal{G} \otimes \mathcal{H}$. Then the following two measurability holds:*

$$\{y \in H; (x, y) \in E\} \in \mathcal{H},$$
$$\{x \in G; (x, y) \in E\} \in \mathcal{G}.$$

Definition 8 (Product measure space)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $E = [0, \infty)$ or $\mathbb{R}^n$ with the natural topology, and $(E, \mathcal{B}(E), \mu)$ be the measure space with the Lebesgue measure μ defined in Definition 4. It is well known that there is a unique measure $\mathbb{P} \otimes \mu$ on $(\Omega \times E, \mathcal{F} \otimes \mathcal{B}(E))$ such that*

$$\mathbb{P} \otimes \mu(A \times B) = \mathbb{P}(A) \times \mu(B) \tag{2.2}$$

for every $A \in \mathcal{F}$ and $B \in \mathcal{B}(E)$. We call such measure the product measure *of $\mathbb{P}$ and μ. The resulting measure space is denoted as $(\Omega \times E, \mathcal{F} \otimes \mathcal{B}(E), \mathbb{P} \otimes \mu)$.*

Definition 9 (Measurable function)**.** *Let $(G, \mathcal{G})$ and $(H, \mathcal{H})$ be two measurable spaces. A function $X : G \to H$ is said to be $\mathcal{G}/\mathcal{H}$–measurable if $X^{-1}(B) \in \mathcal{G}$ for every $B \in \mathcal{H}$.*

Definition 10 (Random variable)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \mathcal{B}(E))$ be the Borel space on a topological space E. An E–valued function X on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a random variable if X is $\mathcal{F}/\mathcal{B}(E)$–measurable.*

If there is no ambiguity, an E-valued random variable is simply called a random variable.

Definition 11 (Law of a random variable)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(E, \mathcal{B}(E))$ be the Borel space on a topological space E, and X be an E-valued random variable. It is well-known that $\sigma(X) := \{X^{-1}(B); B \in \mathcal{B}(E)\}$ is a sub σ–algebra of $\mathcal{F}$, and thus the mapping $B \mapsto \mathbb{P}(X^{-1}(B))$ is a probability measure on $(E, \mathcal{B}(E))$. We refer to this mapping as* the law of X, *often denoted by $\mathbb{P}^X$.*

Definition 12 (Lebesgue integral and expected value)**.** *Let* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *be the Borel space on* $\mathbb{R}$*. Suppose that* $(G, \mathcal{G}, \mu)$ *is a measure space and* $f : G \to \mathbb{R}$ *be a* $\mathcal{G}/\mathcal{B}(\mathbb{R})$*–measurable function. The* Lebesgue integral *is defined as the following four steps:*

1. *Let* $B \in \mathcal{G}$ *and define* $\mathbb{1}_B : G \to \{0, 1\}$ *as*

$$\mathbb{1}_B(x) := \begin{cases} 1, & \text{if } x \in B \\ 0, & \text{if } x \notin B. \end{cases} \tag{2.3}$$

   We call $f : G \to [0, \infty)$ Simple function *if*

$$f = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{B_i}, \tag{2.4}$$

   where $B_1, B_2, \cdots, B_n$ *are elements in* $\mathcal{G}$ *and* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *are nonnegative real numbers. Let* $A \in \mathcal{G}$*. Define* the Lebesgue integral of the simple function f with respect to μ *as*

$$\int_A f d\mu := \sum_{i=1}^{n} \alpha_i \mu(B_i \cap A). \tag{2.5}$$

2. *Let* $A \in \mathcal{G}$ *and* $f : G \to \mathbb{R}$ *be a nonnegative* $\mathcal{G}/\mathcal{B}(\mathbb{R})$*–measurable function.* The Lebesgue integral *of* f *with respect to* μ *is defined as follows:*

$$\int_A f d\mu := \sup \left\{ \int_A g d\mu \ ; \ g \text{ is simple function such that } g \leq f \right\}. \tag{2.6}$$

3. *Let* $A \in \mathcal{G}$ *and* $f : G \to \mathbb{R}$ *be a* $\mathcal{G}/\mathcal{B}(\mathbb{R})$*–measurable function. Let* $f^+ := \max\{f, 0\}$ *and* $f^- := -\min\{f, 0\}$*. It is well known that* f^+ *and* f^- *are* $\mathcal{G}/\mathcal{B}(\mathbb{R})$*–measurable and then* $\int_A f^+ d\mu$ *and* $\int_A f^- d\mu$ *can be defined. When* $\int_A f^+ d\mu < \infty$ *and* $\int_A f^- d\mu < \infty$*, we define* the Lebesgue integral *with respect to* μ *as*

$$\int_A f d\mu := \int_A f^+ d\mu - \int_A f^- d\mu. \tag{2.7}$$

   If $A = G$*, then we denote* $\int_A f d\mu$ *as* $\int f d\mu$*.*

4. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* X *be a* $\mathbb{R}$*–valued random variable such that* $\int_\Omega X^+ d\mathbb{P} < \infty$ *and* $\int_\Omega X^- d\mathbb{P} < \infty$*. Then* $\int_\Omega X d\mathbb{P}$ *is called* the expected value *of* X*, denoted by* $\mathbb{E}[X]$*.*

Remark 3. *In this chapter, we use another type of notation for the Lebesgue integral to accommodate different situations:*

$$\int_A f d\mu = \int_A f(x)\mu(dx) \tag{2.8}$$

Especially, if μ *is a Lebesgue measure, we denote the integral of* $x \mapsto f(x)$ *as following:*

$$\int_A f(x) dx \tag{2.9}$$

Definition 13 (Density of random variable and absolute continuity)**.**

(i) *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space,* $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ *be a Borel space, and* X *be an* $\mathbb{R}^n$*–valued random variable on* $(\Omega, \mathcal{F}, \mathbb{P})$*. Let* $([0,\infty), \mathcal{B}([0,\infty)))$ *is a Borel space on* $[0,\infty)$ *with the natural topology. A* $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}([0,\infty))$*–measurable function* f *is called* the density *of* X *if the following holds:*

$$\mathbb{P}(X^{-1}(B)) = \int_B f(x)dx, \quad \forall B \in \mathcal{B}(\mathbb{R}^n).$$

If there exists such a function f*, we say that* X *has a density.*

(ii) *If* X *has a density, we say that the law of* X *is* absolutely continuous *with respect to the Lebesgue measure.*

In the following sections, we frequently use the notion of almost sure convergence of random variables:

Definition 14. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, and let* (E, d) *be a metric space. Let* X *and* $X_n; n = 1, 2, \cdots$ *be* E*–valued random variables. We say* X_n converges almost surely *to* X *if there exists a measurable set* $N \in \mathcal{F}$ *such that* $\mathbb{P}(N) = 0$ *and*

$$X_n(\omega) \stackrel{n\to\infty}{\longrightarrow} X(\omega)$$

for every $\omega \notin N$*.*

2.5 Stochastic process

Definition 15. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* E *be a Polish space. A family of* E*–valued random variables* $X := \{X_t\}_{t\geq 0}$ *indexed by time parameter* t *is called a* stochastic process*:*

$$\begin{array}{ccc} \Omega & \stackrel{X_t}{\longrightarrow} & E \\ \Cup & & \Cup \\ \omega & \longmapsto & X_t(\omega) \end{array}$$

Following the convention of stochastic analysis, we say X is *measurable* if it satisfies the following assumption:

Assumption 1. *The function* $(\omega, t) \mapsto X_t(\omega)$ *is* $\mathcal{F}\otimes\mathcal{B}([0,\infty))$*-measurable, which means that the inverse image* $\{(\omega, t); X_t(\omega) \in B\}$ *belongs to* $\mathcal{F} \otimes \mathcal{B}([0,\infty))$ *whenever* B *is a Borel set in* E*.*

Remark 4. *Under Assumption 1,* X_t *is* $\mathcal{F}/\mathcal{B}(E)$*-measurable for all* $t \in [0,\infty)$*.*

We denote a path $t \mapsto X_t(\omega)$ of $\{X_t\}_{t\geq 0}$ as $X(\omega)$ for every $\omega \in \Omega$:

$$\begin{array}{ccc} [0,\infty) & \overset{X(\omega)}{\longrightarrow} & E \\ \Cup & & \Cup \\ t & \longmapsto & X_t(\omega) \end{array}$$

Remark 5. *Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space. If X_t is $\mathcal{F}$-measurable for all $t \in [0,\infty)$ and the path $X(\omega)$ is right- or left-continuous almost surely, then X is measurable (see 1.1.14 in [KS91]).*

2.5.1 Brownian motion

When studying properties of distributions of general continuous stochastic processes and topics such as convergence of discretization, including numerical computations, it is common to first discuss examples related to a Brownian motion as it is the most representative continuous stochastic process. Now, we present the definition of the Brownian motion:

Definition 16. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A d–dimensional stochastic process*

$$X := \{X_t\}_{t\geq 0} = \{(X_t^{(1)}, \cdots, X_t^{(d)})\}_{t\geq 0}$$

with state space $\mathbb{R}^d$ is called standard d–dimensional Brownian motion starting at $x \in \mathbb{R}^d$ if

(i) $\mathbb{P}(X_0 = x) = 1$,

(ii) The path $t \mapsto X_t(\omega)$ is continuous with probability one,

(iii) For any $s, t \geq 0$, $t > s$ implies that $X_t - X_s \sim \mathcal{N}(\mathbf{0}, (t-s)I_d)$ i.e., $X_t - X_s$ has normal distribution with mean zero and covariance matrix $(t-s)I_d$, where I_d is the $(d \times d)$ identity matrix.

(iv) If $s \leq t \leq u$, then $X_u - X_t$ is independent of $\sigma(X_v; v \leq s)$, where $\sigma(X_v; v \leq s)$ is the smallest sigma algebra which contains $\sigma(X_v)$ for all $v \leq s$.

The existence of a Brownian motion is established in Section 2.2 of [KS91], relying on the richness of the underlying probability space.

Next, we introduce the absolute continuity of the hitting time of the one–dimensional Brownian motion: We show that the hitting time has positive density.

Fact 2 (See 2.8.11 in [KS91])**.** *Let $\{X_t\}_{t\geq 0}$ be a one–dimensional Brownian motion starting at $x \in (0,\infty)$. Choose $a > x$ and define two hitting times as follows:*

$$T_0 := \inf\{t \geq 0; X_t = 0\},$$
$$T_a := \inf\{t \geq 0; X_t = a\}.$$

Then for $t > 0$:

$$\mathbb{P}[T_0 \wedge T_a \in dt] = \frac{1}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{\infty} \left[(2na+x) \exp\left\{ -\frac{(2na+x)^2}{2t} \right\} + (2na+a-x) \exp\left\{ -\frac{(2na+a-x)^2}{2t} \right\} \right] dt, \tag{2.10}$$

$$\mathbb{P}[T_0 \in dt, T_0 < T_a] = \frac{1}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{\infty} (2na+x) \exp\left\{ -\frac{(2na+x)^2}{2t} \right\} dt, \tag{2.11}$$

$$\mathbb{P}[T_a \in dt, T_a < T_0] = \frac{1}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{\infty} (2na+a-x) \exp\left\{ -\frac{(2na+a-x)^2}{2t} \right\} dt. \tag{2.12}$$

Theorem 1. *Let $\Gamma_1(t)$, $\Gamma_2(t)$ and $\Gamma_3(t)$ be density functions in (2.10), (2.11) and (2.12), respectively. Then $\Gamma_1(t)$, $\Gamma_2(t)$ and $\Gamma_3(t)$ are positive for almost all $t \in (0, \infty)$.*

Proof. Let $\mathbb{C}_+ := \{t \in \mathbb{C}; \mathrm{Re}\, t > 0\}$. Consider the mappings

$$\Gamma_1^{(n)}(t) := \frac{1}{\sqrt{2\pi t^3}} \sum_{k=-n}^{n} \left[(2na+x) \exp\left\{ -\frac{(2na+x)^2}{2t} \right\} + (2na+a-x) \exp\left\{ -\frac{(2na+a-x)^2}{2t} \right\} \right],$$

$$\Gamma_2^{(n)}(t) := \frac{1}{\sqrt{2\pi t^3}} \sum_{k=-n}^{n} (2ka+x) \exp\left\{ -\frac{(2ka+x)^2}{2t} \right\}$$

$$\Gamma_3^{(n)}(t) := \frac{1}{\sqrt{2\pi t^3}} \sum_{k=-n}^{n} (2ka+a-x) \exp\left\{ -\frac{(2ka+a-x)^2}{2t} \right\}$$

as functions on $\mathbb{C}_+$ to $\mathbb{C}$. Then $\Gamma_1^{(n)}(t)$, $\Gamma_2^{(n)}(t)$ and $\Gamma_3^{(n)}(t)$ are holomorphic on the domain $\mathbb{C}_+$. Since $\Gamma_1^{(n)}(t)$, $\Gamma_2^{(n)}(t)$ and $\Gamma_3^{(n)}(t)$ converge uniformly on every compact subsets of the domain, $\Gamma_1(t)$, $\Gamma_2(t)$ and $\Gamma_3(t)$ are holomorphic on the domain $\mathbb{C}_+$ (see Theorem 10.28 in [Rud17]).

Let us show that there is no limit point of $\{t \in \mathbb{C}_+; \Gamma_i(t) = 0\}$ for $i = 1, 2, 3$. If there exists some limit point, then $\Gamma_1(t) = \Gamma_2(t) = \Gamma_3(t) = 0$ for all $t \in \mathbb{C}_+$ (see Theorem 10.18 in [Rud17]). However, we obtain from 2.8.13 in [KS91] that

$$\int_0^\infty \Gamma_2(t)dt = \mathbb{P}[T_0 < T_a] = \frac{a-x}{a} > 0,$$
$$\int_0^\infty \Gamma_3(t)dt = \mathbb{P}[T_a < T_0] = \frac{x}{a} > 0.$$

Therefore, all elements in $\{t \in \mathbb{C}_+; \Gamma_i(t) = 0\}$ $(i = 1, 2, 3)$ are isolated in itself. Thus we can conclude that $\Gamma_i(t)$, $(i = 1, 2, 3)$ are positive for almost all $t \in \mathbb{C}_+$. ☐

2.5.2 Stochastic differential equation(SDE)

Let us now proceed to define time–homogeneous stochastic differential equations in a rigorous manner:

Definition 17. *Let* $b_j(x), \sigma_{i,j}(x);\ 1 \le i, j \le d$ *be Borel functions from* $[0, \infty) \times \mathbb{R}^d$ *into* $\mathbb{R}$*, and define the* $(d \times 1)$ drift vector $b(x) = \{b_i(x)\}_{1 \le i \le d}$ *and the* $(d \times d)$ dispersion matrix $\sigma(x) = \{\sigma_{i,j}(x)\}_{1 \le i,j \le d}$*. Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space that supports a d–dimensional Brownian motion* $\{W_t\}_{t \ge 0} = \{(W_t^{(1)}, \cdots, W_t^{(d)})\}_{t \ge 0}$*. We call a d–dimensional stochastic process* $\{X_t\}_{t \ge 0} = \{(X_t^{(1)}, \cdots, X_t^{(d)})\}_{t \ge 0}$ *a* **strong solution of the** d**–dimensional stochastic differential equation (SDE)**

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t, \\ X_0 = \xi \in \mathbb{R}. \end{cases} \tag{2.13}$$

if $\{X_t\}_{t \ge 0}$ *satisfies the following four properties:*

(i) $\{X_t\}_{t \ge 0}$ *is adapted to the filtration induced by Brownian motion (see 1.1.9 and 5.2.1 in [KS91]),*

(ii) $\mathbb{P}[\omega \in \Omega; X_0(\omega) = \xi] = 1$*,*

(iii) $\mathbb{P}[\omega \in \Omega; \int_0^t \{|b_i(X_s(\omega))| + \sigma_{i,j}^2(X_t(\omega))\} ds < \infty] = 1$ *holds for every* $1 \le i, j \le d$ *and* $0 \le t < \infty$*, and*

(iv) the integral version of (2.41)

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s; \quad 0 \le t < \infty,$$

or equivalently,

$$X_t^{(i)} = X_0^{(i)} + \int_0^t b_i(X_s)ds + \sum_{j=1}^{d} \int_0^t \sigma_{i,j}(X_s)dW_s^{(j)}, \quad 0 \le t < \infty,\ 1 \le i \le d,$$

holds almost surely. Here, the term $\int_0^t \sigma_{i,j}(X_s)dW_s$ *represents the Itô's stochastic integral, defined as the limit of the following stochastic process (refer to Chapter 3 in [KS91]):*

$$\sum_{k=1}^{\infty} \sigma_{i,j}(X_{\frac{k}{n}})(W_{\frac{k+1}{n} \wedge t} - W_{\frac{k}{n} \wedge t}). \tag{2.14}$$

Remark 6. *Brownian motion* $\{W_t\}_{t \ge 0}$ *starting at* $x \in \mathbb{R}^d$ *itself is a solution* $\{X_t\}_{t \ge 0}$ *of following d–dimensional SDE:*

$$\begin{cases} X_t = \int_0^t I_d dW_s \\ X_0 = x, \end{cases}$$

where I_d *is the d–dimensional identity matrix.*

Remark 7. *Brownian motions and SDEs are well-known to be continuous stochastic processes. Therefore they satisfy Assumption 1.*

2.6 Metric Temporal Logic (MTL) and its semantics

In this section, we introduce the formal definition of MTL (Metric Temporal Logic) formulas for a given path. We begin by assuming a set of atomic propositions, denoted as AP, which is typically defined as a finite set. The MTL formulas are then defined as follows:

Definition 18 (Syntax of MTL-formulas)**.** *We define MTL-formulas for a continuous stochastic process $\{X_t\}_{t\geq 0}$ using the following grammar:*

1. *Every atomic proposition $a \in AP$ is an MTL formula.*

2. *If ϕ is an MTL–formula, then $\neg\phi$ is also an MTL–formula.*

3. *If ϕ_1 and ϕ_2 are MTL-formulas, then the conjunction of ϕ_1 and ϕ_2, denoted as $\phi_1 \wedge \phi_2$, is also an MTL formula.*

4. *If ϕ_1 and ϕ_2 are MTL–formulas, and I represents an interval on the domain $[0, \infty)$, then the formula $\phi_1 \mathcal{U}_I \phi_2$ is an MTL formula.*

The grammar above can be conveniently represented using the Backus–Naur form*:*

$$\phi ::= a \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid \phi_1 \mathcal{U}_I \phi_2,$$

Remark 8 ("Until" operator)**.** *In (3) in the definition 18, $\mathcal{U}_I$ is called an* until operator. *The interval I appearing in the until operator $\mathcal{U}_I$ can be closed, left open, right open, or purely open. This means that I can take the form $I = [a, b]$, $(a, b]$, $[a, b)$, or (a, b), respectively. Furthermore, when I is unbounded, it can only be of the form $I = (a, b)$ or $I = [a, b)$, where b can take the value ∞.*

We proceed to define two types of semantics for the previously presented syntax: one for the continuous time domain and the other for the discrete-time domain.

Definition 19 (Continuous Semantics of MTL-Formulas)**.** *Consider a path $X(\omega)$ of the stochastic process $\{X\}_{t\geq 0}$ with a fixed $\omega \in \Omega$. Additionally, for each atomic proposition $a \in \mathrm{AP}$, let us assign a Borel set B_a on the domain E. The continuous semantics of MTL formulas is recursively defined as follows:*

$$\begin{aligned}
X(\omega), t \models a &\iff && X_t(\omega) \in B_a \\
X(\omega), t \models \neg\phi &\iff && \textit{not } [X(\omega), t \models \phi] \\
X(\omega), t \models \phi_1 \wedge \phi_2 &\iff && X(\omega), t \models \phi_1 \textit{ and } X(\omega), t \models \phi_2 \\
X(\omega), t \models \phi_1 \mathcal{U}_I \phi_2 &\iff && \exists s \in I \textit{ s.t.: } X(\omega), t + s \models \phi_2 \textit{ and} \\
& && \forall s' \in [t, t + s),\ X(\omega), s' \models \phi_1
\end{aligned}$$

Definition 20 (Time set)**.** *We introduce the notations* $[\![\phi]\!]$*,* $[\![\phi]\!](t)$*, and* $[\![\phi]\!]_\omega$ *as follows:*

$$\begin{aligned}[\![\phi]\!] &:= \{(\omega,t); X(\omega),t\models\phi\},\\ [\![\phi]\!](t) &:= \{\omega; X(\omega),t\models\phi\},\\ [\![\phi]\!]_\omega &:= \{t\geq 0; X(\omega),t\models\phi\}.\end{aligned}$$

In particular, we refer to $[\![\phi]\!]_\omega$ *as the "time set" associated with* ϕ*.*

Definition 21 (Discrete Semantics of MTL-Formulas)**.** *Let us consider the path* $X(\omega)$ *of* $\{X_t\}_{t\geq 0}$ *and the assignment* B_a *for an atomic proposition* $a\in \mathrm{AP}$*, as well as Definition 19. For any* $n\in\mathbb{N}$*, we denote* $\{k/n; k\in\mathbb{N}\}$ *as* $\mathbb{N}/n$*. The discrete semantics of MTL formulas for any* $t\in\mathbb{N}/n$ *is defined recursively as follows:*

$$\begin{aligned}X(\omega),t\models_n a &\iff && X_t(\omega)\in B_a\\ X(\omega),t\models_n \neg\phi &\iff && \textit{not } [X(\omega),t\models_n\phi]\\ X(\omega),t\models_n \phi_1\wedge\phi_2 &\iff && X(\omega),t\models_n\phi_1 \textit{ and } X(\omega),t\models_n\phi_2\\ X(\omega),t\models_n \phi_1\mathcal{U}_I\phi_2 &\iff && \exists s\in I\cap\mathbb{N}/n \textit{ s.t.: } X(\omega),t+s\models_n\phi_2 \textit{ and}\\ & && \forall s'\in[t,t+s)\cap\mathbb{N}/n,\ X(\omega),s'\models_n\phi_1\end{aligned}$$

For $t\in\mathbb{N}/n$, we denote by $[\![\phi]\!]_n(t)$ the set $\{\omega; X(\omega),t\models_n\phi\}$.

Remark 9 (Constants, Disjunction, Diamond operator and Box Operator)**.** *We often use two constants of propositional logic* $\top$ (top) *and* $\bot$ (bottom)*. Top means undoubted tautology whose truth nobody could ever question, while bottom means undoubted contradiction. These two constants can be represented as*

$$\begin{aligned}\top &= \phi\vee\neg\phi\\ \bot &= \phi\wedge\neg\phi\end{aligned}$$

by arbitrary MTL–formula ϕ*. We often use the following notation:*

$$\begin{aligned}\phi_1\vee\phi_2 &= \neg((\neg\phi_1)\wedge(\neg\phi_2)),\\ \Diamond_I\phi &= \top\mathcal{U}_I\phi,\\ \Box_I\phi &= \neg(\Diamond_I\neg\phi),\end{aligned}$$

We refer to $\Diamond_I$ *and* $\Box_I$ *as the diamond and box operators, respectively. In the continuous and discrete semantics, the following equivalences hold:*

$$\begin{aligned}X(\omega),t\models\Diamond_I\phi &\Leftrightarrow(\exists s\in I)[X(\omega),t+s\models\phi],\\ X(\omega),t\models\Box_I\phi &\Leftrightarrow(\forall s\in I)[X(\omega),t+s\models\phi].\end{aligned}$$

$$\begin{aligned}X(\omega),t\models_n\Diamond_I\phi &\Leftrightarrow(\exists s\in I\cap\mathbb{N}/n)[X(\omega),t+s\models_n\phi],\\ X(\omega),t\models_n\Box_I\phi &\Leftrightarrow(\forall s\in I\cap\mathbb{N}/n)[X(\omega),t+s\models_n\phi].\end{aligned}$$

2.7 Proof of Measurability

As introduced in Definition 3, the probability $\mathbb{P}(F)$ can only be defined for a measurable set F. Therefore, in order to define $\mathbb{P}(\omega \in \Omega; X(\omega), t \models \phi)$ or $\mathbb{P}(\omega \in \Omega; X(\omega), t \models_n \phi)$, it is necessary to show the measurability of $[\![\phi]\!](t) = \{\omega \in \Omega; X(\omega), t \models \phi\}$ or $[\![\phi]\!]_n(t) = \{\omega \in \Omega; X(\omega), t \models_n \phi\}$, respectively. Since the definition of the discrete semantics of MTL involves the intersection or union of at most countably many sets, the measurability of $[\![\phi]\!]_n(t)$ follows immediately. Then $\mathbb{P}(\omega \in \Omega; X(\omega), t \models_n \phi)$ can be defined. However, the measurability of $[\![\phi]\!](t)$ is not straightforward. Then it is not obvious whether $\mathbb{P}(-omega \in \Omega; X(\omega), t \models \phi)$ can be defined or not.

To illustrate the difficulty, let X be an E–valued stochastic process, a, b be atomic propositions, and I be an interval on $[0, \infty)$. Then $X(\omega), t \models a$ is equivalent to $X_t(\omega) \in B_a$ for some Borel set B_a, and $X(\omega), t \models b$ is equivalent to $X_t(\omega) \in B_b$ for some Borel set B_b. Since X_t is $\mathcal{F}/\mathcal{B}(E)$–measurable, then $[\![a]\!](t) = \{\omega \in \Omega; X_t(\omega) \in B_a\}$ belongs to $\mathcal{F}$ and hence $\mathbb{P}(\omega \in \Omega; X(\omega), t \models a)$ can be defined. $\mathbb{P}(\omega \in \Omega; X(\omega), t \models b)$ can be defined for the same reason.

However, can we define $\mathbb{P}(\omega \in \Omega; X(\omega), t \models a\mathcal{U}_I b)$? From the definition of the until operator, $X(\omega), t \models a\mathcal{U}_I b$ is equivalent to the following:

$$(\exists s \in I)[X_{t+s}(\omega) \in B_b \text{ and } (\forall s' \in [0, s))[X_{t+s'} \in B_a]].$$

Therefore,

$$[\![a\mathcal{U}_I b]\!](t) = \bigcup_{s \in I} \bigcap_{s' \in [0,s)} \{\omega \in \Omega; X_{t+s}(\omega) \in B_b\} \cap \{\omega \in \omega; X_{t+s'}(\omega) \in B_a\}.$$

Although the measurability of $\{\omega \in \Omega : X_{t+s}(\omega) \in B_b\} \cap \{\omega \in \Omega : X_{t+s'}(\omega) \in B_a\}$ follows from the $\mathcal{F}/\mathcal{B}(E)$–measurability of X_{t+s} and $X_{t+s'}$, the representation of $[\![a\mathcal{U}_I b]\!](t)$ involves uncountable intersections and unions of these sets. Since measurability is guaranteed to preserve under countable unions or intersections, the measurability of $[\![a\mathcal{U}_I b]\!](t) = \{\omega \in \Omega; X(\omega), t \models a\mathcal{U}_I b\}$ is not obvious. Thus, we have observed that the challenge arises when dealing with the until formula $\mathcal{U}_I$.

In this section, we show the measurability of MTL assuming that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete. To show the measurability of the event $[\![\phi]\!](t) = \{\omega, \in \Omega; X(\omega), t \models \phi\}$ for arbitrary MTL formula ϕ under the completeness, we utilize a profound theorem from the theory of capacity. By employing capacity theory, we show the measurability of the projection. Since we can represent the until operator $\mathcal{U}_I$ using an inverse image of the reaching time of an MTL formula, the inverse image is represented as the projection of measurable set on $\Omega \times [0, \infty)$ to Ω.

Throughout this section, suppose that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete. In order to show the measurability of until formulas, we introduce *reaching time* of a set B on $\Omega \times [0, \infty]$:

Definition 22. *Consider a subset B of $\Omega\times[0,\infty)$. The* reaching time *or* debut $\tau_B(\omega,t)$ *of B is defined for each $\omega\in\Omega$ as the first time $s>t$ at which (ω,s) reaches B, given by:*

$$\tau_B(\omega,t) := \inf\{s>t; (\omega,s)\in B\},$$

where $\tau_B(\omega,t):=\infty$ if $\{s>t;(\omega,s)\in B\}=\emptyset$.

Lemma 1. *For any subset B of $\Omega\times[0,\infty)$, the reaching time $\tau_B(\omega,t)$ is right-continuous with respect to $t\in[0,\infty)$.*

Proof. Assume $\tau_B(\omega,t)>t$. We can express $\tau_B(\omega,t)$ as $t+\alpha$ for some $\alpha>0$. According to the definition of $\tau_B(\omega,t)$, for every s in the interval $(t,t+\alpha)$, it holds that $(\omega,s)\notin B$. Therefore, we have $\tau_B(\omega,s)=t+\alpha$ for such s, and as a result, $\lim_{s\downarrow t}\tau_B(\omega,s)=t+\alpha=\tau_B(\omega,t)$.

Assume $\tau_B(\omega,t)=t$. For every $\epsilon>0$, there exists $\delta\in(0,\epsilon)$ such that $(\omega,t+\delta)\in B$. Therefore, $\tau_B(\omega,s)\le t+\delta<t+\epsilon$ for every $s\in(t,t+\delta)$, which implies $\lim_{s\downarrow t}\tau_B(\omega,s)=t=\tau_B(\omega,t)$.

□

The following lemma serves as an abstract version of Proposition 1.1.13 in [KS91].

Lemma 2. *If a stochastic process $\{Y_t\}_{t\ge0}$ is $[0,\infty]$-valued and right-continuous, then the mapping $(\omega,t)\mapsto Y_t(\omega)$ is $\mathcal{F}\otimes\mathcal{B}([0,\infty))/\mathcal{B}([0,\infty])$-measurable.*

Proof. For $t>0$, $n\ge1$, and $k=0,1,\ldots$, we define $Y_t^{(n)}(\omega)=Y_{(k+1)/2^n}(\omega)$ for $\frac{k}{2^n}<t\le\frac{k+1}{2^n}$, and $Y_0^{(n)}=Y_0(\omega)$. The mapping $(\omega,t)\mapsto Y_t^{(n)}(\omega)$ from $\Omega\times[0,\infty)$ to $[0,\infty]$ is demonstrably $\mathcal{F}\otimes\mathcal{B}([0,\infty))/\mathcal{B}([0,\infty])$-measurable. Furthermore, by right-continuity, we have $Y_t^{(n)}(\omega)\to Y_t(\omega)$ if $n\to\infty$ for any $(\omega,t)\in[0,\infty)\times\Omega$. Consequently, the limit mapping $(\omega,t)\mapsto Y_t(\omega)$ is also $\mathcal{F}\otimes\mathcal{B}([0,\infty))/\mathcal{B}([0,\infty])$-measurable. □

Now let us show the measurability of the reaching time when considering it as a stochastic process. This result is derived from capacity theory, which guarantees the measurability of the projection (of a well-behaved set).

Lemma 3. *If B belongs to $\mathcal{F}\otimes\mathcal{B}([0,\infty))$, the mapping $(\omega,t)\mapsto\tau_B(\omega,t)$ is $\mathcal{F}\otimes\mathcal{B}([0,\infty))/\mathcal{B}([0,\infty])$-measurable.*

Proof. From Lemma 1, the reaching times $\tau_B(t,\omega):=\inf\{s>t;(\omega,s)\in B\}$ are right-continuous with respect to $t\in[0,\infty)$. Then it is enough to show that $\omega\mapsto\tau_B(\omega,t)$ is $\mathcal{F}/\mathcal{B}([0,\infty])$–measurable because of Lemma 2. From the definition of $\tau_B(\omega,t)$, we can represent $\{\omega;\tau_B(\omega,t)<u\}$ by using projection mapping $\pi:\Omega\times[0,\infty]\to\Omega$ as

$$\{\omega;\tau_B(\omega,t)<u\}=\pi(B\cap\{\Omega\times(t,u)\}),\quad\forall u\in[0,\infty].$$

Since $[0,\infty]$ is a locally compact space with countable basis, $\mathcal{F}$ is complete, and the set $B\cap\{\Omega\times(t,u)\}$ belongs to $\mathcal{F}\otimes\mathcal{B}([0,\infty))$, we can apply *Theorem I-4.14 in [RYs91]* to show $\pi(B\cap\{\Omega\times(t,u)\})\in\mathcal{F}$. Therefore, $\{\omega\in\Omega;\tau_B(\omega,t)<u\}\in\mathcal{F}$ for all $u\ge0$, which implies the $\mathcal{F}/\mathcal{B}([0,\infty])$–measurability of the map $\omega\mapsto\tau_B(\omega,t)$.

□

The subsequent lemma regarding the two types of subsets follows directly from basic arguments in measure theory.

Lemma 4. *Let $B \in \mathcal{F} \otimes \mathcal{B}([0,\infty))$. Let f and g be functions from $\Omega \times [0,\infty)$ to $[0,\infty]$, which are $\mathcal{F} \otimes \mathcal{B}([0,\infty))/\mathcal{B}([0,\infty])$-measurable. Then, the following sets are $\mathcal{F} \otimes \mathcal{B}([0,\infty))$-measurable.*

$$\{(\omega,t) \in \Omega \times [0,\infty); f(\omega,t) \geq g(\omega,t)\}, \tag{2.15}$$
$$\{(\omega,t) \in \Omega \times [0,\infty); (\omega, f(\omega,t)) \in B\}. \tag{2.16}$$

Proof. Since f and g are $\mathcal{F} \otimes \mathcal{B}([0,\infty))/\mathcal{B}([0,\infty])$–measurable, the set (2.15) is $\mathcal{F} \otimes \mathcal{B}([0,\infty))$–measurable.

Because f is a $\mathcal{F} \otimes \mathcal{B}([0,\infty))/\mathcal{B}([0,\infty])$ measurable function, $\tilde{f}(\omega,t) = (\omega, f(\omega,t))$ is $\mathcal{F} \otimes \mathcal{B}([0,\infty))/\mathcal{F} \otimes \mathcal{B}([0,\infty])$–measurable function. By considering B as a subset of $\Omega \times [0,\infty]$, it becomes $\mathcal{F} \otimes \mathcal{B}([0,\infty])$–measurable. Consequently, $\tilde{f}^{-1}(B)$ is $\mathcal{F} \otimes \mathcal{B}([0,\infty))$–measurable. Because

$$\{(\omega,t) \in \Omega \times [0,\infty); (\omega, f(\omega,t)) \in B\} = \tilde{f}^{-1}(B) \in \mathcal{F} \otimes \mathcal{B}([0,\infty)),$$

The lemma holds. □

Now, we can proceed to prove the measurability of $[\![\phi]\!]$ and $[\![\phi]\!](t)$.

Lemma 5. *Let ϕ_1 and ϕ_2 be two MTL-formulas and suppose that both $[\![\phi_1]\!]$ and $[\![\phi_2]\!]$ are in $\mathcal{F} \otimes \mathcal{B}([0,\infty))$. Then $\{(\omega,t); X(\omega), t \models \phi_1 \mathcal{U}_I \phi_2\}$ belongs to $\mathcal{F} \otimes \mathcal{B}([0,\infty))$.*

Proof. In order to prove this lemma, we put

$$\tau_1(\omega,t) := \tau_{[\![\phi_1]\!]^C}(\omega,t) = \inf\{s > t; X(\omega), s \not\models \phi_1\},$$
$$\tau_2(\omega,t) := \tau_{[\![\phi_2]\!]}(\omega,t) = \inf\{s > t; X(\omega), s \models \phi_2\}.$$

By Lemma 2, τ_1 and τ_2 are $\mathcal{F} \otimes \mathcal{B}([0,\infty))/\mathcal{B}([0,\infty])$-measurable.

We only prove the case where $I = [a,b]$. The cases for other forms of I can be proved in similar ways. For simplicity, suppose that $a > 0$. Then $X(\omega), t \models \phi_1 \mathcal{U}_I \phi_2$ holds if and only if $X(\omega), t \models \phi_1$ holds and one of the following possibilities holds:

1. $X(\omega), t+a \models \phi_2$ holds and $\tau_1(\omega,t) \geq t+a$
2. $X(\omega), t+b \models \phi_2$ holds and $\tau_1(\omega,t) \geq t+b$ holds
3. $\tau_2(\omega,t+a) < t+b$, $X(\omega), \tau_2(\omega,t+a) \models \phi_2$ and $\tau_1(\omega,t) \geq \tau_2(\omega,t+a)$ hold
4. $\tau_2(\omega,t+a) < t+b$, $X(\omega), \tau_2(\omega,t+a) \not\models \phi_2$ and $\tau_1(\omega,t) > \tau_2(\omega,t+a)$ hold

By $\mathcal{F} \otimes \mathcal{B}([0,\infty))/\mathcal{B}([0,\infty])$–measurability of τ_1 and τ_2,

$$\begin{aligned}&\{(\omega,t);\tau_1(\omega,t) \geq t+a\},\\&\{(\omega,t);\tau_1(\omega,t) \geq t+b\}, \text{ and}\\&\{(\omega,t);\tau_2(\omega,t+a) < t+b\}\end{aligned}$$

are in $\mathcal{F} \otimes \mathcal{B}([0,\infty))$. Thanks to Lemma 4,

$$\begin{aligned}&\{(\omega,t);\tau_1(\omega,t) \geq \tau_2(\omega,t+a)\} \text{ and}\\&\{(\omega,t);\tau_1(\omega,t) > \tau_2(\omega,t+a)\}\end{aligned}$$

are in $\mathcal{F} \otimes \mathcal{B}([0,\infty))$. Since $\tau_2(\omega,t+a)$ is $\mathcal{F} \otimes \mathcal{B}([0,\infty))/\mathcal{B}([0,\infty])$–measurable and then

$$\begin{aligned}\{(\omega,t);X(\omega),\tau_2(\omega,t+a) \models \phi_2\} &= \{(\omega,t);(\omega,\tau_2(\omega,t+a)) \in [\![\phi_2]\!]\},\\\{(\omega,t);X(\omega),\tau_2(\omega,t+a) \not\models \phi_2\} &= \{(\omega,t);(\omega,\tau_2(\omega,t+a)) \notin [\![\phi_2]\!]\}.\end{aligned}$$

From Lemma 4, both sets are in $\mathcal{F} \otimes \mathcal{B}([0,\infty))$. This completes the proof of $\mathcal{F} \otimes \mathcal{B}([0,\infty))$-measurability of $X(\omega), t \models \phi_1 \mathcal{U}_I \phi_2$.

□

Theorem 2. *For each MTL-formula ϕ, $[\![\phi]\!]$ is $\mathcal{F} \otimes \mathcal{B}([0,\infty))$-measurable and $[\![\phi]\!](t)$ is $\mathcal{F}$-measurable for all $t \geq 0$.*

Proof. We can prove the measurability of $[\![\phi]\!]$ by induction on ϕ.

- Atomic Formula: If ϕ is an atomic formula, then $[\![\phi]\!]$ belongs to $\mathcal{F} \otimes \mathcal{B}([0,\infty))$ because the mapping $(\omega,t) \mapsto X_t(\omega)$ is $\mathcal{F} \otimes \mathcal{B}([0,\infty))/\mathcal{B}(E)$-measurable.
- Negation: If $[\![\phi]\!]$ belongs to $\mathcal{F} \otimes \mathcal{B}([0,\infty))$, then $[\![\neg\phi]\!] = [\![\phi]\!]^C$ clearly belongs to $\mathcal{F} \otimes \mathcal{B}([0,\infty))$.
- Conjunction: Suppose ϕ_1 and ϕ_2 are two MTL formulas, and $[\![\phi_i]\!]$ is $\mathcal{F} \otimes \mathcal{B}([0,\infty))$-measurable for $i = 1, 2$. Then it is straightforward to show that $[\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \cap [\![\phi_2]\!]$ is also $\mathcal{F} \otimes \mathcal{B}([0,\infty))$-measurable.
- Until Operator: From Lemma 5, we can obtain $\mathcal{F} \otimes \mathcal{B}([0,\infty))$-measurability of $[\![\phi_1 \mathcal{U}_I \phi_2]\!]$.

Once we have shown that $[\![\phi]\!]$ belongs to $\mathcal{F} \otimes \mathcal{B}([0,\infty))$, the fact that $[\![\phi]\!](t) \in \mathcal{F}$ follows. □

Since the domain of $\mathbb{P}$ is $\mathcal{F}$, we can define $\mathbb{P}(\omega; X(\omega), t \models \phi)$ for all ϕ and $t \in [0,\infty)$.

2.8 Discretization of MTL formula: Counterexamples

Fu and Ufuk [FT15] proposed a methodology for approximating the probability that the solution of a controlled stochastic differential equation (SDE) satisfies an MTL formula. Their approach involves discretizing both the time and state space of the SDE. By utilizing a reachability problem for a timed automaton generated by the SDE and MTL formula, they derive probabilities based on this discretized semantics.

The authors argue that the convergence of their simulation is a result of the convergence in distribution of the approximated SDE, whose state space has been discretized. They claim that this probability obtained from the discretized approach converges to the probability derived from the continuous-time semantics of the original SDE.

However, in this chapter, we show that for a one–dimensional Brownian motion, denoted as X, the probability obtained using the discretized semantics does not necessarily converge to the probability obtained using continuous semantics. This failure arises because a reaching time of the Brownian motion may have positive density.

It is worth noting that a Brownian motion can be viewed as a solution of the stochastic differential equation (SDE) (see Remark 6). Furthermore, every SDE without control can be regarded as a special case of controlled SDEs. Consequently, Brownian motion can serve as an illustrative example of a solution to a controlled SDE. Hence, our counterexample aligns with the scenario presented in [FT15].

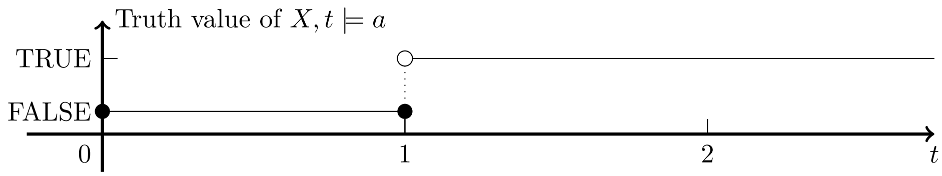
2.9 Deterministic case

Before going to the probabilistic example, we start from a deterministic case. Using the deterministic case, we express the discrepancy between the continuous semantics and the limit of the discrete semantics. In Fu and Ufuk [FT15], they defined the continuous time semantics as Definition 19 (see Definition 3 in [FT15]), while claiming that the continuous semantics is equivalent to the limit of discrete semantics (see Section D in [FT15]). However, we can see that the limit of discrete semantics does not coincide with continuous semantics.

To see this, let $t \mapsto X_t$ a deterministic time evolution which satisfies $X_t(\omega) = t$ for all $t \geq 0$ and $\omega \in \Omega$. Since the path $X(\omega)$ and the state $X_t(\omega)$ does not depend on ω, we omit the argument ω. Define an atomic formula a as

$$X, t \models a \Leftrightarrow X_t > 1.$$

Then clearly $X, t \not\models a$ for $t \leq 1$ and $X, t \models a$ for $t > 1$. Note that the truth value of $X, t \models a$ is left continuous on $(0, 2)$.



Let $I := [0, 2]$ and $\phi := \neg a \mathcal{U}_I a$. Now we show that the discrete semantics of ϕ does not converge to the continuous semantics of it. In particular, we show $X, t \not\models \phi$ for all $t \in [0, \infty)$, while $X, t \models_n \phi$ for all $t \in 0, 1/n, \cdots, 1$ if $n \geq 2$. Indeed, because of the left–continuity of the truth value of $X, t \models a$ on $(0, 2]$, we have for every $t \in [0, 2]$ that $X, t \models a$ leads the existence of some $t' < t$ such that $X, t' \models a$. This contradicts the definition of $X, t \models \neg a \mathcal{U}_I a$. On the other hand, suppose that $t \in \{0, 1/n, \cdots, 1\}$. Then

$$\begin{cases} X_{t+s} \geq 1, & \text{for all } s = 0, 1/n, \cdots, 1-t \\ X_{t+s} > 1, & \text{for all } s = 1-t+1/n, 1-t+1/n, \cdots, \end{cases}$$

which implies

$$\begin{cases} X, t+s \not\models_n a, & \text{for all } s = 0, 1/n, \cdots, 1-t \\ X, t+s \models_n a, & \text{for all } s = 1-t+1/n, 1-t+1/n, \cdots .] \end{cases}$$

Taking $s = 1 - t + 1/n$, we can see that

- $s \in [0, 2] \cap \mathbb{N}$,
- $X, t+s \models_n a$, and
- $X, t+s' \models_n \neg a$ for $s' = 0, 1/n, \cdots, s - 1/n$.

Then we obtain $X, t \models_n \neg a \mathcal{U}_I a$.

2.10 Probabilistic case: The case of one–dimensional Brownian motion

Now let us extend the previous deterministic example to the stochastic case. First of all, there is an evident example such that the probability of the discrete MTL–formula does not converge to that of the corresponding continuous MTL–formula:

Theorem 3. *Let $X := \{X_t\}_{t \geq 0}$ be one-dimensional Brownian motion starting at zero. Consider the case that a, b are atomic formulas such that $X(\omega), \models a \Leftrightarrow X_t \in B_a$ and $X(\omega), t \models B_b$, where $B_a := (-\infty, 1)$ and $B_b := (1, \infty)$. Let $p := \neg(a \wedge b)$ and $\phi := \Diamond_{\langle S,T \rangle} p$. Then $\mathbb{P}(\omega; X(\omega), 0 \models_n \phi)$ does not converge to $\mathbb{P}(\omega; X(\omega), 0 \models \phi)$.*

Proof. Since $X_0 \neq 1$ and X_t has density for all $t > 0$, $\mathbb{P}(\omega; X_t(\omega) = 1) = 0$ for all $t \in [0, \infty) \cap \mathbb{Q}$. Hence the sigma-additivity of probability measure implies $\mathbb{P}(\exists t \in [0, \infty) \cap \mathbb{Q},\ X_t = 1) = 0$. Then $\mathbb{P}(\omega; X(\omega), \Lambda_n(0) \models_n \Diamond_{\langle S,T \rangle} p) = 0$ for every n, S, T and then $\mathbb{P}(\omega; X(\omega), \Lambda_n(0) \models_n \phi) = 0$. On the other hand, if $\tau_1(\omega) := \inf\{t \geq 0; X_t(\omega) = 1\} \in (S,T)$ then $X(\omega), 0 \models \phi$. From Theorem 2 and the monotonicity of probability measures, then $\mathbb{P}(\omega; X(\omega), 0 \models \phi) \geq \mathbb{P}(\tau_1 \in (S,T)) > 0$ and then $\mathbb{P}(\omega; X(\omega), 0 \models_n \phi)$ does not converge to $\mathbb{P}(\omega; X(\omega), 0 \models \phi)$. □

This counterexample comes from the fact that we can make the propositional formula p such that the corresponding set B_p has Lebesgue measure zero. When we consider the convergence of an MTL–formula with the diamond operator in such a setting, the counterexample above definitely arises. Moreover, such a counterexample is avoided in applications.

Therefore, from now on, we consider only the case that all propositional formula has the corresponding set with at least a positive Lebesgue measure.

2.10.1 The case of the single until formula $\mathcal{U}_I$

Condition 1. *Let $I = \langle T_1, T_2 \rangle$ be a positive interval on $[0,\infty)$. In other words, T_1, T_2 are positive constants with $0 \leq T_1 < T_2 \leq \infty$ and I be an interval with endpoint T_1, T_2, i.e., $I = [T_1,T_2], [T_1,T_2), (T_1,T_2]$, or (T_1,T_2).*

Now we show the following statement:

Theorem 4. *Let $\{X_t\}_{t\geq 0}$ be standard one-dimensional Brownian motion starting at $x \in (0,a)$. we assign an closed set $B_p := [0,a]$ with $a > 0$ for an atomic proposition p, i.e., define $X(\omega), t \models p \Leftrightarrow X_t(\omega) \in B_p$. If we put Condition 1 on an interval I in $[0,\infty)$: Then $\mathbb{P}(\omega; X(\omega), 0 \models_n p\mathcal{U}_I \neg p)$ does not converges to $\mathbb{P}(\omega; X(\omega), 0 \models p\mathcal{U}_I \neg p)$ even if $n \to \infty$.*

To show this statement, we list some facts about one-dimensional Brownian motion. Firstly, the following statement is obvious because $\{X_t\}_{t\geq 0}$ is continuous with respect to $t \in [0,\infty)$ and B_p is a closed set:

Proposition 1.

$$\begin{aligned}
&\mathbb{P}(\omega; X(\omega), 0 \models p\mathcal{U}_I \neg p) \\
=&\mathbb{P}(\omega; \exists t \in I \ s.t. \ X_t(\omega) \notin [0,a] \ and \ \forall t' \in [0,t), X_t(\omega) \in [0,a]) \\
=&0.
\end{aligned}$$

Therefore it is enough to show that $\mathbb{P}(\omega; X(\omega), 0 \models_n p\mathcal{U}_I \neg p)$ converges to a positive number.

Fact 3 (see 2.7.18 in [KS91])**.** *With probability one, the path of a standard one-dimensional Brownian motion changes its sign infinitely many times in any time interval $[0,\epsilon]$, $\epsilon > 0$.*

Remark 10. *Put $B_p := [0,\infty)$. Define an atomic formula p as $X(\omega), t \models p \Leftrightarrow X_t(\omega) \in B_p$. Then the truth value of $X(\omega), t \models p$ changes infinitely in $[0,\varepsilon]$ for any $\varepsilon > 0$, almost surely. When we see the truth value as the timed behavior of $X(\omega)$, then the timed behavior has non–Zenoness almost surely.*

Fact 4 (Strong Markov property of Brownian motion). *Let $\{X_t\}_{t\geq 0}$ be one-dimensional Brownian motion and Γ be a Borel set of $\mathbb{R}$. Define a random time $\tau_p := \inf\{t \geq 0; X_t \in \Gamma\}$. Then $\{X_{\tau_p+t} - X_{\tau_p}\}_{t\geq 0}$ is also one dimensional Brownian motion starting at zero.*

Fact 5 (see 2.6–2.8 in [KS91]). *Let us define $\tau_p := \inf\{t \geq 0; X_t \in [0, a]\}$. Then τ_p has a density function on $[0, \infty)$.*

The next lemma follows from the similar discussion as the proof of Proposition 2.8.10 in [KS91]:

Lemma 6. *If T_1, T_2 satisfies $0 < T_1 < T_2 < \infty$, then $\mathbb{P}(\tau_p \in (T_1, T_2)) > 0$.*

Proof. We can see the conclusion immediately from Theorem 1. □

The following statement can be shown using Fact 3, Fact4, and Fact 5.

Lemma 7.

$$\mathbb{P}(\omega; \forall t \in [0, T], X_t(\omega) \in [0, a]; \exists t \in [0, T], X_t(\omega) \in \{0, a\}) = 0, \tag{2.17}$$

Proof. By the subadditivity of probability, the left–hand side of (2.17) is bounded by

$$\mathbb{P}(\omega; \forall t \in [0, T], X_t(\omega) \in [0, a]; \exists t \in [0, T), X_t(\omega) \in \{0, a\}) \tag{2.18}$$
$$+\mathbb{P}(\omega; \forall t \in [0, T], X_t(\omega) \in [0, a]; X_T(\omega) \in \{0, a\}) \tag{2.19}$$

(2.19) is clearly zero because X_T has density. Then it suffices to show that the probability (2.18) is zero. Set τ_p as well as Fact 4 and $\tilde{X}_t := X_{\tau_p+t} - X_{\tau_p}$ for $t \geq 0$, then $\{\tilde{X}_t\}$ is standard Brownian motion starting at zero. Since it follows from Fact 3 that $\tilde{X}_t$ changes its sign in any time interval $[0, \epsilon]$, $\epsilon > 0$ with probability one, there almost surely exists $t \in [0, \epsilon]$ such that $X_{\tau_p+t} \notin [0, a]$. Then we conclude that there is $t \in [0, T]$ such that $X_t(\omega) \notin [0, a]$ and that $\exists t \in [0, T), X_t(\omega) \in \{0, a\}$ is equivalent to $\tau_p < T$ with probability one. Since τ_p has a density, the left hand side of (2.17) equals to

$$\begin{aligned}&\mathbb{P}(\omega; \forall t \in [0, T], X_t(\omega) \in [0, a]; \tau_p < T)\\ =&\mathbb{P}(\omega; \forall t \in [\tau_b, T], X_t(\omega) \in [0, a]; \tau_p < T).\end{aligned} \tag{2.20}$$

With probability one, $\tilde{X}_t$ changes sign in interval $[\tau_b, T)$ and therefore (2.20) equals to zero. □

The following proposition is shown by proof similar to Proposition 1.1 in [Gob00].

Theorem 5. *Let $\tau_p^{(n)} := \inf\{t \in \mathbb{N}/n; X_t \notin [0, a]\}$. Then $1\!\!1_{\{T_1 < \tau_p^{(n)} \leq T_2\}}(\omega)$ converges almost surely to $1\!\!1_{\{T_1 < \tau_p \leq T_2\}}(\omega)$ as $n \to \infty$. In particular, $\mathbb{P}(\omega; X(\omega), 0 \models_n p\mathcal{U}_I \neg p)$ converges to $\mathbb{P}(\tau_p \in (T_1, T_2])$ as $n \to \infty$.*

Proof. From the definition of $\tau_p^{(n)}$, $X(\omega), 0 \models_n p\mathcal{U}_I\neg p$ if and only if $T_1 < \tau_p^{(n)} \leq T_2$. The convergence of $\mathbb{P}(\omega; X(\omega), 0 \models_n p\mathcal{U}_I\neg p)$ to $\mathbb{P}(\tau_p \in (T_1, T_2])$ follows from the almost sure convergence of $\mathbb{1}_{\{T_1<\tau_p^{(n)}\leq T_2\}}(\omega)$ to $\mathbb{1}_{\{T_1<\tau_p\leq T_2\}}(\omega)$ and *bounded convergence theorem.* Therefore it is sufficient to show that $\mathbb{1}_{\{T_1<\tau_p^{(n)}\leq T_2\}}(\omega)$ converges to $\mathbb{1}_{\{T_1<\tau_p\leq T_2\}}(\omega)$ almost surely. Since $\tau_p > T$ implies $\tau_p^{(n)} > T$ for any $n \in \mathbb{N}$ and $T > 0$, we obtain

$$\begin{aligned}
&\mathbb{1}_{\{\tau_p^{(n)}>T\}}(\omega) - \mathbb{1}_{\{\tau_p>T\}}(\omega) \\
=&\mathbb{1}_{\{\tau_p^{(n)}>T\}}(\omega)\mathbb{1}_{\{\tau_p>T\}}(\omega) + \mathbb{1}_{\{\tau_p^{(n)}>T\}}(\omega)\mathbb{1}_{\{\tau_p=T\}}(\omega) \\
&+ \mathbb{1}_{\{\tau_p^{(n)}>T\}}(\omega)\mathbb{1}_{\{\tau_p<T\}}(\omega) - \mathbb{1}_{\{\tau_p>T\}}(\omega) \\
=&\mathbb{1}_{\{\tau_p^{(n)}>T\}}(\omega)\mathbb{1}_{\{\tau_p=T\}}(\omega) + \mathbb{1}_{\{\tau_p^{(n)}>T\}}(\omega)\mathbb{1}_{\{\tau_p<T\}}(\omega)
\end{aligned} \tag{2.21}$$

The first term in the last line of (2.21) equals zero almost surely because τ_p has a density function on $[0, \infty)$. Then it remains to show that the second term goes to zero as $n \to \infty$. From Lemma 3, when $\tau_p < T$, there exists some $t \in (\tau_p, T)$ such that $X_t > a$. Since the path of $\{X_t\}_{t\geq 0}$ is continuous almost surely, there almost surely exists some $\delta > 0$ such that $X_{t'} > b$ for $t' \in [t-\delta, t+\delta]$. Then $\tau_p^{(n)} < T$ for any $n > 1/\delta$. Thus we have shown that the first line of (2.21) converges to zero almost surely. The statement of the theorem can be obtained by $\mathbb{1}_{\{T_1<\tau_p^{(n)}\leq T_2\}}(\omega) = \mathbb{1}_{\{T_1<\tau_p^{(n)}\}}(\omega) - \mathbb{1}_{\{T_2<\tau_p^{(n)}\}}(\omega)$ and $\mathbb{1}_{\{T_1<\tau_p\leq T_2\}}(\omega) = \mathbb{1}_{\{T_1<\tau_p\}}(\omega) - \mathbb{1}_{\{T_2<\tau_p\}}(\omega)$. □

The next two lemmas follow immediately from Fact 5 that τ_p has a density.

Lemma 8. *Put Condition 1. Then* $\mathbb{P}(\tau_p \in I) = \mathbb{P}(\tau_p \in (T_1, T_2))$.

Lemma 9. *For every* $T \in [0, \infty)$, $\mathbb{1}_{\{\tau_p^{(n)}=T\}}$ *converges to zero almost surely.*

Proof. Note that

$$\mathbb{1}_{\{\tau_p^{(n)}=T\}} = \mathbb{1}_{\{\tau_p^{(n)}=T\}}\mathbb{1}_{\{\tau_p=T\}} + \mathbb{1}_{\{\tau_p^{(n)}=T\}}\mathbb{1}_{\{\tau_p>T\}} + \mathbb{1}_{\{\tau_p^{(n)}=T\}}\mathbb{1}_{\{\tau_p<T\}}.$$

From Fact 5, the first term on the right–hand side equals zero almost surely. Since p is atomic, $\{t \in \mathbb{N}/n; X(\omega), t \models_n p\} \subset \{t \in [0, \infty); X(\omega), t \models p\}$ and hence $\tau_p > T$ implies $\tau_p^{(n)} > T$. Then we obtain that the second term equals zero. For the last term, suppose $\tau_p < T$. Then as in the proof of Theorem 5, Lemma 3 implies that $\tau_p^{(n)} < T$ for sufficiently large n and hence the last term converges to zero almost surely. □

Theorem 6. *Under Condition 1, it holds that*

$$\mathbb{P}(\tau_p^{(n)} \in I) \to \mathbb{P}(\tau_p \in (T_1, T_2)), \quad n \to \infty.$$

Proof. Condition 1 implies

$$|\mathbb{1}_{\{\tau_p^{(n)}\in I\}} - \mathbb{1}_{\{\tau_p^{(n)}\in(T_1,T_2]\}}| \leq \mathbb{1}_{\{\tau_p^{(n)}=T_1\}} + \mathbb{1}_{\{\tau_p^{(n)}=T_2\}}.$$

From Lemma 9, the right–hand side converges to zero almost surely. Then Theorem it follows from 5 and Lemma 8, $\mathbb{P}(\tau_p^{(n)} \in I) \to \mathbb{P}(\tau_p \in (T_1, T_2]) = \mathbb{P}(\tau_p \in (T_1, T_2)) > 0$. □

Therefore we can conclude Theorem 4 because Lemma 6 states $\mathbb{P}(\tau_p \in (T_1, T_2)) > 0$, while $\mathbb{P}(\omega; X(\omega), 0 \models \neg p\mathcal{U}_I p) = 0$.

2.10.2 The case of the diamond operators $\Diamond_I$ with the open intervals I

In the previous section, we show the counterexample of the discretization caused by the semantics of until formula:

$$X(\omega), t \models \phi_1 \mathcal{U}_I \phi_2 \iff \exists s \in I \text{ s.t.: } X(\omega), t+s \models \phi_2 \text{ and } \forall s' \in [t, t+s),\ X(\omega), s' \models \phi_1. \tag{2.22}$$

Here, remark that ϕ_1 holds *until just before* ϕ_2 happens and it does not have to hold at the timing that ϕ_2 happens.

Then it is natural to consider whether the same discretization converges when we define the until formula as follows:

$$X(\omega), t \models \phi_1 \mathcal{U}_I \phi_2 \iff \exists s \in I \text{ s.t.: } X(\omega), t+s \models \phi_2 \text{ and } \forall s' \in [t, t+s],\ X(\omega), s' \models \phi_1. \tag{2.23}$$

The only difference between the previous definition is that here we require ϕ_1 to hold at the time ϕ_2 happens. Note that we can give the semantics (2.23) of the until formulas a representation by the semantics (2.22):

$$\begin{aligned} &\exists s \in I \text{ s.t.: } X(\omega), t+s \models \phi_2 && \text{and } \forall s' \in [t, t+s],\ X(\omega), s' \models \phi_1 \\ \iff &\exists s \in I \text{ s.t.: } X(\omega), t+s \models \phi_2 \wedge \phi_1 && \text{and } \forall s' \in [t, t+s),\ X(\omega), s' \models \phi_1. \end{aligned}$$

Then the definition of MTL using the semantics (2.23) of the until formulas is a subclass of the semantics using (2.22) in the sense that every semantics of MTL–formula given by (2.23) can be represented as an MTL–formula given by (2.22).

Actually, in this section, we show that there is a counterexample of the discretization of whichever semantics we choose. We make a counterexample of an MTL–formula in which all temporal operators are of the form $\Diamond_I$ or $\Box_I$. When we restrict all temporal operators to be such forms, the semantics of the until operator is consistent with the choice of (2.22) or (2.23). Indeed, we obtain the following equivalence of diamond operators in both semantics of the until operator.

$$\begin{aligned} X(\omega), t \models \Diamond_I \phi &\iff \exists s \in I \text{ s.t.:} X(\omega), t+s \models \phi, \quad \forall t \in [0, \infty) \\ X(\omega), t \models_n \Diamond_I \phi &\iff \exists s \in I \text{ s.t.:} X(\omega), t+s \models_n \phi \quad \forall t \in \mathbb{N}/n. \end{aligned}$$

Now let us propose the counterexamples. Consider the case that X is one-dimensional Brownian motion starting from 0. Let p be an atomic formula, $B_p := [1, \infty)$ be the set associated with p and $\tau_p(\omega) := \inf\{t \geq 0; X_t(\omega) \in B_p\}$. In other words, $X(\omega), t \models p \Leftrightarrow X_t(\omega) \geq 1$ and $\tau_p(\omega) = \inf\{t \geq 0; X(\omega), t \models p\}$.

Put

$$\phi_1 := \Box_{(1,2)}(\Diamond_{(1,4)}p \wedge \neg\Diamond_{(1,3)}p) \tag{2.24}$$
$$\phi_2 := (\Diamond_{(1,3)}\phi_1) \wedge (\neg\Diamond_{(1,2)}\phi_1) \wedge (\neg\Diamond_{(2,3)}\phi_1), \tag{2.25}$$
$$\phi_3 := \Diamond_{(1,2)}\phi_2, \tag{2.26}$$
$$\psi := (\neg p) \wedge (\neg\Diamond_{(0,8)}p) \wedge \phi_3. \tag{2.27}$$

In one line,

$$\psi := (\neg p) \wedge (\neg\Diamond_{(0,8)}p) \wedge \Diamond_{(1,2)}[(\Diamond_{(1,3)}\Box_{(1,2)}(\Diamond_{(1,4)}p \wedge \neg\Diamond_{(1,3)}p)) \tag{2.28}$$
$$\wedge(\neg\Diamond_{(1,2)}\Box_{(1,2)}(\Diamond_{(1,4)}p \wedge \neg\Diamond_{(1,3)}p)) \tag{2.29}$$
$$\wedge(\neg\Diamond_{(2,3)}\Box_{(1,2)}(\Diamond_{(1,4)}p \wedge \neg\Diamond_{(1,3)}p))]. \tag{2.30}$$

In this setting, the following statements hold.

Proposition 2 (Remark 2.8.3 in [KS91])**.** *$\tau_p(\omega)$ has positive density on $[0, \infty)$.*

We will take ψ as a counterexample that $\mathbb{P}(\omega \in \Omega; X(\omega), 0 \models_n \psi)$ does not converges to $\mathbb{P}(\omega \in \Omega; X(\omega), 0 \models \psi)$. We show that $\mathbb{P}(\omega \in \Omega; X(\omega), 0 \models_n \psi) = 0$ for sufficiently large $n \in \mathbb{N}$, while $\mathbb{P}(\omega \in \Omega; X(\omega), 0 \models \psi) > 0$.

Before showing that ψ is the counterexample, let us describe the meaning of the formula ψ. Note that the following equivalences hold:

$$X(\omega), 0 \models \psi \Leftrightarrow [X(\omega), 0 \models (\neg p) \wedge (\neg\Diamond_{(0,8)}p)] \text{ and } [X(\omega), 0 \models \phi_3],$$
$$X(\omega), 0 \models_n \psi \Leftrightarrow [X(\omega), 0 \models_n (\neg p) \wedge (\neg\Diamond_{(0,8)}p)] \text{ and } [X(\omega), 0 \models_n \phi_3].$$

Let $\tau_p^{(n)}(\omega) := \min\{t \in \mathbb{N}/n; X(\omega), t \models_n p\}$. Then $X(\omega), 0 \models (\neg p) \wedge (\neg\Diamond_{(0,8)}p)$ and $X(\omega), 0 \models_n (\neg p) \wedge (\neg\Diamond_{(0,8)}p)$ means that $\tau_p \geq 8$ and $\tau_p^{(n)} \geq 8$, respectively. By combining these meanings with the semantics of $X(\omega), 0 \models \phi_3$ and $X(\omega), 0 \models_n \phi_3$ respectively, we show that $X(\omega), 0 \models \psi$ is equivalent to $\tau_p \in (8, 9)$ almost surely, while $X(\omega), 0 \models_n \psi$ is equivalent to $X(\omega), 0 \models_n \perp$ almost surely.

Now we estimate the probability $\mathbb{P}(\omega \in \Omega; X(\omega), t \models \psi)$ of continuous semantics of ψ.

Lemma 10. *Suppose that $\tau_p(\omega) \geq 6$. Then $[\![\phi_1]\!]_\omega$ has an isolated point $\tau_p(\omega) - 5$ with positive probability. Moreover, $X(\omega), t \not\models \phi_1$ for $t \in [0, \tau_p(\omega) - 5) \cup (\tau_p(\omega) - 5, \tau_p(\omega) - 3)$. In other words,*

$$\begin{cases} X(\omega), t \not\models \phi_1 & \text{for } 0 \leq t < \tau_p(\omega) - 5, \\ X(\omega), t \models \phi_1 & \text{for } t = \tau_p(\omega) - 5, \\ X(\omega), t \not\models \phi_1 & \text{for } \tau_p(\omega) - 5 < t < \tau_p(\omega) - 3. \end{cases}$$

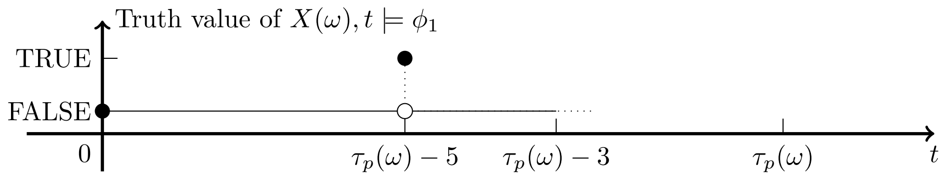
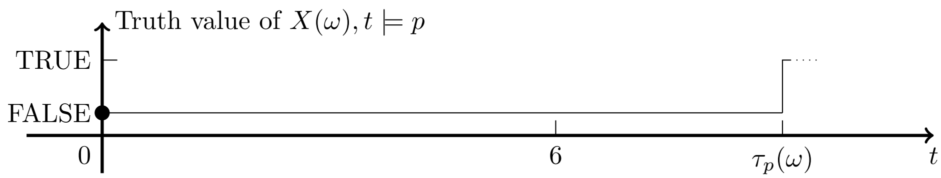


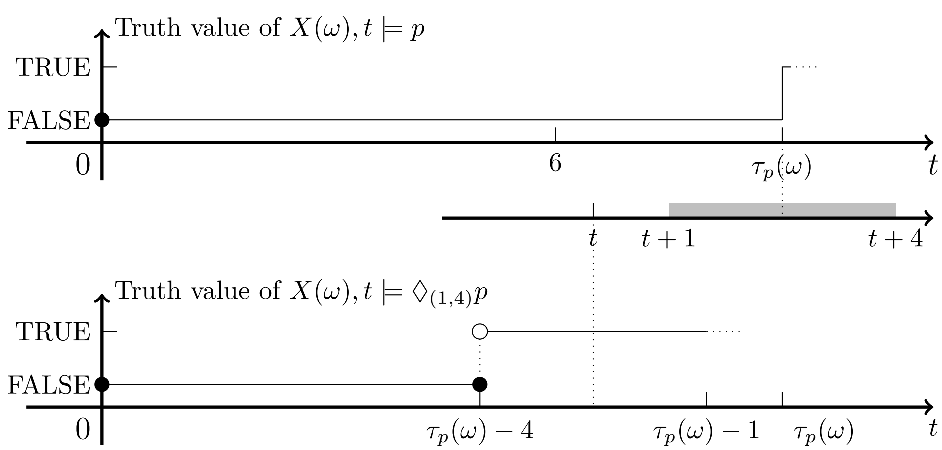
Figure 2.1: The truth value of "$X(\omega), t \models \phi_1$".

Proof. Note that $\tau_p(\omega) < \infty$ almost surely. Suppose $\tau_p(\omega) \geq 6$. Then $X(\omega), t \not\models p$ for $t < \tau_p(\omega)$ and $\inf\{t \geq 0; X(\omega), t \models p\} = \tau_p(\omega)$.



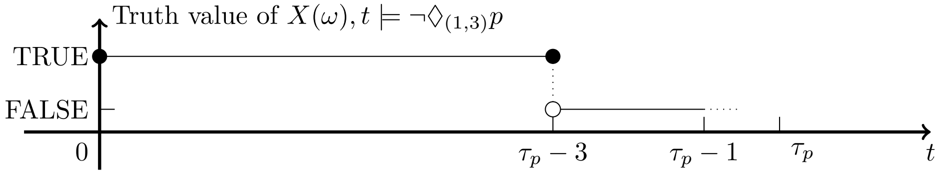
From the definition of $\tau_p(\omega)$, if $t \leq \tau_p(\omega) - 4$, there is no $s \in (t+1, t+4)$ such that $X(\omega), s \models p$, which implies $X(\omega), t \not\models \Diamond_{(1,4)} p$. Again from the definition of $\tau_p(\omega)$, if $t \in (\tau_p(\omega) - 4, \tau_p(\omega) - 1)$, there exists some $s \in (t+1, t+4)$ such that $X(\omega), s \models p$. Thus we obtain

$$\begin{cases} X(\omega), t \not\models \Diamond_{(1,4)} p & \text{for } t \leq \tau_p(\omega) - 4, \\ X(\omega), t \models \Diamond_{(1,4)} p & \text{for } \tau_p(\omega) - 4 < t < \tau_p(\omega) - 1. \end{cases}$$



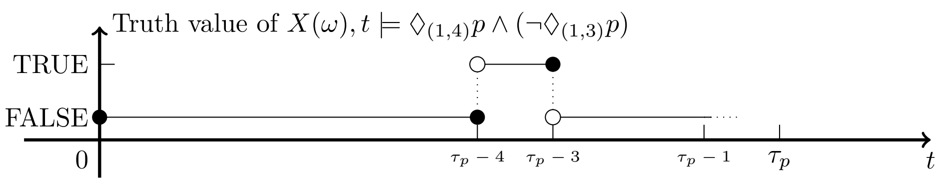
Similarly, we can show that

$$\begin{cases} X(\omega), t \models \neg\Diamond_{(1,3)} p & \text{for } 0 \leq t \leq \tau_p(\omega) - 3, \\ X(\omega), t \not\models \neg\Diamond_{(1,3)} p & \text{for } \tau_p(\omega) - 3 < t < \tau_p(\omega) - 1. \end{cases}$$

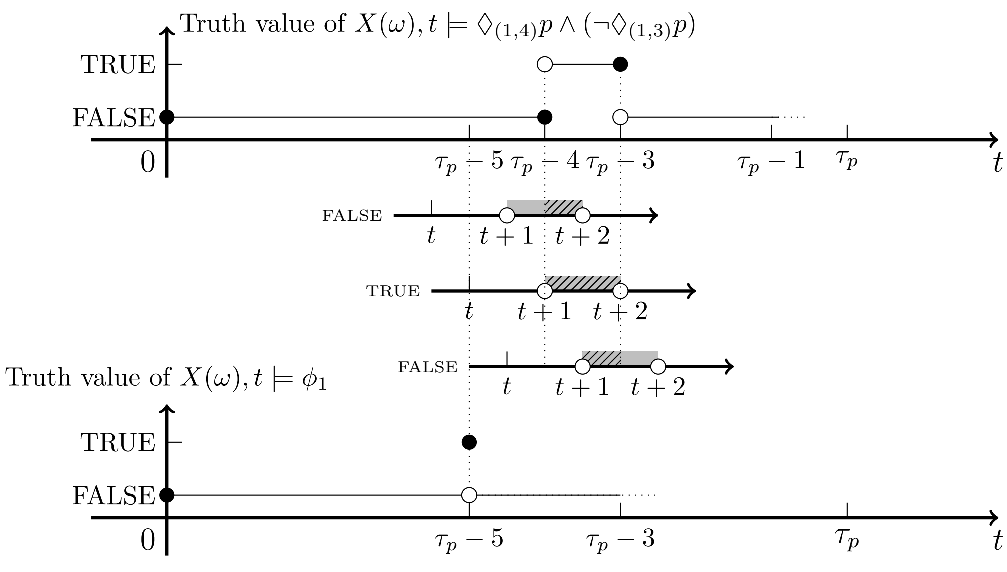


Consequently, $X(\omega), t \models \Diamond_{(1,4)}p \wedge (\neg\Diamond_{(1,3)}p)$ does not hold for any $t \in [0, \tau_p(\omega) - 4]$ and $t \in (\tau_p(\omega) - 3, \tau_p(\omega) - 1)$, but it holds for $(\tau_p(\omega) - 4, \tau_p(\omega) - 3]$. Namely,

$$\begin{cases} X(\omega), t \not\models \Diamond_{(1,4)}p \wedge (\neg\Diamond_{(1,3)}p) & \text{for } 0 \leq t \leq \tau_p(\omega) - 4, \\ X(\omega), t \models \Diamond_{(1,4)}p \wedge (\neg\Diamond_{(1,3)}p) & \text{for } \tau_p - 4(\omega) < t \leq \tau_p(\omega) - 3, \\ X(\omega), t \not\models \Diamond_{(1,4)}p \wedge (\neg\Diamond_{(1,3)}p) & \text{for } \tau_p(\omega) - 3 < t < \tau_p(\omega) - 1. \end{cases}$$



To satisfy $X(\omega), t \models \phi_1$, it must holds that $X(\omega), s \models \Diamond_{(1,4)}p \wedge (\neg\Diamond_{(1,3)}p)$ for every $s \in (t+1, t+2)$. Then $X(\omega), t \models \phi_1$ does not holds for $t \in [0, \tau_p(\omega) - 5)$ and $t \in (\tau_p(\omega) - 5, \tau_p(\omega) - 3)$ and holds on $t = \tau_p(\omega) - 5$. Since such an isolated point occurs whenever $\tau_p(\omega) \geq 6$, we obtain the required claim.



□

Lemma 11. *$X(\omega), 0 \models \psi$ is equivalent to $\tau_p(\omega) \in (8, 9)$ almost surely. In particular, $\mathbb{P}(X(\omega), 0 \models \psi) > 0$.*

Proof. First, we note that $X(\omega), 0 \models (\neg p) \wedge (\neg\Diamond_{(0,8)}p)$ is equivalent to $\tau_p(\omega) \geq 8$. Since $\tau_p(\omega)$ has density, $X(\omega), 0 \models \psi$ implies $\tau_p(\omega) > 8$ almost surely.

Suppose that $\tau_p(\omega) > 8$. Define

$$\tau_1(\omega) := \inf\{t; X(\omega), t \models \phi_1\},$$
$$\tau_2(\omega) := \inf\{t; X(\omega), t \models \phi_2\}.$$

Then, from Lemma 10,

$$\begin{cases} X(\omega), t \not\models \phi_1, & \text{for } t \in [0, \tau_p(\omega) - 5), \\ X(\omega), t \models \phi_1, & \text{at } t = \tau_p(\omega) - 5, \\ X(\omega), t \not\models \phi_1, & \text{for } t \in (\tau_p(\omega) - 5, \tau_p(\omega) - 3). \end{cases} \tag{2.31}$$

Hence $\tau_1(\omega) = \tau_p(\omega) - 5$. Furthermore, $X(\omega), t \models \phi_2$ means

$$\begin{cases} X(\omega), s \not\models \phi_1, & \text{for } s \in (t+1, t+2), \\ X(\omega), s \models \phi_1, & \text{at } s = t+2, \\ X(\omega), s \not\models \phi_1, & \text{for } s \in (t+2, t+3). \end{cases}$$

Then we can conclude from (2.31) that

$$\begin{cases} X(\omega), t \models \neg\phi_2, & \text{for } t \in [0, \tau_p(\omega) - 7) \\ X(\omega), t \models \phi_2, & \text{at } t = \tau_p(\omega) - 7, \\ X(\omega), t \models \neg\phi_2, & \text{for } t \in (\tau_p(\omega) - 7, \tau_p(\omega) - 5). \end{cases}$$

This means exactly $\tau_2(\omega) = \tau_p(\omega) - 7$.

Suppose that $\tau_p(\omega) \geq 9$. Then $\tau_2(\omega) = \tau_p(\omega) - 7 \geq 2$. Since $X(\omega), 0 \models \phi_3$ is equivalent to $\tau_2(\omega) \in (1, 2)$, $X(\omega), 0 \models \phi_3$ does not hold. Then $X(\omega), 0 \models \psi$ implies $\tau_p(\omega) < 9$. Consequently, we obtain that $X(\omega), 0 \models \psi$ implies $\tau_p(\omega) \in (8, 9)$ almost surely.

Conversely, as we have seen that $\tau_p(\omega) > 8$ implies $\tau_2(\omega) = \tau_p(\omega) - 7$, $\tau_p(\omega) \in (8, 9)$ implies $\tau_2(\omega) \in (1, 2)$. Then $X(\omega), 0 \models \phi_3$, and together with $\tau_p > 8$, we can conclude that $X(\omega), 0 \models \psi$. □

Lemma 12. *Let $n \geq 2$, $\tau_p^{(n)}(\omega) := \inf\{t \in \mathbb{N}/n; X(\omega), t \models_n p\}$, and suppose that $\tau_p^{(n)}(\omega) \geq 6$. Then it holds that*

$$\begin{cases} X(\omega), t \not\models_n \phi_1 & \textit{for } t = 0, 1/n, \cdots, \tau_p^{(n)}(\omega) - 5 - 1/n, \\ X(\omega), t \models_n \phi_1 & \textit{for } t = \tau_p^{(n)}(\omega) - 5, \tau_p^{(n)}(\omega) - 5 + 1/n, \\ X(\omega), t \not\models_n \phi_1 & \textit{for } t = \tau_p^{(n)}(\omega) - 5 + 2/n, \cdots, \tau_p^{(n)}(\omega) - 2 - 2/n. \end{cases}$$

Proof. By the definition of diamond operator, $X(\omega), t \models_n \Diamond_{(1,4)} p$ is equivalent to

$$(\exists s \in \{t + 1 + 1/n, t + 1 + 2/n \cdots t + 4 - 1/n\})[X, s \models_n p].$$

Then we observe from the definition of $\tau_p^{(n)}(\omega)$ that

$$\begin{cases} X(\omega), t \not\models_n \Diamond_{(1,4)} p & \text{for } t = 0, 1/n, \cdots, \tau_p^{(n)}(\omega) - 4 \\ X(\omega), t \models_n \Diamond_{(1,4)} p & \text{for } t = \tau_p^{(n)}(\omega) - 4 + 1/n, \tau_p^{(n)}(\omega) - 4 + 2/n, \cdots, \tau_p^{(n)}(\omega) - 1 - 1/n. \end{cases}$$

Similarly, we have

$$\begin{cases} X(\omega), t \models_n \neg\Diamond_{(1,3)} p & \text{for } t = 0, 1/n, \cdots, \tau_p^{(n)}(\omega) - 3 \\ X(\omega), t \not\models_n \neg\Diamond_{(1,3)} p & \text{for } t = \tau_p^{(n)}(\omega) - 3 + 1/n, \tau_p^{(n)}(\omega) - 3 + 2/n, \cdots, \tau_p^{(n)}(\omega) - 1 - 1/n. \end{cases}$$

Then we obtain

$$\begin{cases} X(\omega), t \not\models_n \Diamond_{(1,4)} p \wedge \neg\Diamond_{(1,3)} p & \text{for } t = 0, \cdots, \tau_p^{(n)}(\omega) - 4, \\ X(\omega), t \models_n \Diamond_{(1,4)} p \wedge \neg\Diamond_{(1,3)} p & \text{for } t = \tau_p^{(n)}(\omega) - 4 + 1/n, \cdots \tau_p^{(n)}(\omega) - 3, \\ X(\omega), t \not\models_n \Diamond_{(1,4)} p \wedge \neg\Diamond_{(1,3)} p & \text{for } t = \tau_p^{(n)}(\omega) - 3 + 1/n, \cdots, \tau_p^{(n)}(\omega) - 1 - 1/n. \end{cases}$$

From the definition of Box operator, $X, t \models_n \phi_1$ is equivalent to

$$(\forall s \in \{t + 1 + 1/n, \cdots, t + 2 - 1/n\})[X(\omega), t \models_n \Diamond_{(1,4)} p \wedge \neg\Diamond_{(1,3)} p].$$

Then we observe that

$$\begin{cases} X(\omega), t \not\models_n \phi_1 & \text{for } t = 0, 1/n, \cdots, \tau_p^{(n)}(\omega) - 5 - 1/n, \\ X(\omega), t \models_n \phi_1 & \text{for } t = \tau_p^{(n)}(\omega) - 5, \tau_p^{(n)}(\omega) - 5 + 1/n, \\ X(\omega), t \not\models_n \phi_1 & \text{for } t = \tau_p^{(n)}(\omega) - 5 + 2/n, \cdots, \tau_p^{(n)}(\omega) - 2 - 2/n. \end{cases}$$

□

Lemma 13. *Let* $n \geq 2$*. Then* $X(\omega), 0 \not\models_n \psi$ *for every* $\omega \in \Omega$*.*

Proof. Define $\tau_p^{(n)}(\omega)$ as Lemma 12. Since $X(\omega), 0 \models_n \psi$ implies $X(\omega), 0 \models_n (\neg p) \wedge (\neg\Diamond_{(0,8)} p)$, $\tau_p^{(n)}(\omega) \geq 8$. Then we obtain from Lemma 12 that

$$\begin{cases} X(\omega), t \not\models_n \phi_1 & \text{for } t = 0, 1/n, \cdots, \tau_p^{(n)}(\omega) - 5 - 1/n, \\ X(\omega), t \models_n \phi_1 & \text{for } t = \tau_p^{(n)}(\omega) - 5, \tau_p^{(n)}(\omega) - 5 + 1/n, \\ X(\omega), t \not\models_n \phi_1 & \text{for } t = \tau_p^{(n)}(\omega) - 5 + 2/n, \cdots, \tau_p^{(n)}(\omega) - 2 - 2/n. \end{cases}$$

From the definition of the discrete semantics, $X(\omega), t \models_n \phi_2$ is equivalent to

$$\begin{cases} X(\omega), s \not\models_n \phi_1 & \text{for } s = t + 1 + 1/n, \cdots, t + 2 - 1/n, \\ X(\omega), s \models_n \phi_1 & \text{for } s = t + 2, \\ X(\omega), s \not\models_n \phi_1 & \text{for } s = t + 2 + 1/n, \cdots, t + 3 - 1/n. \end{cases}$$

In other words, for $X(\omega), t \models_n \phi_2$ to hold, $X(\omega), s \models_n \phi_1$ must hold exactly on $s = t+2$, and $X(\omega), s \models_n \phi_1$ must not hold for other $s \in \mathbb{N}$ such that $t+1+1/n \leq s \leq t+3-1/n$.

However, $X(\omega), s \models_n \phi_1$ holds at two adjacent s values, namely $s = \tau_p^{(n)}(\omega) - 5$ and $s = \tau_p^{(n)}(\omega) - 5 + 1/n$, and does not hold for other s values such that $0 \leq s \leq \tau_p^{(n)}(\omega) - 2 - 2/n$. Therefore, $X(\omega), t \models_n \phi_2$ does not hold as long as $t + 2 \leq \tau_p^{(n)}(\omega) - 2 - 2/n$, which implies $t \leq \tau_p^{(n)}(\omega) - 4 - 2/n$. Since $\tau_p(\omega) \geq 8$, $X(\omega), t \models_n \phi_2$ does not hold as long as $t \leq 4 - 2/n$ and hence $X(\omega), 0 \not\models_n \Diamond_{(1,2)}\phi_2$. □

Theorem 7. *Put ψ as Lemma 11. Then $\mathbb{P}(\omega; X(\omega), 0 \models_n \phi_3)$ does not converges to $\mathbb{P}(\omega; X(\omega), 0 \models \phi_3)$.*

Proof. From Lemma 11, we have $\mathbb{P}(\omega; X(\omega), 0 \models \psi) = \mathbb{P}(\omega; \tau_p(\omega) \in (8, 9)) > 0$. On the other hand, from Lemma 13, we have $\mathbb{P}(\omega; X(\omega), 0 \models_n \psi) = 0$ for every n larger than 2. Therefore, $\mathbb{P}(\omega; X(\omega), 0 \models_n \psi)$ never converges to $\mathbb{P}(\omega; X(\omega), 0 \models \psi)$. □

2.10.3 The case of the diamond operator $\Diamond_I$ with the half–open intervals I

Consider the case that X is one-dimensional Brownian motion starting from 0 Let p be an atomic formula, $B_p := [1, \infty)$ be the set associated with p and $\tau_p(\omega) := inf\{t \geq 0; X_t(\omega) \in B_p\}$.

Definition 23. *Define*

$$\phi_1 := \Diamond_{[1,4)}p \wedge \neg\Diamond_{[1,3)}p, \tag{2.32}$$
$$\phi_2 := \Box_{[1,2)}\phi_1, \tag{2.33}$$
$$\phi_3 := (\neg\Diamond_{[0,6)}p) \wedge \Diamond_{[1,2)}\phi_2. \tag{2.34}$$

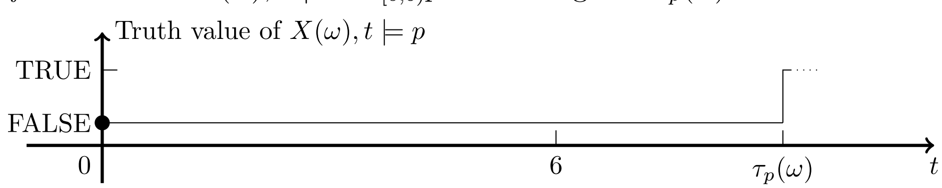
In one line,

$$\phi_3 = (\neg\Diamond_{[0,6)}p) \wedge \Diamond_{[1,2)}\Box_{[1,2)}(\Diamond_{[1,4)}p \wedge \neg\Diamond_{[1,3)}p). \tag{2.35}$$

Define $\tau_p(\omega) := \inf\{t \geq 0; X(\omega), t \models p\} = \inf\{t \geq 0; X_t(\omega) \in B_p\}$.

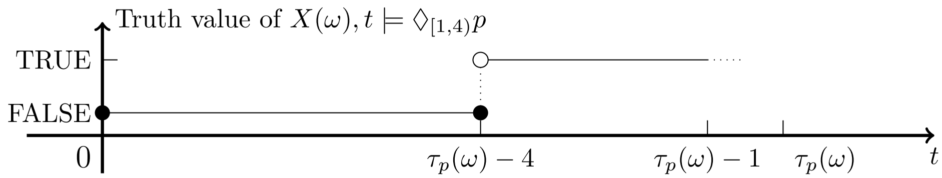
Lemma 14. *Put p, ϕ_1, ϕ_2 and ϕ_3 as Definition 23. Then $\mathbb{P}(X(\omega), 0 \models \phi_3) = 0$.*

Proof. Note that $X(\omega), 0 \models \neg\Diamond_{[0,6)}p$ is nothing but $\tau_p(\omega) \geq 6$.



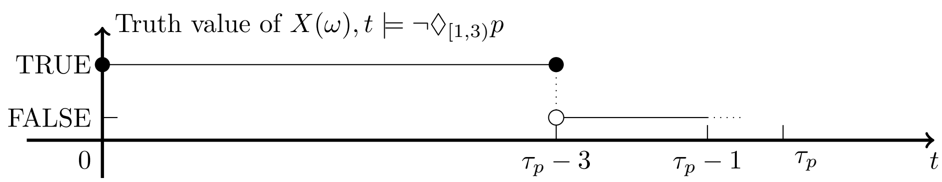
From the definition of $\tau_p(\omega)$, if $t \leq \tau_p(\omega) - 4$, there is no $s \in [t+1, t+4)$ such that $X(\omega), s \models p$, which implies $X(\omega), t \not\models \Diamond_{[1,4)}p$. Again from the definition of $\tau_p(\omega)$, if $t \in (\tau_p(\omega) - 4, \tau_p(\omega) - 1]$, there exists some $s \in [t+1, t+4)$ such that $X(\omega), s \models p$. Thus we obtain

$$\begin{cases} X(\omega), t \not\models \Diamond_{[1,4)}p & \text{for } t \leq \tau_p(\omega) - 4, \\ X(\omega), t \models \Diamond_{[1,4)}p & \text{for } \tau_p(\omega) - 4 < t \leq \tau_p(\omega) - 1. \end{cases}$$



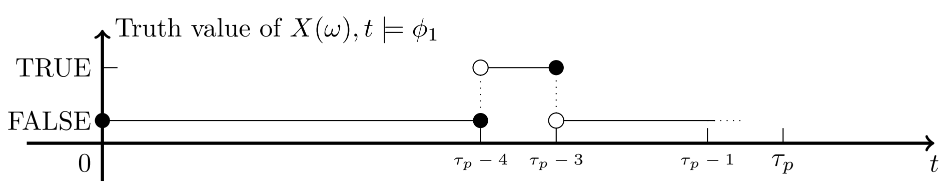
Similarly, we can show that

$$\begin{cases} X(\omega), t \models \neg\Diamond_{[1,3)} p & \text{for } 0 \le t \le \tau_p(\omega) - 3, \\ X(\omega), t \not\models \neg\Diamond_{[1,3)} p & \text{for } \tau_p(\omega) - 3 < t \le \tau_p(\omega) - 1. \end{cases}$$

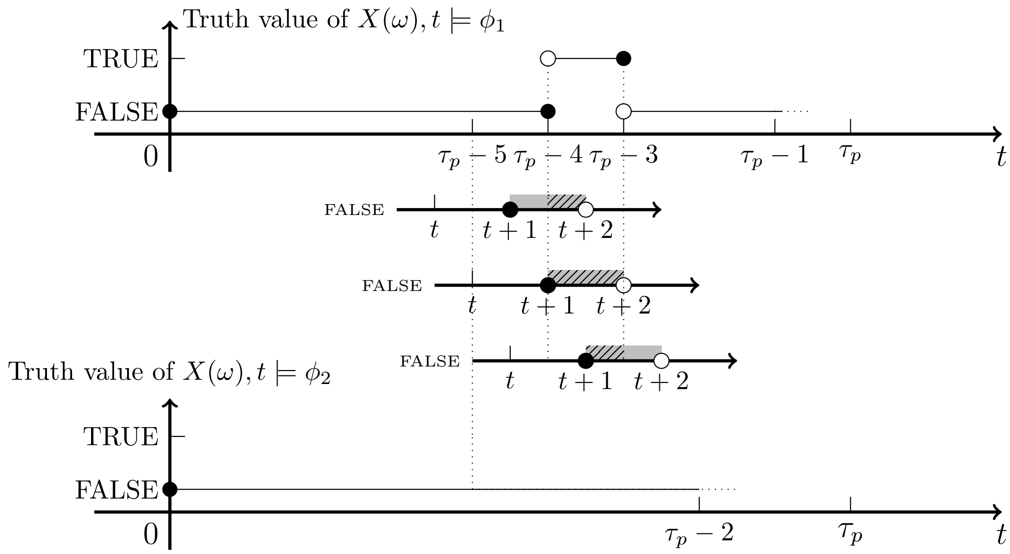


Consequently, $X(\omega), t \models \Diamond_{[1,4)} p \wedge (\neg\Diamond_{[1,3)} p)$ does not hold for any $t \in [0, \tau_p(\omega) - 4]$ and $t \in (\tau_p(\omega) - 3, \tau_p(\omega) - 1]$, but it holds for $(\tau_p(\omega) - 4, \tau_p(\omega) - 3]$. Namely,

$$\begin{cases} X(\omega), t \not\models \phi_1 & \text{for } 0 \le t \le \tau_p(\omega) - 4, \\ X(\omega), t \models \phi_1 & \text{for } \tau_p - 4(\omega) < t \le \tau_p(\omega) - 3, \\ X(\omega), t \not\models \phi_1 & \text{for } \tau_p(\omega) - 3 < t \le \tau_p(\omega) - 1. \end{cases}$$



To satisfy $X(\omega), t \models \phi_2$, it must holds that $X(\omega), s \models \phi_1$ for every $s \in [t+1, t+2)$. However, there is no such t in $[0, \tau_p(\omega) - 2]$. Indeed, for all such t, $[t+1, t+2) \setminus (\tau_p(\omega) - 4, \tau_p - 3(\omega)]$ and $[t+1, t+2) \setminus (\tau_p(\omega) - 2, \infty)$ are not empty set and any element s in it does not satisfy $X(\omega), s \models \phi_1$. Since $\tau_p(\omega) - 2 \ge 4$, there is no t in $[1, 2)$ such that $X(\omega), t \models \phi_2$ and hence $X(\omega), 0 \not\models \Diamond_{[1,2)} \phi_2$.



□

Next, we discretize the semantics of ϕ_3 and give a representation of it.

Lemma 15. *Put*

$$\tau_p^{(n)} := \inf\{t \in \mathbb{N}/n; X(\omega), t \models_n p\}.$$

Then $\tau_p^{(n)}(t) \to \tau_p(t)$ *for all* $t \in [0, \infty)$.

Proof. Since $X(\omega), t \models_n p$ implies $X(\omega), t \models p$ for $t \in \mathbb{N}/n$, $\tau_p^{(n)} \geq \tau_p$. Define $\tilde{X}_t := X_{\tau_p+t} - 1$ for $t \geq 0$. Then $\{\tilde{X}_t\}$ is standard Brownian motion and Fact 3 implies that $\tilde{X}_t$ changes its sign in any time interval $[0, \epsilon]$, $\epsilon > 0$ with probability one. Since the path of $\{X_t\}_{t\geq 0}$ is continuous, there exists some $\delta > 0$ such that $X_s > 1$ for $s \in [t-\delta, t+\delta]$. Then $\tau_p^{(n)} < \tau_p + \varepsilon$ for any $n > 1/\delta$. Therefore we conclude the statement. □

Lemma 16. *Define* ϕ_1, ϕ_2, ϕ_3 *as Definition 23. Then* $X(\omega), 0 \models \phi_3$ *is equivalent to* $\tau_p^{(n)} \in \{6, 6+1/n, \cdots, 7-3/n\}$.

Proof. From the definition of $\tau_p^{(n)}$, $X(\omega), 0 \models \neg\Box_{[0,6)} p$ is equivalent to $\tau_p^{(n)} \geq 6$. Then, if $X(\omega), 0 \models_n \phi_3$, then it holds that

$$\begin{cases} X(\omega), t \not\models_n \Diamond_{[1,4)} p & \text{for } t = 0, 1/n, \cdots, \tau_p^{(n)} - 4, \\ X(\omega), t \models_n \Diamond_{[1,4)} p & \text{for } t = \tau_p^{(n)} - 4 + 1/n, \cdots, \tau_p^{(n)}, \end{cases}$$

and

$$\begin{cases} X(\omega), t \not\models_n \Diamond_{[1,3)} p & \text{for } t = 0, 1/n, \cdots, \tau_p^{(n)} - 3, \\ X(\omega), t \models_n \Diamond_{[1,3)} p & \text{for } t = \tau_p^{(n)} - 3 + 1/n, \cdots, \tau_p^{(n)}. \end{cases}$$

Since $X(\omega), t \models_n \phi_1$ is $X(\omega), t \models_n \Diamond_{[1,4)} p$ and $X(\omega), t \not\models_n \Diamond_{[1,3)} p$, it holds that

$$\begin{cases} X(\omega), t \not\models_n \phi_1 & \text{for } t = 0, 1/n, \cdots, \tau_p^{(n)} - 4, \\ X(\omega), t \models_n \phi_1 & \text{for } t = \tau_p^{(n)} - 4 + 1/n, \cdots, \tau_p^{(n)} - 3, \\ X(\omega), t \not\models_n \phi_1 & \text{for } t = \tau_p^{(n)} - 3 + 1/n, \cdots, \tau_p^{(n)} - 1. \end{cases}$$

Now from the definition of Box operator, it holds that

$$\begin{aligned} & X(\omega), t \models_n \phi_2, \\ \Leftrightarrow & X(\omega), t \models_n \Box_{[1,2)} \phi_1, \\ \Leftrightarrow & X(\omega), s \models_n \phi_1 \text{ for all } s \in [t+1, t+2) \cap \mathbb{N}/n. \end{aligned}$$

Then the following time constraint for ϕ_2 holds:

$$\begin{cases} X(\omega), t \not\models_n \phi_2 & \text{for } t = 0, 1/n, \cdots, \tau_p^{(n)} - 5, \\ X(\omega), t \models_n \phi_2 & \text{for } t = \tau_p^{(n)} - 5 + 1/n, \\ X(\omega), t \not\models_n \phi_2 & \text{for } t = \tau_p^{(n)} - 5 + 2/n, \cdots, \tau_p^{(n)} - 3. \end{cases} \tag{2.36}$$

Then together with $\tau_p^{(n)} \geq 6$,

$$\begin{aligned} & X(\omega), 0 \models_n \phi_2 \\ \Rightarrow & \tau_p^{(n)} - 5 + 2/n \in [1, 2) \text{ and } \tau_p^{(n)} \geq 6 \\ \Rightarrow & \tau_p^{(n)} = 6, 6 + 1/n, \cdots, 7 - 3/n. \end{aligned}$$

Conversely, suppose that $\tau_p^{(n)} \in \{6, 6 + 1/n, \cdots, 7 - 3/n\}$. Then again since $\tau_p^{(n)} \geq 6$, then clearly $X(\omega), 0 \models_n \Diamond_{[0,6)} p$ holds and the same discussion as above can be applied to derive (2.36). Therefore $X(\omega), t \models_n \phi_2$ for some $t \in [1, 2)$ and hence $X(\omega), 0 \models_n \Diamond_{[1,2)} \phi_2$. □

We show the counterexample that the probability of discrete semantics for a stochastic process does not converge to that of continuous semantics as the time step goes to zero.

Theorem 8. *Let $p, \phi_1, \phi_2, \phi_3$ be defined as Definition 23. Then $\mathbb{P}(\omega; X(\omega), 0 \models_n \phi_3)$ does not converge to $\mathbb{P}(\omega; X(\omega), 0 \models \phi_3)$ even if $n \to \infty$.*

Proof. From Lemma 14, $\mathbb{P}(\omega; X(\omega), 0 \models \phi_3) = 0$. Hence it is enough to show that $\mathbb{P}(\omega; X(\omega), 0 \models_n \phi_3)$ converges to some positive number. Now define τ_p and $\tau_p^{(n)}$ as Definition 23 and Lemma 16. Since τ_p has positive probability density function on $[0, \infty)$ (see Remark 2.8.3 in [KS91]), $\mathbb{P}(\tau_p = 6) = \mathbb{P}(\tau_p = 7) = 0$. Then from Lemma 16,

$$\begin{aligned} & \mathbb{P}(\omega; X(\omega), 0 \models_n \phi_3) \\ = & \mathbb{P}(\tau_p^{(n)} \in \{6, \cdots, 7 - 3/n\}) \\ = & \mathbb{P}(\tau_p < 6 \text{ and } \tau_p^{(n)} \in \{6, \cdots, 7 - 3/n\}) \\ & + \mathbb{P}(\tau_p \in (6, 7) \text{ and } \tau_p^{(n)} \in \{6, \cdots, 7 - 3/n\}) \\ & + \mathbb{P}(\tau_p > 7 \text{ and } \tau_p^{(n)} \in \{6, \cdots, 7 - 3/n\}). \end{aligned}$$

From Lemma 15, $\tau_p^{(n)}$ converges almost surely to τ_p. Then it holds that

$$\mathbb{1}_{\{\tau_p<6 \text{ and } \tau_p^{(n)}\in\{6,\cdots,7-3/n\}\}} \to 0, \text{ a.s.},$$
$$\mathbb{1}_{\{\tau_p>7 \text{ and } \tau_p^{(n)}\in\{6,\cdots,7-3/n\}\}} \to 0, \text{ a.s.}.$$

while $\mathbb{1}_{\{\tau_p\in(6,7) \text{ and } \tau_p^{(n)}\in\{6,\cdots,7-3/n\}\}}$ converges $\mathbb{1}_{\{\tau_p\in(6,7)\}}$ almost surely. Then *Bounded Convergence Theorem* implies that $\mathbb{P}(\omega; X(\omega), 0 \models_n \phi_3)$ converges to $\mathbb{P}(\tau_p \in (6,7))$ as $n \to \infty$. Since τ_p has positive density function, $\mathbb{P}(\tau_p \in (6,7)) > 0$ and the statement holds. □

Just in the same way, we can show the following statement.

Theorem 9. *Define*

$$\phi_1 := \Diamond_{(1,4]}p \wedge \neg\Diamond_{(1,3]}p, \tag{2.37}$$
$$\phi_2 := \Box_{(1,2]}\phi_1, \tag{2.38}$$
$$\phi_3 := (\neg\Diamond_{(0,6]}p) \wedge \Diamond_{(1,2]}\phi_2. \tag{2.39}$$

In one line,

$$\phi_3 = (\neg\Diamond_{[0,6)}p) \wedge \Diamond_{(1,2]}\Box_{(1,2]}(\Diamond_{(1,4]}p \wedge \neg\Diamond_{(1,3]}p). \tag{2.40}$$

Then $\mathbb{P}(\omega; X(\omega), 0 \models_n \phi_3)$ *does not converge to* $\mathbb{P}(\omega; X(\omega), 0 \models \phi_3)$ *even if* $n \to \infty$.

Remark 11. *In Theorem 7, the subscripted interval for every diamond operator is restricted to be open. In Theorem 8–9, we restrict the intervals to be left– or right–open, and we constructed similar counterexamples. These results show that we can make a counterexample of convergence when we allow the subscripted intervals to be open, left–open, or right open in Definition 18. However, it remains an open problem whether we can create a counterexample when we restrict all intervals to be closed.*

2.11 Discretization of MTL formula: Convergence Result of ♭MTL formula

In the previous section, we presented a counterexample of an MTL formula, illustrating a case where its probability in discrete semantics does not converge to that in continuous semantics. In contrast, in this section, we establish the convergence of such probabilities by restricting MTL formulas. Specifically, we ensure that we only use diamond operators that do not nest. We refer to these restricted formulas as ♭MTL formulas. While we discussed MTL formulas for a Brownian motion in the preceding sections, we will now present the convergence result for general one-dimensional stochastic differential equations:

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t, \\ X_0 = \xi \in \mathbb{R}. \end{cases} \tag{2.41}$$

To establish the convergence result for $\flat$MTL formulas, we impose the following conditions on the SDE (2.41). These conditions are also sufficient to ensure the existence, uniqueness, and absolute continuity of the solution.(see Appendix A):

Assumption 2.

(i) *For every compact set $K \subset \mathbb{R}$, $\inf \sigma(K) > 0$.*

(ii) *σ is Lipschitz continuous.*

(iii) *b is bounded and Borel measurable.*

Now let us define $\flat$MTL–formulas rigorously.

Definition 24 (Syntax of $\flat$MTL formula)**.** *Let AP be a finite set of atomic formulas. We define the syntax of $\flat$MTL by the following induction.*

(i) *All atomic formulas are $\flat$MTL formulas.*

(ii) *If ϕ is an $\flat$MTL formula, $\neg\phi$ is an $\flat$MTL formula.*

(iii) *If ϕ_1 and ϕ_2 are $\flat$MTL formulas, then $\phi_1 \wedge \phi_2$ is an $\flat$MTL formula.*

(iv) *If p is a propositional formula and I is a positive interval on $[0, \infty)$, then $\Diamond_I p$ is an $\flat$MTL formula.*

Here we define propositional formulas as follows:

(a) *All atomic formulas are propositional formulas.*

(b) *If p is a propositional formula, $\neg p$ is a propositional formula.*

(c) *If p_1 and p_2 are prpositional formulas, then $p_1 \wedge p_2$ is a propositional formula.*

The semantics of $\flat$MTL are given in the same way as MTL formulas (see Section 2.6).

Definition 25 (Semantics of $\flat$MTL formulas)**.** *Let B_i, $i = 1, \cdots, k$ be Borel sets on $\mathbb{R}$ and $AP = \{a_i; i = 1, \cdots, k\}$ be the set of k atomic formulas. The semantics of $\flat$MTL formulas are defined inductively as follows.*

(i) $X(\omega), t \models a_i \Leftrightarrow X_t(\omega) \in B_i$ *for* $i = 1, \cdots, k$.

(ii) $X(\omega), t \models \phi_1 \wedge \phi_2$ *is equivalent to* $X(\omega), t \models \phi_1$ *and* $X(\omega), t \models \phi_2$.

(iii) $X(\omega), t \models \Diamond_I p$ *is equivalent to* $(\exists s \in I)[X(\omega), t + s \models p]$.

Remark 12. *Let us set $\sigma \equiv 1$ and $b \equiv 0$. In this case, both σ and b satisfy Assumption 2. As mentioned in Remark 6, the solution to the SDE (2.41) is given by the one-dimensional Brownian motion $\{W_t\}_{t\geq 0}$ itself. Therefore, the convergence result for probabilities discussed in this section can be applied to the case of one-dimensional Brownian motion.*

Additionally, as mentioned in Remark 9, the diamond operator $\Diamond_I$ and the box operator $\Box_I$ can be represented using the until operator. This allows us to represent every $\flat$MTL formula as an MTL formula without nesting of until operators.

Considering the counterexample presented in Section 2.8 and the discussion about nesting of temporal operators, we can observe the impact of nesting on the convergence of probabilities.

We will show the result of convergence for $\flat$MTL in Section 2.13. We show the convergence of the probability, by showing the convergence of the indicator function of $\flat$MTL formulas. Let us define the indicator functions for MTL formulas as follows:

Definition 26. *Let ϕ be an $\flat$–MTL formula and define random indicator functions $\chi_\phi(\omega, t)$ and $\chi_\phi^{(n)}(t)$ as*

$$\chi_\phi(\omega, t) := \begin{cases} 1 & \text{if } X(\omega), t \models \phi \\ 0 & \text{if } X(\omega), t \not\models \phi, \end{cases}$$

$$\chi_\phi^{(n)}(\omega, t) := \begin{cases} 1 & \text{if } X(\omega), \Lambda_n(t) \models_n \phi \\ 0 & \text{if } X(\omega), \Lambda_n(t) \not\models_n \phi, \end{cases}$$

where $\Lambda_n(t) := \frac{\lfloor nt \rfloor}{n}$.

The convergence of the indicator function for a formula implies the convergence of the probability of the formula. More precisely, our proof of the convergence is based on the following lemma:

Lemma 17. *Suppose that $\chi_\phi^{(n)}(\omega, t) \to \chi_\phi(\omega, t)$ almost surely. Then $\mathbb{P}(\omega; X(\omega), \Lambda_n(t) \models_n \phi) \to \mathbb{P}(\omega; X(\omega), t \models \phi)$ as $n \to \infty$.*

Proof. From the definition of $\chi_\phi(\omega, t)$ and $\chi_\phi^{(n)}(\omega, t)$, $\chi_\phi(\omega, t) = 1$ and $\chi_\phi^{(n)}(\omega, t) = 1$ is equivalent to $X(\omega), t \models \phi$ and $X(\omega), \Lambda_n(t) \models_n \phi$, respectively. Then $\mathbb{P}(\omega \in \Omega; X(\omega), t \models \phi) = \mathbb{E}[\chi_\phi(\omega, t)]$ and $\mathbb{P}(\omega \in \Omega; X(\omega), \Lambda_n(t) \models_n \phi) = \mathbb{E}[\chi_\phi^{(n)}(\omega, t)]$. Since $\chi_\phi(\omega, t) \leq 1$, $\chi_\phi^{(n)}(\omega, t) \leq 1$, and $\mathbb{E}[1] = 1$, we can apply *Lebesgue's dominated convergence theorem* (see Theorem 1.34 in [Rud66]) to observe $\mathbb{E}[\chi_\phi^{(n)}(\omega, t)] \to \mathbb{E}[\chi_\phi(\omega, t)]$. □

2.12 The case of $\Diamond_{\langle S,T\rangle} p$ with p corresponding to a union of intervals

Before proving the convergence of general $\flat$MTL formulas, we first show the convergence for a special type of $\flat$MTL formulas. In the subsequent subsection, we will present the proof for the convergence of $\flat$MTL formulas in the general case. In this subsection, we consider $\flat$MTL formulas of the form $\Diamond_{\langle S,T\rangle} p$, where p is a propositional formula. Here, $\langle S,T\rangle$ denotes a positive interval on $[0,\infty)$, specifically, $\langle S,T\rangle$ represents an interval with endpoints S and T such that $0 \leq S < T$. Note that the interval $\langle S,T\rangle$ can be open, left open, right open, or closed. In the proof of convergence in this case, we utilize deep insights from stochastic calculus, namely, the notion of *local maxima and local minima* of SDE (see Definition 28), and *the dense property of the zero set* of SDE (see Lemma 19).

To prove the convergence in the special case of $\Diamond_{\langle S,T\rangle} p$, we will introduce certain conditions on the propositional formula p.

Definition 27. *Let $B_1, \cdots, B_n$ be a finite family of Borel sets on $\mathbb{R}$. A pair B_i, B_j is said to be* separated *when $\overline{B_i} \cap \overline{B_j} = \emptyset$. We say the set $\{B_1, \cdots, B_n\}$ is* pairwise separated *when all pairs of different elements are separated.*

Now we prove the following theorem:

Theorem 10. *Let X be the strong solution of (2.41) satisfying Assumption 2. Let p be an MTL formula such that $X(\omega), t \models p$ is equivalent to $X_t \in B_p$, where*

$$B_p := \bigcup_{i=1}^{k} \langle x_i, y_i\rangle \tag{2.42}$$

is a union of pairwise separated positive intervals $\{\langle x_i, y_i\rangle; i = 1, \cdots, k\}$ on $\mathbb{R}$. Here B_p possibly equals the empty set or $\mathbb{R}$. Define $X(\omega), t \models_n p$ similarly. Define $\phi := \Diamond_{\langle S,T\rangle} p$, and $\psi := \Box_{\langle S,T\rangle} p$ where $\langle S,T\rangle$ is a positive interval on $[0,\infty)$. Then the following statements hold:

$$\chi_\phi^{(n)}(\omega) \overset{n\to\infty}{\longrightarrow} \chi_\phi(\omega), \quad a.s. \ ,$$
$$\chi_\psi^{(n)}(\omega) \overset{n\to\infty}{\longrightarrow} \chi_\psi(\omega), \quad a.s.$$

In particular,

$$\mathbb{P}(\omega; X(\omega), \Lambda_n(t) \models_n \phi) \overset{n\to\infty}{\to} \mathbb{P}(\omega; X(\omega), t \models \phi),$$
$$\mathbb{P}(\omega; X(\omega), \Lambda_n(t) \models_n \psi) \overset{n\to\infty}{\to} \mathbb{P}(\omega; X(\omega), t \models \psi).$$

The key to proving the convergence of MTL formulas with the diamond operator lies in the following inclusions:

$$[\![\phi]\!]_\omega \subset \overline{\mathrm{int}[\![\phi]\!]_\omega} \text{ almost surely,} \tag{2.43}$$
$$[\![\neg\phi]\!]_\omega \subset \overline{\mathrm{int}[\![\neg\phi]\!]_\omega} \text{ almost surely,} \tag{2.44}$$

where the time set $[\![\phi]\!]_\omega$ of MTL formula ϕ is defined in Definition 20.

In order to show Theorem 10, we will first prove a simplified version of the theorem in which the propositional formula corresponds to an interval. First, let us show (2.43) and (2.44) in this case:

Lemma 18. *Put Assumption 2. Let p be a propositional formula defined as $X(\omega), t \models p \Leftrightarrow X_t(\omega) \in \langle y_1, y_2 \rangle$, where $\langle y_1, y_2 \rangle$ is a positive interval. Then p satisfies (2.43) and (2.44) almost surely. Namely,*

$$[\![p]\!]_\omega \subset \overline{\mathrm{int}[\![p]\!]_\omega}, \quad a.s. \ ,$$
$$[\![\neg p]\!]_\omega \subset \overline{\mathrm{int}[\![\neg p]\!]_\omega}, \quad a.s.$$

To prove this lemma, we have to introduce *local minima* and *local maxima* of X:

Definition 28.

(i) Let $f : [0, \infty) \to \mathbb{R}$ be a given function. A number $t \geq 0$ is called a point of local maximum, if there exists a number $\delta > 0$ with $f(s) \leq f(t)$ valid for every $s \in [(t-\delta)^+, t+\delta]$; and a point of strict local maximum, if there exists a number $\delta > 0$ with $f(s) < f(t)$ valid for every $s \in [(t-\delta)^+, t+\delta] \setminus \{t\}$.

(ii) Let $f : [0, \infty) \to \mathbb{R}$ be a given function. A number $t \geq 0$ is called a point of local minimum, if there exists a number $\delta > 0$ with $f(s) \geq f(t)$ valid for every $s \in [(t-\delta)^+, t+\delta]$; and a point of strict local minimum, if there exists a number $\delta > 0$ with $f(s) > f(t)$ valid for every $s \in [(t-\delta)^+, t+\delta] \setminus \{t\}$.

Lemma 19. *Let $X = \{X_t\}_{t \geq 0}$ be the strong solution of the SDE (2.41) satisfying Assumption 2. Then, the following statements hold (see A for the proof):*

(i) Put

$$\mathcal{L}^a_\omega := \{t \geq 0; X_t(\omega) = a\}, \quad a \in \mathbb{R}, \ \omega \in \Omega. \tag{2.45}$$

Then $\mathcal{L}^a_\omega$ is dense–in–itself almost surely, for all $a \in \mathbb{R}$.

(ii) For almost every $\omega \in \Omega$, the set of points of local maximum and local minimum for the path $t \mapsto X(\omega)$ is dense in $[0, \infty)$, and all local maxima and local minima are strict.

Lemma 20. *Put Assumption 2. Let us define an atomic formula a as $X(\omega), t \models a \Leftrightarrow X_t(\omega) \in \langle y, \infty)$, where $\langle y, \infty)$ is half-line with open or closed endpoint $y \in \mathbb{R}$. Then a satisfies (2.43) and (2.44) almost surely. Namely,*

$$[\![a]\!]_\omega \subset \overline{\mathrm{int}[\![a]\!]_\omega}, \quad a.s. \ ,$$
$$[\![\neg a]\!]_\omega \subset \overline{\mathrm{int}[\![\neg a]\!]_\omega}, \quad a.s.$$

Proof. Put $\tilde{\Omega}$ be the set of $\omega \in \Omega$ with the following properties:

(i) The map $t \mapsto X_t(\omega)$ is continuous,

(ii) $\mathcal{L}^y_\omega := \{t \geq 0; X_t(\omega) = y\}$ is dense–in–itself,

(iii) the set of local maximum and local minimum of $t \mapsto X_t(\omega)$ is dense in $[0, \infty)$, and

(iv) all the local minima and the local maxima are strict

Then there exists some $\hat{\Omega} \in \mathcal{F}$ such that $\hat{\Omega} \subset \tilde{\Omega}$ and $\mathbb{P}(\hat{\Omega}) = 1$ because of Definition 17, Lemma 19. Indeed, let $\Omega_1, \Omega_1, \Omega_3, \Omega_4 \in \mathcal{F}$ be the sets of ω such that (i)–(iv) holds respectively and $\mathbb{P}(\Omega_1) = \mathbb{P}(\Omega_2) = \mathbb{P}(\Omega_3) = \mathbb{P}(\Omega_4) = 1$, then $\mathbb{P}(\bigcap_{i=1}^4 \Omega_i) = 1$ follows from Remark 2. From now on, let us prove that the formula a satisfies (2.43) and (2.44) for all $\omega \in \hat{\Omega}$.

Let $\langle y, \infty)$ be the left-closed interval $[y, \infty)$. The statement $t \in [\![\neg a]\!]_\omega$ is equivalent to $X_t(\omega) < y$. Since $t \mapsto X_t(\omega)$ is continuous, the set

$$[\![\neg a]\!]_\omega = \{t \geq 0; X_t(\omega) < y\}$$

is an open set, and therefore inclusion (2.44) holds clearly. On the other hand, due to the continuity of $t \mapsto X_t(\omega)$, if $X_t(\omega) > y$, then it implies that $t \in \overline{\text{int}\,[\![a]\!]_\omega}$. Hence, it remains to show that $X_t(\omega) = y$ implies $t \in \overline{\text{int}[\![a]\!]_\omega}$. Suppose $X_t(\omega) = y$ and $t \notin \overline{\text{int}[\![a]\!]_\omega}$. Then, there exists $\varepsilon > 0$ such that $(t - \varepsilon, t + \varepsilon) \cap \text{int}[\![a]\!]_\omega = \emptyset$. Since $(\exists s \in (t - \varepsilon, t + \varepsilon))[X_s(\omega) > y]$ implies $(t - \varepsilon, t + \varepsilon) \cap \text{int}[\![a]\!]_\omega \neq \emptyset$, it follows that $(\forall s \in (t - \varepsilon, t + \varepsilon))[X_s(\omega) \leq y]$. By applying (iii), we can conclude that t is a strict local maximum, i.e., $(\forall s \in (t - \varepsilon, t + \varepsilon) \setminus \{t\})[X_s(\omega) < y]$, and thus, t is an isolated point of $\{t \geq 0; X_t(\omega) = y\}$. However, this contradicts (ii). Therefore, we obtain the inclusion (2.43) .

On the other hand, consider $\langle y, \infty)$ as the left-open interval (y, ∞). Now, $t \in [\![a]\!]_\omega$ is equivalent to $X_t(\omega) > y$. Since $t \mapsto X_t(\omega)$ is continuous, the set

$$[\![a]\!]_\omega = \{t \geq 0; X_t(\omega) > y\}$$

is an open set, and thus inclusion (2.43) holds clearly. Moreover, due to the continuity of $t \mapsto X_t(\omega)$, if $X_t(\omega) < y$, then it implies $t \in \text{int}\,[\![\neg a]\!]_\omega$. Therefore, we need to show the inclusion (2.44) when $t \in [\![\neg a]\!]_\omega$ and $X_t(\omega) = y$. Suppose $t \notin \overline{\text{int}[\![\neg a]\!]_\omega}$, which implies the existence of $\varepsilon > 0$ such that $(t - \varepsilon, t + \varepsilon) \cap \text{int}[\![\neg a]\!]_\omega = \emptyset$. If $(\exists s \in (t - \varepsilon, t + \varepsilon))[X_s(\omega) < y]$, then it implies $(t - \varepsilon, t + \varepsilon) \cap \text{int}[\![\neg a]\!]_\omega \neq \emptyset$. Consequently, it holds that $(\forall s \in (t - \varepsilon, t + \varepsilon))[X_s(\omega) \geq y]$. By applying 19–(ii), we can deduce that t is a strict local minimum, i.e., $(\forall s \in (t - \varepsilon, t + \varepsilon) \setminus \{t\})[X_s(\omega) > y]$, which means t is an isolated point of $[\![\neg a]\!]_\omega$. However, this contradicts (ii). Thus, we obtain (2.44). □

Proof of Lemma 18. Let us define atomic formulas a, b as

$$X(\omega), t \models a \Leftrightarrow X_t(\omega) \in \langle y_1, \infty), \tag{2.46}$$

$$X(\omega), t \models b \Leftrightarrow X_t(\omega) \in \langle y_2, \infty), \tag{2.47}$$

where left endpoints y_1, y_2 can be open or closed and satisfy $y_1 < y_2$. Then we can define $X(\omega), t \models p$ is equivalent to $X(\omega), t \models a \wedge \neg b$ and hence it is enough to show that $a \wedge \neg b$ satisfies (2.43) and (2.44) almost surely. To see this, let $\tilde{\Omega}$ be the set of $\omega \in \Omega$ with the following properties

(i) The map $t \mapsto X_t(\omega)$ is continuous,

(ii) the formula a satisfies (2.43) and (2.44), and

(iii) the formula b satisfies (2.43) and (2.44).

Then Definition 17 and Lemma 20 imply that there exists some $\hat{\Omega} \in \mathcal{F}$ such that $\mathbb{P}(\hat{\Omega}) = 1$ and every $\omega \in \hat{\Omega}$ satisfies (i)–(iii). Now let us show (2.43) and (2.44) for every $\omega \in \hat{\Omega}$.

(2.43): Given that

$$[\![a \wedge \neg b]\!]_\omega \subset (\mathrm{int}[\![a]\!]_\omega \cap \mathrm{int}[\![\neg b]\!]_\omega) \cup \partial[\![a]\!]_\omega \cup \partial[\![\neg b]\!]_\omega,$$

and

$$\mathrm{int}[\![a]\!]_\omega \cap \mathrm{int}[\![\neg b]\!]_\omega = \mathrm{int}[\![a \wedge \neg b]\!]_\omega \subset \overline{\mathrm{int}[\![a \wedge \neg b]\!]_\omega},$$

then it is enough to show

$$\begin{aligned} [\![a \wedge \neg b]\!]_\omega \cap \partial[\![a]\!]_\omega &\subset \overline{\mathrm{int}[\![a \wedge \neg b]\!]_\omega} \\ [\![a \wedge \neg b]\!]_\omega \cap \partial[\![\neg b]\!]_\omega &\subset \overline{\mathrm{int}[\![a \wedge \neg b]\!]_\omega}. \end{aligned}$$

Suppose that $t \in [\![a \wedge \neg b]\!]_\omega \cap \partial[\![a]\!]_\omega$. If O is a neighborhood of t, $O \cap \mathrm{int}[\![a]\!]_\omega \neq \emptyset$ because $O \cap [\![a]\!]_\omega \neq \emptyset$ and (ii) hold. Since the path $t \mapsto X(\omega)$ is continuous, $\partial[\![a]\!]_\omega$ implies $X_t(\omega) = y_1 < y_2$ and hence $t \in \mathrm{int}[\![\neg b]\!]_\omega$. Let $\varepsilon > 0$. Since $(t-\varepsilon, t+\varepsilon) \cap \mathrm{int}[\![\neg b]\!]_\omega$ is a neighborhood of t,

$$(t-\varepsilon, t+\varepsilon) \cap \mathrm{int}[\![a]\!]_\omega \cap \mathrm{int}[\![\neg b]\!]_\omega = (t-\varepsilon, t+\varepsilon) \cap \mathrm{int}[\![a \wedge \neg b]\!]_\omega \neq \emptyset, \quad \text{a.s. },$$

and hence $t \in \overline{\mathrm{int}[\![a \wedge \neg b]\!]_\omega}$.

When $t \in [\![a \wedge \neg b]\!]_\omega \cap \partial[\![\neg b]\!]_\omega$, the same argument can be applied. Thus we have shown (2.43).

(2.44): Suppose that $t \in [\![\neg a \vee b]\!]_\omega$ and let O be a neighborhood of t. Since $t \in [\![\neg a]\!]_\omega$ or $t \in [\![b]\!]_\omega$, (ii) and (iii) imply that $O \cap \mathrm{int}[\![\neg a]\!]_\omega \neq \emptyset$ or $O \cap \mathrm{int}[\![b]\!]_\omega \neq \emptyset$ holds. Since $\mathrm{int}[\![\neg a]\!]_\omega \cup \mathrm{int}[\![b]\!]_\omega \subset \mathrm{int}([\![\neg a]\!]_\omega \cup [\![b]\!]_\omega)$, it holds that $O \cap \mathrm{int}[\![\neg a \vee b]\!]_\omega = O \cap \mathrm{int}([\![\neg a]\!]_\omega \cup [\![b]\!]_\omega) \neq \emptyset$. Then $t \in \overline{\mathrm{int}[\![\neg a \vee b]\!]_\omega}$.

□

We can interpret the boundary $\partial[\![\phi]\!]_\omega$ of time set $[\![\phi]\!]_\omega$ as the time that the indicator function $\chi_\phi(\omega, t)$ in Definition 26 changes its value. The next lemma shows that the boundary $\partial[\![\phi]\!]_\omega$ of every MTL formula ϕ has Lebesgue measure zero almost surely if the stochastic process X has a density and AP is distinct in the sense of the following definition.

Definition 29. *We say that a Borel set B on $\mathbb{R}$ is* distinct *if its boundary ∂B has Lebesgue measure zero. We say an atomic formula $a \in AP$ is* distinct *when the corresponding set B_a is distinct. The set of AP of atomic formulas is said to be* distinct *when all $a \in AP$ are distinct.*

Lemma 21. *Consider the case of the distinct set AP of atomic formulas. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Suppose that X is an almost surely continuous stochastic process such that X_t has a density for every $t \in (0, \infty)$. Then, for every MTL formula ϕ there exists some measurable set $K \in \mathcal{F} \otimes \mathcal{B}([0, \infty))$ such that*

(i) $\{t; (\omega, t) \in K\}$ *is almost surely closed,*

(ii) $\{t; (\omega, t) \in K\}$ *has Lebesgue measure zero almost surely,*

(iii) $\mathbb{P}(\{\omega; (\omega, t) \in K\}) = 0$ *for every $t \in (0, \infty)$, and*

(iv) $\partial[\![\phi]\!]_\omega \subset \{t; (\omega, t) \in K\}$.

For this proposal, in the following lemma, we show that the boundary of the time set of the form $\Diamond_{\langle S,T \rangle}\phi$ is restricted to the shift of the boundary of the form ϕ.

Lemma 22. *Let ϕ be an MTL formula and $\langle S, T\rangle$ be a positive interval on $[0, \infty)$. Then it holds almost surely that*

$$\partial[\![\Diamond_{\langle S,T\rangle}\phi]\!]_\omega \subset [(\partial[\![\phi]\!]_\omega \ominus S) \cup (\partial[\![\phi]\!]_\omega \ominus T)], \tag{2.48}$$

where $\partial[\![\phi]\!]_\omega \ominus S := \{t - S; t \in \partial[\![\phi]\!]_\omega\} \cap [0, \infty)$ and $\partial[\![\phi]\!]_\omega \ominus T := \{t - T; t \in \partial[\![\phi]\!]_\omega\} \cap [0, \infty)$.

Proof. Let $\langle S, T\rangle$ be closed interval $[S, T]$. Suppose that $t \in \partial[\![\Diamond_{\langle S,T\rangle}\phi]\!]_\omega$. Then it is clear that $(t + S, t + T) \cap [\![\phi]\!]_\omega = \emptyset$. If not, there exists some neighborhood of t whose every element s satisfies $(s + S, s + T) \cap [\![\phi]\!]_\omega \neq \emptyset$ and hence $t \notin \partial[\![\Diamond_{\langle S,T\rangle}\phi]\!]_\omega$. Again from $t \in \partial[\![\Diamond_{\langle S,T\rangle}\phi]\!]_\omega$, one of the following two statement holds:

(i) There exist some sequence $t_n,\ n = 1, 2, 3, \cdots$ in $[\![\Diamond_{[S,T]}\phi]\!]_\omega$ such that $\sup_n t_n = t$.

(ii) There exist some sequence $t_n,\ n = 1, 2, 3, \cdots$ in $[\![\Diamond_{[S,T]}\phi]\!]_\omega$ such that $\inf_n t_n = t$.

It is enough to show $S + t \in \partial[\![\phi]\!]_\omega$ or $T + t \in \partial[\![\phi]\!]_\omega$ for (i) and (ii).

(i) Since $(S+t, T+t) \subset [\![\neg\phi]\!]_\omega$, $(S+t, S+t+\varepsilon) \cap [\![\neg\phi]\!]_\omega \neq \phi$ for every positive ε. Together with $[\![\phi]\!]_\omega \cap [S+t_n, T+t_n] \neq \emptyset$, $(S+t, T+t) \subset [\![\neg\phi]\!]_\omega$ also implies $[S+t_n, S+t] \cap [\![\phi]\!]_\omega \neq \emptyset$ for every $n \in \mathbb{N}$. Then $(S+t-\varepsilon, S+t] \cap [\![\phi]\!]_\omega \neq \emptyset$ for every positive ε.

(ii) We can show $(T+t-\varepsilon, T+t) \cap [\![\neg\phi]\!]_\omega \neq \emptyset$ and $[T+t, T+t+\varepsilon) \cap [\![\phi]\!]_\omega \neq \emptyset$ by showing $[T+t, T+t_n] \cap [\![\phi]\!]_\omega \neq \emptyset$. Indeed, $(S+t, T+t) \cap [\![\phi]\!]_\omega = \emptyset$ and $[S+t_n, T+t_n] \cap [\![\phi]\!]_\omega \neq \emptyset$ implies $[T+t, T+t_n] \cap [\![\phi]\!]_\omega \neq \emptyset$.

Thus we show the statement when $\langle S, T\rangle$ is closed. We can prove the case of $\langle a, b\rangle = (S,T), [S,T), [S,T)$ in the same way. □

Proof of Lemma 21. Since the map $t \mapsto X_t(\omega)$ is almost surely continuous, there exists some $N \in \mathcal{F}$ such that $\mathbb{P}(N) = 0$ and $t \mapsto X_t(\omega)$ is continuous whenever $\omega \notin N$. Suppose a is an atomic formula and $t \in \partial[\![a]\!]_\omega$. For any positive ε, we can find s and s' in the interval $(t-\varepsilon, t+\varepsilon)$ such that $X_s(\omega) \in B_a$ and $X_{s'}(\omega) \notin B_a$. This is because t is a boundary point of the satisfaction set $[\![a]\!]_\omega$. Since the mapping $t \mapsto X_t(\omega)$ is continuous for $\omega \notin N$, it follows that $X_t(\omega)$ lies on the boundary ∂B_a. In other words, $X_t(\omega)$ is located on the boundary of the set defined by the atomic formula a. Put $K := \{(\omega, t); X_t(\omega) \in \partial B_a\} \cup N \times [0,\infty)$ and $K_\omega := \{t; (\omega,t) \in K\}$. Hence we get $\partial[\![a]\!]_\omega \subset K_\omega$. Since $t \mapsto X_t(\omega)$ is continuous almost surely and X_t has a density for every $t > 0$, K is measurable, K_ω is almost surely closed,

$$\mathbb{P}(\{\omega; (\omega,t) \in K\}) \leq \mathbb{P}(\omega; X_t(\omega) \in \partial B_a) + \mathbb{P}(N) = 0 \quad \forall t \in (0,\infty).$$

Then it holds that

$$\int_{[0,\infty)} \left\{ \int_\Omega \mathbb{1}_K(\omega,t)\mathbb{P}(d\omega) \right\} dt = 0. \tag{2.49}$$

By using Fubini's Theorem (see Theorem 8.8 in [Rud66]), we have

$$\int_\Omega \left\{ \int_{[0,\infty)} \mathbb{1}_K(\omega,t)dt \right\} \mathbb{P}(d\omega) = 0,$$

which implies that K_ω has Lebesgue measure zero almost surely (see (b) of Theorem 1.39 in [Rud66]). When K corresponds to a formula ϕ with (i)-(iv), then K also satisfies (i)-(iv) for$\neg\phi$, since $\partial[\![\neg\phi]\!]_\omega = \partial[\![\phi]\!]_\omega$. When K_1 and K_2 satisfy (i)-(iv) for ϕ_1 and ϕ_2 respectively, $K_1 \cup K_2$ satisfies (i)-(iv) for $\phi_1 \wedge \phi_2$, since $\{t; (\omega,t) \in K_1 \cup K_2\}$ is closed, $\mathbb{P}(\omega; (\omega,t) \in K_1 \cup K_2) = 0$ for $t \in (0,\infty)$, and $\partial[\![\phi_1 \wedge \phi_2]\!]_\omega \subset \{t; (\omega,t) \in K_1 \cup K_2\}$. Suppose that K satisfies (i)-(iv) for ϕ. We show that $\{(\omega,t); t \in [(K_\omega \ominus S) \cup (K_\omega \ominus T)]\}$ satisfies (i)–(iv) for $\Diamond_{\langle S,T\rangle}\phi$.

(i) Since K_ω is closed almost surely, $(K_\omega \ominus S)$ and $(K_\omega \ominus T)$ are almost surely closed and then $(K_\omega \ominus S) \cup (K_\omega \ominus T)$ is closed almost surely.

(ii) From (2.49), it holds that

$$\int_{[0,\infty)}\left\{\int_{\Omega}\mathbb{1}_{\{t\in K_\omega\ominus S\}}(\omega,t)\mathbb{P}(d\omega)\right\}dt=\int_{[S,\infty)}\left\{\int_{\Omega}\mathbb{1}_{\{t\in K_\omega\}}(\omega,t)\mathbb{P}(d\omega)\right\}dt=0,$$
$$\int_{[0,\infty)}\left\{\int_{\Omega}\mathbb{1}_{\{t\in K_\omega\ominus T\}}(\omega,t)\ \mathbb{P}(d\omega)\right\}dt=\int_{[T,\infty)}\left\{\int_{\Omega}\mathbb{1}_{\{t\in K_\omega\}}(\omega,t)\mathbb{P}(d\omega)\right\}dt=0.$$

By Fubini's theorem, we have

$$\int_{\Omega}\left\{\int_{[0,\infty)}\mathbb{1}_{\{t\in K_\omega\ominus S\}}(\omega,t)dt\right\}\mathbb{P}(d\omega)=0,$$
$$\int_{\Omega}\left\{\int_{[0,\infty)}\mathbb{1}_{\{t\in K_\omega\ominus T\}}(\omega,t)dt\right\}\mathbb{P}(d\omega)=0,$$

which implies that $\{(\omega,t);t\in(K_\omega\ominus S)\cup(K_\omega\ominus T)\}$ has Lebesgue measure zero almost surely.

(iii) When $t>0$, we have

$$\begin{aligned}&\mathbb{P}(\omega;t\in(K_\omega\ominus S)\cup(K_\omega\ominus T))\\ \leq&\mathbb{P}(\omega;t\in(K_\omega\ominus S))+\mathbb{P}(\omega;t\in(K_\omega\ominus T))\\ \leq&\mathbb{P}(\omega;t+S\in K_\omega)+\mathbb{P}(\omega;t+T\in K_\omega)=0.\end{aligned}$$

(iv) From Lemma 22, $\partial[\![\Diamond_{\langle S,T\rangle}\phi]\!]_\omega\subset[(\partial[\![\phi]\!]_\omega\ominus S)\cup(\partial[\![\phi]\!]_\omega\ominus T)]\subset[(K_\omega\ominus S)\cup(K_\omega\ominus T)]$ almost surely.

□

In the next lemma, we give a sufficient condition for convergence of the indicator function of the formula with a diamond or box operator.

Lemma 23. *Let X be the solution of SDE (2.41) satisfying Assumption 2. Define an MTL formula p as*

$$X(\omega),t\models p\Leftrightarrow X_t(\omega)\in B_p$$

for some positive interval B_p on $\mathbb{R}$. Let $\langle S,T\rangle$ be a positive interval on $[0,\infty)$. If p satisfies (2.43) and (2.44), the following statements hold:

(i) Define $\phi:=\Diamond_{\langle S,T\rangle}p$. Then $\chi_\phi^{(n)}(\omega,t)\to\chi_\phi(\omega,t)$ for every $t\in[0,\infty)$.

(ii) Define $\psi:=\Box_{\langle S,T\rangle}p$. Then $\chi_\psi^{(n)}(\omega,t)\to\chi_\psi(\omega,t)$ for every $t\in[0,\infty)$.

Here, $\chi_\phi^{(n)}(\omega,t)$, $\chi_\psi^{(n)}(\omega,t)$, $\chi_\phi(\omega,t)$, and $\chi_\psi(\omega,t)$ are the indicator functions defined in Definition 26.

Proof. First, let us show that $\langle t+S, t+T\rangle \cap [\![p]\!]_\omega \neq \emptyset$ implies $(t+S, t+T)\cap \mathrm{int}[\![p]\!]_\omega \neq \emptyset$ almost surely. If $\partial[\![p]\!]_\omega \notin \langle t+S, t+T\rangle$ and $\langle t+S, t+T\rangle \cap [\![p]\!]_\omega \neq \emptyset$, then it follows that $X(\omega), s \models p$ for all $s \in \langle t+S, t+T\rangle$, and therefore, $\langle t+S, t+T\rangle \subset [\![p]\!]_\omega$. Consequently, $(t+S, t+T)\cap \mathrm{int}[\![p]\!]_\omega \neq \emptyset$. Next, suppose $\partial[\![p]\!]_\omega \in \langle t+S, t+T\rangle$ and $\langle t+S, t+T\rangle \cap [\![p]\!]_\omega \neq \emptyset$. Since $\{X_t\}_{t\geq 0}$ satisfies Assumption 2, X_t has a density for $t > 0$ by 5. When $t+S > 0$, we have $t+T > t+S > 0$, and Lemma 21 implies that $t+S$ and $t+T$ do not belong to $\partial[\![p]\!]_\omega$ almost surely. Thus, we have $\partial[\![p]\!]_\omega \cap (t+S, t+T) \neq \emptyset$, which implies $[\![p]\!]_\omega \cap (t+S, t+T) \neq \emptyset$. Therefore, we conclude from (2.43) that $(t+S, t+T)\cap \mathrm{int}[\![p]\!]_\omega \neq \emptyset$. If $t+S=0$, $\partial[\![p]\!]_\omega$ intersects the open set of the form $[0, t+T)$ or $(0, t+T)$ on $[0,\infty)$. Then, from (2.43), we can conclude $\mathrm{int}[\![p]\!]_\omega \cap (t+S, t+T) \neq \emptyset$.

We can show in similar way that $\langle t+S, t+T\rangle \cap [\![\neg p]\!]_\omega \neq \emptyset$ implies $(t+S, t+T)\cap \mathrm{int}[\![\neg p]\!]_\omega \neq \emptyset$ almost surely.

Now let us prove (i) and (ii). Suppose $X(\omega), t \models \phi$. Since $(t+S, t+T)\cap \mathrm{int}[\![p]\!]_\omega$ is a nonempty open set, there exists $s \in (\Lambda_n(t)+S, \Lambda_n(t)+T)\cap \mathbb{N}/n$ such that $X(\omega), s \models_n p$ for sufficiently large n. Hence, $X, \Lambda_n(t) \models_n \phi$. By applying the same argument, we can show from (2.44) that if $X(\omega), t \not\models \psi$, then $X(\omega), \Lambda_n(t) \not\models_n \psi$ for sufficiently large n.

On the other hand, suppose $X(\omega), t \not\models \phi$. Then $[\![p]\!]_\omega \cap (S,T) = \emptyset$ and $\partial[\![p]\!]_\omega \cap (S,T) = \emptyset$. If $t+S>0$, according to 5 and Lemma 21, $t+S$ and $t+T$ do not belong to $\partial[\![p]\!]_\omega$ almost surely. Thus, there exists $\varepsilon > 0$ such that $(t+S-\varepsilon, t+T-\varepsilon) \subset [\![\neg p]\!]_\omega$, and hence $(\Lambda_n(t)+S, \Lambda_n(t)+T) \cap [\![p]\!]_\omega = \emptyset$ for sufficiently large n. If $t+S=0$, since $\Lambda_n(t) = t = 0$, it holds that

$$
\begin{aligned}
&X(\omega), \Lambda_n(t) \not\models_n \phi \Leftrightarrow X(\omega), 0 \not\models_n \Diamond_{\langle 0,T\rangle} p, \\
&X(\omega), t \not\models \phi \Leftrightarrow X(\omega), 0 \not\models \Diamond_{\langle 0,T\rangle} p.
\end{aligned}
$$

Then it is clear that $X(\omega), t \not\models \phi$ implies $X(\omega), \Lambda_n(t) \not\models_n \phi$. Now we have shown that $X(\omega), t \not\models_n \phi$ for sufficiently large n. The same argument can be applied to prove that if $X(\omega), t \models \psi$, then $X(\omega), \Lambda_n(t) \models_n \psi$ for sufficiently large n. □

Lemma 24. *Suppose that a propositional formula p satisfies the conditions introduced in the statement of Theorem 10. Specifically, let $\langle x_i, y_i\rangle$, $i = 1, \cdots, k$ be pairwise disjoint positive intervals, and define $B_p := \bigcup_{i=1}^k \langle x_i, y_i\rangle$. Define a propositional formula $p, p_1, \cdots, p_k$ by*

$$
\begin{aligned}
&X(\omega), t \models p \Leftrightarrow X_t(\omega) \in B_p, \\
&X(\omega), t \models_n p \Leftrightarrow X_t(\omega) \in B_p, \\
&X(\omega), t \models p_i \Leftrightarrow X_t(\omega) \in \langle x_i, y_i\rangle \quad \textit{for } i = 1\cdots, k, \\
&X(\omega), t \models_n p_i \Leftrightarrow X_t(\omega) \in \langle x_i, y_i\rangle \quad \textit{for } i = 1\cdots, k.
\end{aligned}
$$

Then p satisfies (2.43) and (2.44). Namely,

$$
\begin{aligned}
[\![p]\!]_\omega &\subset \overline{\mathrm{int}[\![p]\!]_\omega}, \quad a.s. \ , \\
[\![\neg p]\!]_\omega &\subset \overline{\mathrm{int}[\![\neg p]\!]_\omega}, \quad a.s.
\end{aligned}
$$

Proof. First note that

$$X(\omega), t \models p \Leftrightarrow X(\omega), t \models \bigvee_{i=1}^{k} p_i,$$

$$X(\omega), t \models_n p \Leftrightarrow X(\omega), t \models_n \bigvee_{i=1}^{k} p_i,$$

where $\bigvee_{i=1}^{k} p_i = p_1 \vee p_2 \vee \cdots \vee p_k$. If $B_p = \emptyset$ or $B_p = \mathbb{R}$, clearly $[\![p]\!]_\omega = \emptyset$ or $[\![p]\!]_\omega = [0, \infty)$, respectively. Hence (2.43) and (2.44) holds. Otherwise, From Lemma 18, every p_i satisfies (2.43) and (2.44) almost surely. Now we show that $[\![p]\!]_\omega (= [\![\bigvee_{i=1}^{k} p_i]\!]_\omega)$ satisfies (2.43) and (2.44).

(2.43): Let $t \in [\![\bigvee_{i=1}^{k} p_i]\!]_\omega$ and O be a neighborhood of t. Since $[\![\bigvee_{i=1}^{k} p_i]\!]_\omega = \bigcup_{i=1}^{k} [\![p_i]\!]_\omega$, there exists some $i \in \{1, \cdots, k\}$ such that $t \in [\![p_i]\!]_\omega$. Since $[\![p_i]\!]_\omega$ satisfies (2.43) almost surely and $\mathrm{int}[\![p_i]\!]_\omega \subset \mathrm{int}[\![\bigvee_{i=1}^{k} p_i]\!]_\omega$, $O \cap \mathrm{int}[\![\bigvee_{i=1}^{k} p_i]\!]_\omega \neq \emptyset$ almost surely. Then $\bigvee_{i=1}^{k} p_i$ satisfies (2.43) almost surely.

(2.44): Let $t \in [\![\neg \bigvee_{i=1}^{k} p_i]\!]_\omega$. If $t \in \mathrm{int}[\![\neg \bigvee_{i=1}^{k} p_i]\!]_\omega$, then for any neighborhood O of t, we have $O \cap \mathrm{int}[\![\neg \bigvee_{i=1}^{k} p_i]\!]_\omega \neq \emptyset$. Thus, it suffices to show that $O \cap \mathrm{int}[\![\neg \bigvee_{i=1}^{k} p_i]\!]_\omega \neq \emptyset$ for any neighborhood O of t whenever $t \in \partial[\![\neg \bigvee_{i=1}^{k} p_i]\!]_\omega$. Since $\mathrm{int}[\![\neg \bigvee_{i=1}^{k} p_i]\!]_\omega = \mathrm{int}(\bigcap_{i=1}^{k} [\![\neg p_i]\!]_\omega) = \bigcap_{i=1}^{k} \mathrm{int}[\![\neg p_i]\!]_\omega$, there must exist some $i \in 1, \cdots, k$ such that $t \in \partial[\![\neg p_i]\!]_\omega$. Indeed, if $t \in \mathrm{int}[\![\neg p_i]\!]_\omega$ for every i, then $t \in \mathrm{int}[\![\neg \bigvee_{i=1}^{k} p_i]\!]_\omega$. Since $t \mapsto X_t(\omega)$ is continuous almost surely and $\langle x_1, y_1 \rangle, \cdots \langle x_k, y_k \rangle$ are pairwise separated, we have $X_t(\omega) \in [x_j, y_j]^C$ when $j \neq i$ almost surely, and hence $t \in \mathrm{int}[\![\neg p_j]\!]_\omega$ for $i \neq j$ almost surely. Therefore, $t \in \bigcap_{j \neq i} \mathrm{int}[\![\neg p_j]\!]_\omega$. Then, $(t-\delta, t+\delta) \cap [0, \infty) \subset \bigcap_{j \neq i} \mathrm{int}[\![\neg p_j]\!]_\omega$ for sufficiently small $\delta > 0$. Now, since p_i satisfies (2.43), we have $(t-\delta, t+\delta) \cap \mathrm{int}[\![\neg p_i]\!]_\omega \neq \emptyset$. Hence, $(t-\delta, t+\delta) \cap \mathrm{int}[\![\bigvee_{j=1}^{k} \neg p_j]\!]_\omega = (t-\delta, t+\delta) \cap \bigcap_{j=1}^{k} \mathrm{int}[\![\neg p_j]\!]_\omega = (t-\delta, t+\delta) \cap \mathrm{int}[\![\neg p_i]\!]_\omega \neq \emptyset$.

□

Proof of Theorem 10. From the condition on B_p, it is a distinct set. Then Lemma 23 and Lemma 24 implies the almost sure convergence of $\chi_\phi^{(n)}(\omega, t)$ and $\chi_\psi^{(n)}(\omega, t)$ for every $t \in [0, \infty)$. Finally, Lemma 17 can be employed to show the convergence of the corresponding probability.

□

2.13 The case of ♭MTL formulas

Now we prove the convergence result for general ♭MTL formulas. Let X be the solution of SDE (2.41) with Assumption 2. Henceforth, we discuss under the following assumption:

Assumption 3. *For every propositional formula p,*

$$X(\omega), t \models p \Leftrightarrow X_t(\omega) \in B_p, \tag{2.50}$$

for some B_p which is a finite union of pairwise separated positive intervals on $\mathbb{R}$ (see Definition 27). Here B_p may possibly be $\emptyset$ or $\mathbb{R}$.

Remark 13. *We give some examples of setting so that every propositional formula satisfies Assumption 3. Let $B_1, \cdots, B_k$ be positive intervals on $\mathbb{R}$ such that*

(i) $\bigcup_{i=1}^{k} B_i = \mathbb{R}$.

(ii) $B_i \cap B_j = \emptyset$ *if* $i \neq j$.

We define the semantics of atomic formulas $AP := \{a_1, \cdots, a_k\}$ as $X(\omega), t \models a_i \Leftrightarrow X_t(\omega) \in B_i$ for $i = 1, \cdots, k$. Then every propositional formula satisfies Assumption 3.

Under these settings, we show the following statement.

Theorem 11. *Suppose that $\{X_t\}_{t \geq 0}$ is the solution of SDE (2.41) with Assumption 2. Let AP be the set of atomic formulas such that every propositional formula satisfies Assumption 3. Let ϕ be a $\flat$MTL formula. Then $\chi_\phi^{(n)}(\omega, t) \to \chi_\phi(\omega, t)$ almost surely for every $t \in [0, \infty)$. In particular, $\mathbb{P}(\omega; X(\omega), \Lambda_n(t) \models_n \phi) \to \mathbb{P}(\omega; X(\omega), t \models \phi)$ for all $t \in [0, \infty)$.*

Lemma 25. *Put Assumption 2 and Assumption 3. Let p be a propositional formula. Then $\chi_p^{(n)}(\omega, t) \to \chi_p(\omega, t)$ almost surely, for every $t \in (0, \infty)$. In particular, $\mathbb{P}(\omega; X(\omega), \Lambda_n(t) \models_n p) \to \mathbb{P}(\omega; X(\omega), t \models p)$.*

Proof. First note that $X(\omega), t \models p$ is equivalent to $X_t(\omega) \in B_p$ for some $B_p \subset \mathbb{R}$. Let $t = 0$. Then $\Lambda_n(0) = 0$ and hence $X(\omega), \Lambda_n(0) \models_n p$ is equivalent to $X(\omega), 0 \models B_p$. Next, let $t > 0$. By the definition of indicator functions , $\chi_p(\omega, t) = 1$ is equivalent to $X_t(\omega) \in B_p$ and $\chi_p^{(n)}(\omega, t) = 1$ is equivalent to $X_{\Lambda_n(t)}(\omega) \in B_p$. From Assumption 3, B_p is distinct, then Lemma 21 implies that $t \notin \partial [\![p]\!]_\omega$ almost surely. Then almost surely there exists some $\varepsilon > 0$ such that $\chi_p(\omega, s) = \chi_p(\omega, t)$ for every $s \in (t - \varepsilon, t + \varepsilon) \cap [0, \infty)$. Then it holds almost surely that $\chi_p^{(n)}(\omega, t) = \chi_p(\omega, \Lambda_n(t)) = \chi_p(\omega, t)$ for sufficiently large n. ☐

Proof of Theorem 11. Fix $t \in [0, \infty)$. It is clear that ϕ is Boolean combination of $\{\phi_i, i = 1, \cdots, k\}$, where ϕ_i is a propositional formula or formula of the form $\Diamond_{\langle S,T \rangle} p$ where $\langle S, T \rangle$ is positive interval and p is propositional formula. Then there exists some function $\bigodot_{i=1}^{k} : \{0, 1\}^k \longrightarrow \{0, 1\}$ such that

$$\chi_\phi(\omega, t) = \bigodot_{i=1}^{k} \chi_{\phi_i}(\omega, t), \tag{2.51}$$

$$\chi_\phi^{(n)}(\omega, t) = \bigodot_{i=1}^{k} \chi_{\phi_i}^{(n)}(\omega, t). \tag{2.52}$$

From Assumption 3 and Lemma 24, every propositional formula satisfies (2.43) and (2.44). Then we can apply Lemma 23 and Lemma 25 to show that $\chi^{(n)}_{\phi_i}(\omega, t)$ converges almost surely to $\chi_{\phi_i}(\omega, t)$ for every $i = 1, \cdots, k$. Then, almost surely, there exists some large $N \in \mathbb{N}$ such that $\chi^{(n)}_{\phi_i}(\omega, t) = \chi_{\phi_i}(\omega, t)$ for $n \geq N$ and $i = 1, \cdots, k$. Therefore the lefthand side of (2.52) converges to the left side of (2.51) almost surely. Once we have shown the almost sure convergence of (2.52) to (2.51), one can apply Lemma 17 to see the convergence of the probability. □

2.14 Approximation the probability by discretization of SDE

In the preceding sections, we have explored the approximation of probability through the discretization of the semantics of MTL. However, given the model of a stochastic system as an SDE, we cannot compute the probability that the system satisfies an MTL formula solely by providing the discretization of its semantics. This is because, in general, we cannot obtain the trajectory of the SDE in an a priori or analytical manner. Then we approximate the trajectory of SDE by discretization such as *Euler's method*:

$$\begin{cases} dX^{(n)}_t = b(X^{(n)}_{\Lambda_n(t)})dt + \sigma(X^{(n)}_{\Lambda_n(t)})dW_t, \\ X_0 = \xi \in \mathbb{R}. \end{cases} \tag{2.53}$$

Using such an approximation, we show the following convergence result.

Theorem 12. *Suppose that $\{X_t\}_{t\geq 0}$ is the solution of SDE (2.41) with Assumption 2. Let $\{X^{(n)}_t\}_{t\geq 0}$ be a stochastic process satisfying*

$$(\forall T \geq 0) \left[\sup_{t\leq T} |X^{(n)}_t - X_t| \overset{n\to\infty}{\longrightarrow} 0, \qquad a.s. \right]. \tag{2.54}$$

Let AP be the set of atomic formulas such that every propositional formula satisfies Assumption 3. Let ϕ be a $\flat$MTL formula. Then, $\mathbb{P}(\omega; X^{(n)}(\omega), \Lambda_n(t) \models_n \phi) \to \mathbb{P}(\omega; X(\omega), t \models \phi)$ for all $t \in [0, \infty)$.

Remark 14. *The almost sure convergence (2.54) holds when σ and b satisfies Lipschitz condition:*

$$|b(x) - b(y)| \leq C_1|x - y|,$$
$$|\sigma(x) - \sigma(y)| \leq C_2|x - y|.$$

for every $x, y \in \mathbb{R}$ and some positive constant C_1, C_2 (see Theorem 2.3 in [Gyö98]).

Lemma 26. *Let X be the strong solution of (2.41) with Assumption 2. Put Assumtion 3 on all propositional formula p. Let $\langle S, T\rangle$ be a positive interval on $[0, \infty)$ such that $0 \leq S < T$. Then the following statement holds almost surely:*

If

$$X(\omega), t \models \Diamond_{\langle S,T\rangle} p, \tag{2.55}$$

for some $t \geq 0$, *then*

$$(\exists s \in (t+S, t+T))[X_s(\omega) \in \mathrm{int} B_p]. \tag{2.56}$$

Proof. To avoid the trivial case, let $B_p \neq \emptyset$. Note that Assumption 3 makes ∂B_p at most countable set on $\mathbb{R}$. Let $\tilde{\Omega}$ ($\hat{\Omega}$ respectively) be the sets of $\omega \in \Omega$ such that the following (a)–(d) ((a)–(e) respectively) hold.

(a) $t \mapsto X_t(\omega)$ is continuous,

(b) $X_{t+T}(\omega) \notin \partial B_p$,

(c) $\mathcal{L}^a_\omega$ is dense in itself for every $a \in \partial B_p$, and

(d) all the local maxima and the local minima of $t \mapsto X_t(\omega)$ are strict.

(e) $X_{t+S}(\omega) \notin \partial B_p$.

From Proposition 4, Proposition 5, Lemma 19 and Theorem 18, we obtain $\mathbb{P}(\tilde{\Omega}) = 1$. When $t + S > 0$, we obtain $\mathbb{P}(\hat{\Omega}) = 1$ in the same way. We show the lemma following these two cases:

(i) Assume that $t + S = 0$ and $\omega \in \tilde{\Omega}$. If $X_{t+T}(\omega) \in \mathrm{int} B_p$, the continuity (a) implies that there exists some $s \in (S, T)$ such that $X_{t+s}(\omega) \in \mathrm{int} B_p$. Then suppose that $X_{t+T}(\omega) \notin \mathrm{int} B_p$. Then (2.55) implies that there exist some $y \in \partial B_p$ and $s \in \langle S, T\rangle$ such that $X_{t+s}(\omega) = y$. If $X_{t+s'}(\omega) \notin \mathrm{int} B_p$ for all $s' \in \langle S, T\rangle$, we can see from (a) and the assumption on B_p that $X_{t+s}(\omega)$ must be either a local maximum or a local minimum. Then (d) implies that $X_{t+s}(\omega)$ is the strict local maximum or the local minimum. Therefore $t + s$ is not a limit point of $\mathcal{L}^y_\omega$. However, this contradicts the self–dense property of $\mathcal{L}^y_\omega$. Thus we conclude that there exists some $s' \in \langle S, T\rangle$ such that $X_{t+s'}(\omega) \in \mathrm{int} B_p$.

(ii) Assume that $t + S > 0$ and $\omega \in \hat{\Omega}$. If $X_{t+S}(\omega) \in \mathrm{int} B_p$ or $X_{t+T}(\omega) \in \mathrm{int} B_p$ holds, the continuity (a) implies that there exists some $s \in (S, T)$ such that $X_{t+s}(\omega) \in \mathrm{int} B_p$. Then, because of (b) and (e), it is enough to show (2.56) when $X_{t+S}(\omega), X_{t+T}(\omega) \in \overline{B_p}$. In this case, (2.55) implies that there exist some $y \in \partial B_p$ and $s \in (S, T)$ such that $X_{t+s}(\omega) = y$. If $X_{t+s'}(\omega) \notin \mathrm{int} B_p$ for all $s' \in (S, T)$, we can see from (a) and the assumption on B_p that $X_{t+s}(\omega)$ must be either a local maximum or a local minimum. Then (d) implies that $X_{t+s}(\omega)$ is the strict local maximum or the local minimum. Therefore $t + s$ is not a limit point of $\mathcal{L}^y_\omega$. However, this contradicts the self–dense property of $\mathcal{L}^y_\omega$. Thus we conclude (2.56).

□

Lemma 27. *Let X be the strong solution of (2.41) with Assumption 2. Define a propositional formula p as*

$$X(\omega), t \models p \Leftrightarrow X_t(\omega) \in B_p$$

for some B_p on $\mathbb{R}$ such that $\mathrm{int} B_p \neq \emptyset$*. Then the following holds:*

$$(\forall x \in \mathbb{R})(\forall T > 0)(\exists t \geq T)[X_t(\omega) = x].$$

In particular, it follows that

$$(\forall x \in \mathbb{R})(\forall T > 0)(\exists t \geq T)[X_t(\omega) \in \mathrm{int} B_p].$$

Proof. It is clear from the time–change representation in Theorem 18 and the recurrent property of one–dimensional Brownian motion (see 2.9.7 in [KS91]). □

Lemma 28. *Let X be the strong solution of (2.41) with Assumption 2. Put Assumption 3 on every propositional formula p. Let $\langle S, T\rangle$ be a positive interval on $[0, \infty)$. Then the following statement holds almost surely:*

If

$$X(\omega), t \not\models \Diamond_{\langle S,T\rangle} p, \tag{2.57}$$

for some $t \geq 0$, then

$$\inf_{s \in \langle S,T\rangle} \{\mathrm{dist}(X_{t+s}(\omega), B_p)\} > 0. \tag{2.58}$$

Proof. For simplicity, let B_p be a nonempty set.

Suppose that $\langle S, T\rangle$ be a unbounded right–open interval, i.e., $\langle S, T\rangle \subset [0, \infty)$ and $T = \infty$. Then Lemma 27 clearly leads to $X(\omega), t \models \Diamond_{\langle S,T\rangle} p$. Since the requirement (2.57) violates almost surely, the statement of this lemma holds almost surely.

Next, let us suppose that $\langle S, T\rangle$ be a positive bounded interval on $[0, \infty)$. First let $S + t = 0$ and suppose that $X_0 \in \overline{B_p}$. From Theorem 18 and Lemma 19, there exists some $s \in \langle t + S, t + T\rangle$ such that $X_s(\omega) \in B_p$. Then the requirement (2.57) violates almost surely, and the statement of this lemma holds almost surely. Next let us suppose that $S + t = 0$ and $X_0 \notin \overline{B_p}$. Let $\tilde{\Omega}$ be the set of $\omega \in \Omega$ with following properties:

(i) $t \mapsto X_t(\omega)$ is continuous,

(ii) $X_{t+T}(\omega) \notin \partial B_p$,

(iii) $\mathcal{L}^a_\omega$ is dense in itself for every $a \in \partial B_p$, and

(iv) all the local maxima and the local minima of $t \mapsto X_t(\omega)$ are strict.

Then Proposition 5 and Lemma 19 implies that $\mathbb{P}(\tilde{\Omega}) = 1$. Now let $\omega \in \tilde{\Omega}$. If $\inf_{s\in\langle S,T\rangle}\{\mathrm{dist}(X_{t+s}(\omega), B_p)\} = 0$, then the continuity of $t \mapsto X_t(\omega)$ implies that the existence of $s \in (t+S, t+T)$ such that $X_s(\omega) = y \in B_p$. If $X_s(\omega) = y$ is either a local maximum or a local minimum, since $\{s \in [0,\infty); X_s(\omega)\}$ is dense–in–itself, it is not strict, which violates (iv). Therefore $\inf_{s\in\langle S,T\rangle}\{\mathrm{dist}(X_{t+s}(\omega), B_p)\} > 0$.

It remains to consider the case when $S + t > 0$. Let $\tilde{\Omega}$ be the set of $\omega \in \Omega$ with the following properties:

(i) $t \mapsto X_t(\omega)$ is continuous,

(ii) $X_{t+S}(\omega) \notin \partial B_p$ and $X_{t+T}(\omega) \notin \partial B_p$,

(iii) $\mathcal{L}_\omega^a$ is dense in itself for every $a \in \partial B_p$, and

(iv) all the local maxima and the local minima of $t \mapsto X_t(\omega)$ are strict.

Then Proposition 5 and Lemma 19 implies that $\mathbb{P}(\tilde{\Omega}) = 1$. Let $\omega \in \tilde{\Omega}$. Then we can conclude $\inf_{s\in\langle S,T\rangle}\{\mathrm{dist}(X_{t+s}(\omega), B_p)\} > 0$ by the same argument as above. □

Lemma 29. *Let (E,d) be a metric space and $f : [0,\infty) \longrightarrow E$ be a continuous function. Suppose that a sequence of functions $f_n : [0,\infty) \longrightarrow E$ converges to f locally uniformly, i.e.,*

$$\sup_{t\leq T} d\left(f(t), f_n(t)\right) > 0, \quad \forall T \geq 0. \tag{2.59}$$

Then the following hold:

(i) Let $B_p \subset E$ and $\langle S,T\rangle$ be an interval on $[0,\infty)$. If $f(t) \in \mathrm{int}B_p$ for some $t \in (S,T)$, there exists some $N \in \mathbb{N}$, $\varepsilon > 0$ and $\delta > 0$ such that $[t-\delta, t+\delta] \subset (S,T)$, and $n \geq N$ implies

$$\inf_{s\in[t-\delta,t+\delta]} \left\{\mathrm{dist}\left(f_n(s), B_p^C\right)\right\} \geq \varepsilon. \tag{2.60}$$

(ii) Let $B_p \subset E$ and $\langle S,T\rangle$ be a bounded interval on $[0,\infty)$ and suppose that

$$\inf_{t\in\langle S,T\rangle}\{\mathrm{dist}(f(t), B_p)\} > 0. \tag{2.61}$$

Then there exists some $\delta > 0$ such that

$$\inf_{t\in[(S-\delta)\wedge 0,T+\delta]}\{\mathrm{dist}(f_n(t), B_p)\} > 0 \tag{2.62}$$

for sufficiently large n.

Proof. (i) Suppose that $f(t) \in \mathrm{int} B_p$ for some $t \in (S, T)$. From the continuity of f, there exists some $\varepsilon > 0$ and $\delta > 0$ such that $\{y \in \mathbb{R}; |f(s) - y| < 2\varepsilon\} \in \mathrm{int} B_p$ for $s \in [t - \delta . t + \delta]$. If $[t - \delta . t + \delta] \not\subset (S, T)$, retake δ as

$$\frac{\delta - \max\{(t + \delta - T) \vee 0, (S - t + \delta) \vee 0\}}{2}.$$

then $[t - \delta, t + \delta] \subset (S, T)$ and $\{y \in \mathbb{R}; |f(s) - y| < 2\varepsilon\} \subset \mathrm{int} B_p$ for $s \in [t - \delta . t + \delta]$. Now let $N \in \mathbb{N}$ be sufficiently large such that $n \geq N$ implies that

$$\max_{s \in [t - \delta, t + \delta]} \{|f_n(s) - f(s)| < \varepsilon\}.$$

Then $\{y \in \mathbb{R}; |f_n(s) - y| < \varepsilon\} \subset \mathrm{int} B_p$ for every $s \in [t - \delta, t + \delta]$. Therefore the statement holds.

(ii) Suppose that

$$\inf_{t \in \langle S, T \rangle} \{\mathrm{dist}(f(t), B_p)\} > 0.$$

From the continuity of f and, there exists some $\varepsilon > 0$ and $\delta > 0$ such that

$$\inf_{t \in [(S - \delta) \wedge 0, T + \delta]} \{\mathrm{dist}(f(t), B_p)\} = \varepsilon > 0$$

Since $T < \infty$, (2.59) implies that there exists some $N \in \mathbb{N}$ such that (2.62) holds for $n \geq N$.

□

Lemma 30. *Let $\langle S, T \rangle$ be a positive interval. Let X be the strong solution of (2.41) with Assumption 2 and $\{X_t^{(n)}\}_{t \geq 0}$ be a stochastic process satisfying (2.54). Define p and B_p as Theorem 10. Then the followings hold almost surely.*

(i) If $X(\omega), t \models \Diamond_{\langle S, T \rangle} p$, then $X^{(n)}(\omega), \Lambda_n(t) \models \Diamond_{\langle S, T \rangle} p$ for sufficiently large n.

(ii) If $X(\omega), t \not\models \Diamond_{\langle S, T \rangle} p$, then $X^{(n)}(\omega), \Lambda_n(t) \not\models \Diamond_{\langle S, T \rangle} p$ for sufficiently large n.

Proof. (i) Let $\tilde{\Omega}$ be the set of ω such that

- (2.54) holds, and
- (2.55) implies (2.56).

Then $\mathbb{P}(\tilde{\Omega}) = 1$. Now suppose that $\omega \in \tilde{\Omega}$. Then Lemma 29 implies that there exists $s \in (t + S, t + T)$, $\delta > 0$ and $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\begin{cases} [s - \delta, s + \delta] \subset (t + S, t + T), \\ (\forall u \in [s - \delta, s + \delta])[X_u^{(n)}(\omega) \in B_p]. \end{cases}$$

Take $M = \max\{N, \lfloor 1/\delta \rfloor + 1\}$. Then $n \geq M$ implies that

$$[s-\delta, s+\delta] \cap (t+S, t+T) \cap \mathbb{N}/n \neq \emptyset$$
$$\Rightarrow [s-\delta, s+\delta] \cap (\Lambda_n(t)+S, \Lambda_n(t)+T) \cap \mathbb{N}/n \neq \emptyset.$$

Therefore there exists some $u \in (\Lambda_n(t)+S, \Lambda_n(t)+T) \cap \mathbb{N}/n$ such that

$$X^{(n)}(\omega), u \models_n p,$$

which implies immediately $X^{(n)}(\omega), t \models_n \Diamond_{\langle S,T\rangle} p$.

(ii) Let $\tilde{\Omega}$ be the set of ω such that

- (2.54) holds, and
- (2.57) implies (2.58).

Then $\mathbb{P}(\tilde{\Omega}) = 1$. Now suppose that $\omega \in \tilde{\Omega}$.

If $B_p = \emptyset$, $X(\omega), t \not\models \Diamond_{\langle S,T\rangle} p$ and $X^{(n)}(\omega), \Lambda_n(t) \not\models \Diamond_{\langle S,T\rangle} p$ clearly holds. Next let $T = \infty$. Then Lemma 27 implies that $X(\omega), t \not\models \Diamond_{\langle S,T\rangle} p$ violates almost surely, which leads us to the conclusion. Therefore it remains to consider the case that $\langle S, T\rangle$ is positive and bounded. Then we can use Lemma 28 that

$$\inf_{s \in \langle S,T\rangle} \{\mathrm{dist}(X_{t+s}(\omega), B_p)\} > 0.$$

Then (2.54) and (ii) of Lemma 29 implies that there exists some $\varepsilon > 0$, $\delta > 0$ and $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\inf_{u \in [(t+S-\delta)\wedge 0, t+T+\delta]} \{\mathrm{dist}(X^{(n)}_u(\omega), B_p)\} = \varepsilon > 0$$

By take $N' := \max\{N, 1/\delta, 1/(T-S)\}$, we can conclude that

$$\inf\{\mathrm{dist}(X^{(n)}_u(\omega), B_p) \mid u \in [\Lambda_n(t)+S, \Lambda_n(t)+T] \cap \mathbb{N}/n\} > \varepsilon.$$

for $n \geq N'$. Therefore we can conclude that $X^{(n)}(\omega), \Lambda_n(t) \not\models \Diamond_\langle S, T\rangle p$. □

Now let us show Theorem 12.

Proof of Theorem 12. Let ϕ be a ♭MTL–formula. Define $\chi_\phi(\omega, t)$ and $\tilde{\chi}^{(n)}_\phi(\omega, t)$ as following:

$$\chi_\phi(\omega, t) = \begin{cases} 1, & \text{if } X(\omega), t \models \phi \\ 0, & \text{if } X(\omega), t \not\models \phi \end{cases}$$
$$\tilde{\chi}^{(n)}_\phi(\omega, t) = \begin{cases} 1, & \text{if } X^{(n)}(\omega), \Lambda_n(t) \models_n \phi \\ 0, & \text{if } X^{(n)}(\omega), \Lambda_n(t) \not\models_n \phi \end{cases}$$

Since ♭MTL–formula is a Boolean combination of propositional formulas and formulas with diamond operators, we can write

$$\chi_\phi(\omega,t) = \bigodot_{i=1}^{k} \chi_{\phi_i}(\omega,t),$$
$$\tilde{\chi}_\phi^{(n)}(\omega,t) = \bigodot_{i=1}^{k} \tilde{\chi}_{\phi_i}^{(n)}(\omega,t),$$

where ϕ_i is a propositional formula or $\phi_i = \Diamond_{\langle S_i,T_i\rangle} p_i$ with a propositional formula p_i and a positive interval $\langle S_i, T_i\rangle$. Here $\bigodot_{i=1}^{k}$ is a function on $\{0,1\}^k$ to $\{0,1\}$. From Lemma 30, we have shown that

$$\tilde{\chi}_{\phi_i}^{(n)}(\omega,t) \overset{n\to\infty}{\longrightarrow} \chi_{\phi_i}(\omega,t) \quad \text{a.s. }, \tag{2.63}$$

when ϕ_i is of the form $\phi_i = \Diamond_{\langle S_i,T_i\rangle} p_i$. Therefore it remains to show (2.63) when ϕ_i is a propositional formula p. If $t=0$, $X_t^{(n)}(\omega) = X_t(\omega)$ and then $X_t^{(n)}(\omega) \in B_p$ is equivalent to $X_t(\omega) \in B_p$ for every propositional formula p. Therefore $\chi_p(\omega,t) = \tilde{\chi}^{(n)}(\omega,t)$ for every n. Next let $t>0$ and $\tilde{\Omega}$ be the set of $\omega \in \Omega$ such that

(i) $t \mapsto X_t(\omega)$ is continuous,

(ii) $X_t(\omega) \notin \partial B_p$, and

(iii) (2.54) holds.

Form Assumption 2 and Proposition 5, X_t has a density. Since Assumption 3, ∂B_p has the Lebesgue measure zero and then $\{\omega \in \Omega; X_t(\omega) \in \partial B_p\}$ has probability zero. Therefore (ii) holds almost surely. In addition to the fact that $t \mapsto X_t(\omega)$ is continuous almost surely, we have $\mathbb{P}(\tilde{\Omega}) = 1$. From now suppose that $\omega \in \tilde{\Omega}$. If $\chi_p(\omega,t) = 1$, $X_t(\omega) \in \mathrm{int}B_p$. From the continuity of $t \mapsto X_t(\omega)$, there exists some $\varepsilon, \delta > 0$ such that

$$(\forall s \in [t-\delta, t+\delta])[\{y \in \mathbb{R}; |X_s(\omega) - y| < \varepsilon\} \in \mathrm{int}B_p].$$

From (iii), $t - \Lambda_n(t) < \delta$ and $|X_{\Lambda_n(t)}^{(n)}(\omega) - X_t(\omega)| < \varepsilon/2$ hold for sufficiently large n. Therefore $X_{\Lambda_n(t)}^{(n)}(\omega) \in \mathrm{int}B_p$ for sufficiently large n, which implies that $\tilde{\chi}_p^{(n)}(\omega,t) = 1$ for sufficiently large n. By the same argment, we can show that $\chi_p(\omega,t) = 0$ implies that $\tilde{\chi}_p^{(n)}(\omega,t) = 0$ for sufficiently large n.

Thus we have shown that $\tilde{\chi}_{\phi_i}^{(n)}(\omega,t)$ converges to $\chi_{\phi_i}(\omega,t)$ almost surely for fixed t and i. Therefore $\bigodot_{i=1}^{k} \tilde{\chi}_{\phi_i}^{(n)}(\omega,t)$ converges almost surely to $\bigodot_{i=1}^{k} \chi_{\phi_i}(\omega,t)$ for fixed t, and we obtain the statement of this theorem. □

Theorem 13. *Let X be the solution of the SDE 2.41 with Assumption 2 and ϕ be a ♭MTL formula with Assumption 3. Let $\{X^{(n)}\}_{n\in\mathbb{N}}$ be a sequence of stochastic processes which converges to X in probability with the locally uniform metric, i.e.,*

$$\mathbb{P}(\sup_{t\le T} |X_t^{(n)} - X_t| > \varepsilon) \overset{n\to\infty}{\longrightarrow} 0, \quad \forall \varepsilon > 0,\ T > 0 \tag{2.64}$$

Then

$$\mathbb{P}(\omega \in \Omega; X^{(n)}(\omega), \Lambda_n(t) \models_n \phi) \overset{n\to\infty}{\longrightarrow} \mathbb{P}(\omega \in \Omega; X(\omega), t \models \phi)$$

Proof. Let d be the locally uniform metric between functions on $[0, \infty)$ to $\mathbb{R}$;

$$d(f, g) = \sum_{n=1}^{\infty} \sup_{t \le n} |f(t) - g(t)| \wedge 1,$$

which implies $X^{(n)}$ converges to X in probability with respect to the locally uniform metric d.

Assume that (2.66) does not hold. Then there exists some increasing natural sequence $\{n_k\}$ and $\varepsilon > 0$ such that

$$\left|\mathbb{P}(\omega \in \Omega; X^{(n_k)}(\omega), \Lambda_{n_k}(t) \models_{n_k} \phi) - \mathbb{P}(\omega \in \Omega; X(\omega), t \models \phi)\right| > \varepsilon$$

for all k. From (2.64), $\{X^{(n_k)}\}_{k\in\mathbb{N}}$ converges to X in probability with respect to the metric d. Then there exists a sub–sequence m_k of n_k such that $X^{(m_k)}$ converges to X almost surely with respect to the metric d, which implies (2.66). This is a contradiction to the assumption. □

The next corollary follows from the fact that L^p convergence implies the convergence in probability.

Corollary 1. *Let X be the solution of the SDE 2.41 with Assumption 2 and ϕ be a $\flat$MTL formula with Assumption 3. Let $\{X^{(n)}\}_{n\in\mathbb{N}}$ be a sequence of stochastic processes which converges to X strongly with the locally uniform metric, i.e.,*

$$\left\{\mathbb{E}\left[\sup_{t\le T} |X_t^{(n)} - X_t|^p\right]\right\}^{1/p} \overset{n\to\infty}{\longrightarrow} 0, \quad \forall T > 0 \tag{2.65}$$

for some $p > 0$. Then

$$\mathbb{P}(\omega \in \Omega; X^{(n)}(\omega), \Lambda_n(t) \models_n \phi) \overset{n\to\infty}{\longrightarrow} \mathbb{P}(\omega \in \Omega; X(\omega), t \models \phi) \tag{2.66}$$

Chapter 3

Discretization of SDE: Time change of the Brownian motion

3.1 Introduction

In this article, we provide a numerical method for approximating a weak solution of a one–dimensional stochastic differential equation. There are many studies on the numerical approximation of SDEs which converge strongly to the solution. There are a variety of applications, including path-dependent option pricing in financial engineering. In this work, we focus on the following one-dimensional SDE:

$$dX_t = \sigma(t, X_t)dW_t. \tag{3.1}$$

This kind of SDE model is called a *local volatility model* and is popular in financial practice. Although (3.1) does not include a drift term, we note that under appropriate conditions, a general one-dimensional SDE with drift can be reduced to (3.1). Time–homogeneous one-dimensional SDEs can be transformed to not have a drift term by using a *scale function* in the pathwise sense, and time inhomogeneous SDEs can also be transformed to (3.1) by using the Girsanov–Maruyama transformation in the sense of law.

To study numerical schemes of the SDE (3.1), we need to discuss the conditions under which the existence and uniqueness of the solution hold in various different senses: *strong uniqueness*, *pathwise uniqueness*, and *uniqueness in the sense of probability law.* Many researchers have studied the unique existence of the solution to SDEs for a long time. The most famous condition for the strong unique existence of a solution is the Lipschitz continuity of the drift and diffusion coefficients (see [KS91]).

According to Bru and Yor [BM02], W. Doeblin wrote a paper about this issue before many facts about the structure of martingale were found. He showed that a diffusion process can be represented by some stochastic process driven by a time-changed Brownian motion. Although this work of Doeblin from 1940 was only made public in 2000, the idea was rediscovered and extended in stochastic calculus, and was already in a textbook [IW89] by Ikeda and Watanabe in 1984, where it was shown that a certain class of one-dimensional SDE of the form (3.1) has a unique solution represented by a time changed Brownian motion, where the time change is given as the solution of a random ordinary differential equation, as we discuss in the next section in more detail. We use this representation to construct a new approximation scheme for one-dimensional SDEs. For the time–homogeneous case, namely, $\sigma(t,x) = \sigma(x)$ in (3.1), Engelbert and Schmidt [ES05] gave an an equivalent condition for weak existence and uniqueness in the sense of probability law, under which the weak solution is represented by a time-changed Brownian motion. For time-homogeneous SDEs, an excellent survey [Tag17] about the existence and uniqueness of SDEs is available.

The most famous numerical scheme for SDEs is the *Euler–Maruyama scheme*. This method approximates a solution of an SDE in a very a similar way to the Euler scheme for ordinary differential equations. We define the Euler–Maruyama approximations of $X_t,\ t \in [0,T)$ as the solutions of

$$dX_t^{(n)} = \sigma(t, X^{(n)}_{\frac{\lfloor nt \rfloor}{n}})dW_t, \tag{3.2}$$

where X_0 is a given initial value. It is well known that the Euler–Maruyama approximation converges to the strong solution of a corresponding SDE uniformly in the sense of L^p with convergence rate $n^{-1/2}$ when the diffusion coefficient is Lipschitz continuous [KP09]. Under β-Hölder continuity of the diffusion coefficient $\sigma(t,x)$, where $1/2 \le \beta \le 1$, Gyöngy and Rásonyi [GR11] showed that for any $T > 0$ there exists a constant $C > 0$ such that

$$\left\{ E\left[\sup_{0\le t\le T} |X_t - X_t^{(n)}|^p \right] \right\}^{1/p} \le \begin{cases} \left(\frac{C}{\ln n}\right)^{1/2p} & \text{if } \beta = 1/2 \\ Cn^{-(\beta-1/2)/p} & \text{if } \beta \in (1/2, 1) \\ Cn^{-(\beta/2-1/4)} & \text{if } \beta = 1 \end{cases} \tag{3.3}$$

for any $n \ge 2$ and $p \ge 2$, where X_t is the strong solution of the SDE (3.1) and $X_t^{(n)}$ is the corresponding Euler–Maruyama approximation for step size $1/n$.

When $\beta < 1/2$, a strong solution does not exist in general [Bar82] and no numerical schemes have been proposed. Note that this kind of rough diffusion coefficient appears when we deal with random medium. For example, Brox considered in [Bro86] a one-dimensional diffusion process in which the drift coefficient is an independent white noise. As discussed in [Bro86] and [HLM17], we can remove the distributional drift coefficient by scale transformation and obtain an SDE of the form (3.1) with $\sigma(t,x) = e^{B(s^{-1}(x))}$, where B is an independent two-sided Brownian motion and s is the scale function.

In Section 3.2, we propose a new method of approximating the SDE (3.1). In Section 3.3, we provide the convergence rates of our method under the β-Hölder condition with $0 < \beta \le 1$ or under a certain smoothness condition. One advantage of our approach is that we approximate the weak solution, which enables us to treat an SDE that does not have a strong solution. Our scheme is the first to achieve strong convergence for

$0 < \beta < 1/2$, and provides a better convergence rate than in [GR11] for $1/2 \leq \beta < 2/3$. In Section 4, we provide some numerical experiments on one-dimensional SDEs with a Hölder continuous diffusion coefficient.

3.2 Discretization with time change

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered probability space. We consider a new discretization scheme for the one-dimensional SDE (3.1) on this space. Our method is based on the following theorem from [IW89].

Theorem 14 (Preliminary theorem)**.** *Let $\{b_t\}$ be a one-dimensional $\{\mathcal{F}_t\}$-Brownian motion with $b_0 = 0$ on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ and let X_0 be an $\mathcal{F}_0$-measurable random variable. Define a continuous process $\{\xi_t\}$ by $\xi_t = X_0 + b_t$. Let $\{\varphi(t)\}$ be a.s. a solution of the ODE*

$$\varphi(t) = \int_0^t \frac{\mathrm{d}s}{\sigma^2(\varphi(s), \xi_s)} \tag{3.4}$$

If we then define $X_t = \xi_{\varphi^{-1}(t)} = X_0 + b_{\varphi^{-1}(t)}$ and $\tilde{\mathcal{F}}_t = \mathcal{F}_{\varphi^{-1}(t)}$, there exists an $\{\tilde{\mathcal{F}}_t\}$-Brownian motion $\{W_t\}$ such that $(\{X_t\}, \{W_t\})$ is a weak solution of (3.1) on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\tilde{\mathcal{F}}_t\}$. Moreover, if the solution of the ODE (3.4) is unique a.s., then the solution of (3.1) is unique in law.

Remark 15. *A sufficient condition for the ODE (3.4) to be well-posed for a fixed $\omega \in \Omega$, is that $\sigma(y, \xi_t(\omega))$ is locally Lipschitz continuous in y and satisfies the inequality*

$$|\sigma^{-2}(y, \xi_t(\omega))| \leq a(t)|y| + b(t) \tag{3.5}$$

for all $t \in [0, \infty)$ and $y \in \mathbb{R}$, where $a(t)$ and $b(t)$ are some continuous non-negative (and possibly random) functions of t (refer to [Gri07]). In the next section, we will show the convergence rate of our method, where we assume the boundedness of the diffusion coefficient (see Condition 2). Then, it is easy to verify that the local Lipschitz continuity of $\sigma(y, \xi_t(\omega))$ in y is sufficient because the condition (3.5) follows from the boundedness of $\sigma^{-2}(y, \xi_t(\omega))$.

The main goal of this chapter is to build a numerical approximation of a solution $\{X_t\}$ of the SDE (3.1) using Theorem 14. To approximate this time-changed Brownian motion, we first make an approximation of Brownian motion $\{\xi_t\}$ by $\{\xi_t^{(n)}\}$ that is a linear interpolation of a random walk generated by normally distributed random variables, that is,

$$\xi_t^{(n)} := \xi_{\lfloor nt \rfloor/n} + (t - \frac{\lfloor nt \rfloor}{n})(\xi_{(\lfloor nt \rfloor+1)/n} - \xi_{\lfloor nt \rfloor/n}) \tag{3.6}$$

where $(\xi_{(\lfloor nt\rfloor+1)/n}-\xi_{\lfloor nt\rfloor/n})\sim\mathcal{N}(0,1/n)$. Second, we approximate $\{\varphi(t)\}$ by $\{\varphi_n(t)\}$, the Euler method for ordinary differential equation, namely,

$$\varphi_n(0)=\varphi(0)=0, \tag{3.7}$$

$$\varphi_n(t)=\varphi_n(\frac{k}{n})+(t-\frac{k}{n})\frac{1}{\sigma^2\left(\varphi_n(k/n),\xi_{k/n}\right)},\quad t\in(\frac{k}{n},\frac{k+1}{n}] \tag{3.8}$$

Third, we make the inverse function $\tau_n(t)$ of $t\longmapsto\varphi_n(t)$ by

$$\tau_n(t)=\frac{k}{n}+\frac{t-\varphi_n(\frac{k}{n})}{\varphi_n(\frac{k+1}{n})-\varphi_n(\frac{k}{n})}\frac{1}{n} \tag{3.9}$$

where $t\in\left[\varphi_n(\frac{k}{n}),\varphi_n(\frac{k+1}{n})\right)$. We can easily check that $\tau_n(t)$ is the inverse function of $\varphi_n(t)$ by its definition.

Let $t_j, j=0,1,2,\cdots$ be defined by $t_j=j/n$. The full algorithm for this method is as follows.

STEP1 Construct $\xi_{t_j},\ j=0,1,2,\cdots$ using a normal distributed random sequence $\{\xi_k-\xi_{k-1}\}_{k=1,\cdots,j}$ with $\xi_0=X_0$ and compute the $\varphi_n(t_j)$ for each j. As we prove later, under Condition 2, $\varphi_n(t)$ is strictly increasing and $\varphi(t_j)$ goes to infinity as $j\to\infty$. This makes the next step valid.

STEP2 The first time $\varphi_n(t_j)$ crosses t (i.e., at the first step j such that $\varphi_n(t_j)>t$), calculate $\tau_n(t)$ using the formula (3.9) where we select k in (3.9) to be $k=j-1$.

STEP3 Using $\tau_n(t)$ and (3.6), calculate $\xi^{(n)}_{\tau_n(t)}$ i.e.,

$$\xi^{(n)}_{\tau_n(t)}=\xi_{t_{j-1}}+(\tau_n(t)-t_{j-1})(\xi_{t_j}-\xi_{t_{j-1}}). \tag{3.10}$$

We thus obtain a path of $\xi^{(n)}_{\tau_n(t)}$ using [STEP1]-[STEP3]. The main result of this chapter is the discretization error of $\{\xi^{(n)}_{\tau_n(t)}\}$ in the sense of L^p under the Hölder condition of $\sigma(t,x)$, which is given in the next section.

3.3 Rate of convergence

In this section, we show the convergence rates of our approximation scheme. In the rest of this chapter, we assume the following condition.

Condition 2. *There are positive constants C_1, C_2 such that $C_1\le\sigma(t,x)\le C_2$ for all $(t,x)\in[0,\infty)\times\mathbb{R}$.*

Theorem 15 states that under the β-Hölder continuity of $\sigma(t,x)$ with $\beta\in(0,1]$ our numerical approximation converges to the exact solution in the sense of L^p uniformly, and the convergence rate is $n^{-\alpha^2\beta}$, where α is an arbitrary value smaller than 1/2. Theorem 16 provides a more precise convergence rate $n^{-\alpha}$ when σ is sufficiently smooth.

Theorem 15. *Let $T > 0$. Suppose that $\sigma(t, x)$ satisfies Condition 2 and that there exist constants $\beta \in (0, 1]$, $C_\beta > 0$, and $L_T > 0$ such that for $s, t \leq T$,*

$$|\sigma(s, x) - \sigma(t, y)| \leq L_T|s - t| + C_\beta|x - y|^\beta. \tag{3.11}$$

Let $\xi_t, \xi_t^{(n)}, \tau(t), \tau_n(t)$ be defined as in the previous section. Then, for any $p \geq 1$ and $\alpha \in [0, 1/2)$, there exists a positive constant $\tilde{K}_T$ such that

$$\left\{ E\left[\sup_{t\leq T} \left| \xi^{(n)}_{\tau_n(t)} - \xi_{\tau(t)} \right|^p \right] \right\}^{1/p} \leq \tilde{K}_T \ n^{-\alpha^2\beta}. \tag{3.12}$$

Remark 16. *The conditions in the theorem above are sufficient to guarantee (3.5), which means that a solution to the differential equation (3.4) uniquely exists.*

We use the following lemma that is an immediate consequence of Theorem (2.1) in [RYs91].

Lemma 31. *Let $\{\xi_t\}$ be Brownian motion and denote*

$$H_{\alpha,T} := \sup_{\substack{s\neq t \\ s,t\leq T}} \frac{|\xi_t - \xi_s|}{|t - s|^\alpha}. \tag{3.13}$$

Then the function $T \mapsto H_{\alpha,T}$ is increasing and

$$E[(H_{\alpha,T})^\gamma] < \infty$$

for any $\alpha \in [0, 1/2)$ and $\gamma > 0$.

Lemma 32. *Suppose that $\sigma(t, x)$ satisfies Condition 2 and that $\varphi(t)$ satisfies (3.4). Let $\varphi_n(t)$ be defined as (3.8). Then $\varphi, \varphi_n, \tau, \tau_n$ is continuous and strictly increasing. Furthermore, for each $\gamma > 0$ and $T > 0$ the following holds*

$$\sup_{t\leq T} |\tau_n(t) - \tau(t)| \leq C_2^2 \sup_{t\leq C_2^2 T} |\varphi_n(t) - \varphi(t)|$$

Proof. It follows from Condition 2 and equation (3.4) that φ and φ_n are continuous and strictly increasing. Furthermore, it follows that

$$\varphi(t) \geq \int_0^t \frac{ds}{C_2^2} = C_2^{-2} t, \tag{3.14}$$

$$\varphi_n(t) \geq \sum_{j=0}^{k-1} \frac{1}{C_2^2}\frac{1}{n} + C_2^{-2}(t - \frac{k}{n}) = C_2^{-2} t \tag{3.15}$$

for $t \in [\frac{k}{n}, \frac{k+1}{n})$. Therefore $\varphi(t), \varphi_n(t) \to \infty$ as $t \to \infty$, which implies the existence, continuity, and strictly increasing property of τ and τ_n. It also follows that $\tau(t), \tau_n(t) \to$

∞ as $t \to \infty$. Because of these properties, $\varphi_n(t)$ is a bijection. Then by (3.15), for any $t \le T$, there exists $t' \le C_2^2 T$ such that $t = \varphi_n(t)$, and therefore

$$\sup_{t\le T} \left|\varphi_n^{-1}(t) - \varphi^{-1}(t)\right| \le \sup_{t\le C_2^2 T} \left|(\varphi_n^{-1}(\varphi_n(t)) - \varphi^{-1}(\varphi_n(t))\right| = \sup_{t\le C_2^2 T} \left|t - \varphi^{-1}(\varphi_n(t))\right|. \tag{3.16}$$

Because of Condition2 and (3.4), we obtain

$$|\varphi(\tilde{s}) - \varphi(\tilde{t})| \ge C_2^{-2}|\tilde{s} - \tilde{t}|,$$

for every $\tilde{s}, \tilde{t}$. By taking $\tilde{s} = \varphi^{-1}(s), \tilde{t} = \varphi^{-1}(t)$, we get C_2^{-2}-Lipschitz continuity of φ^{-1}. Therefore, (3.16) with the Lipschitz continuity of φ^{-1} implies

$$\sup_{t\le C_2^2 T} \left|t - \varphi^{-1}(\varphi_n(t))\right| = \sup_{t\le C_2^2 T} \left|\varphi^{-1}(\varphi(t)) - \varphi^{-1}(\varphi_n(t))\right| \le C_2^2 \sup_{t\le C_2^2 T} \left|\varphi(t) - \varphi_n(t)\right|.$$

Thus we obtain the assertion of the lemma using the last inequality and (3.16). □

Proof of Theorem 15. First, from *Minkowski's inequality*, we have

$$\begin{aligned} &\left\{\mathbb{E}\left[\sup_{t\le T}\left|\xi^{(n)}_{\tau_n(t)} - \xi_{\tau(t)}\right|^p\right]\right\}^{1/p} \\ &\le \left\{\mathbb{E}\left[\sup_{t\le T}\left|\xi^{(n)}_{\tau_n(t)} - \xi_{\frac{\lfloor n\tau_n(t)\rfloor}{n}}\right|^p\right]\right\}^{1/p} + \left\{\mathbb{E}\left[\sup_{t\le T}\left|\xi_{\frac{\lfloor n\tau_n(t)\rfloor}{n}} - \xi_{\tau(t)}\right|^p\right]\right\}^{1/p}, \end{aligned}$$

where $\lfloor t\rfloor$ is the largest integer less than t. Since $\xi^{(n)}_t$ is the interpolation of the sequence $\{\xi_{j/n}\}_{j=0,1,2,\cdots}$, it follows that

$$\left|\xi^{(n)}_t - \xi_{\frac{\lfloor nt\rfloor}{n}}\right| \le \left|\xi^{(n)}_{\frac{\lfloor nt\rfloor+1}{n}} - \xi_{\frac{\lfloor nt\rfloor}{n}}\right| = \left|\xi_{\frac{\lfloor nt\rfloor+1}{n}} - \xi_{\frac{\lfloor nt\rfloor}{n}}\right|$$

Therefore, using Minkowski's inequality again, we obtain

$$\left\{E\left[\sup_{t\le T}\left|\xi^{(n)}_{\tau_n(t)} - \xi_{\tau(t)}\right|^p\right]\right\}^{1/p}$$

$$\le \left\{E\left[\sup_{t\le T}\left|\xi_{\frac{\lfloor n\tau_n(t)\rfloor+1}{n}} - \xi_{\frac{\lfloor n\tau_n(t)\rfloor}{n}}\right|^p\right]\right\}^{1/p} \tag{3.17}$$

$$+ \left\{E\left[\sup_{t\le T}\left|\xi_{\frac{\lfloor n\tau_n(t)\rfloor}{n}} - \xi_{\tau_n(t)}\right|^p\right]\right\}^{1/p} \tag{3.18}$$

$$+ \left\{E\left[\sup_{t\le T}\left|\xi_{\tau_n(t)} - \xi_{\tau(t)}\right|^p\right]\right\}^{1/p}. \tag{3.19}$$

Let us provide the desired conclusion by estimating the convergence rate of (3.17)-(3.19) as $n \to \infty$. Define $H_{\alpha,T}$ as (3.13) for $(\alpha, T) \in (0, 1/2) \times [0, \infty)$ and set $T' :=$

$\max\{T, C_2^2 T + 1/n\}$, $\tilde{H} := H_{\alpha,T'}$ $(\geq H_{\alpha,T})$. Because H_t is monotonically increasing with respect to t, Lemma 31 implies

$$\left\{E\left[\sup_{t\leq T}\left|\xi_{(\lfloor n\tau_n(t)\rfloor+1)n^{-1}} - \xi_{\lfloor n\tau_n(t)\rfloor n^{-1}}\right|^p\right]\right\}^{1/p} \leq \left\{E\left[\tilde{H}^p\right]\right\}^{1/p} n^{-\alpha} \tag{3.20}$$

$$\left\{E\left[\sup_{t\leq T}\left|\xi_{\lfloor n\tau_n(t)\rfloor n^{-1}} - \xi_{\tau_n(t)}\right|^p\right]\right\}^{1/p} \leq \left\{E\left[\tilde{H}^p\right]\right\}^{1/p} n^{-\alpha} \tag{3.21}$$

This gives us the rate of convergence of the terms (3.17) and (3.18). It remains to prove that the convergence rate of the term (3.19) is $n^{-\alpha^2\beta}$. From Lemma 31, Lemma 32 and Hölder's inequality, we have

$$\begin{aligned}\left\{E\left[\sup_{t\leq T}\left|\xi_{\varphi_n^{-1}(t)} - \xi_{\varphi^{-1}(t)}\right|^p\right]\right\}^{1/p} &\leq \left\{E\left[\tilde{H}^p C_2^{2p\alpha}\sup_{t\leq C_2^2 T}|\varphi(t)-\varphi_n(t)|^{p\alpha}\right]\right\}^{1/p} \\ &\leq \left\{E\left[\tilde{H}^{2p}\right]\right\}^{1/2p} C_2^{2\alpha}\left\{E\left[\sup_{t\leq T'}|\varphi(t)-\varphi_n(t)|^{2p}\right]\right\}^{1/2p}.\end{aligned} \tag{3.22}$$

We obtain the convergence rate of (3.22) by estimating the error function $e_n(t) := \varphi_n(t) - \varphi(t)$. For a positive number h, define a function $\psi_h : [0,\infty) \to \mathbb{R}$ as

$$\psi_h(t) := \frac{1}{h}\int_t^{t+h}\left(\sigma^{-2}(\varphi(s),\xi_s) - \sigma^{-2}(\varphi(t),\xi_t)\right)ds.$$

From Lemma 31, condition (3.11) and the fact that $|a^{-2}-b^{-2}| = |a^{-1}+b^{-1}||a^{-1}b^{-1}||a-b|$ for $t \leq T' - h$, we obtain

$$\begin{aligned}|\psi_h(t)| \leq &\frac{1}{h}\left|\int_t^{t+h}\left\{\sigma^{-2}(\varphi(s),\xi_s) - \sigma^{-2}(\varphi(t),\xi_t)\right\}ds\right| \\ \leq &\frac{1}{h}\int_t^{t+h} 2C_1^{-3}L_{T'}|\varphi(s)-\varphi(t)| + C_\beta|\xi_t - \xi_s|^\beta ds \\ \leq &\frac{1}{h}\int_t^{t+h} 2C_1^{-3}(L_{T'}+C_\beta)\left|\int_t^s |C_1^{-2}|du + |\xi_t-\xi_s|^\beta\right| ds \\ \leq &\frac{1}{h}\int_t^{t+h} 2C_1^{-3}(L_{T'}+C_\beta)\left|\int_t^s |C_1^{-2}|du + \tilde{H}|t-s|^{\alpha\beta}\right| ds.\end{aligned}$$

From Lemma 31, there is a random variable R depending on T' that has moments of all orders and satisfies the following

$$|\psi_h(t)| \leq \frac{1}{h}\int_t^{t+h} Rh^{\alpha\beta}ds = Rh^{\alpha\beta}.$$

However, from the definition of $\varphi(t)$,

$$\varphi(t) = \varphi(s) + h\sigma^{-2}(\varphi(s), \xi_s) + h\psi_h(s) \quad \text{for } t > s,$$

where $h = t - s$. Then for $t \in (t_i, t_{i+1}]$,

$$e_n(t) = e_n(t_i) + (t - t_i)\{\sigma^{-2}(\varphi^{(n)}(t_i), \xi_{t_i}) - \sigma^{-2}(\varphi(t_i), \xi_{t_i})\} + (t - t_i)\psi_{t-t_i}(t_i),$$

and by the Lipschitz continuity of $\sigma(t, x)$ with respect to t,

$$\begin{aligned}|e_n(t)| \leq& |e_n(t_i)| + (t - t_i)L_{T'}|e_n(t_i)| + (t - t_i)|\psi_{t-t_i}(t_i)| \\ \leq& (1 + hL_{T'})|e_n(t_i)| + Rh^{\alpha\beta+1}.\end{aligned}$$

We repeat this calculation for i and then by using the standard result for a geometric series and the fact that $1 + L_{T'}h < \mathrm{e}^{L_{T'}h}$, we have

$$\begin{aligned}&\sup_{s\leq t} |e_n(s)| \\ \leq& |e_n(t_{i+1})| \leq \sum_{j=0}^{i} (1 + hL_{t'})^j Rh^{\alpha\beta+1} = \frac{Rh^{\alpha\beta}}{L_{T'}}\{(1 + hL_{T'})^{i+1} - 1\} \leq \frac{Rh^{\alpha\beta}}{L_{T'}}\{\mathrm{e}^{L_{T'}(T'+1)} - 1\}.\end{aligned} \tag{3.23}$$

Because of the fact that the inequality (3.23) holds for $t \leq T'$, the integrable property of the random variable R, and the *Cauchy–Schwartz's inequality*, there exists a positive number K depending on T' such that

$$\left\{E\left[\sup_{t\leq T'} |\varphi(t) - \varphi_n(t)|^{2p\alpha}\right]\right\}^{1/2p} \leq K\left(\frac{1}{n}\right)^{\alpha^2\beta}.$$

This completes the proof.

□

Remark 17. *We have already seen that our approximation converges to the solution of (3.1) and the rate of convergence is* $n^{-\alpha^2\beta}$. *Let us compare our result with (3.3) by Gyöngy and Rásonyi [GR11] which provide the rate of strong convergence of the Euler–Maruyama scheme under similar conditions to ours. When* $p \geq 2$ *and* $1/2 < \beta \leq 1$, *it is easy to see that*

$$-\frac{1}{4}\beta < -(\beta - \frac{1}{2})\frac{1}{2} \leq -(\beta - \frac{1}{2})\frac{1}{p}.$$

For a given β, *we can then take* $\alpha \in (0, 1/2)$ *sufficiently close to* $1/2$ *such that*

$$-\frac{1}{4}\beta < -\alpha^2\beta < -(\beta - \frac{1}{2})\frac{1}{p}.$$

Therefore, our method gives a better estimate of the convergence rate than the Euler–Maruyama scheme for $\beta \in [1/2, 1)$. *For* $\beta \in (0, 1/2)$, *the convergence of the Euler–Maruyama approximation is not known. Furthermore, as we prove later, Theorem 16 below implies that the estimated rate in Theorem 15 is not sharp when* σ *is smooth.*

Remark 18. *Giles [Gil08] introduced the Multi-level Monte Carlo method for reducing the computational complexity to achieve a given mean-square-error tolerance level:*

$$E\left[\left(\hat{Y} - E[f(X_T)]\right)^2\right] < \epsilon,$$

where $\hat{Y}$ is the estimator of $E[f(X_T)]$ by combination of the Euler–Maruyama method and the Multi-level Monte Carlo method. It is proven in [Gil08] that a better strong rate of convergence can reduce the computational complexity. We therefore expect our discretization scheme to be more effective than the Euler–Maruyama method for the multilevel Monte Carlo method when the diffusion coefficient is irregular. We leave this as work for the future.

We now provide a better estimate of the convergence rate of our scheme when σ is smooth. Denote by $\mathcal{L}^q$ the class of stochastic processes $\{X_t\}$ and $q \in \mathbb{N}$ such that

$$E[\int_0^t |X_s|^q ds] < \infty, \quad 0 \le t < \infty,$$

and by $\sigma_t, \sigma_x, \sigma_{x,x}$ the partial derivatives of σ :

$$\sigma_t(t,x) := \frac{\partial \sigma}{\partial t}(t,x), \ \ \sigma_x(t,x) := \frac{\partial \sigma}{\partial x}(t,x), \ \ \sigma_{xx}(t,x) := \frac{\partial^2 \sigma}{\partial x^2}(t,x). \tag{3.24}$$

Theorem 16. *Suppose that $\sigma : [0,\infty) \times \mathbb{R} \mapsto \mathbb{R}$ belongs to $C^{2,2}$ and satisfies the following conditions in addition to Condition 2:*

(i) For any $T > 0$, there exists a constant $L_T > 0$ such that

$$|\sigma(s,x) - \sigma(t,x)| \le L_T|s-t|, \quad \forall x \in \mathbb{R}, \ \forall s,t \in [0,T]. \tag{3.25}$$

(ii) There exists some positive constants C_3, C_4 such that

$$|\sigma_{xx}(t,x)| + |\sigma_t(t,x)| \le C_3 \exp\{C_4(t+|x|)\}, \quad \forall x \in \mathbb{R}, \ \forall t \in [0,T]. \tag{3.26}$$

Then for all $T > 0$, $\alpha \in (0,1/2)$ and $p \ge 1$, there exists some constant $K_T > 0$ such that

$$\left\{E\left[\sup_{t\le T}\left|\xi^{(n)}_{\tau_n(t)} - \xi_{\tau(t)}\right|^p\right]\right\}^{1/p} \le K_T n^{-\alpha}$$

Remark 19. *Since $\varphi(t)$ satisfies (3.4), by Condition 2, $\varphi(t)$ is bounded by $C_1^{-2}t$. Then, under the condition (ii) of Theorem 16, $\sigma_t(\varphi(t),\xi_t)$, $\sigma_x(\varphi(t),\xi_t)$ and $\sigma_{x,x}(\varphi(t),\xi_t)$ belong to $\mathcal{L}^q$ for any $q \in \mathbb{N}$.*

Proof of Theorem 16. Under the assumptions of this theorem, (3.17)-(3.22) continue to hold. Therefore, it remains only to estimate the convergence rate of (3.22). More precisely, it remains to prove that for $\alpha \in (0, 1/2)$ and $T > 0$ there exists a constant $K_T > 0$ such that

$$\left\{ E\left[\sup_{t \le T'} |\varphi(t) - \varphi_n(t)|^{2p\alpha} \right] \right\}^{1/2p} \le K_T n^{-\alpha}.$$

We denote by C a generic constant which depends on p, α, and T, and may change line by line. Note that $\xi_t = \xi_0 + b_t$, where b_t is a standard Brownian motion, and let us write $X_t := \sigma^{-2}(\varphi(t), \xi_t)$. Since $\sigma(t, x)$ belongs to $C^{2,2}$, X_t is a semimartingale and can be written as

$$X_t = X_0 + M_t + B_t,$$

where

$$X_0 := \sigma^{-2}(0, \xi_0), \quad M_t := \int_0^t \gamma_s db_s, \quad B_t := \int_0^t \delta_s ds, \tag{3.27}$$

and $\{\gamma_t\}$ and $\{\delta_t\}$ are in $\mathcal{L}^q$ for any $q \in \mathbb{N}$.

Note that

$$\varphi_n(t) - \varphi(t) \tag{3.28}$$

$$= \int_0^t \sigma^{-2}(\varphi_n(\frac{\lfloor ns \rfloor}{n}), \xi_{\frac{\lfloor ns \rfloor}{n}}) ds - \int_0^t X_s ds \tag{3.29}$$

$$= \int_0^t \left\{ X_{\frac{\lfloor ns \rfloor}{n}} - X_s \right\} ds + \int_0^t \left\{ \sigma^{-2}(\varphi_n(\frac{\lfloor ns \rfloor}{n}), \xi_{\frac{\lfloor ns \rfloor}{n}}) - \sigma^{-2}(\varphi(\frac{\lfloor ns \rfloor}{n}), \xi_{\frac{\lfloor ns \rfloor}{n}}) \right\} ds \tag{3.30}$$

Since $\sigma^{-2}(t, x)$ is locally Lipschitz continuous in t uniformly w.r.t. x, for $t \le T'$ we have

$$E\left[\sup_{s \le t} |\varphi_n(s) - \varphi(s)|^{2p\alpha} \right] \le CE\left[\sup_{s \le t} \left| \int_0^s X_{\frac{\lfloor nu \rfloor}{n}} - X_u du \right|^{2p\alpha} \right] \tag{3.31}$$

$$+ C \int_0^t E\left[\sup_{u \le s} |\varphi_n(u) - \varphi(u)|^{2p\alpha} \right] ds \tag{3.32}$$

Here we use the fact that there is a positive constant C depending on $2p\alpha$ such that $|x+y|^{2p\alpha} \le C\{|x|^{2p\alpha} + |y|^{2p\alpha}\}$. In fact, when $2p\alpha \ge 1$ it is clear from the convex property of $|\cdot|^{2p\alpha}$, and $C = 1$ when $0 < 2p\alpha < 1$ by a simple discussion of the derivative of $|\cdot|^{2p\alpha}$. Since we can regard the two expectations in the left- and right-hand side of (3.31) as functions of t, by *Gronwall's lemma,* we get

$$E\left[\sup_{s \le t} |\varphi_n(s) - \varphi(s)|^{2p\alpha} \right] \le CE\left[\sup_{t \le T'} \left| \int_0^t X_{\frac{\lfloor ns \rfloor}{n}} - X_s ds \right|^{2p\alpha} \right]. \tag{3.33}$$

Using *by parts formula* for tX_t and sX_s $(s < t)$,

$$\int_s^t (X_u - X_s)\, du = \int_s^t (t-u)\ dX_u, \quad a.s.$$

and letting $t_i = i/n, i = 0, 1, \cdots$, we obtain

$$\begin{aligned}
E\left[\sup_{t\le T'}\left|\int_0^t X_{\frac{\lfloor ns\rfloor}{n}} - X_s ds\right|^{2p\alpha}\right] =& E\left[\sup_{t\le T'}\left|\sum_{i=1}^{\lfloor nt\rfloor}\int_{t_{i-1}}^{t_i} X_s - X_{t_{i-1}} ds + \int_{\frac{\lfloor nt\rfloor}{n}}^t X_s - X_{\frac{\lfloor nt\rfloor}{n}} ds\right|^{2p\alpha}\right] \\
=& E\left[\sup_{t\le T'}\left|\sum_{i=1}^{\lfloor nt\rfloor}\int_{t_{i-1}}^{t_i} (t_i - s) dX_s + \int_{\frac{\lfloor nt\rfloor}{n}}^t (t-s) dX_s\right|^{2p\alpha}\right] \\
=& E\left[\sup_{t\le T'}\left|\int_0^{\frac{\lfloor nt\rfloor}{n}} (\frac{\lfloor ns\rfloor + 1}{n} - s) dX_s + \int_{\frac{\lfloor nt\rfloor}{n}}^t (t-s) dX_s\right|^{2p\alpha}\right].
\end{aligned}$$

This implies that

$$E\left[\sup_{t\le T'}\left|\int_0^t X_{\frac{\lfloor ns\rfloor}{n}} - X_s ds\right|^{2p\alpha}\right] \le CE\left[\sup_{t\le T'}\left|\int_0^t \left(\frac{\lfloor ns\rfloor + 1}{n} - s\right) dX_s\right|^{2p\alpha}\right] \tag{3.34}$$

$$+ CE\left[\sup_{t\le T'}\left|\int_0^t \left(\frac{\lfloor ns\rfloor + 1}{n} \wedge t - \frac{\lfloor ns\rfloor + 1}{n}\right) dX_s\right|^{2p\alpha}\right]. \tag{3.35}$$

Because of the fact that $\frac{\lfloor ns\rfloor+1}{n} \wedge t - \frac{\lfloor ns\rfloor+1}{n} = 0$ for $s < \frac{\lfloor nt\rfloor}{n}$, the definition $X_s = \sigma^{-2}(\varphi(s), \xi_s)$ and Condition 2, we can estimate the second term in (3.35) as follows.

$$\begin{aligned}
& E\left[\sup_{t\le T}\left|\int_0^t \left(\frac{\lfloor ns\rfloor + 1}{n} \wedge t - \frac{\lfloor ns\rfloor + 1}{n}\right) dX_s\right|^{2p\alpha}\right] \\
=& E\left[\sup_{t\le T}\left|\int_{\frac{\lfloor nt\rfloor}{n}}^t \left(t - \frac{\lfloor nt\rfloor + 1}{n}\right) dX_s\right|^{2p\alpha}\right] \\
=& E\left[\sup_{t\le T}\left|(t - \frac{\lfloor nt\rfloor + 1}{n})(X_{\frac{\lfloor nt\rfloor}{n}} - X_t)\right|^{2p\alpha}\right] \\
\le& E\left[\frac{1}{n^{2p\alpha}}\sup_{t\le T}\left|X_{\frac{\lfloor nt\rfloor}{n}} - X_t\right|^{2p\alpha}\right] \le \frac{C}{n^{2p\alpha}}
\end{aligned} \tag{3.36}$$

To estimate the first term, we recall the notation (3.27) and obtain

$$\begin{aligned}
&E\left[\sup_{t\le T}\left|\int_0^t\left(\frac{\lfloor ns\rfloor+1}{n}-s\right)dX_s\right|^{2p\alpha}\right]\\
\le& C\left\{E\left[\sup_{t\le T}\left|\int_0^t\left(\frac{\lfloor ns\rfloor+1}{n}-s\right)\ \delta_s ds\right|^{2p\alpha}\right]+E\left[\sup_{t\le T}\left|\int_0^t\left(\frac{\lfloor ns\rfloor+1}{n}-s\right)\gamma_s db_s\right|^{2p\alpha}\right]\right\}
\end{aligned}\tag{3.37}$$

Recalling Remark 19, the $\mathcal{L}^{2p\alpha\vee 1}$ property of δ_t implies that

$$E\left[\sup_{t\le T}\left|\int_0^t\left(\frac{\lfloor ns\rfloor+1}{n}-s\right)\delta_s ds\right|^{2p\alpha}\right]\le\frac{1}{n^{2p\alpha}}E\left[\left|\int_0^T|\delta_s|ds\right|^{2p\alpha}\right]\le\frac{C}{n^{2p\alpha}}.\tag{3.38}$$

Here we have used the fact that

$$E\left[\left|\int_0^T|\delta_s|ds\right|^{2p\alpha}\right]\le T^{2p\alpha-1}E\left[\int_0^T|\delta_s|^{2p\alpha}ds\right]<\infty$$

if $2p\alpha\ge 1$ and

$$E\left[\left|\int_0^T|\delta_s|ds\right|^{2p\alpha}\right]\le E\left[\int_0^T|\delta_s|ds\right]^{2p\alpha}<\infty$$

if $2p\alpha<1$.

By using *the Burkholder–Davis–Gundy inequality* and the $\mathcal{L}^{2p\alpha}$ property of γ_t, we obtain the following for the second term of (3.37):

$$E\left[\sup_{t\le T}\left|\int_0^t\left(\frac{\lfloor ns\rfloor+1}{n}-s\right)\gamma_s db_s\right|^{2p\alpha}\right]\le CE\left[\left|\int_0^T\left|\frac{\lfloor ns\rfloor+1}{n}-s\right|^2\gamma_s^2 ds\right|^{p\alpha}\right]\le\frac{C}{n^{2p\alpha}}.\tag{3.39}$$

From (3.35) and (3.36)-(3.39), it follows that

$$E\left[\sup_{t\le T'}\left|\int_0^t X_{\frac{\lfloor ns\rfloor}{n}}-X_s ds\right|^{2p\alpha}\right]\le\frac{C}{n^{2p\alpha}},$$

which concludes the proof. □

3.4 Numerical example

In this section, we provide some numerical examples for the cases where the diffusion coefficients of (3.1) are 1/3,1/2 and 3/4-Hölder continuous. As a specific example of this kind of coefficient, we choose the *Weierstrass function*

$$\sum_{n=0}^{\infty}a^n\cos(b^n\pi x),\tag{3.40}$$

where $0 < a < 1, b > 1$ and $ab > 1$. According to [Har16], this function is $(-\log a/\log b)$-Hölder continuous. Therefore, by choosing parameters $b = 3$ and $a = 3^{-1/3}, 3^{-1/2}, 3^{-3/4}$, we can construct $1/3, 1/2, 3/4$-Hölder continuous functions respectively. For the diffusion coefficient to satisfy Condition 2, we set

$$\sigma(t,x) = \sigma(x) = \varepsilon + \frac{a}{1-a} + \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \tag{3.41}$$

with some positive constant $\varepsilon > 0$. Let us consider the SDE (3.1) with diffusion coefficient σ as (3.41). Since the infinite sum (3.40) is not implementable, we approximate it by the sum of the first 1000 terms. We now compare the numerical simulation by root square approximation error

$$\{E[|X_T - Y_T^{(n)}|^2]\}^{1/2}. \tag{3.42}$$

where $Y_T^{(n)}$ is discretization of solution to the SDE (3.1) by the Euler–Maruyama method or our method. The integer $n \in \mathbb{N}$ is regarded as the same variable as n in (3.6) when we consider the rate of convergence for the new method. When we consider the rate for the Euler–Maruyama approximation, n is the same variable as n in (3.2). We let $T = 1$, $X_0 = 0$, and $n \in \{2^3, 2^4, \cdots, 2^9\}$. Taking $\{Y_t^{(10)}\}_{t \le T}$ as the exact solution of the SDE (3.1), we consider the strong approximation error (3.42). Figure 3.1 shows the error convergence of both methods.

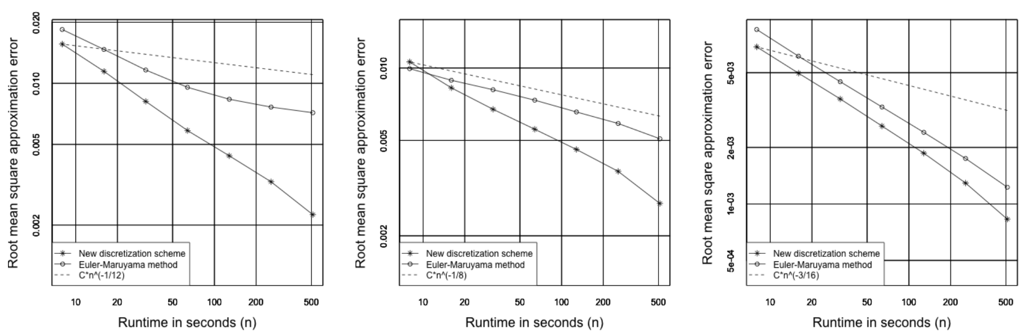


Figure 3.1: Root mean square approximation error of the Euler–Maruyama method and our new discretization method when $n \in \{2^3, 2^4, \cdots, 2^9\}$, with sample size 10^5. The above figures illustrate, from left to right, the cases of $1/3, 1/2, 3/4$–Hölder continuous diffusion coefficient. The broken line in the figure on the left is $Cn^{-1/12}$, where C is a positive constant such that $C(2^3)^{-1/12}$ equals the error of our discretization method with $n = 2^3$. By this line, we attempt to visualize the theoretical decay in the results of Theorem 3.2 when the Hölder index β is $1/3$. The same applies to the remaining figures.

In all cases, the proposed method appears to achieve a faster rate of convergence than guaranteed by Theorem 15. When the Hölder exponent is $1/3$, the numerical approximation error of our method appears to decay linearly in log–log graph, while the error of the Euler–Maruyama method appears to decay logarithmically. Here we remark on the fact that the convergence of the Euler–Maruyama method is not guaranteed when the Hölder index is less than 1/2. When the Hölder exponent is 1/2, we note that the numerical result of the Euler–Maruyama method suggests a faster rate than the estimate (3.3) of logarithmic convergence. Nevertheless, we observe that the speed of convergence for the new method is faster than the Euler–Maruyama method. When the Hölder index is 3/4, there is no apparent difference in the speed of convergence, but the numerical approximation error of both methods seems to decay linearly in log-log graph, as the theoretical results suggest.

3.5 Application to ♭MTL

In Section 2.11, we have shown that the approximation of the probability of ♭MTL works when the stochastic process is a time–homogeneous SDE (2.41):

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t, \\ X_0 = \xi \in \mathbb{R}. \end{cases}$$

with Assumption 2 and Assumption 3. More precisely, under these assumptions, the probability that the exact solution of SDE (2.41) satisfies a ♭MTL–formula based on the discrete semantics converges to the probability based on the continuous semantics. We showed the convergence using the representation of the solution by time–changed Brownian motion.

In addition, we considered in Section 2.14 the approximation of the probability based on continuous semantics by the probability that discretization of SDE satisfies the discrete semantics. We have shown the convergence of the probability when the discretization of SDE converges to the exact solution in probability for the locally uniform topology.

Then we can expect that the new discretization scheme can be applied to the approximation of the probability of a ♭MTL event, since the exact solution of (3.1):

$$\begin{cases} dX_t = \sigma(t, X_t)dW_t. \\ X_0 = x \in \mathbb{R}. \end{cases}$$

can be represented as a time–changed Brownian motion (see Theorem 14), and since the new discretization scheme converges strongly to the exact solution with respect to the locally uniform topology. Thus it is possible to approximate the probability of a ♭MTL–formula for the case of the time–inhomogeneous SDE.

Theorem 17. *Let $X := \{\xi_{\tau(t)}\}_{t\geq 0}$ be the exact solution of SDE (3.1). Let $X^{(n)} := \{\xi^{(n)}_t\}_{t\geq 0}$, $n = 1, 2, \cdots$ be the approximation of X defined in (3.10), respectively. Suppose that $\sigma(t, x)$ satisfies (3.11) with $\beta = 1$ and Condition 2. If the all propositional* *formulas satisfy Assumption 3. Then the following holds for every ♭MTL formula ϕ and $t \geq 0$:*

$$\mathbb{P}(\omega; X^{(n)}(\omega), \Lambda_n(t) \models_n \phi) \overset{n\to\infty}{\longrightarrow} \mathbb{P}(\omega; X(\omega), t \models \phi) \tag{3.43}$$

Note that (3.11) with $\beta = 1$ assures the existence of the density of X (see Theorem IV–2.1.3 in [BH91]). Moreover, $t \mapsto X_t(\omega)$ is continuous almost surely, since $\sigma(t, x)$ is bounded and satisfies (3.11).

Therefore, once we obtain the following lemma, the remaining proof of this theorem is exactly the same as Section 2.11.

Lemma 33. *Let $X = \{X_t\}_{t\geq 0}$ be the strong solution of the SDE (3.1) satisfying (3.11). Then, the following statements hold (see A for the proof):*

(i) Put

$$\mathcal{L}^a_\omega := \{t \geq 0; X_t(\omega) = a\}, \quad a \in \mathbb{R},\ \omega \in \Omega. \tag{3.44}$$

Then $\mathcal{L}^a_\omega$ is dense–in–itself almost surely, for all $a \in \mathbb{R}$.

(ii) For almost every $\omega \in \Omega$, the set of points of local maximum and local minimum for the path $t \mapsto X(\omega)$ is dense in $[0, \infty)$, and all local maxima and local minima are strict.

Proof. Since $\{\xi_t\}_{t\geq 0}$ is a Brownian motion starting at x and $\tau(t)$ is a continuous and strictly increasing process, we can apply the last proof in Appendix A with $p(y) = y$. □

Corollary 2. *Let $X := \{\xi_{\tau(t)}\}_{t\geq 0}$ be the exact solution of SDE (3.1). Let $X^{(n)} := \{\xi^{(n)}_t\}_{t\geq 0}$, $n = 1, 2, \cdots$ be the approximation of X defined in (3.10), respectively. Suppose that $\sigma(t, x)$ satisfies (3.11) with $\beta = 1$ and Condition 2. If the all propositional formulas satisfy Assumption 3. Then the following holds for every ♭MTL formula ϕ and $t \geq 0$:*

$$\mathbb{P}(\omega; X^{(n)}(\omega), \Lambda_n(t) \models_n \phi) \overset{n\to\infty}{\longrightarrow} \mathbb{P}(\omega; X(\omega), t \models \phi) \tag{3.45}$$

Proof. Since we have shown that the $X^{(n)}$ converges strongly to X with respect to the locally uniform topology, we can apply Theorem 13 immediately. □

Chapter 4

Conclusion

Our contribution in this thesis is summarized as follows:

(i) We examined the measurability of events defined by continuous MTL formulas under the assumption of the measurability of the underlying stochastic process as a mapping from sample and time.

(ii) We demonstrated a counterexample that highlights the lack of convergence of the probability derived from discrete semantics to that derived from continuous semantics, specifically when the intervals within diamond operators are allowed to be bounded open or half–open. Then it remains to discuss the case that all the intervals in diamond operators are unbounded or closed.

(iii) We explored the case of $\flat$MTL formulas, which only have $\Box$ or $\Diamond$ without nest as modalities, and demonstrated that the probability obtained from discrete semantics converges to the probability obtained from continuous semantics for every formula within this framework. This finding suggests that $\flat$MTL formulas exhibit a desirable convergence property, highlighting their applicability and reliability in capturing system behaviors.

(iv) We showed the convergence of the probability of discrete semantics for the approximation of the solution to the probability of continuous semantics for the exact solution.

(v) We proposed a new discretization scheme of stochastic differential equations without a drift term. As a result, we showed that the new discretization converges strongly to the solution at a finer rate compared to Euler's method when the diffusion coefficient is β–Hölder continuous with $\beta < 1/2$. Moreover, we showed that this scheme can be applied to the approximation of the probability of $\flat$MTL.

In light of these results, future research efforts should focus on understanding the underlying factors and mechanisms that contribute to the convergence or divergence of probability between discrete and continuous semantics in various formula contexts. By gaining deeper insights into these dynamics, researchers can enhance the effectiveness and accuracy of probability simulations and predictions within the realm of formal verification and system analysis.

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Appendix A

Appendices for proof of Lemma 19

In this section, we prove Lemma 19. Let us recall the claim:

Lemma 34. *Let X be a strong solution of SDE (2.41) on $(\Omega, \mathcal{F}, \mathbb{P})$. Put Assumption 2. Then, the following statements hold:*

(i) Put

$$\mathcal{L}^a_\omega := \{t \geq 0; X_t(\omega) = a\}, \quad a \in \mathbb{R}, \ \omega \in \Omega. \tag{A.1}$$

Then $\mathcal{L}^a_\omega$ is dense in itself almost surely, for all $a \in \mathbb{R}$.

(ii) Almost surely, the set of points of local maximum and local minimum for the path $t \mapsto X(\omega)$ is dense in $[0, \infty)$, and all local maxima and local minima are strict.

Remark 20. *We can write (i) of Lemma 19 as follows:*

$$\mathbb{P}(\omega; \mathcal{L}^a_\omega \text{ is dense in itself }) = 1, \quad \forall a \in \mathbb{R}. \tag{A.2}$$

However, the following probability is not equal to one:

$$\mathbb{P}(\omega; (\forall a \in \mathbb{R})[\mathcal{L}^a_\omega \text{ is dense in itself }]). \tag{A.3}$$

In fact, this equals zero. This is because every local maxima is strict almost surely. Indeed, let $\tilde{\Omega} := \{\omega; \text{local maxima of } t \mapsto X_t(\omega) \text{ is strict }\}$. Then we obtain from (ii) of Lemma 19 that $\mathbb{P}(\tilde{\Omega}) = 1$. Suppose that $\omega \in \tilde{\Omega}$ and $X_t(\omega) = a$ is a strict local maximum, i.e., there exists some $\delta > 0$ such that $a > X_s(\omega)$ for all $s \in [(t-\delta)^+, t+\delta]$. Then $\mathcal{L}^a_\omega$ has isolated point t on $s \in [(t-\delta)^+, t+\delta]$.

On the other hand, the following extension of (i) as follows:

$$\mathbb{P}(\omega; (\forall a \in A)[\mathcal{L}^a_\omega \text{ is dense in itself }]) = 1, \tag{A.4}$$

if A is at most countable set of real numbers.

Toward this goal, we cite a similar fact about Brownian motion:

Proposition 3 (2.9.7 and 2.9.12 in [KS91])**.** *Let X be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Then the following statements hold:*

(i) Put

$$\mathcal{L}_\omega^a := \{t \geq 0; X_t(\omega) = a\}, \quad a \in \mathbb{R}, \ \omega \in \Omega. \tag{A.5}$$

Then $\mathcal{L}_\omega^a$ is dense in itself almost surely, for all $a \in \mathbb{R}$.

(ii) Almost surely, the set of points of local maximum for the Brownian path is dense in $[0, \infty)$, and all local maxima are strict.

The next statement follows directly from Proposition 3 and the rotational invariance of Brownian motion:

Lemma 35. *Almost surely, the set of points of local minimum for the Brownian path $t \mapsto W_t(\omega)$ is dense in $[0, \infty)$, and all local minima are strict.*

Proof. Let X be Brownian motion and define $\tilde{X} := \{\tilde{X}_t\}_{t \geq 0}$ by $\tilde{X}_t(\omega) := -X_t(\omega)$. By rotational invariance (see 3.3.18 in [KS91]), $\tilde{X}$ is also Brownian motion and we can apply Theorem 3 so that the set of local maximum for $\tilde{X}(\omega)$ is dense, and all the local maxima are strict, almost surely. Now since all the local minima of $X(\omega)$ are local maxima of $\tilde{X}(\omega)$, then our statement holds. □

Now we have shown Lemma 19 for the case of Brownian motion. It remains to extend this statement to the case of SDE (2.41) under Assumption 2. Before showing the solution of such an SDE, we have to guarantee the existence and uniqueness of SDE:

Proposition 4 (5.5.17 in [KS91])**.** *Assume that $b : \mathbb{R} \to \mathbb{R}$ is bounded and $\sigma : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with σ^2 bounded away from zero on every compact subset of $\mathbb{R}$. Then, for every initial $\xi \in \mathbb{R}$, equation (2.41) has unique strong solution.*

Furthermore, we show the convergence of the probability of the MTL formula for SDEs that have density functions. The following proposition assures the existence of density for SDE (2.41):

Proposition 5 (Theorem 2.1 in [FP10])**.** *Let ξ be a constant value in $\mathbb{R}$. Assume that σ is Hölder continuous with exponent $\theta \in [1/2, 1]$ and that b is measurable and at most linear growth. Consider a continuous solution $\{X_t\}_{t \geq 0}$ to (2.41). Then, for all $t > 0$, the law of X_t has a density on the set $\{x \in \mathbb{R}; \sigma(x) \neq 0\}$.*

It is an easy task to make sure that the above two propositions can be applied to the unique existence and absolute continuity of X under Assumption 2.

To extend Theorem 3 and Lemma 35 to the case of the stochastic differential equation (2.41), we can make use of the following representation of the solution X as a time change of Brownian motion:

Proposition 6 (5.5.13 in [KS91]). *Assume Assumption 2. Fix a number $c \in \mathbb{R}$ and define the scale function*

$$p(x) := \int_c^x \exp\left\{-2\int_c^\xi \frac{b(\zeta)d\zeta}{\sigma^2(\zeta)}\right\} d\xi; \quad x \in \mathbb{R}$$

and inverse $q : (p(-\infty), p(\infty)) \to \mathbb{R}$ of p. A process $X = \{X_t\}_{t\geq 0}$ is a strong solution of equation (2.41) if and only if the process $Y := \{Y_t = p(X_t)\}_{t\geq 0}$ is a strong solution of

$$Y_t = Y_0 + \int_0^t \tilde{\sigma}(Y_s)dW_s; \quad 0 \leq t < \infty, \tag{A.6}$$

where

$$p(-\infty) < Y_0 < p(\infty) \qquad a.s. \ ,$$

$$\tilde{\sigma}(y) = \begin{cases} p'(q(y))\sigma(q(y)); & p(\infty) < y < p(\infty), \\ 0; & \textit{otherwise.} \end{cases}$$

Fact 6. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.*

(i) In equation (A.6), the Brownian motion $\{W_t\}_{t\geq 0}$ is known to be a continuous local martingale. *The definition of a continuous local martingale can be found in 1.5.15 of the reference [KS91]. Furthermore, the stochastic integral $\{\int_0^t \tilde{\sigma}(Y_s)dW_s\}_{t\geq 0}$ is also a continuous local martingale, provided that the condition*

$$\mathbb{P}\left(\omega; \int_0^t \tilde{\sigma}^2(Y_s(\omega))ds < \infty\right) = 1 \quad \textit{for every } t \in [0, \infty)$$

is satisfied. This fact is also mentioned in Section 3.2.D of [KS91].

(ii) The stochastic process $\{\int_0^t \tilde{\sigma}^2(Y_s)ds\}_{t\geq 0}$ is indeed referred to as the quadratic variation *of the continuous local martingale $\{\int_0^t \tilde{\sigma}(Y_s)dW_s\}_{t\geq 0}$. The definition of quadratic variation can be found in 1.5.18 of the reference [KS91]. This fact is also mentioned in Section 3.2.D of [KS91].*

(iii) Let $M := \{M_t\}_{t\geq 0}$ be a continuous local martingale starting at zero and $\langle M \rangle := \{\langle M \rangle_t\}_{t\geq 0}$ be the quadratic variation of M. Consider a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ in which a Brownian motion exists. According to the results in 3.4.6 and 3.4.7 of the reference [KS91], there exists a Brownian motion $\tilde{B} := \{\tilde{B}_t\}_{t\geq 0}$ defined on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) = (\Omega \times \hat{\Omega}, \mathcal{F} \otimes \hat{\mathcal{F}}, \mathbb{P} \otimes \hat{\mathbb{P}})$ such that

$$M_t(\omega) = B_{\langle M \rangle_t}(\omega, \tilde{\omega}) \quad \textit{almost surely } \tilde{\mathbb{P}}.$$

This provides a representation of the continuous local martingale M in terms of the Brownian motion $\tilde{B}$.

The next lemma gives a representation of SDE (2.41) by time change of Brownian motion.

Theorem 18. *Suppose that $\sigma : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ satisfies Assumption 2. Then there exists a unique strong solution of (2.41). Moreover, there exist*

- *a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$,*
- *a Brownian motion B and an nonnegative continuous strictly increasing process$\{Z_t\}_{t\geq 0}$ with $Z_0 = 0$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) := (\Omega \times \hat{\Omega}, \mathcal{F} \otimes \hat{\mathcal{F}}, \mathbb{P} \otimes \hat{\mathbb{P}})$, and*
- *a strictly increasing continuous function $p : \mathbb{R} \to \mathbb{R}$*

such that

$$X_t(\omega) = p^{-1}(p(\xi) + B_{Z_t}(\omega, \hat{\omega})) \quad 0 \leq t < \infty, \tag{A.7}$$

a.s. $\tilde{\mathbb{P}}$. Here $\xi = X_0$ is the constant initial value of the SDE (2.41).

Proof. The existence and uniqueness of the solution follow from Proposition 4. Since σ and b satisfy Assumption 2, we can deduce from Theorem 6 that there exists a continuous injection $p : \mathbb{R} \to \mathbb{R}$ and a continuous function $\tilde{\sigma} : \mathbb{R} \to \mathbb{R}$ such that $\{Y_t(\omega)\}_{t\geq 0} = \{p(X_t(\omega))\}_{t\geq 0}$ is a strong solution of the SDE (A.6). From the Definition 17 of the strong solution, $\mathbb{P}[\int_0^t \tilde{\sigma}^2(Y_s(\omega))ds < \infty] = 1$ for every t. Then, by (i) of Fact 6, we know that $Y_t(\omega) - Y_0(\omega)$ is a continuous local martingale starting at zero. From (ii) of Fact 6, the quadratic variation $\langle Y \rangle$ of Y is given by $\{\int_0^t \tilde{\sigma}^2(Y_s)ds\}_{t\geq 0}$. Moreover, from the definition $Y_t(\omega) = p(X_t(\omega))$ and the construction of $\tilde{\sigma}$ in (A.6), we have $\tilde{\sigma}^2(Y_t(\omega)) > 0$ for every $t \in [0, \infty)$. Therefore, $t \mapsto \langle Y \rangle_t(\omega)$ is strictly increasing almost surely. Now, by (iii) of Fact 6, there exists a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ such that there exists a Brownian motion B on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) = (\Omega \times \hat{\Omega}, \mathcal{F} \otimes \hat{\mathcal{F}}, \mathbb{P} \otimes \hat{\mathbb{P}})$ such that

$$Y_t(\omega) = Y_0(\omega) + B_{\langle Y \rangle_t}(\omega, \hat{\omega}) \quad \text{almost surely on } \tilde{\mathbb{P}}.$$

Setting $Z_t := \langle Y \rangle_t$, we obtain the desired result. □

Proof of Lemma 19. From Theorem 18, we have $X_t(\omega) = p^{-1}(p(\xi) + B_{Z_t}(\omega, \hat{\omega}))$ for all $t \in [0, \infty)$ almost surely in the extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) = (\Omega \times \hat{\Omega}, \mathcal{F} \otimes \hat{\mathcal{F}}, \mathbb{P} \otimes \hat{\mathbb{P}})$, where $\{B_t\}_{t\geq 0}$ is a Brownian motion and $\{Z_t\}_{t\geq 0}$ is a continuous strictly increasing process.

(i) Set

$$\mathcal{L}^{p(a)-p(\xi)}(\omega, \hat{\omega}) := \{t \geq 0; B_{Z_t}(\omega, \hat{\omega}) = p(a) - p(\xi)\}, \quad a \in \mathbb{R}, \ (\omega, \hat{\omega}) \in \Omega \times \hat{\Omega}.$$

Since $t \mapsto Z_t(\omega, \hat{\omega})$ is strictly increasing and continuous almost surely $\tilde{\mathbb{P}}$, Theorem 3-(i) implies that $\mathcal{L}^{p(a)}(\omega, \hat{\omega})$ is dense in itself almost surely $\tilde{\mathbb{P}}$. Since $X_t(\omega) = p^{-1}(p(\xi) + B_{Z_t}(\omega, \hat{\omega}))$ almost surely $\tilde{\mathbb{P}}$, the set $\mathcal{L}^a_\omega := \{t \geq 0; X_t(\omega) = a\}$ is dense in itself almost surely $\mathbb{P}$.

(ii) Since p^{-1} is strictly increasing, the points of local maximum and local minimum of X are the same as those of $\{B_{Z_t}\}_{t\geq 0}$. Since $t \mapsto Z_t(\omega, \hat{\omega})$ is strictly increasing $\tilde{\mathbb{P}}$-almost surely, every point of the local maximum and the local minimum of $t \mapsto B_{Z_t}(\omega, \hat{\omega})$ is strict. Since $X_t(\omega) = p^{-1}(B_{Z_t}(\omega, \hat{\omega}))$ almost surely $\tilde{\mathbb{P}}$, every point of the local maximum and the local minimum of $t \mapsto X_t$ is strict almost surely $\mathbb{P}$.

□

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List of Works

The results in this thesis are based on the following list of works:

Publications

1. Masaaki Fukasawa, and Mitsumasa Ikeda. *A new discretization scheme for one–dimensional stochastic differential equations using time change method.*, Electronic Communications in Probability **26**, 1 – 12 (2021), doi:10.1214/21-ECP420.

Preprints

1. Mitsumasa Ikeda, Yoriyuki Yamagata, and Takayuki Kihara. *On the Metric Temporal Logic for Continuous Stochastic Processes*, arXiv preprint arXiv: 2308.00984 (2023).

List of talks

1. "A New Discretizatin scheme for SDE with irregular diffusion coefficient", Mitsumas Ikeda, Probability Young Summer Seminar 2019, Aug. 2019.
2. "A New Discretizatin scheme for SDE with irregular diffusion coefficient", Mitsumas Ikeda, The 7th Mathematical Finance Camp–tipe Seminar, Nov. 2019.
3. "A New Discretizatin scheme for SDE with irregular diffusion coefficient", Mitsumas Ikeda, The 11th Shirahama Workshop, Dec. 2019.
4. "A New Discretization scheme for 1–dimensional SDE using time change method", Mitusmasa Ikeda, Probability Spring Seminar, Feb. 2020.
5. "A New Discretization scheme for 1–dimensional SDE using time change method", Mitusmasa Ikeda, Osaka Probability Seminar, Jul. 2020.
6. "A New Discretization scheme for 1–dimensional SDE using time change method", Mitusmasa Ikeda, MSJ Anual Conference, Aug. 2020.

7. “A New Discretization scheme for 1–dimensional SDE using time change method”, Mitusmasa Ikeda, JSIAM Annual Conference, Sep. 2020.

8. “On the probability that a stochastic process satisfies LTL–formula”, Mitsumasa Ikeda, Symbolic Logic and Computer Science in 2021(SLACS2021), Aug. 2021.

9. “On the probability that a stochastic process satisfies MTL–formula”, Mitsumasa Ikeda, Okayama Stochastic Analysis Workshop, Feb. 2022.

10. “On the model ckecking for stochastic processes”, Mitsumasa Ikeda, Probability Young Summer Seminar (YSS2022), Sep. 2022.