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# COMPLEX STRUCTURES OF TORIC HYPERKÄHLER MANIFOLDS

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## 1. Introduction

A Riemannian manifold is hyperkähler if it has three complex structures  $\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3$  satisfying the quaternionic relations  $\mathbf{I}_1\mathbf{I}_2 = -\mathbf{I}_2\mathbf{I}_1 = \mathbf{I}_3$  and if the Riemannian metric is Kähler for each of  $\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3$ . The basic example of a hyperkähler manifold is the quaternionic space  $\mathbf{H}^N$ . The hyperkähler quotient method of Hitchin, Karlhede, Lindström, and Roček is known as a technique for constructing such manifolds [6, §3.(D)]. Bielawski and Dancer studied a hyperkähler quotient of a quaternionic space by a subtorus of a real torus, which they call a toric hyperkähler manifold [1]. Let  $K$  be a subtorus of  $T^N$ . We have a right diagonal action of  $K$  on  $\mathbf{H}^N$  with the hyperkähler moment map  $\mu_K: \mathbf{H}^N \rightarrow \mathfrak{k}^* \otimes \mathbf{R}^3$ . If  $\nu \in \mathfrak{k}^* \otimes \mathbf{R}^3$  is a regular value of  $\mu_K$  and if  $K$  acts freely on  $\mu_K^{-1}(\nu)$ , then we have a toric hyperkähler manifold  $X(\nu) = \mu_K^{-1}(\nu)/K$ . There exists a canonically induced action of  $T^n = T^N/K$  on the  $4n$ -dimensional manifold  $X(\nu)$ , which preserves its hyperkähler structure.

In this paper we study complex structures of a toric hyperkähler manifold. Konno recently studied the variation of its complex structures [9].

We start out in Section 2 with a review of the definition of our manifold. If  $p = (p_1, p_2, p_3)$  is a unit vector in  $\mathbf{R}^3$ , then  $\mathbf{I}_p := \sum_{i=1}^3 p_i \mathbf{I}_i$  is its complex structure. Thus we obtain a family of complex structures parametrized by the 2-sphere. In Section 3 we determine precisely which of these structures have compact complex submanifolds (Theorem 3.3). We can find such structures as follows: let  $\iota: \mathfrak{k} \rightarrow \mathbf{R}^N$  be the inclusion map and  $\{e_1, \dots, e_N\}$  the standard basis of  $\mathbf{R}^N$ . We assume that  $\{\iota^*e_j \mid j \in J\}$  forms a basis of  $\mathfrak{k}^*$ , where  $J \subset \{1, \dots, N\}$ . If  $\nu = \sum_{j \in J} \iota^*e_j \otimes u_j$ , where  $u_j \in \mathbf{R}^3$  for each  $j \in J$ , then  $\mathbf{CP}^1$  is embedded in  $(X(\nu), \mathbf{I}_{u_j/\|u_j\|})$  for each  $j \in J$  (Proposition 3.4). Bielawski and Dancer discussed when our manifold is an affine variety [1, Theorem 5.1]. By their result, we find that the manifold has the structure of an affine variety with respect to each of the other complex structures. In Section 4 we give two examples; one is the case  $\dim K = 1$  (Example 4.1) and the other is the case  $\dim T^n = 1$  (Example 4.2). We apply results in Section 3 to the manifolds in these examples. In the final section we discuss whether complex structures of the family are equivalent to each other. We assume that there exist exactly two complex structures that have compact complex submanifolds. Then it follows that the other complex struc-

tures are all equivalent (Theorem 5.2 (1)). Its proof runs parallel to the proof given in [5, Proposition 9.1 (i)]: we use a circle action on the twistor space of the manifold that preserves the complex structure. It is still open whether in the general case, a similar result holds.

## 2. Toric hyperkähler manifolds

In this section we review the definition of a toric hyperkähler manifold. Let  $\{1, i, j, k\}$  be the standard basis of  $\mathbf{H}$ . Let  $g_0$  be the standard metric on  $\mathbf{H}^N$ . The Riemannian manifold  $(\mathbf{H}^N, g_0)$  is a hyperkähler manifold with complex structures  $\mathbf{I}$ ,  $\mathbf{J}$ ,  $\mathbf{K}$  given by left multiplication by  $i$ ,  $j$ ,  $k$ . We identify  $\sqrt{-1} \in \mathbf{C}$  with  $i \in \mathbf{H}$  and identify  $\xi \in \mathbf{H}$  with  $(z, w) \in \mathbf{C} \oplus \mathbf{C}$  by  $\xi = z + wj$ . Under this identification the complex structures can be written as

$$\begin{aligned}\mathbf{I}(z, w) &= (\sqrt{-1}z, \sqrt{-1}w), \\ \mathbf{J}(z, w) &= (-\bar{w}, \bar{z}), \\ \mathbf{K}(z, w) &= (-\sqrt{-1}\bar{w}, \sqrt{-1}\bar{z}),\end{aligned}$$

where  $z = (z_1, \dots, z_N)$ ,  $w = (w_1, \dots, w_N) \in \mathbf{C}^N$ . The real torus

$$T^N = \{\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{C}^N \mid |\alpha_i| = 1 \text{ for each } 1 \leq i \leq N\}$$

acts on  $\mathbf{H}^N$  by right diagonal multiplication. The action can be written as

$$(z, w) \cdot \alpha = (z\alpha, w\alpha^{-1}).$$

This action preserves the hyperkähler structure.

The hyperkähler moment map for this action is defined by

$$\mu_{T^N} = (\mu_{T^N,1}, \mu_{T^N,2}, \mu_{T^N,3}) : \mathbf{H}^N \rightarrow \mathbf{R}^N \otimes \mathbf{R}^3,$$

where  $\mu_{T^N,1}$ ,  $\mu_{T^N,2}$ ,  $\mu_{T^N,3}$  are the Kähler moment maps corresponding to the complex structures  $\mathbf{I}$ ,  $\mathbf{J}$ ,  $\mathbf{K}$ , respectively. The three moment maps  $\mu_{T^N,1}$ ,  $\mu_{T^N,2}$ ,  $\mu_{T^N,3}$  can be written as

$$\begin{aligned}\mu_{T^N,1}(z, w) &= \frac{1}{2} \sum_{i=1}^N (|z_i|^2 - |w_i|^2) e_i, \\ (\mu_{T^N,2} + \sqrt{-1} \mu_{T^N,3})(z, w) &= -\sqrt{-1} \sum_{i=1}^N (z_i w_i) e_i,\end{aligned}$$

where  $\{e_1, \dots, e_N\}$  is the standard basis of  $\mathbf{R}^N$ .

Let  $K$  be a subtorus of  $T^N$  whose Lie algebra  $\mathfrak{k} \subset \mathbf{R}^N$  is generated by rational vectors. Then we have the torus  $T^n = T^N/K$ . We obtain an exact sequence:

$$0 \longrightarrow \mathfrak{k} \xrightarrow{\iota} \mathbf{R}^N \xrightarrow{\pi} \mathbf{R}^n \longrightarrow 0,$$

and by duality an exact sequence:

$$0 \longrightarrow \mathbf{R}^n \xrightarrow{\pi^*} \mathbf{R}^N \xrightarrow{\iota^*} \mathfrak{k}^* \longrightarrow 0,$$

where  $\iota$  is the inclusion map and  $\pi$  is the projection. The hyperkähler moment map for the action of  $K$  is defined by

$$\mu_K = (\mu_{K,1}, \mu_{K,2}, \mu_{K,3}) : \mathbf{H}^N \rightarrow \mathfrak{k}^* \otimes \mathbf{R}^3,$$

where

$$\mu_{K,i} = \iota^* \circ \mu_{T^n,i} \quad \text{for each } 1 \leq i \leq 3.$$

The following definition is due to Bielawski and Dancer [1, §3].

**DEFINITION 2.1.** Suppose that  $\nu \in \mathfrak{k}^* \otimes \mathbf{R}^3$  is a regular value of the hyperkähler moment map  $\mu_K$  and that  $K$  acts freely on  $\mu_K^{-1}(\nu)$ . Then the hyperkähler quotient

$$X(\nu) = \mu_K^{-1}(\nu)/K$$

is a smooth hyperkähler manifold of dimension  $4n$ . We call  $X(\nu)$  a *toric hyperkähler manifold*.

We denote by  $(g; \mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3)$  the hyperkähler structure and we denote by  $\omega_i$  the Kähler form corresponding to  $\mathbf{I}_i$  for each  $1 \leq i \leq 3$ . There exists a canonically induced action of  $T^n$  on  $X(\nu)$ , which preserves the hyperkähler structure.

Konno discussed when the hyperkähler quotient  $X(\nu)$  is a smooth manifold. Let  $m$  be a non-negative integer. We set

$$\Lambda_m = \{J \subset \{1, \dots, N\} \mid \#J = \dim \text{span}\{\iota^* e_j \mid j \in J\} = m\}.$$

For each  $J \in \Lambda_{\dim K-1}$ , we define a hyperplane  $\mathcal{H}_J$  in  $\mathfrak{k}^*$  by  $\mathcal{H}_J = \text{span}\{\iota^* e_j \mid j \in J\}$ . He obtained the following propositions [8, Proposition 2.1, Proposition 2.2]:

**Proposition 2.2.** *Let  $\nu = (\nu_1, \nu_2, \nu_3) \in \mathfrak{k}^* \otimes \mathbf{R}^3$ . Then the following conditions are equivalent:*

- (1)  *$\nu$  is a regular value of the hyperkähler moment map  $\mu_K$ .*
- (2) *For each  $J \in \Lambda_{\dim K-1}$ , we have  $\nu_1 \notin \mathcal{H}_J$ ,  $\nu_2 \notin \mathcal{H}_J$ , or  $\nu_3 \notin \mathcal{H}_J$ .*

**Proposition 2.3.** *Let  $\nu \in \mathfrak{k}^* \otimes \mathbf{R}^3$  is a regular value of the hyperkähler moment map  $\mu_K$ . Then the following conditions are equivalent:*

- (1) *The action of  $K$  on  $\mu_K^{-1}(\nu)$  is free.*
- (2)  *$\{\pi(e_j) \mid j \in J\}$  is a  $\mathbf{Z}$ -basis of  $\pi(\mathbf{Z}^N)$  for each  $J \subset \{1, \dots, N\}$  such that  $\{\pi(e_j) \mid j \in J\}$  is a basis of  $\mathbf{R}^n$ .*

We remark that the following cases are not essential for our discussion.

REMARKS. (1) Suppose that  $\pi(e_{i_0}) = 0$  for some  $i_0$ ,  $1 \leq i_0 \leq N$ . Then we have

$$\mathfrak{k} = \mathfrak{k}' \oplus \text{span}\{e_{i_0}\}, \quad \text{where } \mathfrak{k}' = \mathfrak{k} \cap \text{span}\{e_i \mid 1 \leq i \leq N, i \neq i_0\}.$$

Let  $K'$  be the Lie group corresponding to  $\mathfrak{k}'$ . Let  $\jmath: \mathfrak{k}' \rightarrow \mathfrak{k}$  be the inclusion map. We set  $\nu' = (\jmath^* \otimes 1_{\mathbf{R}^3})(\nu)$ . The hyperkähler quotient  $X(\nu)$  of  $\mathbf{H}^N$  by  $K$  is just the hyperkähler quotient  $X'(\nu')$  of  $\mathbf{H}^{N-1}$  by  $K'$ , where  $\mathbf{H}^{N-1} = \{(z, w) \in \mathbf{H}^N \mid z_{i_0} = w_{i_0} = 0\}$ .

(2) Suppose that  $\iota^* e_{i_0} = 0$  for some  $i_0$ ,  $1 \leq i_0 \leq N$ . Then we have  $\mathfrak{k} \subset \text{span}\{e_i \mid 1 \leq i \leq N, i \neq i_0\}$ . The hyperkähler quotient  $X(\nu)$  of  $\mathbf{H}^N$  by  $K$  is just the product of the hyperkähler quotient  $X'(\nu)$  of  $\mathbf{H}^{N-1}$  by  $K$  and  $\mathbf{H}$ .

### 3. Main results

In this section we prove our main results. Let  $X(\nu)$  be a toric hyperkähler manifold. If  $p = {}^t(p_1, p_2, p_3)$  is a unit vector in  $\mathbf{R}^3$ , then we have

$$\left( \sum_{i=1}^3 p_i \mathbf{I}_i \right)^2 = -1.$$

Thus we obtain a family of complex structures parametrized by the 2-sphere. We denote by  $\mathbf{I}_p$  the complex structure  $\sum_{i=1}^3 p_i \mathbf{I}_i$ . We determine precisely which of these structures have compact complex submanifolds.

Bielawski and Dancer proved the following proposition [1, Theorem 5.1]:

**Proposition 3.1.** *We set  $\nu = (\nu_1, \nu_2, \nu_3)$ . We assume that for each  $J \in \Lambda_{\dim K-1}$ , we have either  $\nu_2 \notin \mathcal{H}_J$  or  $\nu_3 \notin \mathcal{H}_J$ . Then  $(X(\nu), \mathbf{I}_1)$  is biholomorphic to the affine variety  $\text{Spec } \mathbf{C}[V]^K$ , where  $V$  is defined by the equation*

$$-\sqrt{-1} \sum_{i=1}^N (z_i w_i) \iota^* e_i = \nu_2 + \sqrt{-1} \nu_3,$$

and  $K^{\mathbf{C}}$  is the complexification of  $K$ .

Let  $P = (p_{ij})$  be an element in  $SO(3)$ . For each  $1 \leq i \leq 3$ , we denote by  $p_i$  the

$i$ th row of  $P$ . We set

$$P\nu = \left( \sum_{j=1}^3 p_{1j}\nu_j, \sum_{j=1}^3 p_{2j}\nu_j, \sum_{j=1}^3 p_{3j}\nu_j \right) \quad \text{for each } \nu = (\nu_1, \nu_2, \nu_3) \in \mathfrak{k}^* \otimes \mathbf{R}^3.$$

If  $\nu \in \mathfrak{k}^* \otimes \mathbf{R}^3$  is a regular value of  $\mu_K$ , then so is  $P\nu$ . We prove the following theorem needed later:

**Theorem 3.2.** *There exists a map  $\Psi: X(\nu) \rightarrow X(P\nu)$  that satisfies the following conditions:*

- (1)  $\Psi$  is an isometry.
- (2)  $\Psi$  is a biholomorphic map of  $(X(\nu), \mathbf{I}_{p_i})$  onto  $(X(P\nu), \mathbf{I}_i)$  for each  $1 \leq i \leq 3$ .

Proof. We set

$$\mathbf{J}_i = p_{i1}\mathbf{I} + p_{i2}\mathbf{J} + p_{i3}\mathbf{K} \quad \text{and} \quad \mathbf{j}_i = p_{i1}\mathbf{i} + p_{i2}\mathbf{j} + p_{i3}\mathbf{k} \quad \text{for each } 1 \leq i \leq 3.$$

We consider the hyperkähler structure  $(g_0; \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3)$  on  $\mathbf{H}^N$ . We identify  $\sqrt{-1} \in \mathbf{C}$  with  $\mathbf{j}_1 \in \mathbf{H}$ . Under this identification we define the action (a.2) of  $T^N$  on  $\mathbf{H}^N$  by right diagonal multiplication. Let  $\phi_{T^N}: \mathbf{H}^N \rightarrow \mathbf{R}^N \otimes \mathbf{R}^3$  be the hyperkähler moment map for the action (a.2). We assume that  $\phi_{T^N}(0) = 0$ . We define  $\tilde{\mu}_{T^N}: \mathbf{H}^N \rightarrow \mathbf{R}^N \otimes \mathbf{R}^3$  by

$$\tilde{\mu}_{T^N}(\xi) = P\mu_{T^N}(\xi) \quad \text{for each } \xi \in \mathbf{H}^N.$$

The map  $\tilde{\mu}_{T^N}$  is the hyperkähler moment map for the  $T^N$ -action (a.1) defined in the preceding section. We set

$$\lambda = \begin{cases} \mathbf{j}_2 & \text{if } p_{11} = -1, \\ \frac{p_{11} + 1 - p_{31}\mathbf{j}_2 + p_{21}\mathbf{j}_3}{\sqrt{2(p_{11} + 1)}} & \text{if } p_{11} \neq -1. \end{cases}$$

We define  $\psi: \mathbf{H} \rightarrow \mathbf{H}$  by right multiplication by  $\lambda$ . Let  $\psi^{\oplus N} = \psi \oplus \cdots \oplus \psi$  ( $N$  times). It is easy to verify the following:

CLAIM. (i)  $(a + b\mathbf{i})\lambda = \lambda(a + b\mathbf{j}_1)$  for each  $a, b \in \mathbf{R}$ .  
(ii)  $\psi^{\oplus N}$  is orthogonal.  
(iii)  $\psi^{\oplus N}$  is a biholomorphic map of  $(\mathbf{H}^N, \mathbf{J}_i)$  onto itself for each  $1 \leq i \leq 3$ .

Since  $\phi_{T^N} \circ \psi^{\oplus N}$  is also the hyperkähler moment map for the action (a.1) by this claim, we have

$$\phi_{T^N} \circ \psi^{\oplus N}(\xi) = \tilde{\mu}_{T^N}(\xi) \quad \text{for each } \xi \in \mathbf{H}^N.$$

Thus  $\psi^{\oplus N}$  induces a map  $\Psi: X(\nu) \rightarrow X(P\nu)$  satisfying conditions (1) and (2).  $\square$

Let  $J$  be an element in  $\Lambda_{\dim K}$ . We write

$$\nu = \sum_{j \in J} \iota^* e_j \otimes u_j, \quad \text{where } u_j \in \mathbf{R}^3 \text{ for each } j \in J.$$

Then we set

$$U_J = \left\{ \frac{\pm u_j}{\|u_j\|} \mid j \in J \right\}.$$

Note that from Proposition 2.2, we have  $u_j \neq 0$  for each  $j \in J$ . We set

$$\mathcal{C}_\nu = \{p \in S^2 \mid (X(\nu), \mathbf{I}_p) \text{ has a compact complex submanifold}\},$$

where we regard  $S^2$  as the unit sphere in  $\mathbf{R}^3$ . The main theorem in this section is the following:

**Theorem 3.3.** *Let  $X(\nu)$  be a toric hyperkähler manifold with  $\dim_{\mathbf{R}} X(\nu) > 0$ . Then we have*

$$\mathcal{C}_\nu = \bigcup_{J \in \Lambda_{\dim K}} U_J.$$

Proof. By Remark (1) at the end of Section 2, we may assume that  $\pi(e_i) \neq 0$  for each  $1 \leq i \leq N$ .

We assume that  $\{n+1, n+2, \dots, N\} \in \Lambda_{\dim K}$ . Then  $\{\pi(e_1), \dots, \pi(e_n)\}$  is a basis of  $\mathbf{R}^n$ . Let  $\{v_1, \dots, v_n\}$  be the dual basis corresponding to  $\{\pi(e_1), \dots, \pi(e_n)\}$ . We write

$$\nu = \sum_{i=n+1}^N \iota^* e_i \otimes u_i,$$

where  $u_i \in \mathbf{R}^3$  for each  $n+1 \leq i \leq N$ . Let  $P$  be an element in  $SO(3)$  whose first row equals  ${}^t u_{n+1} / \|u_{n+1}\|$ . We set  $Pu_i = {}^t(\lambda_{1i}, \lambda_{2i}, \lambda_{3i})$  for each  $n+1 \leq i \leq N$ . We may assume that  $\lambda_{1,n+1} > 0$ . Note that  $\lambda_{2,n+1} = \lambda_{3,n+1} = 0$ . By Theorem 3.2 we find that  $(X(\nu), \mathbf{I}_{u_{n+1}/\|u_{n+1}\|})$  is biholomorphic to  $(X(P\nu), \mathbf{I}_1)$ .

We prove that  $(X(P\nu), \mathbf{I}_1)$  has a compact complex submanifold. We set

$$h_i = \sum_{j=n+1}^N \lambda_{ij} e_j \quad \text{for each } 1 \leq i \leq 3.$$

We have  $P\nu = (\iota^* h_1, \iota^* h_2, \iota^* h_3)$ . Let  $\rho: \mu_K^{-1}(P\nu) \rightarrow X(P\nu)$  be the projection. We set  $[z, w] = \rho(z, w)$  for each  $(z, w) \in \mu_K^{-1}(P\nu)$ . The hyperkähler moment map for

the action of  $T^n$  on  $X(P\nu)$  is defined by

$$\mu_{T^n} = (\mu_{T^n,1}, \mu_{T^n,2}, \mu_{T^n,3}): X(P\nu) \rightarrow \mathbf{R}^n \otimes \mathbf{R}^3,$$

where  $\mu_{T^n,i}$  is the Kähler moment map corresponding to the complex structure  $I_i$  for each  $1 \leq i \leq 3$ . For each  $1 \leq i \leq 3$ ,  $\mu_{T^n,i}$  can be written as

$$\langle \mu_{T^n,i}([z, w]), \pi(x) \rangle = \langle \mu_{T^n,i}(z, w), x \rangle - \langle h_i, x \rangle$$

for each  $x \in \mathbf{R}^N$  and for each  $(z, w) \in \mu_K^{-1}(P\nu)$ . We write

$$\pi(e_j) = \sum_{i=1}^n \alpha_{ij} \pi(e_i) \quad \text{for each } 1 \leq j \leq N,$$

where  $\alpha_{ij} \in \mathbf{Z}$  by Proposition 2.3 and [8, Lemma 2.2]. By assumption we have  $(\alpha_{1j}, \dots, \alpha_{nj}) \neq 0$  for each  $n+1 \leq j \leq N$ . There exists  $r$ ,  $1 \leq r \leq n$ , such that  $\alpha_{r,n+1} \neq 0$ . We may assume that  $\alpha_{r,n+1} < 0$ . We set

$$a = \min \left\{ -\frac{\lambda_{1j}}{\alpha_{rj}} \mid n+1 \leq j \leq N, \alpha_{rj} \lambda_{1j} < 0, \lambda_{2j} + \sqrt{-1} \lambda_{3j} = 0 \right\}.$$

We define the closed segment  $\Delta$  in  $\mathbf{R}^n$  by  $\Delta = \{tv_r \mid 0 \leq t \leq a\}$ . We prove that  $\mu_{T^n}^{-1}(\Delta, 0, 0)$  is a compact complex submanifold of  $(X(P\nu), I_1)$ .

(a) We prove that  $\mu_{T^n}^{-1}(\Delta, 0, 0)$  is compact. Let  $[z, w] \in \mu_{T^n}^{-1}(\Delta, 0, 0)$ . Then we have the following:

- (a.i)  $0 \leq \langle \mu_{T^n,1}([z, w]), \pi(e_r) \rangle = \frac{1}{2}(|z_r|^2 - |w_r|^2) \leq a.$
- (a.ii)  $0 = \langle \mu_{T^n,1}([z, w]), \pi(e_i) \rangle = \frac{1}{2}(|z_i|^2 - |w_i|^2) \quad \text{for each } 1 \leq i \leq n, i \neq r.$
- (a.iii)  $0 = \langle (\mu_{T^n,2} + \sqrt{-1} \mu_{T^n,3})([z, w]), \pi(e_i) \rangle = -\sqrt{-1} z_i w_i \quad \text{for each } 1 \leq i \leq n.$

We have from (a.i) and (a.ii),

$$\begin{aligned} \langle \mu_{T^n,1}([z, w]), \pi(e_i) \rangle &= \sum_{j=1}^n \alpha_{ji} \langle \mu_{T^n,1}([z, w]), \pi(e_j) \rangle \\ &= \frac{1}{2}(|z_r|^2 - |w_r|^2) \alpha_{ri} \quad \text{for each } 1 \leq i \leq N. \end{aligned}$$

Thus we have

$$\begin{aligned} (a.iv) \quad \frac{1}{2}(|z_r|^2 - |w_r|^2) \alpha_{ri} + \lambda_{1i} &= \langle \mu_{T^n,1}([z, w]), \pi(e_i) \rangle + \langle h_i, e_i \rangle \\ &= \frac{1}{2}(|z_i|^2 - |w_i|^2) \quad \text{for each } n+1 \leq i \leq N. \end{aligned}$$

Hence we have from (a.i) and (a.iii),

$$(3.1) \quad w_r = 0 \quad \text{and} \quad |z_r|^2 \leq 2a.$$

Furthermore we have from (a.ii) and (a.iii),

$$(3.2) \quad z_i = w_i = 0 \quad \text{for each } 1 \leq i \leq n, \quad i \neq r.$$

Since  $(z, w) \in \mu_K^{-1}(P\nu)$ , we have

$$(3.3) \quad -\sqrt{-1} z_i w_i = \lambda_{2i} + \sqrt{-1} \lambda_{3i} \quad \text{for each } n+1 \leq i \leq N.$$

It follows from (3.1), (3.2), (3.3), and (a.iv) that  $(\mu_{T^n} \circ \rho)^{-1}(\Delta, 0, 0)$  is compact. Hence  $\mu_{T^n}^{-1}(\Delta, 0, 0)$  is compact.

(b) We prove that  $\mu_{T^n}^{-1}(\Delta, 0, 0)$  is a complex submanifold of  $(X(P\nu), \mathcal{I}_1)$ . From the proof of (a), we obtain the following: let  $[z, w] \in \mu_{T^n}^{-1}(\Delta, 0, 0)$ . Let  $i$  be an element in  $\{n+1, \dots, N\}$  such that  $\lambda_{2i} + \sqrt{-1} \lambda_{3i} = 0$ . Then from (3.3) we have  $z_i w_i = 0$ . If  $\alpha_{ri} \leq 0$  and  $\lambda_{1i} < 0$ , then from (a.iv) and (3.1) we have

$$\frac{1}{2}(|z_i|^2 - |w_i|^2) = \frac{1}{2}|z_r|^2 \alpha_{ri} + \lambda_{1i} < 0.$$

Hence  $z_i = 0$ . If  $\alpha_{ri} > 0$  and  $\lambda_{1i} < 0$ , then from (3.1), (a.i), and (a.iv) we have

$$\frac{1}{2}(|z_i|^2 - |w_i|^2) = \alpha_{ri} \left( \frac{1}{2}|z_r|^2 + \frac{\lambda_{1i}}{\alpha_{ri}} \right) \leq \alpha_{ri} \left( \frac{1}{2}|z_r|^2 - a \right) \leq 0.$$

Hence  $z_i = 0$ . Thus we have

$$z_i = 0 \quad \text{for each } n+1 \leq i \leq N \quad \text{such that } \lambda_{1i} < 0 \text{ and } \lambda_{2i} + \sqrt{-1} \lambda_{3i} = 0.$$

Similarly, we have

$$w_i = 0 \quad \text{for each } n+1 \leq i \leq N \quad \text{such that } \lambda_{1i} > 0 \text{ and } \lambda_{2i} + \sqrt{-1} \lambda_{3i} = 0.$$

We denote by  $M$  the set of all points of  $\mathbf{C}^N \times \mathbf{C}^N$  that satisfy the following conditions:

- (b.i)  $w_r = 0$  and  $z_i = w_i = 0$  for each  $1 \leq i \leq n, i \neq r$ .
- (b.ii)  $-\sqrt{-1} z_i w_i = \lambda_{2i} + \sqrt{-1} \lambda_{3i}$  for each  $n+1 \leq i \leq N$  such that  $\lambda_{2i} + \sqrt{-1} \lambda_{3i} \neq 0$ .
- (b.iii)  $z_i = 0$  for each  $n+1 \leq i \leq N$  such that  $\lambda_{1i} < 0$  and  $\lambda_{2i} + \sqrt{-1} \lambda_{3i} = 0$ .
- (b.iv)  $w_i = 0$  for each  $n+1 \leq i \leq N$  such that  $\lambda_{1i} > 0$  and  $\lambda_{2i} + \sqrt{-1} \lambda_{3i} = 0$ .

By an argument similar to that in (a), we have

$$(3.4) \quad (\mu_{T^n} \circ \rho)^{-1}(\Delta, 0, 0) = M \cap \mu_{K,1}^{-1}(\iota^* h_1).$$

Obviously  $M$  is a complex submanifold of  $(\mathbf{H}^N, \mathcal{I})$  and so its induced metric is Kähler. The action of  $K$  on  $M$  preserves the Kähler structure. It is clear that its moment map

is the restriction of  $\mu_{K,1}$  to  $M$ . Since from (3.4) we have

$$\mu_{T^n}^{-1}(\Delta, 0, 0) = \{M \cap \mu_{K,1}^{-1}(\iota^*h_1)\}/K,$$

we find that  $\mu_{T^n}^{-1}(\Delta, 0, 0)$  is a complex submanifold of  $(X(P\nu), \mathbf{I}_1)$ .

Next we prove that  $p \in \mathcal{C}_\nu$  implies  $p \in U_J$  for some  $J \in \Lambda_{\dim K}$ . Let  $Q$  be an element in  $SO(3)$  whose first row equals  ${}^t p$ . Since  $(X(Q\nu), \mathbf{I}_1)$  has a compact complex submanifold by assumption, there exists  $J \in \Lambda_{\dim K}$  satisfying the following condition: if

$$\nu = \sum_{j \in J} \iota^* e_j \otimes a_j, \quad \text{where } a_j \in \mathbf{R}^3 \text{ for each } j \in J,$$

then there exists  $j_0 \in J$  such that the second and the third component of  $Qa_{j_0}$  are equal to zero. Therefore we have  $p \in \text{span}\{a_{j_0}\}$ .  $\square$

**Proposition 3.4.** *The submanifold  $\mu_{T^n}^{-1}(\Delta, 0, 0)$  is biholomorphic to  $\mathbf{CP}^1$ .*

Proof. It is sufficient to show that  $\mu_{T^n}^{-1}(\Delta, 0, 0)$  is homeomorphic to  $\mathbf{CP}^1$ . Let  $S^1$  be the one-dimensional subtorus of  $T^n$  whose Lie algebra  $\mathfrak{s}$  is spanned by  $\pi(e_r)$ . Let  $\iota: \mathfrak{s} \rightarrow \mathbf{R}^n$  be the inclusion map. There exists a canonically induced action of  $S^1$  on  $\mu_{T^n}^{-1}(\Delta, 0, 0)$ . The moment map  $\phi$  for this action is the restriction of  $\iota^* \circ \mu_{T^n,1}$  to  $\mu_{T^n}^{-1}(\Delta, 0, 0)$ .

Let  $t \in (0, a)$ . First we show that  $S^1$  acts freely on  $\phi^{-1}(\iota^*(tv_r))$ . Let  $[z, w] \in \phi^{-1}(\iota^*(tv_r))$ . By (a.i), we have  $z_r \neq 0$ . By (b.ii), we have  $z_i w_i \neq 0$  for each  $n+1 \leq i \leq N$  such that  $\lambda_{2i} + \sqrt{-1} \lambda_{3i} \neq 0$ . We assume that there exists  $i_0$ ,  $n+1 \leq i_0 \leq N$ , such that  $\lambda_{2i_0} + \sqrt{-1} \lambda_{3i_0} = 0$  and  $z_{i_0} = w_{i_0} = 0$ . Since  $\alpha_{ri_0} \neq 0$ , we have  $|z_r|^2/2 = -\lambda_{1i_0}/\alpha_{ri_0}$  by (a.iv). Since  $0 < |z_r|^2/2 < a$  by assumption, this is a contradiction. Hence  $S^1$  acts freely on  $\phi^{-1}(\iota^*(tv_r))$ .

Next we show that  $\phi^{-1}(\iota^*(0))$  and  $\phi^{-1}(\iota^*(av_r))$  are one-point sets. Let  $[z, w], [z', w'] \in \phi^{-1}(\iota^*(av_r))$ . By (a.i), we have

$$(3.5) \quad |z_r| = |z'_r|.$$

By (a.iv) and (3.5), we have

$$|z_i|^2 - |w_i|^2 = |z'_i|^2 - |w'_i|^2 \quad \text{and} \quad z_i w_i = z'_i w'_i$$

for each  $n+1 \leq i \leq N$ . Thus for each  $n+1 \leq i \leq N$ , there exists  $\alpha_i \in T^1$  such that

$$z_i = z'_i \alpha_i \quad \text{and} \quad w_i = w'_i \alpha_i^{-1}.$$

There exists  $j_0$ ,  $n+1 \leq j_0 \leq N$ , such that

$$\alpha_{rj_0} \lambda_{1j_0} < 0, \quad \lambda_{2j_0} + \sqrt{-1} \lambda_{3j_0} = 0, \quad \text{and} \quad a = -\frac{\lambda_{1j_0}}{\alpha_{rj_0}}.$$

Since  $a = |z_r|^2/2 = |z_r'|^2/2$ , we have  $z_{j_0} = w_{j_0} = z_{j_0}' = w_{j_0}' = 0$  by (a.iv). Thus  $[z, w] = [z', w']$ . Similarly, it follows that  $\phi^{-1}(\iota^*(0))$  is a one-point set. Thus  $\phi$  is a Morse function with exactly two critical points. We have thus shown that  $\mu_{T^n}^{-1}(\Delta, 0, 0)$  is homeomorphic to  $\mathbf{CP}^1$ .  $\square$

We obtain the following three corollaries to Theorem 3.3.

**Corollary 3.5.** *Let  $X(\nu)$  be a toric hyperkähler manifold with  $\dim_{\mathbf{R}} X(\nu) > 0$ . If  $p$  is a point of  $S^2 \setminus \mathcal{C}_\nu$ , then  $(X(\nu), \mathbf{I}_p)$  is biholomorphic to an affine variety.*

Proof. Let  $P$  be an element in  $SO(3)$  whose first row equals  ${}^t p$ . Since  $(X(\nu), \mathbf{I}_p)$  is biholomorphic to  $(X(P\nu), \mathbf{I}_1)$  by Theorem 3.2,  $(X(P\nu), \mathbf{I}_1)$  has not a compact complex submanifold. By Theorem 3.3,  $P\nu$  satisfies the condition of Proposition 3.1. Thus  $(X(P\nu), \mathbf{I}_1)$  is biholomorphic to an affine variety.  $\square$

Corollary 3.6 is the converse of Proposition 3.1. Konno also proves this corollary in a different way [9, Corollary 6.12].

**Corollary 3.6.** *Let  $X(\nu)$  be a toric hyperkähler manifold with  $\dim_{\mathbf{R}} X(\nu) > 0$ . Let  $(X(\nu), \mathbf{I}_1)$  be biholomorphic to an affine variety. Then for each  $J \in \Lambda_{\dim K-1}$ , we have either  $\nu_2 \notin \mathcal{H}_J$  or  $\nu_3 \notin \mathcal{H}_J$ .*

Proof. We assume that there exists  $J \in \Lambda_{\dim K-1}$  such that  $\nu_2, \nu_3 \in \mathcal{H}_J$ . Let  $j_0$  be an element in  $\{1, \dots, N\}$  such that  $J \cup \{j_0\} \in \Lambda_{\dim K}$ . If

$$\nu = \sum_{j \in J \cup \{j_0\}} \iota^* e_j \otimes u_j, \quad \text{where } u_j \in \mathbf{R}^3 \text{ for each } j \in J \cup \{j_0\},$$

then the second and the third component of  $u_{j_0}$  are equal to zero. From Theorem 3.3,  $(X(\nu), \mathbf{I}_1)$  has a compact complex submanifold. Thus  $(X(\nu), \mathbf{I}_1)$  is not biholomorphic to an affine variety.  $\square$

**Corollary 3.7.** *Let  $X(\nu)$  be a toric hyperkähler manifold with  $4N > \dim_{\mathbf{R}} X(\nu) > 0$ . Then the cardinality of  $\mathcal{C}_\nu$  is even and we have*

$$1 \leq \frac{\#\mathcal{C}_\nu}{2} \leq \#\{\mathcal{H}_J \mid J \in \Lambda_{\dim K-1}\}.$$

Proof. By Theorem 3.3 we find that  $\#\mathcal{C}_\nu$  is finite and more than zero. If  $X(\nu)$  has a compact complex submanifold for some complex structure, then  $X(\nu)$  also has one for its conjugate. Hence  $\#\mathcal{C}_\nu$  is even.

We prove that  $\#\mathcal{C}_\nu/2$  is less than or equal to  $\#\{\mathcal{H}_J \mid J \in \Lambda_{\dim K-1}\}$ . Let  $I, J \in$

$\Lambda_{\dim K}$ . We assume that there exist  $i_0 \in I$  and  $j_0 \in J$  such that

$$\text{span}\{\iota^*e_i \mid i \in I \setminus \{i_0\}\} = \text{span}\{\iota^*e_j \mid j \in J \setminus \{j_0\}\}.$$

If

$$\nu = \sum_{i \in I} \iota^*e_i \otimes u_i = \sum_{j \in J} \iota^*e_j \otimes v_j, \quad \text{where } u_i, v_j \in \mathbf{R}^3 \text{ for each } i \in I \text{ and } j \in J,$$

then we have  $u_{i_0} \in \text{span}\{v_{j_0}\}$ . Thus we have

$$\frac{\#\mathcal{C}_\nu}{2} \leq \#\{\mathcal{H}_J \mid J \in \Lambda_{\dim K-1}\}. \quad \square$$

#### 4. Examples

In this section we give examples of toric hyperkähler manifolds; one is the case  $\dim K = 1$  and the other is the case  $\dim T^n = 1$ .

In order to obtain a toric hyperkähler manifold, it is sufficient to define a linear map  $\pi: \mathbf{R}^N \rightarrow \mathbf{R}^n$ .

EXAMPLE 4.1. We consider the case  $\dim K = 1$ . Let  $\pi: \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$  be a linear map such that:

- (1)  $\{\pi(e_i) \mid i = 1, \dots, n\}$  forms a basis of  $\mathbf{R}^n$ .
- (2)  $\pi(e_{n+1}) = -\pi(e_1) - \dots - \pi(e_n)$ .

Then the Lie algebra  $\mathfrak{k}$  is spanned by  $e_1 + \dots + e_{n+1}$ . We have  $\iota^*e_1 = \dots = \iota^*e_{n+1}$ . The moment maps are

$$\begin{aligned} \mu_{K,1}(z, w) &= \frac{1}{2} \sum_{i=1}^{n+1} (|z_i|^2 - |w_i|^2) \iota^*e_{n+1}, \\ (\mu_{K,2} + \sqrt{-1}\mu_{K,3})(z, w) &= -\sqrt{-1} \sum_{i=1}^{n+1} (z_i w_i) \iota^*e_{n+1}. \end{aligned}$$

Let  $\nu$  be a nonzero element in  $\mathfrak{k}^* \otimes \mathbf{R}^3$ . From Proposition 2.2,  $\nu$  is a regular value of the hyperkähler moment map  $\mu_K$ . Moreover, from Proposition 2.3, the action of  $K$  on  $\mu_K^{-1}(\nu)$  is free. We write  $\nu = \iota^*e_{n+1} \otimes u$ , where  $u \neq 0 \in \mathbf{R}^3$ . Then we have from Theorem 3.3,

$$\mathcal{C}_\nu = \left\{ \frac{u}{\|u\|}, -\frac{u}{\|u\|} \right\}.$$

Note that  $(X(\nu), \mathbf{I}_{u/\|u\|})$  is biholomorphic to  $(X(\nu), \mathbf{I}_{-u/\|u\|})$ . Let  $P$  be an element in  $SO(3)$  whose first row equals  $^t u/\|u\|$ . From Theorem 3.2,  $(X(\nu), \mathbf{I}_{u/\|u\|})$  is biholomorphic to  $(X(P\nu), \mathbf{I}_1)$ . Thus from [1, Theorem 7.1],  $(X(\nu), \mathbf{I}_{u/\|u\|})$  is biholomorphic to  $T^*\mathbf{CP}^n$  with its natural complex structure.

REMARK. There exist many linear maps of  $\mathbf{R}^{n+1}$  onto  $\mathbf{R}^n$ , but we need only consider  $\pi: \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$  satisfying conditions (1) and (2). Indeed, let  $\pi'$  be a linear map of  $\mathbf{R}^{n+1}$  onto  $\mathbf{R}^n$  such that:

- (i)  $\pi'$  satisfies condition (2) of Proposition 2.3.
- (ii)  $(\iota')^* e_i \neq 0$  for each  $1 \leq i \leq n+1$ , where  $\iota': \ker \pi' \rightarrow \mathbf{R}^{n+1}$  is the inclusion map (recall the remarks at the end of Section 2).

By the conditions above, there exists a map  $\varepsilon: \{1, \dots, n+1\} \rightarrow \{1, -1\}$  such that

$$\varepsilon(n+1) = -1 \quad \text{and} \quad \pi'(e_{n+1}) = \sum_{i=1}^n \varepsilon(i) \pi'(e_i).$$

Let  $K'$  be the Lie group corresponding to  $\ker \pi'$ . We set  $\nu' = (\iota')^* e_{n+1} \otimes u$ . We denote by  $X'(\nu')$  the hyperkähler quotient of  $\mathbf{H}^{n+1}$  by  $K'$ . We define  $F: X(\nu) \rightarrow X'(\nu')$  by

$$F([z, w]) = [u, v],$$

where

$$(u_i, v_i) = \begin{cases} (z_i, w_i) & \text{for each } 1 \leq i \leq n+1 \text{ such that } \varepsilon(i) = -1, \\ (-w_i, z_i) & \text{for each } 1 \leq i \leq n+1 \text{ such that } \varepsilon(i) = 1. \end{cases}$$

Note that the map  $F$  is well-defined. Under this map  $X'(\nu')$  is isomorphic as a hyperkähler manifold to  $X(\nu)$ .

EXAMPLE 4.2. We consider the case  $\dim T^n = 1$ . Let  $\pi: \mathbf{R}^N \rightarrow \mathbf{R}$  be a linear map such that:

- (1)  $\pi(e_N)$  is nonzero.
- (2)  $\pi(e_1) = \dots = \pi(e_{N-1}) = -\pi(e_N)$ .

By an argument similar to that in the remark above, we need only consider  $\pi: \mathbf{R}^N \rightarrow \mathbf{R}$  satisfying conditions (1) and (2). The Lie algebra  $\mathfrak{k}$  is spanned by  $e_1 + e_N, \dots, e_{N-1} + e_N$ . We have  $\iota^* e_N = \sum_{i=1}^{N-1} \iota^* e_i$ . The moment maps are

$$\begin{aligned} \mu_{K,1}(z, w) &= \frac{1}{2} \sum_{i=1}^{N-1} (|z_i|^2 - |w_i|^2 + |z_N|^2 - |w_N|^2) \iota^* e_i, \\ (\mu_{K,2} + \sqrt{-1} \mu_{K,3})(z, w) &= -\sqrt{-1} \sum_{i=1}^{N-1} (z_i w_i + z_N w_N) \iota^* e_i. \end{aligned}$$

Let  $\nu \in \mathfrak{k}^* \otimes \mathbf{R}^3$ . We write

$$\nu = \sum_{i=1}^{N-1} \iota^* e_i \otimes u_i, \quad \text{where } u_i \in \mathbf{R}^3 \text{ for each } 1 \leq i \leq N-1.$$

We assume that

- (i)  $u_i \neq 0$  for each  $1 \leq i \leq N-1$ .
- (ii)  $u_i \neq u_j$  for each  $1 \leq i \neq j \leq N-1$ .

Then, from Proposition 2.2,  $\nu$  is a regular value of the hyperkähler moment map  $\mu_K$ . Moreover, from Proposition 2.3, the action of  $K$  on  $\mu_K^{-1}(\nu)$  is free. From Theorem 3.3, we have

$$\mathcal{C}_\nu = \left\{ \pm \frac{u_i}{\|u_i\|} \mid 1 \leq i \leq N-1 \right\} \cup \left\{ \frac{u_i - u_j}{\|u_i - u_j\|} \mid 1 \leq i \neq j \leq N-1 \right\}.$$

We express  $X(\nu)$  as an affine algebraic set in  $\mathbf{C}^3$ . First we define the map  $\tau_i$  of  $S^2$  into  $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$  for each  $1 \leq i \leq N-1$  as follows:

CASE (a).  $u_i$  and  $u_{N-1}$  are linearly independent.

For each  $p \in S^2$ , we choose points  $p', p'' \in \mathbf{R}^3$  such that  $(p, p', p'') \in SO(3)$ .

We define

$$\tau_i(p) = \frac{\langle p' + \sqrt{-1}p'', u_i \rangle}{\langle p' + \sqrt{-1}p'', u_{N-1} \rangle}.$$

Note that  $\tau_i(p)$  is independant of the choice of  $p', p''$ .

CASE (b).  $u_i$  and  $u_{N-1}$  are linearly dependent.

Then there exists  $\lambda_i \in \mathbf{R}$  such that  $u_i = \lambda_i u_{N-1}$ . We define

$$\tau_i(p) = \lambda_i \quad \text{for each } p \in S^2.$$

We have:

**Proposition 4.3** (cf. [1, Example 5.2]). *Let  $p \in S^2 \setminus \mathcal{C}_\nu$ . Then  $(X(\nu), \mathbf{I}_p)$  is biholomorphic to the affine variety*

$$xy = z \prod_{i=1}^{N-1} (\tau_i(p) - z).$$

**REMARK.** We obtain a family of affine varieties parametrized by  $S^2 \setminus \mathcal{C}_\nu$ . These varieties are known to be diffeomorphic to the minimal resolution of the simple singularity of type  $A_{N-1}$ . It is still open whether these varieties are biholomorphic to each other.

**Proof.** Let  $p \in S^2 \setminus \mathcal{C}_\nu$ . Let  $P$  be an element in  $SO(3)$  such that  $(p, p', p'') \in SO(3)$ , where  $p', p'' \in \mathbf{R}^3$ . From the proof of Corollary 3.5,  $(X(\nu), \mathbf{I}_p)$  is biholomorphic to  $\text{Spec } \mathbf{C}[V]^{K^c}$ , where  $V$  is defined by the following equations:

$$-\sqrt{-1}(z_i w_i + z_N w_N) = \langle p' + \sqrt{-1}p'', u_i \rangle \quad \text{for each } 1 \leq i \leq N-1.$$

The invariant ring  $\mathbf{C}[V]^{K^C}$  is generated as a  $\mathbf{C}$ -algebra by

$$z_N \prod_{i=1}^{N-1} w_i, \quad w_N \prod_{i=1}^{N-1} z_i, \quad z_N w_N \pmod{I(V)}.$$

Thus  $\mathbf{C}[V]^{K^C}$  can be written as

$$\mathbf{C}[x, y, z] \left/ \left( xy - z \prod_{i=1}^{N-1} (\langle \sqrt{-1} p' - p'', u_i \rangle - z) \right) \right..$$

This completes the proof of Proposition 4.3.  $\square$

We identify  $\widehat{\mathbf{C}}$  with  $S^2$  by the stereographic projection from the south pole. We may regard  $\tau_i$  as a map of  $\widehat{\mathbf{C}}$  into itself. We obtain an explicit formula for  $\tau_i$  for each  $1 \leq i \leq N-1$ .

**Proposition 4.4.** *In Case (a), there exist two linear fractional transformations  $S_i$  and  $T_i$  such that*

$$S_i \circ \tau_i \circ T_i(z) = z + \frac{1}{z} \quad \text{for each } z \in \widehat{\mathbf{C}}.$$

Proof. Let  $\theta_i$ ,  $0 < \theta_i < \pi$ , be the angle between  $u_i$  and  $u_{N-1}$ . Let  $v_i = {}^t(0, \sin \theta_i, \cos \theta_i)$ . There exists  $P_i \in SO(3)$  such that

$$P_i v_i = \frac{u_i}{\|u_i\|} \quad \text{and} \quad P_i e_3 = \frac{u_{N-1}}{\|u_{N-1}\|}.$$

We denote by  $T_i$  the linear fractional transformation corresponding to  $P_i$ . Let  $z \in \mathbf{C} \setminus \{0\}$ . We denote by  $p = {}^t(p_1, p_2, p_3)$  the point of  $S^2$  corresponding to  $z$ . Let  $p'$  and  $p''$  be points of  $\mathbf{R}^3$  such that  $(p, p', p'') \in SO(3)$ . By the definition of  $\tau_i$ , we have

$$\tau_i \circ T_i(z) = \frac{\langle P_i(p' + \sqrt{-1} p''), u_i \rangle}{\langle P_i(p' + \sqrt{-1} p''), u_{N-1} \rangle} = \frac{\|u_i\| \langle p' + \sqrt{-1} p'', v_i \rangle}{\|u_{N-1}\| \langle p' + \sqrt{-1} p'', e_3 \rangle}.$$

By direct calculation, we find that the right-hand side is equal to

$$\begin{aligned} & \frac{\|u_i\|}{\|u_{N-1}\|} \left\{ \cos \theta_i - \frac{\sqrt{-1} \sin \theta_i}{1 - p_3^2} (p_1 - \sqrt{-1} p_2 p_3) \right\} \\ &= \frac{\|u_i\|}{\|u_{N-1}\|} \left\{ \cos \theta_i - \frac{\sqrt{-1} \sin \theta_i}{2} \left( z + \frac{1}{z} \right) \right\}. \end{aligned}$$

We define the linear fractional transformation  $S_i$  by

$$S_i(z) = \frac{2\sqrt{-1}}{\sin \theta_i} \left( \frac{\|u_{N-1}\|}{\|u_i\|} z - \cos \theta_i \right) \quad \text{for each } z \in \widehat{\mathbf{C}}.$$

Then we have

$$S_i \circ \tau_i \circ T_i(z) = z + \frac{1}{z} \quad \text{for each } z \in \widehat{\mathbf{C}}. \quad \square$$

## 5. The equivalency of complex structures

In this section we discuss whether complex structures of the family are equivalent to each other. We mainly consider the case  $\#\mathcal{C}_\nu = 2$ .

Let  $X(\nu)$  be a toric hyperkähler manifold. We assume that  $\nu$  is of the form  $(\nu_1, 0, 0)$ . Then we have  $\mathcal{C}_\nu = \{(\pm 1, 0, 0)\}$ . We define the two circle actions on  $X(\nu)$  by

$$(5.1) \quad [z, w] \cdot e^{\sqrt{-1}\theta} = [ze^{\sqrt{-1}\theta}, w]$$

and

$$(5.2) \quad [z, w] * e^{\sqrt{-1}\theta} = [z, we^{\sqrt{-1}\theta}],$$

where  $\theta \in \mathbf{R}$  and we denote by  $[z, w]$  the element in  $X(\nu)$  defined by  $(z, w) \in \mu_K^{-1}(\nu)$ . Since  $\nu = (\nu_1, 0, 0)$ , these actions are well-defined. These actions preserve both the metric and the Kähler form  $\omega_1$ . We denote by  $X_1^\#$  and  $X_2^\#$  the fundamental vector fields corresponding to  $1 \in \mathbf{R}$  for the actions (5.1) and (5.2), respectively. The moment maps for the actions (5.1) and (5.2) with respect to  $\omega_1$  are the maps

$$f_1([z, w]) = \frac{1}{2} \sum_{i=1}^N |z_i|^2 \quad \text{and} \quad f_2([z, w]) = \frac{1}{2} \sum_{i=1}^N |w_i|^2,$$

respectively.

We can easily show the following proposition, and so we omit its proof.

**Proposition 5.1.** *We have*

$$L_{X_i^\#} \omega_1 = 0, \quad L_{X_i^\#} \omega_2 = -\omega_3, \quad L_{X_i^\#} \omega_3 = -\omega_2 \quad \text{for each } i = 1, 2.$$

The main theorem in this section is the following:

**Theorem 5.2.** *Let  $X(\nu)$  be a toric hyperkähler manifold with  $\#\mathcal{C}_\nu = 2$ . Then*

- (1)  $(X(\nu), \mathbf{I}_p)$  and  $(X(\nu), \mathbf{I}_q)$  are biholomorphic for each  $p, q \in S^2 \setminus \mathcal{C}_\nu$ .
- (2)  $(X(\nu), \mathbf{I}_p)$  is a Stein manifold for each  $p \in S^2 \setminus \mathcal{C}_\nu$ .

**REMARK.** By Corollary 3.5, (2) is obvious. In this section we prove it by giving a strictly plurisubharmonic exhaustion function.

Proof. (1) First we consider the case where  $\nu$  is of the form  $(\nu_1, 0, 0)$ . We define the circle action on  $S^2$  by the standard rotation leaving  ${}^t(\pm 1, 0, 0)$  fixed. Then the circle acts diagonally on  $X(\nu) \times S^2$  by the action (5.1) on  $X(\nu)$  and by the action on  $S^2$ . We denote by  $\tilde{X}$  the fundamental vector field corresponding  $1 \in \mathbf{R}$  for the circle action on  $X(\nu) \times S^2$ .

Let  $\mathbf{I}_{S^2}$  be the standard complex structure on  $S^2$ . The product manifold  $X(\nu) \times S^2$  has a natural complex structure  $\tilde{\mathbf{I}}$  defined on the tangent space of  $X(\nu) \times S^2$  at  $([z, w], p)$  as follows [6, §3.(F)]: we express the tangent space as the direct sum  $T_{[z, w]}X(\nu) \oplus T_p S^2$  and define

$$\tilde{\mathbf{I}}_{([z, w], p)} = (\mathbf{I}_p)_{[z, w]}, (\mathbf{I}_{S^2})_p).$$

Since the circle action on  $X(\nu) \times S^2$  preserves the complex structure  $\tilde{\mathbf{I}}$ , we find that  $(X(\nu), \mathbf{I}_p)$  and  $(X(\nu), \mathbf{I}_q)$  are biholomorphic for each  $p, q \in S^2$  whose first component is equal to zero. Hence we must prove that the vector field  $\tilde{\mathbf{I}}\tilde{X}$  is complete.

Let  $c(t): (t_1, t_2) \rightarrow X(\nu) \times S^2$  be a maximal integral curve of  $\tilde{\mathbf{I}}\tilde{X}$ , where  $0 \in (t_1, t_2)$ . Let  $p_1: X(\nu) \times S^2 \rightarrow X(\nu)$  and  $p_2: X(\nu) \times S^2 \rightarrow S^2$  be the projections. We set  $\varphi(t) = p_1 \circ c(t)$  and  $x(t) = p_2 \circ c(t)$ . We assume that the first component  $x_1(t)$  of  $x(t)$  is strictly increasing and that  $x_1(0) = 0$ . Assuming that  $t_2 < \infty$ , we derive a contradiction. We have

$$\frac{d\varphi}{dt}(t) = (\mathbf{I}_{x(t)})_{\varphi(t)}(X_1^\#)_{\varphi(t)} \quad \text{for each } t \in (t_1, t_2).$$

Thus we have

$$\begin{aligned} g\left(\frac{d\varphi}{dt}(t), (\mathbf{I}_1)_{\varphi(t)}(X_1^\#)_{\varphi(t)}\right) &= g((\mathbf{I}_{x(t)})_{\varphi(t)}(X_1^\#)_{\varphi(t)}, (\mathbf{I}_1)_{\varphi(t)}(X_1^\#)_{\varphi(t)}) \\ &= x_1(t)g((X_1^\#)_{\varphi(t)}, (X_1^\#)_{\varphi(t)}) \\ (5.3) \quad &= x_1(t)\|(X_1^\#)_{\varphi(t)}\|^2 \quad \text{for each } t \in (t_1, t_2). \end{aligned}$$

We fix an element  $t_0 \in (0, t_2)$ . Since  $x_1(t)$  is increasing by assumption, we have from (5.3),

$$\begin{aligned} x_1(t_0)\|(X_1^\#)_{\varphi(t)}\|^2 &\leq x_1(t)\|(X_1^\#)_{\varphi(t)}\|^2 \\ (5.4) \quad &= g\left(\frac{d\varphi}{dt}(t), (\mathbf{I}_1)_{\varphi(t)}(X_1^\#)_{\varphi(t)}\right) \quad \text{for each } t \in [t_0, t_2]. \end{aligned}$$

Now for each vector field  $Y$  on  $X(\nu)$ ,

$$g(\text{grad } f_1, Y) = df_1(Y) = -\omega_1(X_1^\#, Y) = -g(\mathbf{I}_1 X_1^\#, Y)$$

and hence

$$\text{grad } f_1 = -\mathbf{I}_1 X_1^\#.$$

Thus from (5.4),

$$\begin{aligned} x_1(t_0) \|(X_1^\#)_{\varphi(t)}\|^2 &\leq g\left(\frac{d\varphi}{dt}(t), -(\text{grad } f_1)_{\varphi(t)}\right) \\ &= -\frac{d}{dt}(f_1 \circ \varphi)(t) \quad \text{for each } t \in [t_0, t_2]. \end{aligned}$$

We set  $x_0 = \sqrt{x_1(t_0)}$ . Since  $x_1(t_0) > 0$ , we have the inequality

$$(5.5) \quad x_0 \|(X_1^\#)_{\varphi(t)}\| \leq \left| \frac{d}{dt}(f_1 \circ \varphi)(t) \right|^{1/2} \quad \text{for each } t \in [t_0, t_2].$$

Let  $s_0$  and  $s$  be two points on  $[t_0, t_2]$ , where  $s_0 < s$ . Let a curve  $\tilde{\varphi}(t) = (z(t), w(t))$  in  $\mu_K^{-1}(\nu)$  be a horizontal lift of  $\varphi|[s_0, s]$ , where  $s_0 \leq t \leq s$ . By definition of the metric on  $X(\nu)$ , we have

$$\begin{aligned} \int_{s_0}^s \left\| \frac{d\varphi}{dt}(t) \right\| dt &= \int_{s_0}^s \left| \frac{d\tilde{\varphi}}{dt}(t) \right| dt \\ (5.6) \quad &\geq |\tilde{\varphi}(s) - \tilde{\varphi}(s_0)|. \end{aligned}$$

We have

$$\begin{aligned} |\tilde{\varphi}(s) - \tilde{\varphi}(s_0)| &= \left( \sum_{i=1}^N |z_i(s) - z_i(s_0)|^2 + \sum_{i=1}^N |w_i(s) - w_i(s_0)|^2 \right)^{1/2} \\ &\geq \left( \sum_{i=1}^N |z_i(s) - z_i(s_0)|^2 \right)^{1/2} \\ &\geq \left| \left( \sum_{i=1}^N |z_i(s)|^2 \right)^{1/2} - \left( \sum_{i=1}^N |z_i(s_0)|^2 \right)^{1/2} \right| \\ (5.7) \quad &= \sqrt{2} \left| \sqrt{f_1 \circ \varphi(s)} - \sqrt{f_1 \circ \varphi(s_0)} \right|. \end{aligned}$$

From (5.6) and (5.7),

$$\int_{s_0}^s \left\| \frac{d\varphi}{dt}(t) \right\| dt \geq \sqrt{2} \left| \sqrt{f_1 \circ \varphi(s)} - \sqrt{f_1 \circ \varphi(s_0)} \right|.$$

Thus if  $f_1 \circ \varphi(s_0) \neq 0$ , then we have

$$(5.8) \quad \left\| \frac{d\varphi}{dt} \Big|_{t=s_0} \right\| \geq \sqrt{2} \left| \frac{d}{dt} \sqrt{f_1 \circ \varphi} \Big|_{t=s_0} \right|.$$

Now

$$\begin{aligned}
 \left\| \frac{d\varphi}{dt}(t) \right\|^2 &= g \left( \frac{d\varphi}{dt}(t), \frac{d\varphi}{dt}(t) \right) \\
 &= g \left( (\mathbf{I}_{x(t)})_{\varphi(t)} (X_1^\#)_{\varphi(t)}, (\mathbf{I}_{x(t)})_{\varphi(t)} (X_1^\#)_{\varphi(t)} \right) \\
 (5.9) \quad &= \left\| (X_1^\#)_{\varphi(t)} \right\|^2 \quad \text{for each } t \in (t_1, t_2).
 \end{aligned}$$

Thus from (5.5) and (5.8),

$$\sqrt{2} \left| \frac{d}{dt} \sqrt{f_1 \circ \varphi} \right|_{t=s_0} \leq \left\| (X_1^\#)_{\varphi(t)} \right\| \leq \frac{1}{x_0} \left| \frac{d}{dt} (f_1 \circ \varphi) \right|_{t=s_0}^{1/2}$$

and hence

$$\left| \frac{d}{dt} (f_1 \circ \varphi) \right|_{t=s_0} \leq \frac{2}{x_0^2} (f_1 \circ \varphi)(s_0).$$

Note that the inequality above is also valid for each  $s_0 \in [t_0, t_1]$  with  $f_1 \circ \varphi(s_0) = 0$ . Hence

$$(5.10) \quad \left| \frac{d}{dt} (f_1 \circ \varphi)(t) \right| \leq \frac{2}{x_0^2} (f_1 \circ \varphi)(t) \quad \text{for each } t \in [t_0, t_2].$$

Thus we have

$$\begin{aligned}
 f_1 \circ \varphi(t) &\leq f_1 \circ \varphi(t_0) \exp \left( \frac{2(t - t_0)}{x_0^2} \right) \\
 &\leq f_1 \circ \varphi(t_0) \exp \left( \frac{2(t_2 - t_0)}{x_0^2} \right) \quad \text{for each } t \in [t_0, t_2].
 \end{aligned}$$

Thus from (5.5), (5.9), and (5.10) we have

$$(5.11) \quad \left\| \frac{d\varphi}{dt}(t) \right\| \leq \frac{\sqrt{2(f_1 \circ \varphi)(t_0)}}{x_0^2} \exp \left( \frac{t_2 - t_0}{x_0^2} \right) \quad \text{for each } t \in [t_0, t_2].$$

We set

$$L = \frac{\sqrt{2(f_1 \circ \varphi)(t_0)}}{x_0^2} \exp \left( \frac{t_2 - t_0}{x_0^2} \right).$$

We fix an element  $(z_0, w_0) \in \mu_K^{-1}(\nu)$  such that  $\varphi(t_0) = [z_0, w_0]$ . We write  $z_0 = (\alpha_1, \dots, \alpha_N)$  and  $w_0 = (\beta_1, \dots, \beta_N)$ . Let  $s_0 = t_0$  and let  $\tilde{\varphi}(t_0) = (z_0, w_0)$ . From (5.6) and (5.11),

$$(t_2 - t_0)L \geq (s - t_0)L$$

$$\begin{aligned}
&\geq \int_{t_0}^s \left\| \frac{d\varphi}{dt}(t) \right\| dt \\
&= \int_{t_0}^s \left| \frac{d\tilde{\varphi}}{dt}(t) \right| dt. \\
&\geq \left( \sum_{i=1}^N |z_i(s) - \alpha_i|^2 + \sum_{i=1}^N |w_i(s) - \beta_i|^2 \right)^{1/2} \\
&\geq \left| \left( \sum_{i=1}^N (|z_i(s)|^2 + |w_i(s)|^2) \right)^{1/2} - \left( \sum_{i=1}^N (|\alpha_i|^2 + |\beta_i|^2) \right)^{1/2} \right| \\
&= \left| |\tilde{\varphi}(s)| - \left( \sum_{i=1}^N (|\alpha_i|^2 + |\beta_i|^2) \right)^{1/2} \right|
\end{aligned}$$

and hence

$$(t_2 - t_0)L + \left( \sum_{i=1}^N (|\alpha_i|^2 + |\beta_i|^2) \right)^{1/2} \geq |\tilde{\varphi}(s)|.$$

Hence  $\{\varphi(t) \mid t \in [t_0, t_2]\}$  is contained in some compact set in  $X(\nu)$ . This is a contradiction. Hence we have  $t_2 = \infty$ . Similarly, we have  $t_1 = -\infty$ .

Next we consider the general case. Since  $\#\mathcal{C}_\nu = 2$ , by Theorem 3.3 there exists  $P \in SO(3)$  such that the second and the third component of  $P\nu$  are equal to zero. Hence the theorem follows from Theorem 3.2.  $\square$

(2). We may assume that  $\nu$  is of the form  $(\nu_1, 0, 0)$ . From (1) it is sufficient to consider the complex structure  $\mathbf{I}_2$ . Now for each vector field  $Y$  on  $X(\nu)$ , we have

$$(5.12) \quad df_i(\mathbf{I}_2 Y) = (\partial f_i + \bar{\partial} f_i)(\mathbf{I}_2 Y) = \sqrt{-1}(\partial f_i - \bar{\partial} f_i)(Y) \quad \text{for each } i = 1, 2,$$

where  $\partial$  and  $\bar{\partial}$  are the (1,0) and (0,1) parts of  $d$  with respect to  $\mathbf{I}_2$ . We have

$$\begin{aligned}
df_i(\mathbf{I}_2 Y) &= -i(X_i^\#)\omega_1(\mathbf{I}_2 Y) \\
&= -g(\mathbf{I}_1 X_i^\#, \mathbf{I}_2 Y) \\
&= -g(\mathbf{I}_3 X_i^\#, Y) \\
&= -\omega_3(X_i^\#, Y) \\
&= -i(X_i^\#)\omega_3(Y) \quad \text{for each } i = 1, 2.
\end{aligned}$$

Thus from (5.12),

$$-i(X_i^\#)\omega_3 = \sqrt{-1}(\partial f_i - \bar{\partial} f_i) \quad \text{for each } i = 1, 2.$$

Thus from Proposition 5.1 we have

$$2\sqrt{-1}\partial\bar{\partial}f_i = L_{X_i^\#}\omega_3 = \omega_2 \quad \text{for each } i = 1, 2$$

and hence

$$\sqrt{-1}\partial\bar{\partial}(f_1 + f_2) = \omega_2.$$

Since  $f_1 + f_2$  is proper and  $\omega_2$  is a Kähler form,  $f_1 + f_2$  provides a strictly plurisubharmonic exhaustion function for  $X(\nu)$  with respect to  $\mathbf{I}_2$ . Hence  $(X(\nu), \mathbf{I}_2)$  is a Stein manifold.  $\square$

**EXAMPLE 5.3.** Let  $X(\nu)$  be a toric hyperkähler manifold in Example 4.1. Since  $\#\mathcal{C}_\nu = 2$ , it follows from Theorem 5.2 (1) that  $(X(\nu), \mathbf{I}_p)$  and  $(X(\nu), \mathbf{I}_q)$  are biholomorphic for each  $p, q \in S^2 \setminus \{\pm u / \|u\|\}$ .

**Problem.** Let  $X(\nu)$  be a toric hyperkähler manifold with  $\#\mathcal{C}_\nu > 2$ . It is still open whether  $(X(\nu), \mathbf{I}_p)$  and  $(X(\nu), \mathbf{I}_q)$  are biholomorphic for each  $p, q \in S^2 \setminus \mathcal{C}_\nu$ .

We give an example of a toric hyperkähler manifold  $X(\nu)$  with  $\#\mathcal{C}_\nu > 2$  such that  $(X(\nu), \mathbf{I}_p)$  and  $(X(\nu), \mathbf{I}_q)$  are biholomorphic for each  $p, q \in S^2 \setminus \mathcal{C}_\nu$ .

**EXAMPLE 5.4.** Let  $X(\nu_i)$  be a hyperkähler quotient of  $\mathbf{H}^{N_i}$  by  $K_i$  for each  $1 \leq i \leq m$ . Suppose that  $\#\mathcal{C}_{\nu_i} = 2$  for each  $1 \leq i \leq m$ . We set  $\nu = (\nu_1, \dots, \nu_m)$ . The product  $X(\nu_1) \times \dots \times X(\nu_m)$  is the hyperkähler quotient  $X(\nu)$  of  $\mathbf{H}^{N_1} \times \dots \times \mathbf{H}^{N_m}$  by  $K_1 \times \dots \times K_m$ . For each  $1 \leq i \leq m$ , there exists  $p_i \in S^2$  such that  $\mathcal{C}_{\nu_i} = \{p_i, -p_i\}$ . We assume that  $p_i$  and  $p_j$  are linearly independent for each  $1 \leq i \neq j \leq m$ . We have from Theorem 3.3,

$$\mathcal{C}_\nu = \{\pm p_i \mid 1 \leq i \leq m\}.$$

By assumption,  $\#\mathcal{C}_\nu$  is equal to  $2m$ . It follows from Theorem 5.2 (1) that  $(X(\nu_i), \mathbf{I}_p)$  and  $(X(\nu_i), \mathbf{I}_q)$  are biholomorphic for each  $p, q \in S^2 \setminus \mathcal{C}_{\nu_i}$  and for each  $1 \leq i \leq m$ . Thus  $(X(\nu), \mathbf{I}_p)$  and  $(X(\nu), \mathbf{I}_q)$  are biholomorphic for each  $p, q \in S^2 \setminus \mathcal{C}_\nu$ .

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