A comprehensive study of edge rings with graph-theoretic insights

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Papers related to the thesis

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Contents

Chapter 1

Introduction

Richard Stanley's affirmative proof of the upper bound conjecture for spheres, utilizing the theory of Cohen–Macaulay rings, marked a significant turning point at the crossroads of combinatorics, commutative algebra, and topology. It was this milestone that gave rise to the concept of Cohen–Macaulay complexes in the mid-1970s, subsequently forming the nucleus of a captivating and intricately connected area known as combinatorial commutative algebra. Notably, this development uncovered the pivotal role of commutative algebra in the algebraic exploration of combinatorics on convex polytopes and simplicial complexes. Stanley pioneered the systematic application of commutative algebra concepts and techniques to investigate simplicial complexes by considering the Hilbert function of Stanley–Reisner rings, whose defining ideals are generated by square-free monomials. Subsequently, the study of square-free monomial ideals, from both algebraic and combinatorial perspectives, became a flourishing area of research within commutative algebra. For a more detailed understanding, one may refer to standard textbooks on combinatorics and commutative algebra, including Bruns–Herzog [8], Hibi [19], Miller–Sturmfels [32] and Stanley [44].

In the mid-1960s, Buchberger introduced the concept of Gröbner bases in his thesis. His work presented a Gröbner basis criterion and an algorithm for computing such bases. By the late 1980s, the theory of Gröbner bases gained popularity and found applications across various mathematical fields including algebraic geometry, cryptography, coding theory, and so on. For instance, Gröbner bases offer algorithmic solutions to problems in polynomial ideal theory, they are integral in the construction and analysis of error-correcting codes and are also widely used in computer algebra systems for polynomial manipulation and solving systems of equations symbolically. Gröbner bases, along with initial ideals, introduced new computational methods and paved the way for theoretical advances in both commutative algebra and combinatorics. For fundamental results on Gröbner basis and initial ideals, advance to Sturmfels [45], which explores the application of Gröbner basis techniques in the study of convex polytopes. In this thesis, we will utilize these techniques as a method for demonstrating our results.

Edge rings: An overview

This research is particularly centered on edge rings. The edge rings are a special class of affine semigroup rings and their connection to another combinatorial object, finite graphs, motivated our exploration of edge rings. For a comprehensive introduction to the edge rings and toric ideals of graphs, one may advance to [18, Section 5] and [49, Section 10]. Throughout the dissertation, we assume all graphs to be finite, connected, and have no loops and multiple edges. Given a finite graph G, its associated edge ring is represented as $\Bbbk[G]$, where $\Bbbk$ is any field. Many researchers have conducted intensive studies on the edge rings and toric ideals of graphs. This research, in particular, centers on characterizing some intriguing algebraic properties of edge rings such as Cohen–Macaulayness, Gorensteinness, almost Gorensteinness, and so on, in terms of the corresponding graph. All the fundamental notions of combinatorial commutative algebra and results on edge rings that we will be dealing within this study have been introduced in the first part of this thesis.

On h-vectors of edge rings

The h-vectors of homogeneous rings are one of the most important invariants that often reflect ring-theoretic properties. The Gorensteinness of homogeneous normal Cohen–Macaulay domains are characterized by the symmetry of their h-vectors (see, [43]). Moreover, there are many other results claiming that the h-vectors of homogeneous (or semi-standard graded) normal Cohen–Macaulay rings (or domains) have some connection with their ring-theoretic properties (see, e.g., [7, 21, 25, 52], and so on). On the other hand, there are a few examples of edge rings of graphs whose h-vectors are explicitly computed.

As far as we know, the h-vectors (or their counterparts) of the edge rings of the following graphs have been computed:

- Let $K_{m,n}$ denote the complete bipartite graph with $m+n$ vertices.

Then

$$h(\Bbbk[K_{m,n}];t) = \sum_{i=0}^{\min\{m,n\}} \binom{m-1}{i}\binom{n-1}{i} t^i.$$

- Let K_m denote the complete graph with m vertices. Then

$$h(\Bbbk[K_m];t) = 1 + \frac{m(m-3)}{2}t + \sum_{i=2}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2i} t^i.$$

- The Hilbert functions of the edge rings of complete multipartite graphs were computed in [35, Theorem 2.6].
- The Hilbert series of the edge rings of bipartite graphs are described using their interior polynomials (see, [27]).

(For further clarification on the notations used above, see Section 2.1.) Regarding the results on $K_{m,n}$ and K_m, see [48], or [49, Section 10].

It is important to note that if the edge rings of the graphs are normal, then the Hilbert functions (resp. h-vectors) of the edge rings agree with the "Ehrhart polynomials (resp. h^*-vectors) of the edge polytopes arising from the graphs". The primary goal of the second part of this thesis is to explicitly compute the h-vector for some specific classes of graphs, thereby offering more examples to enrich the index of edge rings with explicitly known h-vectors.

On the characterization of algebraic properties of edge rings

For an affine semigroup $S \subset \mathbb{N}^d$, let $\Bbbk[S]$ be the affine semigroup ring of S. The affine semigroups and their associated affine semigroup rings have been well-studied by many researchers in several contexts. While studying for a characterization of the Cohen–Macaulay affine semigroup ring, Goto and Watanabe [16] have defined an extension S' of S and claimed that the condition, $S' = S$ is the necessary and sufficient condition for $\Bbbk[S]$ to be Cohen–Macaulay. Trung and Hoa [47] presented a counterexample and also demonstrated that $S' = S$ is insufficient to establish the Cohen–Macaulayness of $\Bbbk[S]$. They have also provided an additional topological condition on the convex rational polyhedral cone spanned by S in $\mathbb{Q}^d$ and characterized the Cohen–Macaulayness of $\Bbbk[S]$. Schäfer and Schenzel [38, Theorem 6.3] claimed that the condition $S' = S$ corresponds to the Serre's condition (S_2). Moreover, Katthän [28] unveils a substantial link between the structural aspects of holes within the affine semigroup S and ring-theoretic properties, including depth, Serre's condition (R_1) and (S_2). For an introduction to Serre's condition (R_1) and (S_2), readers may refer to [8, Section 2] or [30]. Note that the edge rings are affine semigroup rings arising from finite graphs. We will explore this further in subsequent sections of the thesis.

The benefit of characterizing algebraic properties using graphs lies in its ability to provide new perspectives, enhance understanding, and facilitate the development of efficient algorithms for solving problems in diverse mathematical areas. The Cohen–Macaulayness of the edge ring $\Bbbk[G]$ in terms of the corresponding graph G has been the subject of extensive research. Given that the edge ring $\Bbbk[G]$ is an affine semigroup ring, it is known from [26, Theorem 1] that, if $\Bbbk[G]$ is normal then $\Bbbk[G]$ is Cohen–Macaulay. Ohsugi and Hibi [34] have characterized the normality of an edge ring in terms of its graph. At about the same time, Simis–Vasconcelos–Villarreal independently came to the same conclusion and reported it in [41]. Recall from [8, Theorem 2.2.22] that the edge ring $\Bbbk[G]$ is normal if and only if $\Bbbk[G]$ satisfies Serre's conditions (R_1) and (S_2). In [20, Theorem 2.1], Hibi and Katthän have characterized the edge rings satisfying (R_1)-condition. Note that Serre's condition (S_2) is a necessary condition for $\Bbbk[G]$ to be Cohen–Macaulay. Based on these insights, Higashitani and Kimura [22] have provided the necessary condition for an edge ring to satisfy (S_2)-condition.

Through the third part of this study, we anticipate achieving a characterization of edge rings satisfying (S_2)-condition. Furthermore, it would be highly intriguing to investigate the existence of an edge ring that satisfies Serre's condition (S_2) but is not Cohen–Macaulay. While such examples are known for affine semigroup rings (demonstrated by Trung and Hoa [47]), it remains an open question whether such an example can be found as an edge ring. Based on our expectations, it is conjectured that no such example exists in the case of edge rings. This aspect presents an exciting challenge, and resolving it could shed light on the relationship between Serre's condition (S_2) and Cohen–Macaulayness for edge rings.

The notion of almost Gorenstein homogeneous rings was introduced by Goto–Takahashi–Taniguchi in [15], as a new class of graded rings between Cohen–Macaulay rings and Gorenstein rings. After this work, almost Gorenstein homogeneous rings have been studied further, e.g., in [21, 23, 31]. On almost Gorensteinness of edge rings, known examples of almost Gorenstein non-Gorenstein edge rings are presumably rare. However, almost Gorenstein edge rings arising from complete multipartite graphs were completely characterized in [23]. According to [23, Examples 1.5-1.7], we know that $K_{2,m}$, $K_{1,1,m}$, $K_{1,m,m}$ with

$m \geq 3$ and $K_{1,1,m,m}$ with $m \geq 2$ give almost Gorenstein but not Gorenstein edge rings, where $K_{a_1,\ldots,a_r}$ denotes the complete r-partite graph. A connected non-bipartite graph $\mathcal{G}_n$ with n copies of 3-cycles sharing a single vertex is introduced in Chapter 4. By Theorem 4.1.1, we note that the family of graphs $\mathcal{G}_n$, where $n \geq 3$, is a new family of graphs whose edge rings are almost Gorenstein but not Gorenstein.

Thesis objectives on edge rings

While the historical background provides valuable context, our main objective is to extend the current understanding and provide new insights into this area. As previously discussed, this research work has two distinct objectives regarding edge rings. Firstly, to focus on computing the h-vectors of some interesting family of graphs, to make a greater contribution to the index of edge rings whose h-vectors are explicitly known. Secondly, to investigate non-normal edge rings satisfying (S_2)-condition with an expectation to yield new insights into the characterization of (S_2)-condition for edge rings and to provide supporting evidence to the conjecture that there is no difference between satisfying Serre's condition (S_2) and Cohen–Macaulayness for edge rings.

Exploring graph theory: Beyond edge rings

The edge ring associated with a graph on d vertices and d edges, with a single odd cycle is identified as a polynomial ring in d variables over a field $\Bbbk$. It is well-known that a polynomial ring over a field is Cohen–Macaulay and the h-polynomial has to be 1. Therefore, instead of considering such unicyclic graphs in the context of edge rings, we will take a strict graph-theoretical approach in the fourth part of this research.

Some general references for graph-theoretic ideas are Bollobás [5] and Bondy–Murty [6]. The last part of this study focuses on the topic of graph labeling. The roots of graph labeling can be traced back to the mid-1960s, with its initial definition by Sadláček [37], followed by the formulation by Kotzig and Rosa [29] in 1970. Since then, the field has witnessed remarkable growth, with over 200 graph labeling techniques explored in over 3000 research papers. Labeled graphs are becoming an increasingly useful family of mathematical models from a broad range of applications. The importance of graph labeling includes its numerous applications in many areas like circuit design, radar, communication network address, and so on.

Various authors introduced labelings that generalized the idea of a magic square, i.e., a square array of positive integers where the sum of the numbers in every row, every column, and even the diagonals are the same. For a finite simple graph G on vertex set $V(G)$ with edge set $E(G)$, Kotzig and Rosa [29] defined the magic valuation of G as a labeling of both vertices and edges of G, in which the labels are the integers from 1 to $|V(G)| + |E(G)|$. Here, the sum of labels on an edge and its two endpoints is constant and we call it the magic constant of G. In 1996, Ringel and Llado [36] rediscovered this labeling idea and called it edge-magic. We shall use the phrase edge-magic total, as developed by Wallis [50] to distinguish this usage from that of other forms of labelings that utilize the word magic. A graph that has an edge-magic total labeling is called an edge-magic total graph. In [29], Kotzig and Rosa have demonstrated that the n-cycles with $n \geq 3$, and the complete bipartite graphs $K_{m,n}$ with $m, n \geq 1$, are edge-magic total graphs. They also established that a complete graph K_n is an edge-magic total if and only if $1 \leq n \leq 6$, and further proved that the disjoint union of n copies of path graph P_2 has an edge-magic total labeling if and only if n is odd. The bibliography section includes references to several pieces of pertinent literature, including [3, 14, 33, 36, 42].

Enomoto et al. [10] introduced the name super edge-magic total for edge-magic total labelings, with the added property that the vertices of graph G receive the smaller labels, i.e., 1 to $|V(G)|$. As demonstrated in [10], a complete graph K_n is super edge-magic total if and only if $n = 1, 2$, or 3 and a complete bipartite graph $K_{m,n}$ is super edge-magic total if and only if $m = 1$ or $n = 1$. Moreover, n-cycles are super edge-magic total if and only if n is odd (see, [10]). Some further results on the super edge-magic total graph can be found in [12, 13, 14].

Avadayappan–Jeyanthi–Vasuki [2], introduced the concept of the super edge-magic total strength of a graph G as the minimum magic constant, where the minimum is taken over all the super edge-magic total labelings of G. The super edge-magic total strength of an odd cycle of length n equals $\frac{5n+3}{2}$ (see [2, Theorem 3]). Further results on the super edge-magic total strength of specific graphs are available in [2, 46]. More comprehensive results on this topic can be found in Section 8.1. The final part of our study focuses on unicyclic graphs and provides evidence to conjecture that the super edge-magic total strength of a certain family of unicyclic graphs with m edges, consisting of an odd cycle of length n, is equal to $2m + \frac{n+3}{2}$.

Outline of the thesis

A brief structure of this dissertation is as follows. The thesis is divided into four parts. The first three parts consist of two chapters each and the final part has one chapter.

Part I consists of two chapters, where we revisit some basic prerequisite definitions and results that will be encountered throughout this thesis. In Chapter 2, we recall fundamental notions on combinatorial commutative algebra, and in Chapter 3, we prepare materials on the edge rings and toric ideals of graphs for further investigations.

Part II focuses on the explicit computation of h-vectors for certain classes of graphs, particularly those with normal edge rings (square-free initial ideals), by using the technique of initial ideals and the associated simplicial complex. Chapter 4 explores a family of graphs comprising n copies of 3-cycles with a unique common vertex and computes their h-vectors. This analysis sheds light on the algebraic properties of these structured graphs and introduces a new family of almost Gorenstein but not Gorenstein edge rings. Chapter 5 delves into the explicit computation of the h-polynomials of a related family of graphs, containing those studied in Chapter 4. Moreover, in this chapter, we characterize the almost Gorensteinness of their corresponding edge rings.

Part III is dedicated to the study of certain families of graphs whose edge rings are non-normal and satisfy (S_2)-condition. More precisely, we examine the graph formed by the union of two complete graphs that share exactly one common vertex, along with a distinct family of cactus graphs. In Chapter 6, it is demonstrated that, within a specific range of the number of vertices and edges, there exists a certain family of graphs with non-normal edge rings that satisfy Serre's condition (S_2). In Chapter 7, we embark on an exploration of a special family of cactus graphs with non-normal edge rings. It's important to note that the approach used in this chapter to prove the (S_2)-condition differs slightly from the one employed in Chapter 6. We base our method on insights from [28] concerning the interplay between the structural attributes of holes in an affine semigroup and the ring-theoretic properties of their corresponding affine semigroup rings.

Part IV has a single chapter and is exclusively dedicated to some special graph-theoretical approaches. In this segment, our approach differs from the methods employed in our previous studies. Since this part is not focusing on our main object – edge rings, we begin Chapter 8 with Section 8.1 that contains a detailed introduction to the relevant notions and results regarding super edge-magic total labeling of graphs.

In Chapter 8, we look at a special family of unicyclic graphs that consists of an odd cycle of length n and a certain number of pendant vertices adjacent to each of the vertices of the n-cycle. We particularly examine three specific graphs belonging to this family of unicyclic graphs and provide substantial evidence in favor of Conjecture 8.1.8.

Part I

Preliminaries

Chapter 2

Notions on combinatorial commutative algebra

Here, we will review some of the key concepts and findings from combinatorial commutative algebra that we will be using in our investigation. For any undefined terms and notations in this chapter, one may refer to standard texts; like Burns–Herzog [8], Matsumura [30], Stanley [44] and Sturmfels [45].

2.1 Homogeneous rings and Hilbert series

We restrict ourselves to recalling the most important notions and facts. Let R be a Noetherian local ring and M be a nonzero R-module. M is said to be *Cohen–Macaulay* if $\operatorname{depth} M = \dim M$. R is said to be *Cohen–Macaulay ring* if it is a Cohen–Macaulay module over itself, i.e., $\operatorname{depth} R = \dim R$. One can advance to [8], for the systematic study of depth and the Cohen–Macaulay property.

Let R be a Noetherian ring and $\operatorname{Spec}(R)$ denotes the set of all prime ideals of R. Recollect that for an integer n, we say that R satisfies:

- *Serre's condition* (R_n) if $R_\mathfrak{p}$ is a regular local ring, for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R_\mathfrak{p} \leq n$.
- *Serre's condition* (S_n) if $\operatorname{depth} R_\mathfrak{p} \geq \inf\{n, \dim R_\mathfrak{p}\}$, for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

We easily see from the definition that R is Cohen–Macaulay if and only if R satisfies (S_n)-condition for every $n \geq 0$. Recall that a *normal ring* is an integral domain that is integrally closed in its field of fractions (see [30], for detailed discussions of normality).

Serre's normality criterion: A Noetherian ring R is normal if and only if it satisfies Serre's conditions (R_1) and (S_2).

Throughout this doctoral thesis, we only consider homogeneous rings. For detailed information on homogeneous rings, see [8] or [49]. Let R be a Cohen–Macaulay homogeneous ring of dimension d over a field $\Bbbk$. For a graded R-module M, let us denote $\mu(M)$ as the number of elements in a minimal system of generators of M as an R-module and $e(M)$ be the multiplicity of M. Generally, $e(M) \geq \mu(M)$. We denote the *Cohen–Macaulay type* of R by $r(R)$ and let ω_R be a canonical module of R. It is well known that $r(R) = \mu(\omega_R)$.

The *Hilbert function*, $H(M, i)$, measures the dimension of the i^{th} homogeneous piece of the graded module M and the *Hilbert series* is the corresponding generating function, $\mathcal{H}_M(t) = \sum_{i\in\mathbb{Z}} H(M, i)t^i$. They are one of the most important numerical invariants of graded modules and form a bridge between commutative algebra and its combinatorial applications.

For a homogeneous ring R, the Hilbert function $H(R, i) = \dim_\Bbbk R_i$ and the Hilbert series is:

$$\mathcal{H}_R(t) = \sum_{i\geq 0} \dim_\Bbbk R_i t^i = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^d},$$

where R_i denotes the homogeneous part of R of degree i, $\dim_\Bbbk$ denotes the dimension as a $\Bbbk$-vector space, and we assume that $h_s \neq 0$. The rationality of the Hilbert series is already a powerful result, but some of its refined properties reflect homological conditions for the rings and modules under consideration.

We call the polynomial $h_0 + h_1 t + \cdots + h_s t^s$ appearing in the numerator as the *h-polynomial* of R, denoted by $h(R; t)$, and the sequence of the coefficients $(h_0, h_1, \ldots, h_s)$ as the *h-vector* of R, denoted by $h(R)$. The index s is called the *socle degree* of R. Moreover, $e(R) = \sum_{i=0}^{s} h_i$.

A Cohen–Macaulay ring R is called *Gorenstein* if R is isomorphic to its own canonical module, i.e., $R \cong \omega_R$. It follows from [43, Theorem 4.4] that, if R is a normal Cohen–Macaulay homogeneous domain, then R is Gorenstein if and only if $h(R)$ is symmetric, i.e., $h_i = h_{s-i}$ for $i = 0, 1, \ldots, s$.

For a graded R-module M, let $M(-l)$ denote the R-module whose underlying R-module is the same as that of M and whose grading is given by $(M(-l))_n = M_{n-l}$ for all $n \in \mathbb{Z}$. Let $a = a(R)$ be the a-invariant of R. Note that, for a canonical module ω_R of R, we have $a = -\min\{i \colon (\omega_R)_i \neq 0\}$. We say that a Cohen–Macaulay ring R is *almost Gorenstein* if there exists an exact sequence

$$0 \longrightarrow R \xrightarrow{\phi} \omega_R(-a) \longrightarrow C \longrightarrow 0$$

of graded R-modules with $\mu(C) = e(C)$, where ϕ is an injection of degree 0 (see, [15, Definition 1.5]). Let us recall the necessary and sufficient condition for a homogeneous domain to be almost Gorenstein, as stated in [21].

Proposition 2.1.1 ([21, Corollary 2.7])**.** *Let R be a Cohen–Macaulay homogeneous domain of dimension d over a field $\Bbbk$ and let $h(R) = (h_0, \dots, h_s)$ be its h-vector. Then R is almost Gorenstein if and only if the following equality holds:*

$$r(R) - 1 = \sum_{j=0}^{s-1} ((h_s + \cdots + h_{s-j}) - (h_0 + \cdots + h_j)) =: \tilde{e}(R).$$

Note that $r(R) - 1 \leq \tilde{e}(R)$ is always satisfied, so almost Gorensteinness is equivalent to the inequality $r(R) \geq \tilde{e}(R) + 1$.

2.2 Affine semigroup rings

Let us recall some of the basics and useful results on *affine semigroup rings*. One may refer to [8, Chapter 6] for detailed fundamental studies.

A *semigroup* is a set with an associative binary operation. A finitely generated semigroup that is isomorphic to a sub-semigroup of a free abelian group $\mathbb{Z}^d$ for some $d \geq 0$ is known as an *affine semigroup*. Affine semigroups lie in the intersection of algebraic geometry, combinatorics, commutative algebra, convex discrete geometry and number theory.

Let $S \subset \mathbb{Z}^d_{\geq 0}$ be an arbitrary affine semigroup and $\Bbbk$ be any field. The *affine semigroup ring*, denoted as $\Bbbk[S]$ is a $\Bbbk$-algebra with a basis consisting of the symbols $\mathbf{x}^s$, which corresponds to $s \in S$, and the multiplication on $\Bbbk[S]$ is defined by $\mathbf{x}^s\mathbf{x}^{s'} = \mathbf{x}^{s+s'}$. That is, we have $\Bbbk[S] = \Bbbk[\mathbf{x}^s \colon s \in S]$, where for $s = (s_1, \dots, s_d) \in S$, $\mathbf{x}^s := x_1^{s_1} \cdots x_d^{s_d}$. Recall that $\dim \Bbbk[S] = \dim S$.

For an arbitrary affine semigroup $S \subset \mathbb{Z}^d_{\geq 0}$, let $\{s_1, \dots, s_t\}$ be the minimal finite subset of S such that $S = \left\{ \sum_{i=1}^t z_i s_i \colon z_i \in \mathbb{Z}_{\geq 0} \right\}$. Then, $\{s_1, \dots, s_t\} \subset S$ is called the *minimal generating system* of S. Note that for $\mathbb{A} \subset \mathbb{R}$, we denote

$$\mathbb{A}S = \left\{ \sum_{i=1}^t a_i s_i : a_i \in \mathbb{A} \right\}.$$

In this study, we only consider the cases where $\mathbb{A}$ is $\mathbb{Q}_{\geq 0}$ or $\mathbb{Z}$ or $\mathbb{Z}_{\geq 0}$. Let $\mathbb{Z}S$ be the free abelian group generated by S and $\mathbb{Q}_{\geq 0}S$ be the rational polyhedral cone generated by S.

An affine semigroup S is called *normal* if it satisfies the following condition: if $mz \in S$ for some $z \in \mathbb{Z}S$ and $m \in \mathbb{Z}_{>0}$, then $z \in S$. Thus we see that S must be normal if $\Bbbk[S]$ is a normal domain. It is evident from [8, Thoerem 6.1.4.] that $\Bbbk[S]$ is normal if and only if S is a normal affine semigroup. Let $\overline{S} = \mathbb{Q}_{\geq 0}S \cap \mathbb{Z}S$ be the *normalization* of S. Analogous to the definition of normality of S, we say that S is normal if $S = \overline{S}$ holds. (See, e.g., [8, Section 6.1].)

Furthermore, it is well known as Hochster's theorem (see [8, Theorem 6.3.5]) that if S is a normal affine semigroup, then $\Bbbk[S]$ is Cohen–Macaulay.

The set $F \subset S$ is said to be a *face* of S, if the following holds: $s, s' \in S,\ s + s' \in F \iff s \in F$ and $s' \in F$. The dimension of a face F is defined to be the rank of the free abelian group $\mathbb{Z}F$. Throughout our study, we consider only *positive* affine semigroups, i.e., the minimal face of S is $\{0\}$.

The set of holes $\overline{S} \backslash S$ in S was given a geometric description by Katthän in [28], and it was connected to the ring-theoretical features of S. While the affine semigroups examined in [28] were not necessarily positive, in this thesis, we translate these findings into the case of positive affine semigroups.

Theorem 2.2.1 ([28, Theorem 3.1])**.** *Let S be an affine semigroup. Then there exists a (not-necessarily disjoint) decomposition*

$$\overline{S} \backslash S = \bigcup_{i=1}^{l} (s_i + \mathbb{Z}F_i) \cap \mathbb{Q}_{\geq 0}S \tag{2.1}$$

with $s_i \in S$ and faces F_i of S. If no $s_i + \mathbb{Z}F_i$ can be omitted from the union, then the decomposition is unique.

The set $s_i + \mathbb{Z}F_i$ in (2.1) is called a *j-dimensional family of holes* of S, where $j = \dim F_i$.

Theorem 2.2.2 ([28, Theorem 5.2])**.** *Let S be an affine semigroup of dimension d. Then $\Bbbk[S]$ satisfies Serre's condition (S_2) if and only if every family of holes of S is of dimension $d-1$.*

Now, let us consider the set $\{F_1, \dots, F_m\}$ to be the set of all facets of the convex rational polyhedral cone $\mathbb{Q}_{\geq 0}S$. We define

$$\begin{aligned} S_i &:= S - S \cap F_i \\ &= \{\mathbf{x} \in \mathbb{Z}S \colon \exists\ \mathbf{y} \in S \cap F_i \text{ such that } \mathbf{x} + \mathbf{y} \in S\}, \end{aligned}$$

and $S' := \bigcap_{i-1}^{m} S_i$. Note that, the condition $S' = S$ corresponds to the Serre's condition (S_2); see [38, Theorem 6.3].

The primary focus of this study is on *edge rings*, which are affine semigroup rings associated with finite graphs. We will delve deeper into this topic in later sections.

2.3 Gröbner basis and initial ideals

Let $R = \mathbb{k}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{k}$ with each $\deg x_i = 1$. Let $\mathcal{M}(R)$ be the set of all monomials of R. Note that the set $\mathcal{M}(R)$ is a $\mathbb{k}$-basis of R and for any polynomial $f \in R$, we write

$$f = \sum_{r \in \mathcal{M}(R)} a_r r, \text{ with } a_r \in \mathbb{k}.$$

Then the set of all monomials in $\mathcal{M}(R)$ such that $a_r \neq 0$, is called the *support* of f and is denoted by $\mathrm{supp}(f)$.

Given monomials $\mathbf{x^u} = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$ and $\mathbf{x^v} = x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$ in $\mathcal{M}(R)$, we say that $\mathbf{x^u}$ *divides* $\mathbf{x^v}$, denoted by $\mathbf{x^u} \mid \mathbf{x^v}$, if each $u_i \leq v_i$ for $1 \leq i \leq n$. A monomial $\mathbf{x^u}$ in R is called *square-free* if the exponent vector $\mathbf{u} = (u_1, \ldots, u_n)$ is such that $0 \leq u_i \leq 1$ for all $1 \leq i \leq n$.

Recall that a *total order* on a set is a partial order in which any two elements of the set are comparable. A *monomial order* on R is a total order $<$ on $\mathcal{M}(R)$ such that

(i) $1 < r$ for all $1 \neq r \in \mathcal{M}(R)$;

(ii) if $r, r' \in \mathcal{M}(R)$ and $r < r'$, then $rs < r's$ for all $s \in \mathcal{M}(R)$.

Specifically, we encounter the following monomial order in this dissertation. Let $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ be vectors in $\mathbb{Z}^n_{\geq 0}$. We define the monomial order $<_{\text{lex}}$ on $\mathcal{M}(R)$ by arranging $\mathbf{x^u} <_{\text{lex}} \mathbf{x^v}$ if either $\sum_{i=1}^n u_i < \sum_{i=1}^n v_i$; or $\sum_{i=1}^n u_i = \sum_{i=1}^n v_i$ and the leftmost nonzero component of $\mathbf{u} - \mathbf{v}$ is negative. The monomial order $<_{\text{lex}}$ is called the *graded lexicographic order* on R induced by the ordering $x_1 > \cdots > x_n$.

Example 2.3.1. Let us consider the ring $R = \mathbb{k}[x_1, x_2, x_3]$ and some monomials $x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2$ in $\mathcal{M}(R)$. By the graded lexicographic order on R induced by the ordering $x_1 > x_2 > x_3$, we have

$$x_3^2 <_{\text{lex}} x_2x_3 <_{\text{lex}} x_2^2 <_{\text{lex}} x_1x_3 <_{\text{lex}} x_1x_2 <_{\text{lex}} x_1^2.$$

Let us fix a monomial order $<$ on R. Consider a polynomial f of R such that $f = \sum_{r \in \mathcal{M}(R)} a_r r$, with $a_r \in \mathbb{k}$. We define the *initial monomial* of f with respect to $<$, denoted by $\mathrm{in}_<(f)$, as the largest monomial belonging to $\mathrm{supp}(f)$ with respect to $<$. The *leading coefficient* of f is the coefficient of $\mathrm{in}_<(f)$ in f.

Definition 2.3.2. Let I be a nonzero ideal of R. The *initial ideal* of I with respect to $<$, written as $\text{in}_<(I)$, is defined as the monomial ideal of R generated by the initial monomials of nonzero polynomials in I. Thus, we have $\text{in}_<(I) = \langle \text{in}_<(f) \colon 0 \neq f \in I \rangle$.

Now, we recall the definition of *Gröbner basis*. Let I be a nonzero ideal of $R = \Bbbk[x_1, \ldots, x_n]$ and $<$ be a monomial order on R.

Definition 2.3.3. Let $\mathscr{G} = \{g_1, \ldots, g_r\}$ be a finite set of nonzero polynomials with each $g_i \in I$. The set $\mathscr{G}$ is said to be a *Gröbner basis* of I with respect to $<$ if the initial ideal $\text{in}_<(I)$ of I is:

$$\text{in}_<(I) = \langle \text{in}_<(g_1), \ldots, \text{in}_<(g_r) \rangle.$$

Theorem 2.3.4 ([17, Theorem 2.1.8])**.** *Let I be a nonzero ideal of $R = \Bbbk[x_1, \ldots, x_n]$ and $\mathscr{G} = \{g_1, \ldots, g_r\}$ be a Gröbner basis of I with respect to a monomial order $<$ on R. Then, $I = \langle g_1, \ldots, g_r \rangle$, i.e., every Gröbner basis of I is a system of generators of the ideal I.*

Let us recall from [18, Theorem 1.19] that, the set of monomials that do not belong to $\text{in}_<(I)$ form a $\Bbbk$-basis of the residue ring R/I as a vector space over $\Bbbk$. As an immediate consequence of this, for a graded ideal $I \subset R$, the computation of the Hilbert series of R/I can be reduced to the case that I is a monomial ideal.

Proposition 2.3.5 ([18, Proposition 2.6])**.** *Let $<$ be a monomial order on $R = \Bbbk[x_1, \ldots, x_n]$ and $I \subset R$ be a graded ideal. Then*

$$\mathcal{H}_{R/I}(t) = \mathcal{H}_{R/\,\text{in}_<(I)}(t).$$

2.3.1 Buchberger's criterion

We have seen that the Gröbner basis of an ideal is its system of generators. Buchberger's criterion is a highlight of the theory of Gröbner bases since it helps us in determining whether a given system of generators of an ideal form its Gröbner basis or not. As before, we consider $R = \Bbbk[x_1, \ldots, x_n]$ and fix a monomial order $<$ on R. First of all, we recall the division algorithm.

The division algorithm: Let $g_1, \ldots, g_r$ be nonzero polynomials in R. Given a nonzero polynomial f in R, there exist polynomials $f_1, \ldots, f_r$ and f' in R with

$$f = \sum_{i=1}^{r} f_i g_i + f' \tag{2.2}$$

such that the following conditions hold.

- If $f' \neq 0$, then no monomial $u \in \text{supp}(f')$ belongs to the monomial ideal $\langle \text{in}_<(g_i) : 1 \leq i \leq r \rangle$.
- If $f_i g_i \neq 0$, then, then $\text{in}_<(f) \geq \text{in}_<(f_i g_i)$.

Let f and g be nonzero polynomials of R and $\text{lcm}(\text{in}_<(f), \text{in}_<(g))$ denotes the least common multiple of $\text{in}_<(f)$ and $\text{in}_<(g)$. Let c_f be the coefficient of $\text{in}_<(f)$ in f and c_g be the coefficient of $\text{in}_<(g)$ in g. Then the polynomial

$$S(f,g) = \frac{\text{lcm}(\text{in}_<(f), \text{in}_<(g))}{\text{in}_<(f)} f - \frac{\text{lcm}(\text{in}_<(f), \text{in}_<(g))}{\text{in}_<(g)} g$$

is called the *S-polynomial* of f and g.

We say that f *reduces to* 0 with respect to $g_1, \ldots, g_r$ if there exists a standard expression (2.2) of f with respect to $g_1, \ldots, g_r$ with $f' = 0$. Recall that if $\text{in}_<(f)$ and $\text{in}_<(g)$ are relatively prime, then $S(f,g)$ reduces to 0 with respect to f, g (see, e.g., [18, Lemma 1.27]).

Now, let us state the Buchberger's criterion. For the detailed proof, see, e.g., [18, Theorem 1.29].

Theorem 2.3.6 (**Buchberger's criterion).** *Let I be a nonzero ideal of the polynomial ring R and $\mathscr{G} = \{g_1, \ldots, g_r\}$ be a system of generators of I. Then $\mathscr{G}$ is a Gröbner basis of I if and only if for all $i \neq j$, the S-polynomial $S(g_i, g_j)$ reduces to 0 with respect to $g_1, \ldots, g_r$.*

2.4 Stanley–Reisner rings

In this section, we recall some basic definitions and properties of an *abstract simplicial complex* and its associated *Stanley–Reisner ring.* Throughout our study, we will be using the term *simplicial complex* to represent an abstract simplicial complex. In this section, we also emphasize *pure shellable complexes* and recollect some well-known results on them.

Definition 2.4.1. Let $[n] = \{1, \ldots, n\}$ be the *vertex set* and Δ be a subset of the powerset of V. Δ is said to be a *simplicial complex* on $[n]$ if whenever $F \in \Delta$ and $G \subset F$, then $G \in \Delta$.

Any $F \in \Delta$ is called a *face* of Δ and $\dim F = |F| - 1$. The *dimension* of the simplicial complex is given by

$$\dim \Delta = \max\{\dim F : F \in \Delta\}.$$

The maximal faces of Δ under inclusion are called *facets* of Δ. A simplicial complex Δ is determined by its facets. If all the facets of

Δ have the same dimension, then we say that Δ is *pure*. Let F be a subset of $[n]$ with $F \notin \Delta$, then F is said to be a *nonface* of Δ.

Let us consider $\dim \Delta = d-1$ and f_i be the number of i-dimensional faces of Δ. For any non-empty simplicial complex Δ on the vertex set $[n]$, the empty set $\emptyset$ is regarded as a face of Δ. Thus we have $f_{-1} = 1$ and $f_0 = n$. The sequence $f(\Delta) = (f_0, f_1, \dots, f_{d-1})$ is called the *f-vector* of Δ.

Let Δ be a simplicial complex on the vertex set $[n]$. Let $\Bbbk$ be a field and $R = \Bbbk[x_1, \dots, x_n]$ be the standard polynomial ring over $\Bbbk$. The monomial ideal I_Δ, generated by square-free monomials $\mathbf{x}_F = \prod_{i \in F} x_i$ such that $F \notin \Delta$, is called the *Stanley–Reisner ideal* of Δ. In other words, $I_\Delta = \langle \mathbf{x}_F : F \text{ is minimal nonface in } \Delta \rangle$. Recall that the primary decomposition of I_Δ is given by $I_\Delta = \bigcap\limits_{F \in \Delta} P_{\overline{F}}$, where $P_{\overline{F}} = \big\langle x_i \colon i \notin F \big\rangle$.

Definition 2.4.2. Let Δ be a simplicial complex on $[n]$ and $R = \Bbbk[x_1, \dots, x_n]$ be the standard polynomial ring over the field $\Bbbk$. The quotient ring $\Bbbk[\Delta] = R/I_\Delta$ is called the *Stanley–Reisner ring* of Δ.

Recall that the dimension of $\Bbbk[\Delta]$ is given by $\dim \Bbbk[\Delta] = \dim \Delta + 1$. For a simplicial complex Δ of dimension $d-1$, the Hilbert series of the corresponding Stanley–Reisner ring $\Bbbk[\Delta]$ is given by

$$\mathcal{H}_{\Bbbk[\Delta]}(t) = \sum_{i=0}^{d} f_{i-1} \left(\frac{t}{1-t} \right)^i = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d},$$

where $h_i \in \mathbb{Z}$. The $(d+1)$-tuple $h(\Delta) = (h_0, h_1 \dots, h_d)$ is called the *h-vector* of the simplicial complex Δ.

2.4.1 Shellable simplicial complexes

Recall that a simplicial complex Δ is Cohen–Macaulay over any field $\Bbbk$ if the associated Stanley–Reisner ring $\Bbbk[\Delta]$ is a Cohen–Macaulay ring.

Let $F_1, \dots, F_t$ be the facets of a pure simplicial complex Δ of dimension $d-1$. Let $\langle F_1, \dots, F_m \rangle$ be the unique smallest simplicial complex which contains all F_i, $1 \le i \le m$. The ordering of the facets is said to be a *shelling* if it satisfies that $\langle F_i \rangle \cap \langle F_1, \dots, F_{i-1} \rangle$ is generated by a non-empty set of maximal proper faces of F_i with dimension $d-2$, for all $2 \le i \le t$. We say that a pure simplicial complex is *shellable* if it has a shelling. Throughout our further study, we may refer the subcomplex $\langle F_i \rangle \cap \langle F_1, \dots, F_{i-1} \rangle$ as the *intersection subcomplex* corresponding to the i^{th} shelling step.

Consider a shellable simplicial complex Δ and let $F_1, \dots, F_t$ be a shelling of Δ. Let r_i be the number of maximal proper faces of F_i in

$\langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle$ for $2 \leq i \leq t$, and let $r_1 = 0$. Then $h(\Delta) = (h_0, \ldots, h_d)$, is obtained by $h_j = |\{i \colon r_i = j\}|$. This is known as the *McMullen characterization of h-vectors* of a pure shellable simplicial complex. For a detailed approach, see [8, Corollary 5.1.14].

From [8, Theorem 5.1.13], we recall that a shellable simplicial complex is Cohen–Macaulay over any field.

Chapter 3

Edge rings

In this chapter, we will go over pertinent basic definitions, theorems, and well-known findings that are necessary for a thorough comprehension of our main study object, the edge rings.

3.1 Convex polytopes

We limit ourselves to recalling the most significant and relevant notions; for an in-depth study of *convex polytopes*, see [18] or [19]. A *convex polytope* of $\mathbb{R}^n$ is a convex hull of a nonempty finite set of $\mathbb{R}^n$ and a *hyperplane* of $\mathbb{R}^n$ is defined as $\mathcal{H} = \left\{(r_1, \ldots, r_n) \in \mathbb{R}^n \colon \sum_{i=1}^{n} a_i r_i = c\right\}$, where $r_i, c \in \mathbb{R}$. For a given $\mathcal{H} \subset \mathbb{R}^n$, the closed half-spaces $\mathcal{H}^+$ and $\mathcal{H}^-$ of $\mathbb{R}^n$ are defined as follows: $\mathcal{H}^+ = \left\{(r_1, \ldots, r_n) \in \mathbb{R}^n \colon \sum_{i=1}^{n} a_i r_i \geq c\right\}$ and $\mathcal{H}^- = \left\{(r_1, \ldots, r_n) \in \mathbb{R}^n \colon \sum_{i=1}^{n} a_i r_i \leq c\right\}$.

Let $\mathcal{P} \subset \mathbb{R}^n$ be a convex polytope. A hyperplane $\mathcal{H} \subset \mathbb{R}^n$ is said to be a *supporting hyperplane* if $\mathcal{H} \cap \mathcal{P} \neq \emptyset$, $\mathcal{H} \cap \mathcal{P} \neq \mathcal{P}$ and either $\mathcal{P} \subset \mathcal{H}^+$ or $\mathcal{P} \subset \mathcal{H}^-$. If $\mathcal{H}$ is a supporting hyperplane of $\mathcal{P}$, then a subset of $\mathcal{P}$, of the form $\mathcal{H} \cap \mathcal{P}$ is said to be a *face* of $\mathcal{P}$. If $\{\mathbf{v}\}$ is a face of $\mathcal{P}$, then $\mathbf{v} \in \mathcal{P}$ is called a *vertex* of $\mathcal{P}$. A convex polytope $\mathcal{P}$ is called an *integral polytope* if each vertex of $\mathcal{P}$ belongs to $\mathbb{Z}^n$.

3.2 Toric rings and toric ideals

An overview of toric rings and toric ideals will be covered in this section. Let $\mathcal{A} = (a_{i,j})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$ be an $n \times m$-matrix of integer entries and let

$$\mathbf{a}_j = \begin{pmatrix} a_{1,j} \\ \vdots \\ a_{n,j} \end{pmatrix}, \ 1 \leq j \leq m$$

be the column vectors of $\mathcal{A}$. Let $\mathbb{Z}^{n \times m}$ denote the set of matrices $\mathcal{A} = (a_{i,j})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$ with each $a_{i,j} \in \mathbb{Z}$.

Let us recollect that, for any field $\Bbbk$ and the matrix $\mathcal{A} \in \mathbb{Z}^{n \times m}$ with column vectors $\mathbf{a}_j$, we define a $\Bbbk$-algebra homomorphism

$$\pi \colon \Bbbk[x_1, \ldots, x_m] \longrightarrow \Bbbk[t_1, \ldots, t_n] \text{ with } x_j \mapsto \mathbf{t}^{\mathbf{a}_j},$$

such that the image of π denoted by $\Bbbk[\mathcal{A}]$ is called the *toric ring* of $\mathcal{A}$ and the kernel of π denoted by $I_{\mathcal{A}}$ is called the *toric ideal* of $\mathcal{A}$.

Let $\mathbf{k}^t$ be the transpose of vector $\mathbf{k}$ and $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i$ represents the inner product of vectors $\mathbf{a} = (a_1, \ldots, a_n)^t$ and $\mathbf{b} = (b_1, \ldots, b_n)^t$. Recall that a *configuration* is a matrix $\mathcal{A} \in \mathbb{Z}^{n \times m}$, with column vectors $\mathbf{a}_j$, such that there exists $\mathbf{k} \in \mathbb{Q}^n$ with $\mathbf{a}_j \cdot \mathbf{k} = 1$ for $1 \leq j \leq m$. The configuration $\mathcal{A}$ is called *normal* if $\mathbb{Z}_{\geq 0}\mathcal{A} = \mathbb{Z}\mathcal{A} \cap \mathbb{Q}_{\geq 0}\mathcal{A}$. In the language of commutative algebra, a configuration $\mathcal{A} \in \mathbb{Z}^{n \times m}$ is normal if and only if the toric ring $\Bbbk[\mathcal{A}]$ is normal.

Now, let $\mathcal{P} \subset \mathbb{R}^n$ be an integral polytope with $\mathcal{P} \cap \mathbb{Z}^n = \{\mathbf{a}_1 \ldots, \mathbf{a}_m\}$ and $\mathcal{A}_{\mathcal{P}} \in \mathbb{Z}^{(n+1) \times m}$ be the configuration with $(\mathbf{a}_j, 1)^t$ as the column vector for $1 \leq j \leq m$. If the configuration $\mathcal{A}_{\mathcal{P}}$ is normal, then we say that the integral polytope $\mathcal{P}$ is normal.

3.3 Finite graphs

Here we will have a look at the essential definitions and results regarding finite graphs, which will be utilized through our investigation. For more concrete details on graphs, one may refer to the standard texts on graph theory; like Bollobás [5] and Bondy–Murty [6].

All along this thesis, we will only consider *finite simple graphs*. This means that we only take into account graphs with a finite number of vertices and edges, and a maximum of one edge connecting any two vertices. We also forbid loops, which are edges that connect one vertex to another.

Let $[d] = \{1, \ldots, d\}$ and G be a finite simple graph on the vertex set $V(G) = [d]$ with edge set $E(G) = \{e_1, \ldots, e_m\}$. Recall that given an edge $e = \{i, j\} \in E(G)$, the end vertices i and j are called *adjacent* vertices.

If the vertex set $V(G)$ can be partitioned into two disjoint subsets such that no two vertices in the same partition are adjacent, then the graph G is called a *bipartite graph*.

A simple graph in which an edge joins each pair of distinct vertices is called a *complete graph*. A complete graph on n vertices is denoted by K_n.

Recall that in a graph G, a *path* between any two vertices is a sequence of distinct edges that joins the two vertices. The *length* of a path is the total number of edges in it. A path from vertex u to v is the *shortest path* if there is no other path from u to v with a lower length. The *distance* between two vertices is the length of the shortest path between those two vertices. The distance between a graph's most distant vertices is known as its *diameter* and let us denote it as $\operatorname{diam} G$.

A *closed walk* of length r in a graph G is a sequence of edges $\Gamma = (e_{i_1}, \ldots, e_{i_r})$, such that $e_{i_j} = \{i_j, i_{j+1}\}$, for $j = 1, \ldots, r$, where $i_{r+1} = i_1$. A closed walk of even length is called an *even closed walk*.

A *cycle* of a graph G is a closed walk $C = (e_{i_1}, \ldots, e_{i_n})$ in G (with $n \geq 3$) such that $i_k \neq i_l$ for all $1 \leq k < l \leq n$. A cycle of length n is called an *n-cycle*. Recall that, a cycle of odd length is called an *odd cycle* and that of even length is called an *even cycle*. Throughout this dissertation we denote a cycle by only listing its vertices, that is, we denote an n-cycle as $C = (i_1, \ldots, i_n)$. A *chord* of a cycle $C = (i_1, \ldots, i_n)$ is an edge of the form $e = \{i_k, i_l\}$, where $1 \leq k < l \leq n$, such that $l \neq k + 1$ and $(k, l) \neq (1, n)$. An n-cycle is said to be *minimal* in G if there exists no chord in it.

A finite simple graph G is said to be *connected* if there exists a path between any two distinct vertices of G. A connected subgraph of G that is not a part of any other larger connected subgraphs is known as a *connected component* of G.

Let G be a finite simple graph on vertex set $V(G)$. A non-empty subset $T \subset V(G)$ is called an *independent set* if $\{v, w\} \notin E(G)$ for any $v, w \in T$. Now, let us look at some definitions and notations that we will be using in this study.

- Let us consider $W \subset V(G)$, and we define G_W as the *induced subgraph* of G on vertex set $V(G_W) = W$ with edge set $E(G_W) = \{e \in E(G) \colon e \subset W\}$.

- For a subset $W \subset V(G)$, let $G \setminus W$ be the induced subgraph on $V(G) \setminus W$. If $W = \{w\}$, then we write $G \setminus w$ instead of $G \setminus \{w\}$.

- For $v \in V(G)$, let $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$, and for any subset $W \subset V(G)$, let $N_G(W) = \bigcup_{w \in W} N_G(w)$.

- For an independent set $T \subset V(G)$, we define a *bipartite graph induced by T* as the graph on vertex set $T \cup N_G(T)$ with edge set $\big\{\{v,w\} \in E(G)\colon v \in T,\ w \in N_G(T)\big\}$.

3.4 Edge polytopes and edge rings

In this section, we review the concepts of edge rings, toric ideals of graphs, and some basic theorems related to them.

Let G be a graph on the vertex set $[d]$ with edge set $E(G) = \{e_1, \ldots, e_m\}$. Given $e = \{u,v\} \in E(G)$, let $\rho(e) = \mathbf{e}_u + \mathbf{e}_v$, where $\mathbf{e}_1, \ldots, \mathbf{e}_d \in \mathbb{R}^d$ are the unit vectors of $\mathbb{R}^d$. Let $A_G = \{\rho(e) : e \in E(G)\}$. The convex hull of A_G, denoted by $\mathcal{P}_G$ is called the *edge polytope* of G. It is well known from [18, Lemma 5.2], that the set of vertices of $\mathcal{P}_G$ coincides with $\mathcal{P}_G \cap \mathbb{Z}^d$ and $\mathcal{P}_G \cap \mathbb{Z}^d = A_G$.

We define a polynomial ring $R = \Bbbk[t_1, \ldots, t_d]$ in d variables and another one $K = \Bbbk[x_1, \ldots, x_m]$ in m variables, where $\Bbbk$ is a field. Let $\pi : K \to R$ be the ring homomorphism defined by $\pi(x_i) = \mathbf{t}^{e_i}$ for $i = 1, \ldots, m$, where $\mathbf{t}^e := t_u t_v$ for any edge $e = \{u,v\} \in E(G)$. The image $\mathrm{Im}(\pi)$, which is a subalgebra of R, is called the *edge ring* of G, and the kernel $\mathrm{Ker}(\pi)$, an ideal of K, is called the *toric ideal* of G. We denote the edge ring of G by $\Bbbk[G]$ and the toric ideal of G by I_G. Clearly, we have the ring isomorphism $\Bbbk[G] \cong K/I_G$.

We can regard $\Bbbk[G]$ as a semigroup algebra of an affine semigroup $\mathbb{Z}_{\geq 0}A_G$. Stated otherwise, the edge ring $\Bbbk[G]$ is the toric ring of the edge polytope $\mathcal{P}_G$. In particular, the structure of the cone $\mathbb{Q}_{\geq 0}A_G$ plays a crucial role in the study of $\Bbbk[G]$. Later, we'll examine it in further detail. It is known that $\dim \mathcal{P}_G = d - 1$ if G is not bipartite (see, [34, Proposition 1.3]) and thus the edge ring $\Bbbk[G]$ is d-dimensional if G is not bipartite.

Next, from [18, Section 5.3], we shall recall how to describe the generators of the toric ideal of a graph. Let G be a graph. Given an even closed walk $\Gamma = (e_{i_1}, \ldots, e_{i_{2r}})$, we associate a binomial

$$f_\Gamma = f_\Gamma^{(+)} - f_\Gamma^{(-)},$$

where

$$f_\Gamma^{(+)} = \prod_{k=1}^{r} x_{i_{2k-1}} \quad \text{and} \quad f_\Gamma^{(-)} = \prod_{k=1}^{r} x_{i_{2k}}.$$

We say that an even closed walk Γ is *primitive* if there is no even closed walk Γ' of G with $f_{\Gamma'} \neq f_\Gamma$ such that $f_{\Gamma'}^{(+)}$ divides $f_\Gamma^{(+)}$ and $f_{\Gamma'}^{(-)}$ divides $f_\Gamma^{(-)}$.

Lemma 3.4.1 ([18, Lemma 5.10]). *The toric ideal I_G is generated by the binomials f_Γ for all primitive even closed walks Γ.*

Lemma 3.4.2 ([18, Lemma 5.11]). *An even closed walk Γ of G is primitive if it is one the following:*

(i) Γ *is an even cycle of* G*;*

(ii) $\Gamma = (C, C')$*, where* C *and* C' *are odd cycles of* G *with exactly one common vertex;*

(iii) $\Gamma = (C, \Gamma, C', \Gamma')$*, where* C *and* C' *are disjoint odd cycles of* G *with* $V(C) \cap V(C') = \emptyset$ *and* Γ *and* Γ' *are both walks of* G *such that* Γ *connects a vertex* $u \in C$ *to a vertex* $v \in C'$ *and* Γ' *connects* $v \in C'$ *to* $u \in C$*. Moreover, no other vertices of* $V(C) \cap V(C')$*, other than* u *and* v*, appear in any of the edges in* Γ *and* Γ'*.*

Let $S_G := \mathbb{Z}_{\geq 0} A_G$. We have seen that the edge ring $\Bbbk[G]$ is the affine semigroup ring of S_G. We may assume that $\mathbb{Q}_{\geq 0} A_G$, the convex rational polyhedral cone spanned by S_G in $\mathbb{Q}^d$, is of dimension d and let $\mathcal{F}(G)$ be the set of all facets of $\mathbb{Q}_{\geq 0} A_G$. Let us look at the facets of $\mathbb{Q}_{\geq 0} A_G$. For that, we recollect some important definitions and theorems from [34].

Definition 3.4.3. Let G be a finite connected simple graph with vertex set $V(G)$. A vertex $v \in V(G)$ is said to be *regular* in G if every connected component of $G \backslash v$ contains at least one odd cycle. A non-empty set $T \subset V(G)$ is said to be *fundamental* in G if all the conditions below are satisfied by T:

1. T is an independent set;
2. the bipartite graph induced by T is connected;
3. either $T \cup N_G(T) = V(G)$, or each of the connected components of the graph $G \setminus (T \cup N_G(T))$ contains an odd cycle.

Facets of $\mathbb{Q}_{\geq 0} A_G$ are given by the intersection of the half-spaces defined by the supporting hyperplanes of $\mathbb{Q}_{\geq 0} A_G$, and were investigated by Ohsugi and Hibi [34].

Theorem 3.4.4 (From [34, Theorem 1.7 (a)]). *Let G be a finite connected simple graph on the vertex set $[d]$, containing at least one odd cycle. Then, all the supporting hyperplanes of $\mathbb{Q}_{\geq 0} A_G$ are as follows:*

1. $\mathcal{H}_v = \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d \colon x_v = 0\right\}$*, where* v *is a regular vertex in* G*.*

2. $\mathcal{H}_T = \{(x_1,\ldots,x_d) \in \mathbb{R}^d \colon \sum_{i\in T} x_i = \sum_{j\in N_G(T)} x_j\}$, *where T is a fundamental set in G.*

Note that, we denote F_v and F_T as the facets of $\mathbb{Q}_{\geq 0}A_G$ corresponding to the hyperplanes $\mathcal{H}_v$ and $\mathcal{H}_T$ respectively.

Let C and C' be minimal cycles of G with $V(C) \cap V(C') = \emptyset$, if we have $u \in V(C)$ and $v \in V(C')$, then $e = \{u, v\} \in E(G)$ is called a *bridge* between C and C'. A pair of odd cycles (C, C') is called *exceptional* if C and C' are minimal odd cycles in G such that $V(C) \cap V(C') = \emptyset$ and there exists no bridge connecting them.

We say that G satisfies the *odd cycle condition* if for any pair of odd cycles (C, C') in G, either $V(C) \cap V(C') \neq \emptyset$ or there exists a bridge between C and C'. In other words, the graph G has no exceptional pairs.

The normality of affine semigroup rings were studied in Section 2.2. The edge ring $\Bbbk[G]$ being the affine semigroup ring of S_G, let us recall the condition for the normality of $\Bbbk[G]$ from Section 2.2. We have, $\overline{S}_G = \mathbb{Q}_{\geq 0}A_G \cap \mathbb{Z}A_G$. For any facet $F \in \mathcal{F}(G)$, we define

$$S_F := S_G - S_G \cap F = \{\mathbf{x} \in \mathbb{Z}A_G \colon \exists\, \mathbf{y} \in S_G \cap F \text{ such that } \mathbf{x} + \mathbf{y} \in S_G\},$$

and $S'_G := \bigcap_{F\in\mathcal{F}(G)} S_F$. As seen in Section 2.2, the edge ring $\Bbbk[G]$ is normal if and only if

$$\mathbb{Z}_{\geq 0}A_G = \mathbb{Q}_{\geq 0}A_G \cap \mathbb{Z}A_G \tag{3.1}$$

holds. That is, $\Bbbk[G]$ is normal when $S_G = \overline{S}_G$. On the normality of $\Bbbk[G]$, the following combinatorial criterion is known.

Theorem 3.4.5 (From [34, Theorem 2.2] and [41, Theorem 1.1])**.** *A connected graph G satisfies* (3.1) *if and only if G satisfies the odd cycle condition.*

From all of the above observations, we have:

$$S_G = \overline{S}_G \iff \Bbbk[G] \text{ is normal } \iff G \text{ satisfies odd cycle condition.}$$

With reference to the work of Ohsugi and Hibi [34, Theorem 2.2], normalization of the edge ring $\Bbbk[G]$ can be expressed as

$$\overline{S}_G = S_G \ + \ \mathbb{Z}_{\geq 0}\{\mathbb{E}_C + \mathbb{E}_{C'} \colon (C, C') \text{ is exceptional in } G\},$$

where for any odd cycle C, we define $\mathbb{E}_C := \sum_{i\in V(C)} \mathbf{e}_i$. We observe that,

$$2(\mathbb{E}_C + \mathbb{E}_{C'}) = \left(\sum_{e\in E(C)} \rho(e) + \sum_{e'\in E(C')} \rho(e')\right) \in S_G.$$

Given that the edge ring $\Bbbk[G]$ is an affine semigroup ring, it is known from [26, Theorem 1] that, if $\Bbbk[G]$ is normal then $\Bbbk[G]$ is Cohen–Macaulay. Remark that $\Bbbk[G]$ is a normal Cohen–Macaulay homogeneous domain if G satisfies the odd cycle condition.

By [38, Theorem 6.3], $S_G = S'_G$ corresponds to Serre's condition (S_2). In general, $S_G \subset S'_G \subset \overline{S}_G$. Therefore, in order to prove that the edge ring $\Bbbk[G]$ satisfies (S_2)-condition, it is enough to show that for any $\alpha \in \overline{S}_G \backslash S_G$, we have $\alpha \notin S'_G$. This implies that $S'_G \subset S_G$ and therefore $S_G = S'_G$.

In [22], Higashitani and Kimura have provided the necessary condition that a graph G has to hold in order to satisfy (S_2)-condition.

Theorem 3.4.6 (From [22, Theorem 4.1])**.** *Let G be a finite simple connected graph. Suppose that, there exists an exceptional pair (C, C') satisfying the following two conditions:*

1. *for each regular vertex $v \in V(G)\backslash[V(C) \cup V(C')]$ in G, both C and C' belong to the same connected component of the graph $G\backslash v$;*
2. *for each fundamental set $T \in G$ with $[V(C) \cup V(C')] \cap [T \cup N_G(T)] = \emptyset$, both C and C' belong to the same connected component of $G_{V(G)\backslash(T \cup N_G(T))}$.*

Then, $\mathbb{E}_C + \mathbb{E}_{C'} \in S'_G$. In particular, $S_G \neq S'_G$.

Note that Serre's condition (S_2) is a necessary condition for $\Bbbk[G]$ to be Cohen–Macaulay. Later in this thesis, we will investigate some family of graphs with non-normal edge rings that satisfy (S_2)-condition.

Part II

On h-vectors of normal edge rings

Chapter 4

The h-vectors of the edge rings of a special family of graphs

In this chapter, we compute the h-vector of a special family of graphs, by using the technique of initial ideals and the associated simplicial complex. The contents of this chapter are entirely contained in the author's paper [24] with A. Higashitani.

4.1 Graph $\mathcal{G}_n$ and the main theorem

Let $n \geq 2$ be an integer. We introduce a connected non-bipartite graph $\mathcal{G}_n$ as shown in Figure 4.1. Namely, $\mathcal{G}_n$ consists of n copies of 3-cycles where all the copies share a single vertex in common, say w. Clearly, $\mathcal{G}_n$ satisfies the odd cycle condition (see Section 3.4). Therefore, we know that the edge ring $\Bbbk[\mathcal{G}_n]$ is a normal Cohen–Macaulay homogeneous domain.

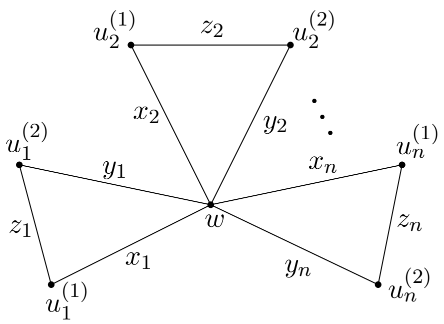

Figure 4.1: The graph $\mathcal{G}_n$

Our main result of this chapter is the following:

Theorem 4.1.1. *The h-polynomial of* $\Bbbk[\mathcal{G}_n]$ *is as follows:*

$$h(\Bbbk[\mathcal{G}_n];t) = \binom{n}{0} + \left(\binom{n}{1} - 1\right)t + \binom{n}{2}t^2 + \cdots + \binom{n}{n}t^n = (1+t)^n - t.$$

Moreover, $\Bbbk[\mathcal{G}_n]$ *is almost Gorenstein but not Gorenstein if and only if* $n \geq 3$.

On almost Gorensteinness of edge rings, known examples of almost Gorenstein non-Gorenstein edge rings are presumably rare. Our main result, Theorem 4.1.1 gives a new family of graphs whose edge rings are almost Gorenstein but not Gorenstein.

4.2 Fundamental properties of $\Bbbk[\mathcal{G}_n]$

Let us now concentrate on our graph $\mathcal{G}_n$ in more detail. We identify the edges of $\mathcal{G}_n$ with the variables of the polynomial ring K, as depicted in Figure 4.1. Namely, we consider

$$K = \Bbbk[x_1, y_1, z_1, \dots, x_n, y_n, z_n]$$

and we regard $I_{\mathcal{G}_n}$ as an ideal of K.

For $\mathcal{G}_n$, we see that, from Lemma 3.4.2, every primitive even closed walk consists of two 3-cycles with exactly one common vertex, and is given by

$$(x_i, z_i, y_i, x_j, z_j, y_j); \quad 1 \leq i < j \leq n.$$

Hence, the toric ideal $I_{\mathcal{G}_n}$ is generated by the binomials:

$$x_i y_i z_j - z_i x_j y_j; \quad 1 \leq i < j \leq n. \tag{4.1}$$

Let $<_{\text{lex}}$ be the graded lexicographic order on K induced by the ordering of the variables

$$x_1 <_{\text{lex}} y_1 <_{\text{lex}} z_1 <_{\text{lex}} \cdots <_{\text{lex}} x_n <_{\text{lex}} y_n <_{\text{lex}} z_n. \tag{4.2}$$

Lemma 4.2.1. *The binomials in* (4.1) *form a Gröbner basis of* $I_{\mathcal{G}_n}$ *with respect to the monomial order* $<_{\text{lex}}$.

Proof. The result follows from the straightforward application of Buchberger's criterion to the set of generators (4.1) of $I_{\mathcal{G}_n}$.

Let $f = x_i y_i z_j - z_i x_j y_j$ and $g = x_p y_p z_q - z_p x_q y_q$ be two generators. If $i \neq p$ and $j \neq q$, then the leading terms of f and g are relatively prime and thus the S-polynomial $S(f, g)$ will reduce to 0 by [18, Lemma 1.27].

Suppose $i = p$. Then

$$\begin{aligned}
S(f,g) &= \frac{\operatorname{lcm}(\mathrm{in}_{<_{\mathrm{lex}}}(f), \mathrm{in}_{<_{\mathrm{lex}}}(g))}{\mathrm{in}_{<_{\mathrm{lex}}}(f)} f - \frac{\operatorname{lcm}(\mathrm{in}_{<_{\mathrm{lex}}}(f), \mathrm{in}_{<_{\mathrm{lex}}}(g))}{\mathrm{in}_{<_{\mathrm{lex}}}(g)} g \\
&= z_q f - z_j g \\
&= z_q(x_i y_i z_j - z_i x_j y_j) - z_j(x_i y_i z_q - z_i x_q y_q) \\
&= z_i(x_q y_q z_j - z_q x_j y_j).
\end{aligned}$$

Note that, up to sign, $x_q y_q z_j - z_q x_j y_j$ is a generator of $I_{\mathcal{G}_n}$ and therefore $S(f,g)$ will reduce to 0. The $j = q$ case is similar. □

Corollary 4.2.2. *The initial ideal $in_{<_{\mathrm{lex}}}\big(I_{\mathcal{G}_n}\big)$ of $I_{\mathcal{G}_n}$ with respect to the monomial order $<_{\mathrm{lex}}$ is generated by the square-free monomials*

$$x_i y_i z_j; \quad 1 \le i < j \le n. \tag{4.3}$$

By this corollary, since the given monomial ideal is square-free, we can associate a simplicial complex whose Stanley–Reisner ideal coincides with the initial ideal generated by (4.3).

4.3 Computation of the h-polynomial of $\mathbb{k}[\mathcal{G}_n]$

Let Δ_n be the simplicial complex whose Stanley–Reisner ideal coincides with the initial ideal of the toric ideal $I_{\mathcal{G}_n}$ with respect to $<_{\mathrm{lex}}$. Let $\mathcal{F}(\Delta_n)$ be the set of all facets of Δ_n. By definition, any facet of our simplicial complex Δ_n can be expressed as the maximal set that does not contain the triplet $\{x_i, y_i, z_j\}$; $1 \le i < j \le n$. Since x_n, y_n and z_1 will be contained in all the facets, without loss of generality, we write the facets without indicating these elements. Therefore, any $F \in \mathcal{F}(\Delta_n)$ can be expressed as:

$$F = \bigcup_{\substack{i \in I \\ 1 \le i \le n-1}} \{x_i\} \cup \bigcup_{\substack{j \in J \\ 1 \le j \le n-1}} \{y_j\} \cup \bigcup_{\substack{k \in K \\ 2 \le k \le n}} \{z_k\},$$

which is maximal and does not contain the triplet $\{x_i, y_i, z_j\}$; $1 \le i < j \le n$. Let us try to get a more concrete representation for the facets in $\mathcal{F}(\Delta_n)$. Consider any $F \in \mathcal{F}(\Delta_n)$.

Case 1:

Let $i \in I \cap J$. This implies $z_{i+1}, \ldots, z_n \notin F$, since F does not contain the triplet $\{x_i, y_i, z_j\}$; $1 \le i < j \le n$.

Case 2:

Let us consider $i \in I$. If there exists some k with $i < k \le n$ such that $z_k \in F$, then we have $y_i \notin F$, that is, $i \in I \backslash J$.

If $z_k \notin F$ for all k with $i < k \leq n$, then by the maximality of F, we have $y_i \in F$ and $i \in I \cap J$.

Case 3:

Let us consider $j \in J$. If there exists some k with $j < k \leq n$ such that $z_k \in F$, then we have $x_j \notin F$, that is, $j \in J \backslash I$.

If $z_k \notin F$ for all k with $j < k \leq n$, then by the maximality of the set F, we have $x_j \in F$ and $j \in I \cap J$.

According to our observations from the three situations above, any facet in $\mathcal{F}(\Delta_n)$ is of the form:

$$\{w_1, \ldots, w_{j-1}\} \cup \{x_j, y_j, \ldots, x_{n-1}, y_{n-1}\} \cup \{z_2, \ldots, z_j\}, \qquad (4.4)$$

where $w_i \in \{x_i, y_i\}$ and $j = 1, \ldots, n$.

Remark 4.3.1. We will use the method of McMullen characterization of h-vectors (see, Section 2.4.1) to compute the h-vector of the shellable simplicial complex Δ_n.

Let us consider an ordering of the facets $F_1^1, \ldots, F_{t_n}^n$ of Δ_n. From the structure of each facet as shown in (4.4), we have $t_n = \sum_{i=0}^{n-1} 2^i$ and $|F_i^n| = 2n - 2$ for all $1 \leq i \leq t_n$. Let us consider each facet as a $(2n-2)$-tuple of $x_1, y_1, x_2, y_2, z_2, \ldots, x_{n-1}, y_{n-1}, z_{n-1}, z_n$. Lexicographic order $<_L$ in $\mathcal{F}(\Delta_n)$ is defined by

$$(a_1, \ldots, a_{2n-2}) <_L (b_1, \ldots, b_{2n-2})$$

if and only if either $a_i \neq b_i$ for some i and $a_i <_{\text{lex}} b_i$ with respect to (4.2). Now, we consider an ordering of the facets $F_1^n, \ldots, F_{t_n}^n$ of Δ_n such that they are arranged in lexicographically increasing order of their corresponding $(2n-2)$-tuple.

Let r_i^n be the number of maximal proper faces of F_i^n that generates the i^{th} intersection subcomplex for $2 \leq i \leq t_n$. We define $\delta_n = \{r_2^n, \ldots, r_{t_n}^n\}$ with $n \geq 2$ as a multi-set.

Lemma 4.3.2. *For each $n \geq 2$, we have*

$$\delta_{n+1} = \{1, \delta_n, 2, \delta_n + 1\},$$

where $\delta_n + 1 = \{\alpha + 1 : \alpha \in \delta_n\}$.

By induction on n, and using Lemma 4.3.2, we can obtain the h-vector of Δ_n. Our goal is to show that

$$h(\Delta_n) = \left(\binom{n}{0}, \binom{n}{1} - 1, \binom{n}{2}, \binom{n}{3}, \ldots, \binom{n}{n}\right) \quad \text{for any } n \geq 2.$$

For $n = 2$, since $\mathcal{F}(\Delta_2) = \{\{x_1, y_1\}, \{x_1, z_2\}, \{y_1, z_2\}\}$, we obtain the h-vector as $(1, 1, 1)$ which is equal to our formula $\left(\binom{2}{0}, \binom{2}{1} - 1, \binom{2}{2}\right)$.

By the hypothesis of induction, assume that our formula holds for an arbitrary n. Therefore, $h(\Delta_n) = \left(\binom{n}{0}, \binom{n}{1} - 1, \binom{n}{2}, \binom{n}{3}, \ldots, \binom{n}{n}\right)$. Let $h(\Delta_{n+1}) = (h_0^{n+1}, h_1^{n+1}, \ldots, h_{n+1}^{n+1})$. By Lemma 4.3.2, we see the following:

$$
\begin{aligned}
h_0^{n+1} &= 1 = \binom{n+1}{0}, \\
h_1^{n+1} &= 1 + h_1^n = 1 + \binom{n}{1} - 1 = \binom{n+1}{1} - 1, \\
h_2^{n+1} &= 1 + h_2^n + h_1^n = 1 + \binom{n}{2} + \binom{n}{1} - 1 = \binom{n+1}{2}, \\
h_i^{n+1} &= h_i^n + h_{i-1}^n = \binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i} \quad \text{for } 2 \leq i \leq n.
\end{aligned}
$$

Hence by induction, the h-vector associated with simplicial complex Δ_n is

$$
\left(\binom{n}{0}, \binom{n}{1} - 1, \binom{n}{2}, \binom{n}{3}, \ldots, \binom{n}{n}\right) \quad \text{for any } n \geq 2.
$$

Since the Hilbert series of K/I coincides with that of $K/\operatorname{in}_{<_{\text{lex}}}(I)$ in general (see, Proposition 2.3.5), we conclude the above as the desired h-vector of $\Bbbk[\mathcal{G}_n]$.

The remaining section is devoted to illustrating that our ordering of the facets gives a shelling of Δ_n and establishing Lemma 4.3.2.

For each $n \geq 2$, we observe that $r_2^n = 1$, $t_n = |\mathcal{F}(\Delta_n)| = \sum_{i=0}^{n-1} 2^i$ and we have $t_{n+1} = 1 + 2t_n$.

As per our ordering, the facets are ordered in such a manner that the facets consisting of x_1 comes first and after the $\left(\frac{t_n+1}{2}\right)^{\text{th}}$ stage, the pattern of facet ordering repeats in the exact same manner as that of $F_2^n, \ldots, F_{\frac{t_n+1}{2}}^n$, and consists of same elements except for x_1 replaced with y_1. More precisely, we have $F_{\frac{t_n+1}{2}+i}^n = (F_{i+1}^n \setminus \{x_1\}) \cup \{y_1\}$ for $1 \leq i \leq \frac{t_n-1}{2}$. Therefore, the corresponding intersection subcomplex for each k^{th} shelling step, $\frac{t_n+3}{2} \leq k \leq t_n$, will definitely contain the maximal face

$$
F_{i+1}^n \setminus \{x_1\} \text{ for any } 1 \leq i \leq \frac{t_n - 1}{2}.
$$

Moreover, when we look at the first $\frac{t_n+1}{2}$ facets, we observe that each of the facets differs from the preceding one by just an element. Thus with these observations, we claim that $F_1^n, \dots, F_{t_n}^n$ is a shelling of Δ_n.

Since the ordering pattern repeats after the $\left(\frac{t_n+1}{2}\right)^{\text{th}}$ facet, for $1 \le i \le \frac{t_n-1}{2}$, we see that r_{i+1}^n maximal proper faces will always be contained in the intersection subcomplex for each k^{th} shelling step, $\frac{t_n+1}{2} + i \le k \le t_n$ and the intersection subcomplex also contains the maximal face $F_{i+1}^n \setminus \{x_1\}$. Therefore, we have

$$r^n_{\frac{t_n+1}{2}+i} = 1 + r^n_{i+1} \text{ for any } 1 \le i \le \frac{t_n-1}{2}.$$

Hence, we can express δ_n as

$$\delta_n = \{1, r_3^n, \dots, r^n_{\frac{t_n+1}{2}}, 2, r_3^n + 1, \dots, r^n_{\frac{t_n+1}{2}} + 1\}.$$

The set $\mathcal{F}(\Delta_n)$ consists of (4.4). Namely, each facet F_k^n, $1 \le k \le t_n$, can be denoted as:

$$\bigcup_{i=1}^{j-1}\{w_i\} \cup \bigcup_{i=j}^{n-1}\{x_i, y_i\} \cup \bigcup_{i=2}^{j}\{z_i\}, \quad j = 1, \dots, n.$$

For each $1 \le j \le n$, we have 2^{j-1} number of facets corresponding to it. We can show that there exists a one-to-one correspondence between the facets F_k^n and F_{k+1}^{n+1} for all $n \ge 2$ and $2 \le k \le t_n$. The one-to-one correspondence is given by

$$\phi\colon \left\{F_k^n \colon 2 \le k \le t_n\right\} \longrightarrow \left\{F_{k'}^{n+1} \colon 3 \le k' \le \frac{t_{n+1}+1}{2}\right\}$$

$$\phi(F_k^n) = F_{k+1}^{n+1}, \quad 2 \le k \le t_n,$$

$$\bigcup_{i=1}^{j-1}\{w_i\}\cup\bigcup_{i=j}^{n-1}\{x_i, y_i\}\cup\bigcup_{i=2}^{j}\{z_i\} \mapsto \{x_1\}\cup\bigcup_{i=1}^{j-1}\{w_{i+1}\}\cup\bigcup_{i=j}^{n-1}\{x_{i+1}, y_{i+1}\}\cup\bigcup_{i=2}^{j+1}\{z_i\},$$

with $j = 1, \dots, n$.

In order to prove this one-to-one correspondence, we show that any facet in the set, $\left\{F_{k'}^{n+1} \colon 3 \le k' \le \frac{t_{n+1}+1}{2}\right\}$ will be of the form

$$\{x_1\} \cup \bigcup_{i=1}^{j-1}\{w_{i+1}\} \cup \bigcup_{i=j}^{n-1}\{x_{i+1}, y_{i+1}\} \cup \bigcup_{i=2}^{j+1}\{z_i\},$$

corresponding to the facet $\bigcup_{i=1}^{j-1}\{w_i\} \cup \bigcup_{i=j}^{n-1}\{x_i, y_i\} \cup \bigcup_{i=2}^{j}\{z_i\}$ $(2 \le j \le n)$ in the set $\left\{F_k^n \colon 2 \le k \le t_n\right\}$.

According to our shelling, any facet of the simplicial complex Δ_{n+1} containing the element $\{x_1\}$ belongs to $\left\{F_k^{n+1}\colon 1 \leq k \leq \frac{t_{n+1}+1}{2}\right\}$ with $1 \leq j \leq n+1$. Therefore, when we consider some $F_{k'}^{n+1} \in \left\{F_k^{n+1}\colon 3 \leq k \leq \frac{t_{n+1}+1}{2}\right\}$, it can be expressed as:

$$F_{k'}^{n+1} = \{x_1\} \cup \bigcup_{i=2}^{j'-1}\{w_i\} \cup \bigcup_{i=j'}^{n}\{x_i, y_i\} \cup \bigcup_{i=2}^{j'}\{z_i\}, \quad 3 \leq j' \leq n+1.$$

We can write it as

$$\begin{aligned} F_{k'}^{n+1} &= \{x_1\} \cup \bigcup_{i=2}^{j'-1}\{w_i\} \cup \bigcup_{i=j'}^{n}\{x_i, y_i\} \cup \bigcup_{i=2}^{j'}\{z_i\} \\ &= \{x_1\} \cup \bigcup_{i=1}^{j'-2}\{w_{i+1}\} \cup \bigcup_{i=j'-1}^{n-1}\{x_{i+1}, y_{i+1}\} \cup \bigcup_{i=2}^{j'}\{z_i\}; \\ &\qquad\qquad\qquad\qquad \text{for } 3 \leq j' \leq n+1. \end{aligned}$$

By expressing it in terms of $2 \leq j \leq n$, we have

$$F_{k'}^{n+1} = \{x_1\} \cup \bigcup_{i=1}^{j-1}\{w_{i+1}\} \cup \bigcup_{i=j}^{n-1}\{x_{i+1}, y_{i+1}\} \cup \bigcup_{i=2}^{j+1}\{z_i\} \ \text{ for } \ 2 \leq j \leq n.$$

Hence, we have the one-to-one correspondence. This one-to-one correspondence guarantees that $r_{i+1}^{n+1} = r_i^n$ for $2 \leq i \leq t_n$.

We also observe that

$$t_{n+1} = 1 + 2t_n \implies t_n = \frac{t_{n+1}-1}{2} \implies t_n + 1 = \frac{t_{n+1}+1}{2}.$$

From all the above observations, we have

$$\begin{aligned} & \delta_{n+1} = \{1, r_3^{n+1}, \ldots, r_{t_{n+1}}^{n+1}, 2, r_3^{n+1}+1, \ldots, r_{t_{n+1}}^{n+1}+1\} \\ \implies & \delta_{n+1} = \{1, r_2^n, \ldots, r_{t_n}^n, 2, r_2^n+1, \ldots, r_{t_n}^n+1\} \\ \implies & \delta_{n+1} = \{1, \delta_n, 2, \delta_n+1\}. \end{aligned}$$

4.4 On the almost Gorensteinness of $\Bbbk[\mathcal{G}_n]$

In this section, we are in the process of obtaining a theoretical proof to state that $\Bbbk[\mathcal{G}_n]$ is almost Gorenstein for all $n \geq 2$. Let us assume that R is a domain. Then there is an ideal I_R which is isomorphic to a canonical module of R as an R-module.

Let us consider $R = \Bbbk[\mathcal{G}_n]$. Note that $\mathcal{G}_n$ satisfies the odd cycle condition, so $\Bbbk[\mathcal{G}_n]$ is normal. By the first part of Theorem 4.1.1, we can compute $\tilde{e}(R)$ as follows:

$$\sum_{j=0}^{n-1} \left\{ \left(\binom{n}{n} + \cdots + \binom{n}{n-j} \right) - \left(\binom{n}{0} + (\binom{n}{1} - 1) + \cdots + \binom{n}{j} \right) \right\}$$
$$= \sum_{j=1}^{n-2} 1 = n-2.$$

Hence by Proposition 2.1.1, it is enough to show that $r(R) \geq n-1$. Furthermore, $r(R)$ is equal to the number of elements of a minimal system of generators of I_R, which is the relative interior of $\mathbb{Q}_{\geq 0} A_{\mathcal{G}_n} \cap \mathbb{Z} A_{\mathcal{G}_n}$ (see [8, Theorem 6.3.5]). Let $\mathcal{C} = \mathbb{Q}_{\geq 0} A_{\mathcal{G}_n} \subset \mathbb{R}^{2n+1}$. In what follows, it suffices to show that we need at least $(n-1)$ elements as a minimal system of generators for the relative interior of the cone $\mathcal{C}$.

Let us denote the vertices of $\mathcal{G}_n$ as follows:

$V(\mathcal{G}_n) = \{u_i^{(1)}, u_i^{(2)} : i = 1, \ldots, n\} \cup \{w\}$ and we let
$x_i = \{w, u_i^{(1)}\}$, $y_i = \{w, u_i^{(2)}\}$ and $z_i = \{u_i^{(1)}, u_i^{(2)}\}$ for $i = 1, \ldots, n$.

We use the following notation for each entry of $\mathbb{R}^{2n+1}$, that is, $\mathbb{R}^{2n+1}$ is equal to

$$\{c_{1,1}\mathbf{e}_{1,1} + \cdots + c_{1,n}\mathbf{e}_{1,n} + c_{2,1}\mathbf{e}_{2,1} + \cdots + c_{2,n}\mathbf{e}_{2,n} + c'\mathbf{e}' : c_{1,i}, c_{2,i}, c' \in \mathbb{R}\},$$

where $\mathbf{e}_{1,i}, \mathbf{e}_{2,i}, \mathbf{e}'$ are the unit vectors of $\mathbb{R}^{2n+1}$, each $\mathbf{e}_{1,i}$ (resp. $\mathbf{e}_{2,i}$) corresponds to $u_i^{(1)}$ (resp. $u_i^{(2)}$) and $\mathbf{e}'$ corresponds to w.

For $j = 1, \ldots, n-1$, let

$$\alpha_j := \sum_{i=1}^{n} (\mathbf{e}_{1,i} + \mathbf{e}_{2,i}) + 2j\mathbf{e}'.$$

In what follows, we verify that $\alpha_j \in \mathcal{C}^\circ \cap \mathbb{Z}^{2n+1}$, where $\mathcal{C}^\circ$ denotes the relative interior of $\mathcal{C}$, and they should be included in a minimal system of generators of I_R.

The first step: We check that $\alpha_j \in \mathcal{C}^\circ$. Here, we see the following:

- Each of $u_i^{(1)}$s and $u_i^{(2)}$s is a regular vertex of $\mathcal{G}_n$, while w is not.
- A subset T of $V(\mathcal{G}_n)$ is fundamental if and only if $T = \{w\}$ or $T = \{u_1, \ldots, u_n\}$, where $u_i \in \{u_i^{(1)}, u_i^{(2)}\}$ for each i.

It follows from Theorem 3.4.4 that, for a graph G, the cone $\mathcal{C} = \mathbb{Q}_{\geq 0}A_G$ consists of the elements $(x_v)_{v \in V(G)} \in \mathbb{Q}^d$ satisfying all the following inequalities:

$$\begin{aligned} x_u \geq 0 \quad & \text{for any regular vertex } u; \\ \sum_{v \in N_G(T)} x_v \geq \sum_{u \in T} x_u \quad & \text{for any fundamental set } T. \end{aligned} \tag{4.5}$$

Hence, it follows from (4.5) that $\sum_{i=1}^{n}(c_{1,i}\mathbf{e}_{1,i} + c_{2,i}\mathbf{e}_{2,i}) + c'\mathbf{e}' \in \mathbb{R}^{2n+1}$, belongs to $\mathcal{C}$ if and only if the following inequalities are satisfied:

$$\begin{aligned} & c_{1,i} \geq 0 \ \text{ and } \ c_{2,i} \geq 0 \quad \text{for any } i = 1, \dots, n, \\ & \sum_{i=1}^{n}(c_{1,i} + c_{2,i}) \geq c', \\ & \sum_{i \in U} c_{i,1} + \sum_{i \in [n] \setminus U} c_{i,2} + c' \geq \sum_{i \in [n] \setminus U} c_{i,1} + \sum_{i \in U} c_{i,2} \quad \text{for any } U \subset [n]. \end{aligned} \tag{4.6}$$

It is straightforward to check that α_j satisfies these inequalities with strict inequalities for each j. This implies that $\alpha_j \in \mathcal{C}^\circ$.

The second step: We prove that α_j cannot be written as a sum of an element in $\mathcal{C}^\circ \cap \mathbb{Z}^{2n+1}$ and an element in $\mathcal{C} \cap \mathbb{Z}^{2n+1} \setminus \{\mathbf{0}\}$.

Suppose that $\alpha_j = \alpha' + \beta$ for some $\alpha' \in \mathcal{C}^\circ \cap \mathbb{Z}^{2n+1}$ and $\beta \in \mathcal{C} \cap \mathbb{Z}^{2n+1} \setminus \{\mathbf{0}\}$. Let

$$\alpha' = \sum_{i=1}^{n} a'_{1,i}\mathbf{e}_{1,i} + \sum_{i=1}^{n} a'_{2,i}\mathbf{e}_{2,i} + a'\mathbf{e}' \ \text{ and } \ \beta = \sum_{i=1}^{n} b_{1,i}\mathbf{e}_{1,i} + \sum_{i=1}^{n} b_{2,i}\mathbf{e}_{2,i} + b\mathbf{e}'.$$

Then we see that $a'_{1,i} \geq 1$ and $a'_{2,i} \geq 1$ for $1 \leq i \leq n$ (see (4.6)). Hence, $b_{1,i} \leq 0$ and $b_{2,i} \leq 0$. On the other hand, $b_{1,i} \geq 0$ and $b_{2,i} \geq 0$ should be also satisfied, so we obtain that $\beta = b\mathbf{e}'$. Since $\beta \neq \mathbf{0}$, by the second inequality of (4.6), we have $b < 0$, a contradiction to the third inequality.

The third step: By the first and second steps, we see that $\alpha_1, \dots, \alpha_{n-1}$ are required for a minimal system of generators of I_R. This shows that $r(R) \geq n - 1$, as required.

Chapter 5

The h-vectors of edge rings of odd cycle compositions

This chapter focuses on the explicit computation of the h-vectors of a certain family of graphs that satisfy the odd cycle condition, generalizing a result of Chapter 4. Furthermore, we obtain a characterization of the graphs in this family whose edge rings are almost Gorenstein. This chapter and all of its contents are part of the author's joint work [4] with K. Bhaskara and A. Higashitani.

5.1 The graph $\mathfrak{g}_{r_1,\ldots,r_m}$ and the main results

The h-polynomials of a certain family of non-bipartite graphs $\mathcal{G}_n$, consisting of 3-cycles that share a single common vertex (see Figure 5.1) were determined in Chapter 4. For this family, we have seen that the edge rings $\Bbbk[\mathcal{G}_n]$ are almost Gorenstein. As a step towards generalizing this result, in this chapter, we first compute the h-polynomials of a related family of graphs, containing those studied in Chapter 4.

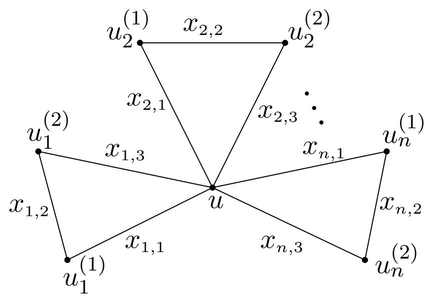

Figure 5.1: The graph $\mathcal{G}_n \coloneqq \mathfrak{g}_{n,0,\ldots,0}$

Let n be a positive integer. We consider a non-bipartite graph $\mathfrak{g}_{r_1,\ldots,r_m}$, consisting of n odd cycles that share a single common vertex (see Figure 5.2). To be precise, for an integer $m \geq 1$ and each $j \in [m] = \{1, \ldots, m\}$, we define $\mathfrak{g}_{r_1,\ldots,r_m}$ to be the graph consisting of r_j cycles of length $2j+1$, such that all $n = \sum_{j=1}^{m} r_j$ odd cycles share a single common vertex.

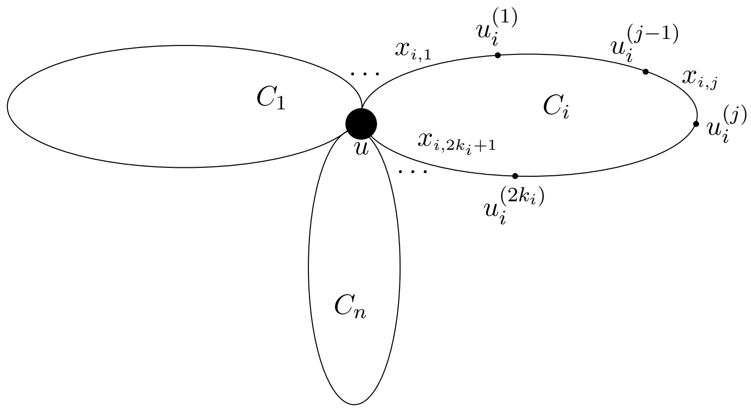

Figure 5.2: The graph $\mathfrak{g}_{r_1,\ldots,r_m}$ with $n = \sum\limits_{j=1}^{m} r_j$

Theorem 5.1.1. *The h-polynomial of* $\Bbbk[\mathfrak{g}_{r_1,\ldots,r_m}]$ *is given by*

$$h(\Bbbk[\mathfrak{g}_{r_1,\ldots,r_m}]; t) = \prod_{j=1}^{m} (1 + \cdots + t^j)^{r_j} - t \prod_{j=1}^{m} (1 + \cdots + t^{j-1})^{r_j}. \quad (5.1)$$

Using Theorem 5.1.1, we are then able to characterize the graphs $\mathfrak{g}_{r_1,\ldots,r_m}$ such that $\Bbbk[\mathfrak{g}_{r_1,\ldots,r_m}]$ is almost Gorenstein.

Theorem 5.1.2. *For the graph* $\mathfrak{g}_{r_1,\ldots,r_m}$ *with* $n = \sum_{j=1}^{m} r_j$ *and* $N = \sum_{j=1}^{m} j r_j$, *the edge ring* $\Bbbk[\mathfrak{g}_{r_1,\ldots,r_m}]$ *is almost Gorenstein if and only if either*

- $n = 1, 2$*; or*
- $n \geq 3$ *and every cycle in* $\mathfrak{g}_{r_1,\ldots,r_m}$ *is a* 3*-cycle, that is,* $N = n$.

5.2 On the edge ring $\Bbbk[\mathfrak{g}_{r_1,\ldots,r_m}]$

In this section, we concentrate on our graph $\mathfrak{g}_{r_1,\ldots,r_m}$ and study some of the fundamental properties of its associated edge ring $\Bbbk[\mathfrak{g}_{r_1,\ldots,r_m}]$.

Note that from here on, we will write $\mathfrak{g}_n := \mathfrak{g}_{r_1,\dots,r_m}$ and $\Bbbk[\mathfrak{g}_n] := \Bbbk[\mathfrak{g}_{r_1,\dots,r_m}]$, where $n = \sum_{p=1}^{m} r_p$. For each $1 \leq i \leq n$, let C_i be the i^{th} cycle in $\mathfrak{g}_n$. The length of C_i is $2k_i + 1$, where $k_i \in [m]$.

Since all the odd cycles in $\mathfrak{g}_n$ share a common vertex, our graph $\mathfrak{g}_n$ satisfies the odd cycle condition, and therefore, the edge ring $\Bbbk[\mathfrak{g}_n]$ is a normal Cohen–Macaulay homogeneous domain.

The vertex set of $\mathfrak{g}_n$ is $\{u\} \cup \{u_i^{(j)} \colon 1 \leq i \leq n, 1 \leq j \leq 2k_i\}$ and we label the edges of $\mathfrak{g}_n$ as $x_{i,j}$ such that

- for $1 \leq i \leq n$, $x_{i,1} = \{u, u_i^{(1)}\}$ and $x_{i,2k_i+1} = \{u, u_i^{(2k_i)}\}$, and
- for $1 \leq i \leq n$ and $2 \leq j \leq 2k_i$, $x_{i,j} = \{u_i^{(j-1)}, u_i^{(j)}\}$. (For illustration, see Figure 5.2.)

Let K be the polynomial ring $\Bbbk[x_{i,j} \colon 1 \leq i \leq n, 1 \leq j \leq 2k_i + 1]$ where $k_i \in [m]$, and we regard $I_{\mathfrak{g}_n}$ as an ideal of K.

By Lemma 3.4.2, every primitive even closed walk of the graph $\mathfrak{g}_n$ is given by

$$(x_{i,1}, x_{i,2}, \dots x_{i,2k_i+1}, x_{j,1}, x_{j,2}, \dots x_{j,2k_j+1}),$$

where $1 \leq i < j \leq n$, and $k_i, k_j \in [m]$. Therefore, we can observe that the toric ideal $I_{\mathfrak{g}_n}$ is generated by the binomials:

$$\prod_{s=0}^{k_i} x_{i,2s+1} \prod_{t=1}^{k_j} x_{j,2t} - \prod_{s=1}^{k_i} x_{i,2s} \prod_{t=0}^{k_j} x_{j,2t+1}; \quad 1 \leq i < j \leq n. \tag{5.2}$$

Let $\prec$ be the graded lexicographic order on K induced by the ordering of the variables

$$x_{1,1} \succ x_{1,2} \succ \cdots \succ x_{1,2k_1+1} \succ \cdots \succ x_{n,1} \succ x_{n,2} \succ \cdots \succ x_{n,2k_n+1}. \tag{5.3}$$

Lemma 5.2.1. *The binomials in* (5.2) *form a Gröbner basis of* $I_{\mathfrak{g}_n}$ *with respect to the monomial order* $\prec$.

Proof. Using the application of Buchberger's criterion to the set of generators (5.2) of $I_{\mathfrak{g}_n}$.

Let

$$f = \prod_{s=0}^{k_i} x_{i,2s+1} \prod_{t=1}^{k_j} x_{j,2t} - \prod_{s=1}^{k_i} x_{i,2s} \prod_{t=0}^{k_j} x_{j,2t+1}$$

and

$$g = \prod_{s=0}^{k_p} x_{p,2s+1} \prod_{t=1}^{k_q} x_{q,2t} - \prod_{s=1}^{k_p} x_{p,2s} \prod_{t=0}^{k_q} x_{q,2t+1}$$

be two generators. If $i \neq p$ and $j \neq q$, then the leading terms of f and g are relatively prime and the S-polynomial $S(f,g)$ reduces to 0.
Let $i = p$. Then

$$\begin{aligned} S(f,g) &= \frac{\operatorname{lcm}(\mathrm{in}_\prec(f), \mathrm{in}_\prec(g))}{\mathrm{in}_\prec(f)} f - \frac{\operatorname{lcm}(\mathrm{in}_\prec(f), \mathrm{in}_\prec(g))}{\mathrm{in}_\prec(g)} g \\ &= f \prod_{t=1}^{k_q} x_{q,2t} - g \prod_{t=1}^{k_j} x_{j,2t} \\ &= \prod_{s=1}^{k_i} x_{i,2s} \left(\prod_{t=1}^{k_j} x_{j,2t} \prod_{t=0}^{k_q} x_{q,2t+1} - \prod_{t=0}^{k_j} x_{j,2t+1} \prod_{t=1}^{k_q} x_{q,2t} \right). \end{aligned}$$

Up to sign, $\prod_{t=1}^{k_j} x_{j,2t} \prod_{t=0}^{k_q} x_{q,2t+1} - \prod_{t=0}^{k_j} x_{j,2t+1} \prod_{t=1}^{k_q} x_{q,2t}$ is a generator of $I_{\mathfrak{g}_n}$ and thus $S(f,g)$ will reduce to 0. Similarly, we can prove this for the $j = q$ case. □

Corollary 5.2.2. *The initial ideal $\mathrm{in}_\prec(I_{\mathfrak{g}_n})$ of $I_{\mathfrak{g}_n}$ with respect to the monomial order $\prec$ is generated by the square-free monomials*

$$\prod_{s=0}^{k_i} x_{i,2s+1} \prod_{t=1}^{k_j} x_{j,2t}; \quad 1 \leq i < j \leq n; \;\; k_i, k_j \in [m]. \tag{5.4}$$

Since the given monomial ideal is square-free, we can associate a simplicial complex whose Stanley–Reisner ideal coincides with the initial ideal generated by (5.4).

5.3 The h-polynomial of $\Bbbk[\mathfrak{g}_{r_1,\ldots,r_m}]$

In this section, we will explore specific constructions and key findings necessary to prove Theorem 5.1.1. Upon conclusion, we will utilize these insights to establish Theorem 5.1.1.

5.3.1 Stanley–Reisner complex $\Delta_{\mathfrak{g}_n}$

Let $\Delta_{\mathfrak{g}_n}$ be the simplicial complex whose Stanley–Reisner ideal coincides with $\mathrm{in}_\prec(I_{\mathfrak{g}_n})$. Let $\mathcal{F}(\Delta_{\mathfrak{g}_n})$ be the set of all facets of $\Delta_{\mathfrak{g}_n}$.

For the graph $\mathfrak{g}_n$, let

$$\mathcal{O}_i := \{x_{i,1}, x_{i,3}, \ldots, x_{i,2k_i+1}\}; \;\; \mathcal{E}_i := \{x_{i,2}, x_{i,4}, \ldots, x_{i,2k_i}\},$$

where $1 \leq i \leq n$ and corresponding $k_i \in [m]$. Therefore by definition, any facet of $\Delta_{\mathfrak{g}_n}$ can be written as a maximal set that does not contain the set $\mathcal{O}_i \cup \mathcal{E}_j$; $1 \leq i < j \leq n$. Note that $\mathcal{O}_n \cup \mathcal{E}_1 \subset F$ for all $F \in \mathcal{F}(\Delta_{\mathfrak{g}_n})$. Hence, without loss of generality, let us express the facets without indicating any elements of $\mathcal{O}_n \cup \mathcal{E}_1$.

Now, let us get a more concrete representation for the facets in $\mathcal{F}(\Delta_{\mathfrak{g}_n})$. Note that if our graph $\mathfrak{g}_n$ consists of n copies of 3-cycles, then it is none other than the graph $\mathcal{G}_n$ illustrated in Figure 5.1. This family of graphs has been thoroughly examined in the previous chapter, and it was proved that any facet in this case is of the form

$$\{z_1, \ldots, z_{j-1}\} \cup \{x_{j,1}, x_{j,3}, \ldots, x_{n-1,1}, x_{n-1,3}\} \cup \{x_{2,2}, \ldots, x_{j,2}\}, \quad (5.5)$$

where $z_i \in \{x_{i,1}, x_{i,3}\}$ and $j = 1, \ldots, n$.

Using similar arguments from Chapter 4 concerning the explicit form of the facets, we can get a concrete representation for the facets of $\Delta_{\mathfrak{g}_n}$. Let $F \in \mathcal{F}(\Delta_{\mathfrak{g}_n})$. We have $\mathcal{O}_p \cap F \neq \emptyset$, for any $1 \leq p \leq n$, and the following cases.

Case 1: For all q with $2 \leq q \leq p$, by the definition of facets, we have $\mathcal{O}_p \cup \mathcal{E}_q \subset F$.

Case 2: If there exists some q with $p < q \leq n$ such that $\mathcal{E}_q \subset F$, then by the definition of facets, $\mathcal{O}_p \not\subset F$. In particular, by maximality, F contains all but one element of $\mathcal{O}_p$.

Case 3: If $\mathcal{E}_q \not\subset F$ for all q with $p < q \leq n$, then by the maximality of F, every element of $\mathcal{E}_q$ excluding one will belong to F and we have $\mathcal{O}_p \subset F$.

Therefore, any facet $F \in \mathcal{F}(\Delta_{\mathfrak{g}_n})$ is of the form:

$$\bigcup_{i=1}^{j-1} \zeta_i \cup \bigcup_{i=2}^{j} \mathcal{E}_i \cup \bigcup_{i=j}^{n-1} \mathcal{O}_i \cup \bigcup_{i=j+1}^{n} \omega_i, \quad (5.6)$$

where $j = 1, \ldots, n$, such that $\zeta_i \in \binom{\mathcal{O}_i}{k_i}$, for all $k_i \geq 1$ (ζ_i is a set containing any k_i elements of $\mathcal{O}_i$), and $\omega_i \in \binom{\mathcal{E}_i}{k_i - 1}$ for all $k_i \geq 2$. By default, let $\omega_i = \emptyset$ for any $k_i = 1$.

Remark 5.3.1. Let us express the facets in (5.5) using the generalized expression (5.6). For the particular case of graph $\mathfrak{g}_n$ in Figure 5.1, $k_i = 1$, $\mathcal{O}_i = \{x_{i,1}, x_{i,3}\}$, $\mathcal{E}_i = \{x_{i,2}\}$, and $\omega_i = \emptyset$, for all $1 \leq i \leq n$. Therefore, $\bigcup_{i=j+1}^{n} \omega_i$ will not occur in the facet expression. In addition to this, $\{z_1, \ldots, z_{j-1}\}$ where $z_i \in \mathcal{O}_i$, corresponds to $\bigcup_{i=1}^{j-1} \zeta_i$, $\{x_{j,1}, x_{j,3}, \ldots, x_{n-1,1}, x_{n-1,3}\} = \bigcup_{i=j}^{n-1} \mathcal{O}_i$, and $\{x_{2,2}, \ldots, x_{j,2}\} = \bigcup_{i=2}^{j} \mathcal{E}_i$. Therefore, for the family of graphs $\mathfrak{g}_n$ with n copies of 3-cycles, any facet of $\Delta_{\mathfrak{g}_n}$ is of the form:

$$\bigcup_{i=1}^{j-1} \zeta_i \cup \bigcup_{i=2}^{j} \mathcal{E}_i \cup \bigcup_{i=j}^{n-1} \mathcal{O}_i, \text{ where } j = 1, \ldots, n.$$

Example 5.3.2. Let us consider the graph $\mathfrak{g}_3 := \mathfrak{g}_{1,1,1}$ in Figure 5.3 with $k_1 = 3$, $k_2 = 2$, $k_3 = 1$. For this graph, the initial ideal $\mathrm{in}_\prec(I_{\mathfrak{g}_3})$ is

$$\mathrm{in}_\prec(I_{\mathfrak{g}_3}) = \langle x_{1,1}x_{1,3}x_{1,5}x_{1,7}x_{2,2}x_{2,4}\ ,\ x_{1,1}x_{1,3}x_{1,5}x_{1,7}x_{3,2}\ ,\ x_{2,1}x_{2,3}x_{2,5}x_{3,2}\rangle.$$

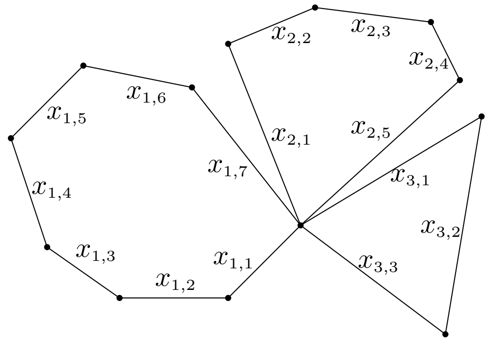

Figure 5.3: The graph $\mathfrak{g}_{1,1,1}$ with $k_1 = 3$, $k_2 = 2$, $k_3 = 1$

Note that since $k_3 = 1$, $\omega_3 = \emptyset$ in (5.6) for the graph $\mathfrak{g}_3$. As per (5.6), the facets of $\Delta_{\mathfrak{g}_3}$ are as follows:

$$j = 1 \qquad \underbrace{\{x_{1,1}, x_{1,3}, x_{1,5}, x_{1,7}\}}_{\mathcal{O}_1} \cup \underbrace{\{x_{2,1}, x_{2,3}, x_{2,5}\}}_{\mathcal{O}_2} \cup \underbrace{\binom{\{x_{2,2}, x_{2,4}\}}{1}}_{\omega_2}$$

$$j = 2 \qquad \underbrace{\binom{\mathcal{O}_1}{3}}_{\zeta_1} \cup \underbrace{\{x_{2,1}, x_{2,3}, x_{2,5}\}}_{\mathcal{O}_2} \cup \underbrace{\{x_{2,2}, x_{2,4}\}}_{\mathcal{E}_2}$$

$$j = 3 \qquad \underbrace{\binom{\mathcal{O}_1}{3}}_{\zeta_1} \cup \underbrace{\binom{\mathcal{O}_2}{2}}_{\zeta_2} \cup \underbrace{\{x_{2,2}, x_{2,4}\}}_{\mathcal{E}_2} \cup \underbrace{\{x_{3,2}\}}_{\mathcal{E}_3}$$

Remark 5.3.3. The Stanley–Reisner complex $\Delta_{\mathfrak{g}_n}$ is shellable and a generalized version of the shelling order described in Chapter 4 can be used to get a shelling. However, since shellability is not used in the proofs of the main theorems of this chapter, we omit the proof of shellability.

5.3.2 Construction of the graph $\mathfrak{g}_n$ for the proof

We will now examine a particular way of constructing our graph $\mathfrak{g}_n$, which contributes to the proof of Theorem 5.1.1.

Consider a graph $\mathfrak{g}'_n$ with n odd cycles C'_i such that length of each C'_i is $2k'_i + 1$, where $1 \leq i \leq n$ and $k'_i \in [m]$. We construct the graph

$\mathfrak{g}_n$ from $\mathfrak{g}'_n$ by extending the odd cycle C'_1 of $\mathfrak{g}'_n$ by two edges (x and y) while leaving the other odd cycles unchanged. For illustration, see Figure 5.4.

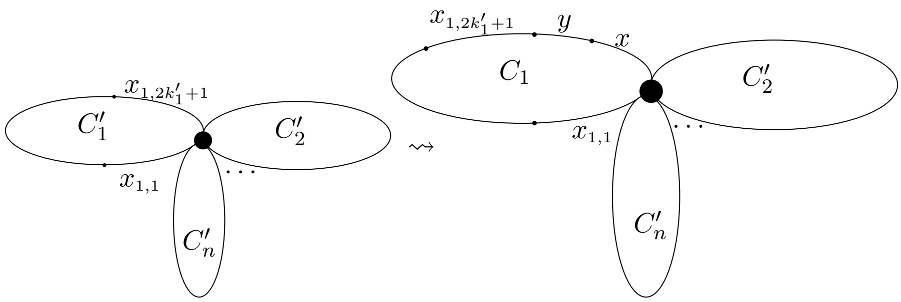

Figure 5.4: Left: Graph $\mathfrak{g}'_n$. Right: Graph $\mathfrak{g}_n$

For the graph $\mathfrak{g}'_n$, we denote $\mathcal{O}'_i \coloneqq \{x_{i,1}, x_{i,3}, \ldots, x_{i,2k'_i+1}\}$, and $\mathcal{E}'_i \coloneqq \{x_{i,2}, x_{i,4}, \ldots, x_{i,2k'_i}\}$ for all $1 \leq i \leq n$, where $k'_i \in [m]$.We define ζ'_i and ω'_i analogously. Recall that for all $1 \leq i \leq n$, the cycles C_i in $\mathfrak{g}_n$ are of length $2k_i + 1$, where $k_i \in [m]$, and by construction we have

- $C_i = C'_i$ for all $i \neq 1$,
- $k_1 = k'_1 + 1$ and $k_i = k'_i$, for all $i \neq 1$,
- $x = x_{1,2k_1+1}$ and $y = x_{1,2k_1}$.

Let $\mathfrak{g}''_n \coloneqq \mathfrak{g}'_n \backslash C'_1$, the subgraph of $\mathfrak{g}'_n$ which contains all the C'_i except C'_1. Figure 5.5 demonstrates the graph $\mathfrak{g}''_n$. As per the construction, the cycles in $\mathfrak{g}''_n$ are $C''_i \coloneqq C'_{i+1}$, with $k''_i \in [m]$ and $k''_i = k'_{i+1}$, for all $1 \leq i \leq n-1$. Moreover, $\mathcal{O}''_i$, $\mathcal{E}''_i$, ζ''_i and ω''_i are the corresponding notations.

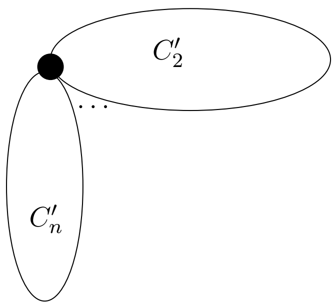

Figure 5.5: The graph $\mathfrak{g}''_n$

5.3.3 Towards the proof of Theorem 5.1.1

Here, we re-examine the simplicial complex $\Delta_{\mathfrak{g}_n}$ in light of our special construction of the graph $\mathfrak{g}_n$. Additionally, we explore the feasible lattice points corresponding to any $\mathfrak{g}_n$. Using these insights and with an inductive approach, we prove Theorem 5.1.1.

For a simplicial complex Δ and any nonempty set X, let $\Delta * X$ represent the simplicial complex whose facets are of the form $F \cup X$, where $F \in \mathcal{F}(\Delta)$. According to our construction of the graph $\mathfrak{g}_n$, we have the following lemma.

Lemma 5.3.4. *For the graph $\mathfrak{g}_n$, we have $\Delta_{\mathfrak{g}_n} = \Delta_{\mathfrak{g}'_n} * \{x\} \cup \Delta'' * \mathcal{O}'_1$, where $\Delta'' = \Delta_{\mathfrak{g}''_n} * \mathcal{E}'_2$.*

Proof. With $k_i \in [m]$, the facets of $\Delta_{\mathfrak{g}_n}$ are of the form (5.6). By construction, we have $\mathcal{O}_1 = \mathcal{O}'_1 \cup \{x\}$ and for all $2 \le i \le n$, $\mathcal{O}_i = \mathcal{O}'_i$, $\mathcal{E}_i = \mathcal{E}'_i$ and $\omega_i = \omega'_i$. Using this fact along with the general form of facets given in (5.6), we have that any facet $F \in \mathcal{F}(\Delta_{\mathfrak{g}_n})$ has one of the following forms:

$$\{x\} \cup \bigcup_{i=1}^{j-1} \zeta'_i \cup \bigcup_{i=2}^{j} \mathcal{E}'_i \cup \bigcup_{i=j}^{n-1} \mathcal{O}'_i \cup \bigcup_{i=j+1}^{n} \omega'_i, \text{ where } 1 \le j \le n, \text{ or} \tag{5.7a}$$

$$\mathcal{O}'_1 \cup \bigcup_{i=2}^{j-1} \zeta'_i \cup \bigcup_{i=2}^{j} \mathcal{E}'_i \cup \bigcup_{i=j}^{n-1} \mathcal{O}'_i \cup \bigcup_{i=j+1}^{n} \omega'_i, \text{ where } 2 \le j \le n. \tag{5.7b}$$

Now, let us focus on the graph $\mathfrak{g}''_n$. Using (5.6) for $\mathfrak{g}''_n$, the facets of $\Delta_{\mathfrak{g}''_n}$ are of the form

$$\bigcup_{i=1}^{j-1} \zeta''_i \cup \bigcup_{i=2}^{j} \mathcal{E}''_i \cup \bigcup_{i=j}^{n-2} \mathcal{O}''_i \cup \bigcup_{i=j+1}^{n-1} \omega''_i,$$

where $1 \le j \le n-1$. We know that the cycles in $\mathfrak{g}''_n$ are given by $C''_j := C'_{j+1}$, for all $1 \le j \le n-1$. Thus, we have $\mathcal{E}''_j = \mathcal{E}'_{j+1}$ and $\mathcal{O}''_j = \mathcal{O}'_{j+1}$, for all $1 \le j \le n-1$. Therefore, the facets of $\Delta_{\mathfrak{g}''_n}$ are of the form $\bigcup_{i=2}^{j} \zeta'_i \cup \bigcup_{i=3}^{j+1} \mathcal{E}'_i \cup \bigcup_{i=j+1}^{n-1} \mathcal{O}'_i \cup \bigcup_{i=j+2}^{n} \omega'_i$, where $1 \le j \le n-1$. Now, let us rewrite this expression such that the facets of $\Delta_{\mathfrak{g}''_n}$ are of the form

$$\bigcup_{i=2}^{j-1} \zeta'_i \cup \bigcup_{i=3}^{j} \mathcal{E}'_i \cup \bigcup_{i=j}^{n-1} \mathcal{O}'_i \cup \bigcup_{i=j+1}^{n} \omega'_i, \text{ where } 2 \le j \le n. \tag{5.8}$$

Let Δ'' be the simplicial complex whose facets are

$$\bigcup_{i=2}^{j-1} \zeta_i' \cup \bigcup_{i=2}^{j} \mathcal{E}_i' \cup \bigcup_{i=j}^{n-1} \mathcal{O}_i' \cup \bigcup_{i=j+1}^{n} \omega_i', \text{ where } 2 \leq j \leq n. \tag{5.9}$$

From (5.8) and (5.9), it is evident that any facet of Δ'' is of the form $F'' \cup \mathcal{E}_2'$, where $F'' \in \mathcal{F}(\Delta_{\mathfrak{g}_n''})$. Therefore, we can express Δ'' as

$$\Delta'' = \Delta_{\mathfrak{g}_n''} * \mathcal{E}_2'. \tag{5.10}$$

Note that any facet $F' \in \mathcal{F}(\Delta_{\mathfrak{g}_n'})$ is of the form

$$\bigcup_{i=1}^{j-1} \zeta_i' \cup \bigcup_{i=2}^{j} \mathcal{E}_i' \cup \bigcup_{i=j}^{n-1} \mathcal{O}_i' \cup \bigcup_{i=j+1}^{n} \omega_i', \text{ where } 1 \leq j \leq n.$$

Consequently, any facet of $\Delta_{\mathfrak{g}_n}$ of the form (5.7a) corresponds to $F' \cup \{x\}$, where $F' \in \mathcal{F}(\Delta_{\mathfrak{g}_n'})$. It is thus possible to state that the facets in (5.7a) correspond to facets of $\Delta_{\mathfrak{g}_n'} * \{x\}$. By comparing (5.7b) and (5.9), we observe that the collection of facets in (5.7b) matches $\Delta'' * \mathcal{O}_1'$. Therefore, we have $\Delta_{\mathfrak{g}_n} = \Delta_{\mathfrak{g}_n'} * \{x\} \cup \Delta'' * \mathcal{O}_1'$. □

By examining the two forms of facets in $\mathcal{F}(\Delta_{\mathfrak{g}_n})$, it is evident that the intersection of any facet in (5.7a) with any other facet of (5.7b) is equal to some facet of $\Delta_{\mathfrak{g}_n'}$. Hence we have

$$\Delta_{\mathfrak{g}_n'} * \{x\} \cap \Delta'' * \mathcal{O}_1' = \Delta_{\mathfrak{g}_n'}.$$

Observe that $\mathcal{E}_2'$ in (5.10) is just a simplex and therefore the h-polynomial of the Stanley–Reisner ring $\Bbbk[\Delta'']$ equals $h(\Bbbk[\Delta_{\mathfrak{g}_n''}]; t)$. By Lemma 5.3.4 we observe that $\Delta_{\mathfrak{g}_n}$, $\Delta_{\mathfrak{g}_n'} * \{x\}$, and $\Delta'' * \mathcal{O}_1'$ have same dimension. Furthermore, we have $\Delta_{\mathfrak{g}_n'} * \{x\} \cap \Delta'' * \mathcal{O}_1' = \Delta_{\mathfrak{g}_n'}$ and $\dim \Delta_{\mathfrak{g}_n'} = \dim \Delta_{\mathfrak{g}_n} - 1$. Let us assume that $\dim \Bbbk[\Delta_{\mathfrak{g}_n}] = d$. Therefore by applying the inclusion-exclusion principle to the Hilbert series of $\Bbbk[\Delta_{\mathfrak{g}_n}]$ in Lemma 5.3.4, we have

$$\frac{h(\Bbbk[\Delta_{\mathfrak{g}_n}]; t)}{(1-t)^d} = \frac{h(\Bbbk[\Delta_{\mathfrak{g}_n'}]; t)}{(1-t)^d} + \frac{h(\Bbbk[\Delta_{\mathfrak{g}_n''}]; t)}{(1-t)^d} - \frac{h(\Bbbk[\Delta_{\mathfrak{g}_n'}]; t)}{(1-t)^{d-1}}.$$

Since the Hilbert series of K/I coincides with that of $K/\operatorname{in}_{\prec}(I)$ in general (see, Proposition 2.3.5), we conclude the h-polynomial of the Stanley–Reisner ring $\Bbbk[\Delta_{\mathfrak{g}_n}]$ gives the desired h-polynomial of corresponding edge ring $\Bbbk[\mathfrak{g}_n]$. As a result, the h-polynomials are as follows:

$$\begin{aligned} h(\Bbbk[\mathfrak{g}_n]; t) &= h(\Bbbk[\mathfrak{g}_n']; t) + h(\Bbbk[\mathfrak{g}_n'']; t) - (1-t)h(\Bbbk[\mathfrak{g}_n']; t) \\ &= t\ h(\Bbbk[\mathfrak{g}_n']; t) + h(\Bbbk[\mathfrak{g}_n'']; t). \end{aligned} \tag{5.11}$$

For any graph $\mathfrak{g}_n = \mathfrak{g}_{r_1,\ldots,r_m}$ $(m \geq 1)$, we can assign the lattice points (n, N) where $n = \sum_{j=1}^{m} r_j$ (total number of odd cycles) and $N = \sum_{j=1}^{m} jr_j$, such that $N \geq n \geq 1$. The feasible lattice points corresponding to any $\mathfrak{g}_n$ are shown in Figure 5.6.

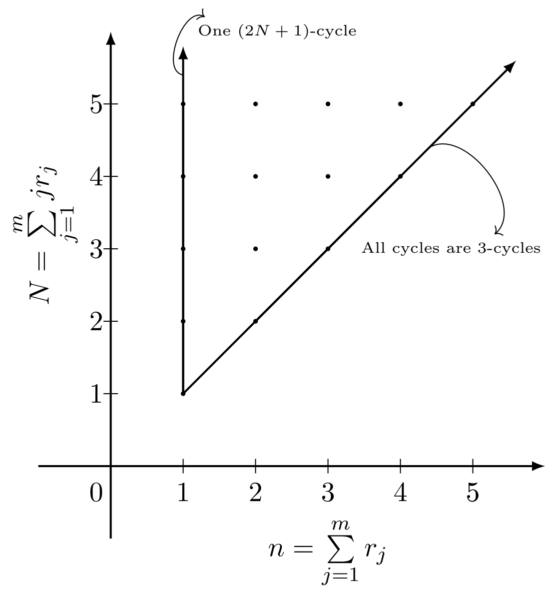

Figure 5.6: Feasible lattice points corresponding to the graph $\mathfrak{g}_{r_1,\ldots,r_m}$

Any graph $\mathfrak{g}_n$ with an arbitrary lattice point (n, N) can be constructed from a base graph with n copies of 3-cycles, i.e., with lattice point (n, n), by a similar method of construction as discussed in Section 5.3.2. Thus, we demonstrate our main theorem using an inductive approach over N in the feasible lattice points (n, N) with $N \geq n \geq 1$.

Proof of Theorem 5.1.1. As earlier, let $\mathfrak{g}_n := \mathfrak{g}_{r_1,\ldots,r_m}$ with each i^{th} cycle of length $2k_i + 1$, where $k_i \in [m]$.

For the base case with $N \geq n = 1$, i.e., lattice points $(1, N)$, the edge ring is isomorphic to a polynomial ring in $2N + 1$ (length of the single odd cycle) variables and the h-polynomial has to be 1. Therefore, (5.1) stands true.

As we can see in Figure 5.6, the graph with n copies of 3-cycles having a unique common point, or $N = n \geq 1$, is another base case. In Chapter 4, this case has been examined. Thus, Thoerem 4.1.1 implies that our main theorem is true for the base case $N = n \geq 1$.

Let us assume that the h-polynomial is given by (5.1), for any graph of our concern with corresponding lattice point (n, N).

As per the discussion in Section 5.3.2, we construct our graph $\mathfrak{g}_n$ from $\mathfrak{g}'_n$ by altering one of the odd cycles of $\mathfrak{g}'_n$. Let $\mathfrak{g}'_n$ be a graph with lattice point (n, N) such that $n = \sum_{j=1}^{m} r'_j$ and $N = \sum_{j=1}^{m} j r'_j$. By construction, the graph $\mathfrak{g}''_n$ has lattice point $(n-1, \widetilde{N})$, where $n - 1 = \sum_{j=1}^{m} r''_j$ and $\widetilde{N} = \sum_{j=1}^{m} j r''_j = \sum_{j=1}^{m} j r'_j - k'_1 < N$. Therefore by the induction hypothesis, the formula (5.1) holds for both $\mathfrak{g}'_n$ and $\mathfrak{g}''_n$. Now, as per the construction, graph $\mathfrak{g}_n$ has a total of $n = \sum_{j=1}^{m} r_j$ odd cycles such that

$$\begin{aligned}\sum_{j=1}^{m} j r_j &= k_1(r'_{k_1} + 1) + (k_1 - 1)(r'_{k_1-1} - 1) + \sum_{\substack{j \geq 1 \\ j \neq k_1, k_1 - 1}} j r'_j \\ &= \sum_{j=1}^{m} j r'_j + 1 = N + 1.\end{aligned}$$

Therefore, the lattice point corresponding to $\mathfrak{g}_n$ is $(n, N + 1)$.

From (5.11), we have $h(\mathbb{k}[\mathfrak{g}_n]; t) = t\, h(\mathbb{k}[\mathfrak{g}'_n]; t) + h(\mathbb{k}[\mathfrak{g}''_n]; t)$. By the induction hypothesis, we can apply (5.1) to $h(\mathbb{k}[\mathfrak{g}'_n]; t)$ and $h(\mathbb{k}[\mathfrak{g}''_n]; t)$, and we have

$$\begin{aligned}h(\mathbb{k}[\mathfrak{g}_n]; t) = t \prod_{j=1}^{m} (1 + \cdots + t^j)^{r'_j} - t^2 \prod_{j=1}^{m} (1 + \cdots + t^{j-1})^{r'_j} \\ + \prod_{j=1}^{m} (1 + \cdots + t^j)^{r''_j} - t \prod_{j=1}^{m} (1 + \cdots + t^{j-1})^{r''_j},\end{aligned}$$

where $k := k'_1$; $r'_j = r''_j$ for $j \neq k$ and $r'_k = r''_k + 1$ (by construction). Therefore we have

$$\begin{aligned}h(\mathbb{k}[\mathfrak{g}_n]; t) &= t(1 + \cdots + t^k) \prod_{j=1}^{m} (1 + \cdots + t^j)^{r''_j} \\ &\quad - t^2 (1 + \cdots + t^{k-1}) \prod_{j=1}^{m} (1 + \cdots + t^{j-1})^{r''_j} \\ &\quad + \prod_{j=1}^{m} (1 + \cdots + t^j)^{r''_j} - t \prod_{j=1}^{m} (1 + \cdots + t^{j-1})^{r''_j} \\ &= (1 + \cdots + t^{k+1}) \prod_{j=1}^{m} (1 + \cdots + t^j)^{r''_j} \\ &\quad - t(1 + \cdots + t^k) \prod_{j=1}^{m} (1 + \cdots + t^{j-1})^{r''_j}.\end{aligned}$$

Furthermore, as per our construction, we can express r_j as:

$$r_j = \begin{cases} r''_j + 1 & \text{for } j = k+1, \\ r''_j & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} h(\Bbbk[\mathfrak{g}_n];t) &= (1+\cdots+t^{k+1}) \prod_{j=1}^{m} (1+\cdots+t^j)^{r''_j} \\ &\quad - t(1+\cdots+t^k) \prod_{j=1}^{m} (1+\cdots+t^{j-1})^{r''_j} \\ &= \prod_{j=1}^{m} (1+\cdots+t^j)^{r_j} - t \prod_{j=1}^{m} (1+\cdots+t^{j-1})^{r_j}. \end{aligned}$$

This concludes our proof. □

5.4 On almost Gorensteinness of $\Bbbk[\mathfrak{g}_{r_1,\ldots,r_m}]$

In this section we characterize the almost Gorensteinness of our edge ring $\Bbbk[\mathfrak{g}_{r_1,\ldots,r_m}]$.

Let R be a Cohen–Macaulay homogeneous domain of dimension d over a field $\Bbbk$ and let $h(R) = (h_0, h_1, \ldots, h_s)$ be its h-vector. Then by Proposition 2.1.1, we recall that R is almost Gorenstein if and only if

$$r(R) - 1 = \sum_{i=0}^{s} ((h_s + \cdots + h_{s-i}) - (h_0 + \cdots + h_i)) = \tilde{e}(R),$$

where $r(R)$ denotes the Cohen–Macaulay type of R. In the case of $\mathfrak{g}_{r_1,\ldots,r_m}$, we have the following.

Proposition 5.4.1 ([51, Theorem 5.7])**.** *For any graph $\mathfrak{g}_{r_1,\ldots,r_m}$ consisting of $n = \sum_{j=1}^{m} r_j$ cycles, $r(\Bbbk[\mathfrak{g}_{r_1,\ldots,r_m}]) = n - 1$.*

Now we are in the position to provide a proof for the second main theorem.

Proof of Theorem 5.1.2. As before, let $\mathfrak{g}_n \coloneqq \mathfrak{g}_{r_1,\ldots,r_m}$. We know that for $n = 1$, the edge ring $\Bbbk[\mathfrak{g}_n]$ is isomorphic to a polynomial ring in $2N+1$ variables. For the case $n = 2$, the toric ideal $I_{\mathfrak{g}_n}$ has only one generator and therefore the corresponding edge ring is a hypersurface. Thus, the edge ring $\Bbbk[\mathfrak{g}_n]$ is Gorenstein for both $n = 1, 2$; which implies that $\Bbbk[\mathfrak{g}_n]$ is almost Gorenstein.

Now, let $n \geq 3$. From Proposition 2.1.1 and Proposition 5.4.1, we have that $\Bbbk[\mathfrak{g}_n]$ is almost Gorenstein if and only if $\tilde{e}(\Bbbk[\mathfrak{g}_n]) = n-2$. Let $h(\Bbbk[\mathfrak{g}_n]) = (h_0, h_1, \ldots, h_s)$ be the h-vector of $\Bbbk[\mathfrak{g}_n]$ and let us consider $h'_i = (h_s + \cdots + h_{s-i}) - (h_0 + \cdots + h_i)$, for $i = 0, 1, \ldots, s$. From the proof of [21, Proposition 2.4], we have

$$\sum_{i=0}^{s} h'_i t^i = \frac{1}{1-t} \sum_{i=0}^{s} (h_{s-i} - h_i) t^i. \tag{5.12}$$

From Theorem 5.1.1, we have $s = \deg h(\Bbbk[\mathfrak{g}_n]; t) = \sum_{j=1}^{m} j r_j = N$. For $f(t) = \sum_{i=0}^{s} a_i t^i$, we have $t^s f(t^{-1}) = \sum_{i=0}^{s} a_{s-i} t^i$. Applying this to (5.12), we can express

$$\sum_{i=0}^{s} h'_i t^i = \frac{t^N h(\Bbbk[\mathfrak{g}_n]; t^{-1}) - h(\Bbbk[\mathfrak{g}_n]; t)}{1-t}.$$

By (5.1), we have

$$t^N h(\Bbbk[\mathfrak{g}_n]; t^{-1}) = t^N \prod_{j=1}^{m} (1 + \cdots + t^{-j})^{r_j} - t^{N-1} \prod_{j=1}^{m} (1 + \cdots + t^{-(j-1)})^{r_j}.$$

For $N = \sum_{j=1}^{m} j r_j$, we can express $N-1$ as $\sum_{j=1}^{m} (j-1) r_j + \sum_{j=1}^{m} r_j - 1$. Therefore,

$$\begin{aligned} t^N h(\Bbbk[\mathfrak{g}_n]; t^{-1}) &= t^N \prod_{j=1}^{m} (1 + \cdots + t^{-j})^{r_j} \\ &\quad - t^{n-1} \prod_{j=1}^{m} t^{(j-1)r_j} \left(1 + \cdots + t^{-(j-1)}\right)^{r_j} \\ &= \prod_{j=1}^{m} (1 + \cdots + t^{j})^{r_j} - t^{n-1} \prod_{j=1}^{m} (1 + \cdots + t^{j-1})^{r_j}. \end{aligned}$$

Hence, we have

$$\begin{aligned} & t^N h(\Bbbk[\mathfrak{g}_n]; t^{-1}) - h(\Bbbk[\mathfrak{g}_n]; t) \\ &\quad = \prod_{j=1}^{m} (1 + \cdots + t^{j})^{r_j} - t^{n-1} \prod_{j=1}^{m} (1 + \cdots + t^{j-1})^{r_j} \\ &\qquad - \prod_{j=1}^{m} (1 + \cdots + t^{j})^{r_j} + t \prod_{j=1}^{m} (1 + \cdots + t^{j-1})^{r_j} \\ &\quad = t(1 - t^{n-2}) \prod_{j=1}^{m} (1 + \cdots + t^{j-1})^{r_j}. \end{aligned}$$

Consequently, (5.12) becomes

$$\sum_{i=0}^{s} h_i' t^i = \frac{t(1-t^{n-2}) \prod_{j=1}^{m} (1+\cdots+t^{j-1})^{r_j}}{1-t}$$
$$= t(1+t+\cdots+t^{n-3}) \prod_{j=1}^{m} (1+\cdots+t^{j-1})^{r_j}.$$

We see that (5.12) evaluated at $t = 1$ gives $\tilde{e}(\Bbbk[\mathfrak{g}_n])$. Therefore, we have $\tilde{e}(\Bbbk[\mathfrak{g}_n]) = (n-2)\prod_{j=1}^{m} j^{r_j}$. Recall that for $n \geq 3$, the edge ring $\Bbbk[\mathfrak{g}_n]$ is almost Gorenstein if and only if $\tilde{e}(\Bbbk[\mathfrak{g}_n]) = n-2$. That is, when $\prod_{j=1}^{m} j^{r_j} = 1$. This occurs exclusively when $r_j = 0$ for all $j > 1$. In other words, every cycle in $\mathfrak{g}_n$ is a 3-cycle ($N = n$). □

From Theorem 5.1.2 in conjunction with Theorem 4.1.1, the following result on $\Bbbk[\mathfrak{g}_{r_1,\ldots,r_m}]$ is derived.

Corollary 5.4.2. *For $\mathfrak{g}_{r_1,\ldots,r_m}$ with $n = \sum_{j=1}^{m} r_j$ and $N = \sum_{j=1}^{m} jr_j$, the edge ring $\Bbbk[\mathfrak{g}_{r_1,\ldots,r_m}]$ is almost Gorenstein but not Gorenstein if and only if $n \geq 3$ and $N = n$.*

Part III

On non-normal edge rings and (S_2)-condition

Chapter 6

Non-normal edge rings satisfying (S_2)-condition

In this chapter, we will prove that, given integers d and m, where $d \geq 7$ and $d+1 \leq m \leq \frac{d^2-7d+24}{2}$, there exists a finite simple connected graph G with $|V(G)| = d$ and $|E(G)| = m$, such that the edge ring $\mathbb{k}[G]$ is non-normal and satisfies (S_2)-condition. The contents of this chapter are entirely contained in the author's paper [39].

6.1 The main theorem and graph $G_{a,b}$

In this section, we will first state the basic theorem of our study and then, explicitly deals with the construction and study of a special graph $G_{a,b}$, whose edge ring $\mathbb{k}[G_{a,b}]$ is non-normal. By stating and proving the proposition that the edge ring $\mathbb{k}[G_{a,b}]$ meets (S_2)-condition, we bring the section to a close.

Recall the notions and notations related to edge rings from Chapter 3. We will follow the same. The main theorem that we will prove in this study is as follows:

Theorem 6.1.1. *Given integers d and m such that, $d \geq 7$ and $d+1 \leq m \leq \frac{d^2-7d+24}{2}$, there exists a finite simple connected graph G with $|V(G)| = d$ and $|E(G)| = m$ such that, the edge ring $\mathbb{k}[G]$ is non-normal and satisfies (S_2)-condition.*

A detailed explanation of why the quadratic expression $\frac{d^2-7d+24}{2}$ appears in the main theorem is provided in Section 6.4.

Let $G_{a,b}$ be a finite simple connected graph with $|V(G_{a,b})| = d = a+b+1$, where $3 \leq a \leq b$. We construct the graph $G_{a,b}$ (Figure 6.1) such that, it is formed by the union of two complete graphs K_{a+1} and

K_{b+1} with exactly one common vertex. Let us consider the vertex set $V(G_{a,b}) = V(K_{a+1}) \cup V(K_{b+1})$, such that $V(K_{a+1}) = \{u_1, \ldots, u_a, w\}$ and $V(K_{b+1}) = \{v_1, \ldots, v_b, w\}$.

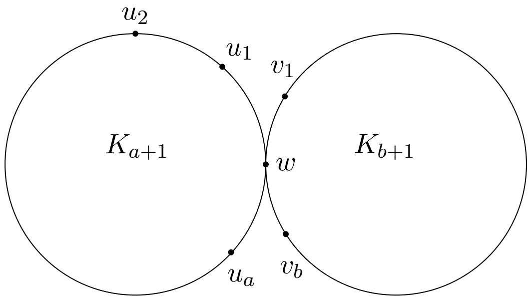

Figure 6.1: The graph $G_{a,b}$

Any vertex $v \in V(G_{a,b})$ is regular in $G_{a,b}$. We observe that the fundamental sets in $G_{a,b}$ are :

$$\{i\},\ \forall\ i \in V(G_{a,b}) \text{ and } \{i,j\},\ \forall\ \{i,j\} \notin E(G_{a,b}).$$

For any fundamental set $T \subset V(G_{a,b})$, let H_T be the bipartite graph induced by T. We consider K_a to be the induced subgraph of $G_{a,b}$ on vertex set $V(K_{a+1})\backslash\{w\}$, that is, the complete graph on $V(K_a) = \{u_1, \ldots, u_a\}$. Similarly, K_b is the induced subgraph of $G_{a,b}$ on the vertex set $V(K_{b+1})\backslash\{w\}$, which is the complete graph on $V(K_b) = \{v_1, \ldots, v_b\}$.

Let $\mathcal{A}_{G_{a,b}} := \{\rho(e) \colon e \in E(G_{a,b})\}$. For $S_{G_{a,b}} = \mathbb{Z}_{\geq 0}\mathcal{A}_{G_{a,b}}$, let $S := S_{G_{a,b}}$ and $\overline{S} := \overline{S_{G_{a,b}}}$. We consider $\mathcal{C}_S$ to be the convex rational polyhedral cone spanned by $\mathcal{A}_{G_{a,b}}$ in $\mathbb{Q}^d$, *i.e.*, $\mathcal{C}_S := \mathbb{Q}_{\geq 0}\mathcal{A}_{G_{a,b}}$. Let $\mathcal{F}(G_{a,b})$ be the set of all facets of $\mathcal{C}_S$ and for any $F_i \in \mathcal{F}(G_{a,b})$,

$$S_i := S - S \cap F_i = \{\mathbf{x} \in \mathbb{Z}\mathcal{A}_{G_{a,b}} \colon \exists\ \mathbf{y} \in S \cap F_i \text{ such that } \mathbf{x} + \mathbf{y} \in S\},$$

$$S' := \bigcap_{F_i \in \mathcal{F}(G_{a,b})} S_i.$$

Now, let us look at the facets of $\mathcal{C}_S$ in detail. As stated in Section 3.4, for any regular vertex $v \in V(G_{a,b})$ and any fundamental set T in $G_{a,b}$, we denote F_v and F_T as the facets of $\mathcal{C}_S$ corresponding to the hyperplanes $\mathcal{H}_v$ and $\mathcal{H}_T$ respectively. We observe that, for any regular vertex $v \in V(G_{a,b})$,

$$S \cap F_v = \mathbb{Z}_{\geq 0}\mathcal{A}_{G_{a,b}\backslash v}.$$

Corresponding to each fundamental set in $G_{a,b}$, we have

$$S \cap F_{\{w\}} = \mathbb{Z}_{\geq 0}\mathcal{A}_{H_{\{w\}}},$$

$$S \cap F_{\{u_i\}} = \mathbb{Z}_{\geq 0}\mathcal{A}_{H_{\{u_i\}} \bigsqcup K_b},$$
$$S \cap F_{\{v_j\}} = \mathbb{Z}_{\geq 0}\mathcal{A}_{K_a \bigsqcup H_{\{v_j\}}},$$
$$S \cap F_{\{u_i, v_j\}} = \mathbb{Z}_{\geq 0}\mathcal{A}_{H_{\{u_i, v_j\}}},$$

where $1 \leq i \leq a$ and $1 \leq j \leq b$. Note that, throughout this investigation, we are only concerned about the description of $S \cap F_w$, for a regular vertex w.

Lemma 6.1.2. *Let the pair of odd cycles (C, C') be exceptional in $G_{a,b}$. Consider vertices $v, w \in V(G_{a,b})$, where w is the common vertex of K_{a+1} and K_{b+1}. Let $\mathbf{e}_v$ and $\mathbf{e}_w$ be the canonical unit coordinate vectors of $\mathbb{R}^d$ corresponding to vertices v and w respectively. Then,*

$$\mathbb{E}_C \ + \ \mathbb{E}_{C'} \ + \ \mathbf{e}_v \ + \ \mathbf{e}_w \in S.$$

Proof. We consider an exceptional pair (C, C') in $G_{a,b}$. Without loss of generality, let $C = \{u_{i_1}, u_{i_2}, u_{i_3}\}$ be a minimal odd cycle in K_{a+1} and $C' = \{v_{j_1}, v_{j_2}, v_{j_3}\}$ be a minimal odd cycle in K_{b+1}.

Since K_{a+1} and K_{b+1} are complete graphs with common vertex w and (C, C') is exceptional in $G_{a,b}$, we have $V(C) \cap V(C') = \emptyset$ and $w \notin V(C) \cup V(C')$.

Without loss of generality, we may assume $v \in V(K_{a+1})$. Given that K_{a+1} and K_{b+1} are complete graphs, for any $u_{i_k} \in V(C)$ that is distinct from v, we have $\{v, u_{i_k}\} \in E(G_{a,b})$ and for any $v_{j_k} \in V(C')$, we have $\{w, v_{j_k}\} \in E(G_{a,b})$. Suppose, we choose $u_{i_1} \neq v$. Then we can express

$$\begin{aligned}
\mathbb{E}_C + \mathbb{E}_{C'} + \mathbf{e}_v + \mathbf{e}_w &= \sum_{k=1}^{3} \mathbf{e}_{u_{i_k}} + \sum_{k=1}^{3} \mathbf{e}_{v_{j_k}} + \mathbf{e}_v + \mathbf{e}_w \\
&= (\mathbf{e}_v + \mathbf{e}_{u_{i_1}}) + \sum_{k=2}^{3} \mathbf{e}_{u_{i_k}} + \sum_{k=1}^{2} \mathbf{e}_{v_{j_k}} + \\
&\quad (\mathbf{e}_{v_{j_3}} + \mathbf{e}_w).
\end{aligned}$$

Therefore, $\mathbb{E}_C + \mathbb{E}_{C'} + \mathbf{e}_v + \mathbf{e}_w$ is equal to:

$$\rho\big(\{v, u_{i_1}\}\big) + \rho\big(\{u_{i_2}, u_{i_3}\}\big) + \rho\big(\{v_{j_1}, v_{j_2}\}\big) + \rho\big(\{v_{j_3}, w\}\big).$$

As we can see, the expression $\mathbb{E}_C + \mathbb{E}_{C'} + \mathbf{e}_v + \mathbf{e}_w$ can be written as a linear combination of some $\rho(e)$, where $e \in E(G_{a,b})$. Therefore, for any $v, w \in V(G_{a,b})$, we have $\big(\mathbb{E}_C + \mathbb{E}_{C'} + \mathbf{e}_v + \mathbf{e}_w\big) \in S$. □

Let us consider $\mathbf{x} = (x_{u_1}, \ldots, x_{u_a}, x_w, x_{v_1}, \ldots, x_{v_b}) \in \mathbb{Z}^d_{\geq 0}$. We define a set,

$$A := \Bigg\{\mathbf{x} : \ x_w = 0, \sum_{u \in \{u_1, \ldots, u_a\}} x_u \text{ is odd }, \sum_{v \in \{v_1, \ldots, v_b\}} x_v \text{ is odd } \Bigg\}.$$

For the finite simple connected graph $G_{a,b}$ (Figure 6.1) and the set A as defined above, we can state the following lemma.

Lemma 6.1.3. $\overline{S} \subset S \cup A$.

Proof. Let α be an arbitrary element in $\overline{S}$. As we have seen in Section 3.4, the normalization of the edge ring $\Bbbk[G_{a,b}]$ can be expressed as

$$\overline{S} = S \ + \ \mathbb{Z}_{\geq 0}\big\{\mathbb{E}_C + \mathbb{E}_{C'} \colon (C, C') \text{ is exceptional in } G_{a,b}\big\}.$$

Therefore, any $\alpha \in \overline{S}$ can be expressed as $\alpha = \beta + \gamma$, where we have $\beta \in S$ and $\gamma \in \mathbb{Z}_{\geq 0}\big\{\mathbb{E}_C + \mathbb{E}_{C'} \colon (C, C') \text{ is exceptional in } G_{a,b}\big\}$. If $\gamma = 0$, then $\alpha \in S$. So, let us consider the non-trivial case where $\gamma \neq 0$. Let α_i, β_i and γ_i represent the i^{th} coordinates of α, β and γ respectively.

Since for any two (possibly identical) exceptional pairs (C, C'), $(\overline{C}, \overline{C}')$, it follows from the completeness of the graphs K_{a+1} and K_{b+1} that $\mathbb{E}_C + \mathbb{E}_{C'} + \mathbb{E}_{\overline{C}} + \mathbb{E}_{\overline{C}'} \in S$. Therefore, without loss of generality, for an exceptional pair (C, C') in $G_{a,b}$, we may assume that $\gamma = \mathbb{E}_C + \mathbb{E}_{C'}$.

Case 1. Let $\alpha_w = 0$.

We have $\alpha_w = 0$ and $\gamma \neq 0$ with $\gamma_w = 0$. Therefore $\beta_w = 0$, that is, we are not considering any edge adjacent to the common vertex w. This assures that, both $\sum_{u \in V(K_a)} \beta_u$ and $\sum_{v \in V(K_b)} \beta_v$ have to be even. Hence, both $\sum_{u \in V(K_a)} \alpha_u$ and $\sum_{v \in V(K_b)} \alpha_v$ will be odd. Thus we have $\alpha \in A$ and therefore, $\alpha \in S \cup A$.

Case 2. Let $\alpha_w > 0$.

Consider an exceptional pair (C, C') in $G_{a,b}$. We have $\gamma = \mathbb{E}_C + \mathbb{E}_{C'}$. The condition $\alpha_w > 0$ implies $\beta_w > 0$. This indicates that there must be at least one edge adjacent to the common vertex w, say $\{v, w\}$. For any exceptional pair (C, C') in $G_{a,b}$, by Lemma 6.1.2, we have $\mathbb{E}_C + \mathbb{E}_{C'} + \mathbf{e}_v + \mathbf{e}_w \in S$. Hence, $\alpha = \beta + \gamma \in S$. Thus, it proves that $\alpha \in S \cup A$. $\square$

All of the observations we have made so far lead us to the conclusion that

$$S \subset S' \subset \overline{S} \subset S \cup A.$$

Proposition 6.1.4. *Let $G_{a,b}$ be a finite simple connected graph with $3 \leq a \leq b$ and $|V(G_{a,b})| = a + b + 1 = d$, such that $G_{a,b}$ (Figure 6.1) consists of two complete graphs K_{a+1} and K_{b+1} joined at a common vertex w. Let $\Bbbk[G_{a,b}]$ be the edge ring of the graph $G_{a,b}$. Then, $\Bbbk[G_{a,b}]$ is non-normal and satisfies (S_2)-condition.*

Proof. Since $G_{a,b}$ is the union of two complete graphs with a common vertex w, it is assured that all the pairs of odd cycles of the form $(\{u_i, u_{i+1}, u_{i+2}\}, \{v_j, v_{j+1}, v_{j+2}\})$ where $1 \leq i \leq a-2$ and $1 \leq j \leq b-2$

are exceptional. Therefore, the graph $G_{a,b}$ does not satisfy the odd cycle condition and by Theorem 3.4.5, we conclude that the edge ring $\Bbbk[G_{a,b}]$ is non-normal.

Let us consider an element $\alpha = (x_{u_1}, \ldots, x_{u_a}, 0, x_{v_1}, \ldots, x_{v_b})$ from the set A, such that $\alpha \in \overline{S} \backslash S$. We have seen that the common vertex w is regular in $G_{a,b}$ and corresponding to this regular vertex we have $S \cap F_w = \mathbb{Z}_{\geq 0}\mathcal{A}_{G_{a,b} \backslash w}$. For any $\beta \in S \cap F_w$, let β_i represent the i^{th} coordinate of β. We observe that $\beta_w = 0$ and both $\sum_{i=1}^{a} \beta_{u_i}$, and $\sum_{j=1}^{b} \beta_{v_j}$ are even. Therefore, for all $\beta \in S \cap F_w$, we have $\alpha + \beta \in A$ and not in S. Thus, there exists no $\beta \in S \cap F_w$, such that $\alpha + \beta \in S$. Therefore, $\alpha \notin S_w$. Hence,

$$\alpha \notin \bigcap_{F_i \in \mathcal{F}(G_{a,b})} S_i = S'.$$

This implies that, for any $\alpha \in \overline{S} \backslash S$, we have $\alpha \notin S'$. Therefore, $(\overline{S} \backslash S) \cap S' = \emptyset$ and $S' \subset S$. We know that $S \subset S'$. Hence, $S = S'$. □

We proved that $\Bbbk[G_{a,b}]$ is non-normal and satisfies (S_2)-condition. Now, we are interested in modifying the graph $G_{a,b}$, to see how the behavior of the corresponding edge ring varies.

6.2 Removing edges of $G_{a,b}$ and (S_2)-condition

In Section 6.1, we have studied the graph $G_{a,b}$ in detail. In this section, we will investigate whether we can remove edges of $G_{a,b}$ to obtain $\widetilde{G}$, a subgraph of $G_{a,b}$, with $V(\widetilde{G}) = V(G_{a,b})$ and $|E(\widetilde{G})| = d + 1$ such that the edge ring $\Bbbk[\widetilde{G}]$ is non-normal and satisfies (S_2)-condition. We will modify a method of eliminating edges of $G_{a,b}$ so that the common vertex w remains regular in any graph created using this method. By the end of this section, we prove that $\Bbbk[\widetilde{G}]$ is non-normal and satisfies (S_2)-condition.

By eliminating one edge from the graph $G_{a,b}$ per step, we will gradually build up $\widetilde{G}$. First of all, we will be removing edges of the graph K_{a+1} and will not alter the graph K_{b+1}. Let us denote $u_0 := w$ and $G_0^{u_1} := G_{a,b}$. We construct a new subgraph of $G_{a,b}$ through an edge removal process such that for each $1 \leq i \leq a - 3$,

- we remove the edge $\{u_0, u_i\}$ from $G_0^{u_i}$ to obtain $G_i^{u_i}$, and
- remove the edge $\{u_i, u_j\}$ from $G_{j-1}^{u_i}$ to obtain $G_j^{u_i}$, $\forall\ i + 1 \leq j \leq a - 1$.

Let us denote $G_0^{u_i} := G_{a-1}^{u_{i-1}}$, $\forall\ 2 \le i \le a-2$. By construction, we observe that

$$E(G_0^{u_i}) = E(G_{a,b}) \setminus \bigcup_{p=1}^{i-1} \Big\{ \{u_p, u_q\} \colon q \neq p, 0 \le q \le a-1 \Big\},$$

where $2 \le i \le a-2$. For $i = a-2$, the construction of the subgraph $G_j^{u_i}$, $a-2 \le j \le a-1$ is as follows:

- the subgraph $G_{a-2}^{u_{a-2}}$ is obtained by removing the edge $\{u_0, u_{a-2}\}$ from $G_0^{u_{a-2}}$;
- the edge $\{u_0, u_{a-1}\}$ is removed from $G_{a-2}^{u_{a-2}}$, to construct the subgraph $G_{a-1}^{u_{a-2}}$.

Remark 6.2.1. For $1 \le i \le a-2$, $i \le j \le a-1$ and $(i,j) \neq (a-2, a-1)$, we have

$$\begin{aligned} E(G_j^{u_i}) &= E(G_{a,b}) \setminus \big\{ \{u_p, u_q\} \colon q \neq p, (p,q) \in [a-1] \times [i-1] \\ &\qquad \cup \{(0,1), \ldots, (0,i), (i,i+1), \ldots, (i,j)\} \big\}. \end{aligned}$$

In particular, if $1 \le p < q \le a$, then $\{u_p, u_q\} \in E(G_j^{u_i})$ if and only if one of the following cases happens:

(i) $p \ge i+1$;

(ii) $p = i$, $j+1 \le q \le a$;

(iii) $p \le i-1$, $q = a$.

If $1 \le p < q < r \le a$, then u_p, u_q, u_r form a 3-cycle in $G_j^{u_i}$ if and only if either of the following cases happens:

(a) $p = i$, $j+1 \le q$;

(b) $p \ge i+1$.

Moreover, for $1 \le t \le a$, we have $\{u_t, w\} \in E(G_j^{u_i})$ if and only if $t \ge i+1$.

Remark 6.2.2. For $(i,j) = (a-2, a-1)$, we have

$$\begin{aligned} E(G_{a-1}^{u_{a-2}}) &= E(G_{a,b}) \setminus \big\{ \{u_p, u_q\} \colon q \neq p, (p,q) \in [a-1] \times [a-3] \\ &\qquad \cup \{(0,1), (0,2), \ldots, (0,a-1)\} \big\}. \end{aligned}$$

In particular, if $1 \le p < q$, then $\{u_p, u_q\} \in E(G_{a-1}^{u_{a-2}})$ if and only if one of the following cases happens:

(i) $p \ge a-1$, $q = a$;

(ii) $(p, q) = (a-2, a-1)$.

If $1 \le p < q < r \le a$, then u_p, u_q, u_r form a 3-cycle in $G_{a-1}^{u_{a-2}}$ if and only if $(p, q, r) = (a-2, a-1, a)$. Moreover, $\{u_t, w\} \in E(G_{a-1}^{u_{a-2}})$ if and only if $t = a$.

An example of the sequence of subgraphs $G_j^{u_i}$; $1 \le i \le 2$ and $i \le j \le 3$ that is constructed from the graph $G_{4,3}$ using the edge removal process defined above is illustrated in Figure 6.2.

$$G_{4,3} =: G_0^{u_1} \xrightarrow{-\{u_0,u_1\}} G_1^{u_1} \xrightarrow{-\{u_1,u_2\}} G_2^{u_1} \xrightarrow{-\{u_1,u_3\}} G_3^{u_1} =: G_0^{u_2} \xrightarrow{-\{u_0,u_2\}} G_2^{u_2} \xrightarrow{-\{u_0,u_3\}} G_3^{u_2}$$

Figure 6.2: A sequence of subgraphs constructed from $G_{4,3}$

As per our construction, $V(G_j^{u_i}) = V(G_{a,b})$, $\forall\ 1 \le i \le a-2$, $i \le j \le a-1$. By the end of this entire process, we construct the subgraph of $G_{a,b}$, as shown in Figure 6.3.

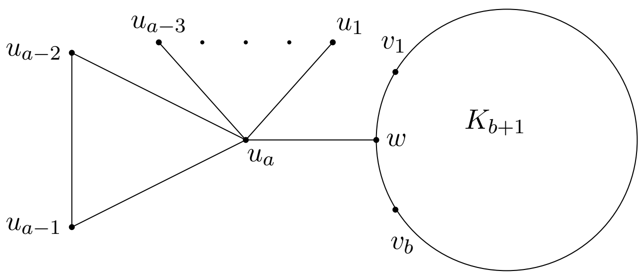

Figure 6.3: The graph $G_{a-1}^{u_{a-2}}$

Let $C = \{u_{i_1}, u_{i_2}, \ldots, u_{i_{2l+1}}\}$ be an odd cycle in $G_j^{u_i}$, such that $2l+1 \ge 5$ and $1 \le i_1 < i_2 < \cdots < i_{2l+1}$. We have $\{u_{i_1}, u_{i_2}\} \in E(G_j^{u_i})$ and $\{u_{i_1}, u_{i_{2l+1}}\} \in E(G_j^{u_i})$. Thus, according to our construction, $\{u_{i_1}, u_k\} \in E(G_j^{u_i})$, for all $i_1 < k \le i_{2l+1}$. Hence, for all $1 \le i \le a-2$; $i \le j \le a-1$, the minimal odd cycles of $G_j^{u_i}$ are cycles of length three.

Remark 6.2.3. Let $1 \le i \le a-2$, $i \le j \le a-1$ be integers. Let $C = \{u_{i_1}, u_{i_2}, u_{i_3}\}$ be a 3-cycle in $G_j^{u_i}$, where $1 \le i_1 < i_2 < i_3$. Then from Remarks 6.2.1 and 6.2.2, w is always adjacent to u_{i_3}. Indeed, if $(i, j) \ne (a-2, a-1)$ then w is even adjacent to both u_{i_2} and u_{i_3}. If $(i, j) = (a-2, a-1)$ then $(i_1, i_2, i_3) = (a-2, a-1, a)$ and w is adjacent to $u_{i_3} = u_a$.

Remark 6.2.4. Let $1 \le i \le a-2$, $i \le j \le a-1$ be integers. Let C and $\overline{C}$ be two (possibly identical) 3-cycles in $G_j^{u_i}$. Then by Remarks 6.2.1 and 6.2.2, there is an edge connecting two vertices of C and $\overline{C}$. Indeed, let $C = \{u_{i_1}, u_{i_2}, u_{i_3}\}$ and $\overline{C} = \{u_{j_1}, u_{j_2}, u_{j_3}\}$ where $1 \le i_1 < i_2 < i_3$, $1 \le j_1 < j_2 < j_3$. Then $i_2, j_2 \ge i+1$, so u_{i_2} is adjacent to either u_{j_2} or u_{j_3}.

Lemma 6.2.5. *Let $1 \le i \le a-2$, $i \le j \le a-1$ be integers. Let (C, C') be an exceptional pair in $G_j^{u_i}$. If $\{w, v\} \in E(G_j^{u_i})$, then*

$$\mathbb{E}_C \;+\; \mathbb{E}_{C'} \;+\; \mathbf{e}_w \;+\; \mathbf{e}_v \in S_{G_j^{u_i}}.$$

Proof. Since (C, C') is an exceptional pair, we have $V(C) \cap V(C') = \emptyset$ and $w \notin V(C) \cup V(C')$. By Remark 6.2.4, we may assume that $V(C) = \{u_{i_1}, u_{i_2}, u_{i_3}\} \subset V(K_a)$ and $V(C') = \{v_{j_1}, v_{j_2}, v_{j_3}\} \subset V(K_b)$, where $i_1 < i_2 < i_3$, $j_1 < j_2 < j_3$.
Case 1. Let $v = u_k \in V(K_a)$. We claim that $\mathbb{E}_C + \mathbf{e}_v$, $\mathbb{E}_{C'} + \mathbf{e}_w \in S_{G_j^{u_i}}$. By Remark 6.2.3, w is adjacent to v_{j_3}, so

$$\mathbb{E}_{C'} + \mathbf{e}_w = \rho\big(\{w, v_{j_3}\}\big) + \rho\big(\{v_{j_1}, v_{j_2}\}\big) \in S_{G_j^{u_i}}.$$

Since $\{u_{i_1}, u_{i_2}\}$, $\{u_{i_1}, u_{i_3}\}$ are edges and $i_1 < i_2 < i_3$, by Remarks 6.2.1 and 6.2.2, $i_1 \ge i$ and hence $i_2 \ge i+1$. Since $\{w, u_k\}$ is an edge, by the same results, $k \ge i+1$. Hence Remarks 6.2.1 and 6.2.2 imply that u_k is adjacent to either u_{i_2} or u_{i_3}. This implies $\mathbb{E}_C + \mathbf{e}_v \in S_{G_j^{u_i}}$.
Case 2. Let $v \in V(K_b)$. We claim that $\mathbb{E}_C + \mathbf{e}_w$, $\mathbb{E}_{C'} + \mathbf{e}_v \in S_{G_j^{u_i}}$. Since K_b is complete, $\mathbb{E}_{C'} + \mathbf{e}_v \in S_{G_j^{u_i}}$. By Remark 6.2.3, w is adjacent to u_{i_3}. Hence $\mathbb{E}_C + \mathbf{e}_w \in S_{G_j^{u_i}}$.
In both cases, we get the desired containment. □

For the graph $G_j^{u_i}$, where $1 \le i \le a-2$, $i \le j \le a-1$ and the set A as defined in Section 6.1, we have the following lemma.

Lemma 6.2.6. $\overline{S_{G_j^{u_i}}} \subset S_{G_j^{u_i}} \cup A$, *for all* $1 \le i \le a-2$ *and* $i \le j \le a-1$.

Proof. Let α be an arbitrary element in $\overline{S_{G_j^{u_i}}}$, where $1 \le i \le a-2$, $i \le j \le a-1$. The normalization of the semigroup $S_{G_j^{u_i}}$ can be expressed as

$$\overline{S_{G_j^{u_i}}} = S_{G_j^{u_i}} \;+\; \mathbb{Z}_{\ge 0}\big\{\mathbb{E}_C + \mathbb{E}_{C'} \colon (C, C') \text{ is exceptional in } G_j^{u_i}\big\}.$$

Therefore, any $\alpha \in \overline{S_{G_j^{u_i}}}$ can be expressed as $\alpha = \beta + \gamma$, where $\beta \in S_{G_j^{u_i}}$ and $\gamma \in \mathbb{Z}_{\ge 0}\big\{\mathbb{E}_C + \mathbb{E}_{C'} \colon (C, C') \text{ is exceptional in } G_j^{u_i}\big\}$. If $\gamma = 0$, then

$\alpha \in S_{G_j^{u_i}}$. So, let us consider the non-trivial case where $\gamma \neq 0$. Let α_k, β_k and γ_k represent the k^{th} coordinates of α, β and γ respectively.

For any two (possibly identical) exceptional pairs $(C, C'), (\overline{C}, \overline{C}')$, using Remark 6.2.4 and from the completeness of K_{b+1}, we have that $\mathbb{E}_C + \mathbb{E}_{C'} + \mathbb{E}_{\overline{C}} + \mathbb{E}_{\overline{C}'} \in S_{G_j^{u_i}}$, for all $1 \leq i \leq a-2$ and $i \leq j \leq a-1$. Therefore, without loss of generality, for an exceptional pair (C, C') in $G_j^{u_i}$, we may assume that $\gamma = \mathbb{E}_C + \mathbb{E}_{C'}$.

Case 1. Let $\alpha_w = 0$.

We have $\gamma_w = 0$ and $\alpha_w = 0$. Therefore $\beta_w = 0$, that is, we are not considering any edge adjacent to the common vertex w. This assures that, both $\sum_{u \in V(K_a)} \beta_u$ and $\sum_{v \in V(K_b)} \beta_v$ have to be even. Hence, both $\sum_{u \in V(K_a)} \alpha_u$ and $\sum_{v \in V(K_b)} \alpha_v$ will be odd. Thus we have $\alpha \in A$.

Case 2. Let $\alpha_w > 0$.

The condition $\alpha_w > 0$ implies $\beta_w > 0$. This indicates that among the edges defining the vector β, there must be at least one edge adjacent to w, say $\{w, v\}$. For any exceptional pair (C, C') in $G_j^{u_i}$, by Lemma 6.2.5, $\mathbb{E}_C + \mathbb{E}_{C'} + \mathbf{e}_w + \mathbf{e}_v \in S_{G_j^{u_i}}$, and thus $\alpha = \beta + \gamma \in S_{G_j^{u_i}}$. □

Therefore, for all $1 \leq i \leq a-2$, and $i \leq j \leq a-1$, we can observe that,

$$S_{G_j^{u_i}} \subset S'_{G_j^{u_i}} \subset \overline{S_{G_j^{u_i}}} \subset S_{G_j^{u_i}} \cup A.$$

Proposition 6.2.7. *The edge ring* $\Bbbk[G_j^{u_i}]$ *of the graph* $G_j^{u_i}$ *is non-normal and satisfies* (S_2)*-condition, for all* $1 \leq i \leq a-2$ *and* $i \leq j \leq a-1$.

Proof. For any $1 \leq k \leq b-2$, the pair $(\{u_{a-2}, u_{a-1}, u_a\}, \{v_k, v_{k+1}, v_{k+2}\})$ is always exceptional in $G_j^{u_i}$. Hence, $\Bbbk[G_j^{u_i}]$ is always non-normal.

Let us consider an element $\alpha \in \overline{S_{G_j^{u_i}}} \setminus S_{G_j^{u_i}}$. By Lemma 6.2.6, we have $\alpha \in A$ and $\alpha_w = 0$. We observe that, the common vertex w is regular in $G_j^{u_i}$. Hence, corresponding to w, we have $S_{G_j^{u_i}} \cap F_w := \mathbb{Z}_{\geq 0}\mathcal{A}_{G_j^{u_i} \setminus w}$. For any $\beta \in S_{G_j^{u_i}} \cap F_w$, let β_k be the k^{th} coordinate of β. We observe that $\beta_w = 0$ and both $\sum_{i=1}^{a} \beta_{u_i}$, and $\sum_{j=1}^{b} \beta_{v_j}$ are even. Therefore, for all $\beta \in S_{G_j^{u_i}} \cap F_w$, we have $\alpha + \beta \in A$ and not in $S_{G_j^{u_i}}$. Thus, there exists no $\beta \in S_{G_j^{u_i}} \cap F_w$, such that $\alpha + \beta \in S_{G_j^{u_i}}$, and this implies that, $\alpha \notin S'_{G_j^{u_i}}$. As a result, we have $(\overline{S_{G_j^{u_i}}} \setminus S_{G_j^{u_i}}) \cap S'_{G_j^{u_i}} = \emptyset$ and $S'_{G_j^{u_i}} \subset S_{G_j^{u_i}}$. Therefore, $S_{G_j^{u_i}} = S'_{G_j^{u_i}}$. □

Now, let us continue a similar edge removal process on K_{b+1} and remove the maximum number of edges from K_{b+1} resulting in the formation of the graph $\widetilde{G}$, as per our requirement. Let $v_0 := w$ and

$\widetilde{G}_0^{v_1} := G_{a-1}^{u_{a-2}}$. We construct a new subgraph of $G_{a,b}$ through an edge removal process such that for each $1 \le i \le b-3$,

- we remove the edge $\{v_0, v_i\}$ from $\widetilde{G}_0^{v_i}$ to obtain $\widetilde{G}_i^{v_i}$, and
- remove the edge $\{v_i, v_j\}$ from $\widetilde{G}_{j-1}^{v_i}$ to obtain $\widetilde{G}_j^{v_i}$, $\forall\ i+1 \le j \le b-1$.

Let us denote $\widetilde{G}_0^{v_i} := \widetilde{G}_{b-1}^{v_{i-1}}$, $\forall\ 2 \le i \le b-2$. By construction, we observe that

$$E(\widetilde{G}_0^{v_i}) = E(\widetilde{G}_0^{v_1}) \backslash \bigcup_{p=1}^{i-1} \Big\{ \{v_p, v_q\} \colon q \ne p, 0 \le q \le b-1 \Big\}, \ \ \forall\, 2 \le i \le b-2.$$

For $i = b-2$, the construction of the subgraph $\widetilde{G}_j^{v_i}$, $b-2 \le j \le b-1$ is as follows:

- the subgraph $\widetilde{G}_{b-2}^{v_{b-2}}$ is constructed by removing the edge $\{v_0, v_{b-2}\}$ from $\widetilde{G}_0^{v_{b-2}}$, and
- the edge $\{v_0, v_{b-1}\}$ is removed from $\widetilde{G}_{b-2}^{v_{b-2}}$, to obtain the subgraph $\widetilde{G}_{b-1}^{v_{b-2}}$.

As per construction, $V(\widetilde{G}_j^{v_i}) = V(G_{a,b}), \forall\, 1 \le i \le b-2,\ i \le j \le b-1$ and by the end of this removal procedure, we construct the graph depicted in Figure 6.4.

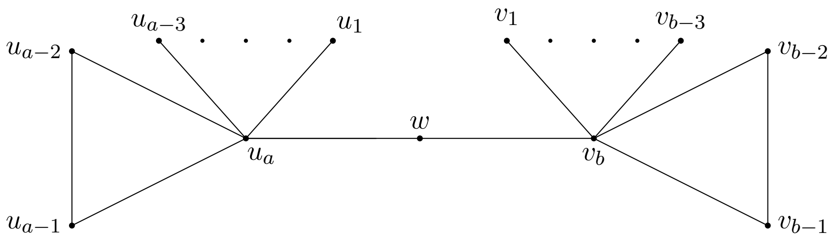

Figure 6.4: The graph $\widetilde{G}$

Remark 6.2.8. For $1 \le i \le b-2$, $i \le j \le b-1$ and $(i,j) \ne (b-2, b-1)$, we have

$$\begin{aligned} E(\widetilde{G}_j^{v_i}) &= E(\widetilde{G}_0^{v_1}) \backslash \big\{ \{v_p, v_q\} \colon q \ne p, (p,q) \in [b-1] \times [i-1] \\ &\qquad \cup \{(0,1), \dots, (0,i), (i,i+1), \dots, (i,j)\} \big\}. \end{aligned}$$

In particular, if $1 \le p < q \le b$, then $\{v_p, v_q\} \in E(\widetilde{G}_j^{v_i})$ if and only if one of the following cases happens:

(i) $p \geq i+1$;

(ii) $p = i$, $j+1 \leq q \leq b$;

(iii) $p \leq i-1$, $q = b$.

If $1 \leq p < q < r \leq b$, then v_p, v_q, v_r form a 3-cycle in $\widetilde{G}_j^{v_i}$ if and only if either of the following cases happens:

(a) $p = i$, $j+1 \leq q$;

(b) $p \geq i+1$.

Moreover, for $1 \leq t \leq b$, we have $\{v_t, w\} \in E(\widetilde{G}_j^{v_i})$ if and only if $t \geq i+1$.

Remark 6.2.9. For $(i,j) = (b-2, b-1)$, we have

$$\begin{aligned} E(\widetilde{G}_{b-1}^{v_{b-2}}) &= E(\widetilde{G}_0^{v_1}) \backslash \big\{\{v_p, v_q\} \colon q \neq p, (p,q) \in [b-1] \times [b-3] \\ &\quad \cup \{(0,1),(0,2),\ldots,(0,b-1)\}\big\}. \end{aligned}$$

In particular, if $1 \leq p < q$, then $\{v_p, v_q\} \in E(\widetilde{G}_{b-1}^{v_{b-2}})$ if and only if one of the following cases happens:

(i) $p \geq b-1$, $q = b$;

(ii) $(p,q) = (b-2, b-1)$.

If $1 \leq p < q < r \leq b$, then v_p, v_q, v_r form a 3-cycle in $\widetilde{G}_{b-1}^{v_{b-2}}$ if and only if $(p,q,r) = (b-2, b-1, b)$. Moreover, $\{v_t, w\} \in E(\widetilde{G}_{b-1}^{v_{b-2}})$ if and only if $t = b$.

Remark 6.2.10. From Remarks 6.2.8 and 6.2.9, we see that the minimal odd cycles of $\widetilde{G}_j^{v_i}$ are 3-cycles. Let C be a 3-cycle of $\widetilde{G}_j^{v_i}$, we claim that a vertex of C is adjacent to w. If $V(C) \subseteq V(K_a)$, as $\widetilde{G}_j^{v_i}$ is a subgraph of $\widetilde{G}_0^{v_1} = G_{a-1}^{u_{a-2}}$, we must have $C = \{u_{a-2}, u_{a-1}, u_a\}$. In this case, w adjacent to u_a.
If C is a subgraph of K_b, let its vertices be $v_{j_1}, v_{j_2}, v_{j_3}$ where $1 \leq j_1 < j_2 < j_3 \leq b$. Then w is adjacent to v_{j_3}, as Remarks 6.2.8 and 6.2.9 implies that $j_1 \geq i$ and $i+1 \leq j_2 < j_3$. In both cases, a vertex of C is adjacent to w.
Moreover, for any two (possibly identical) 3-cycles C and $\overline{C}$ of $\widetilde{G}_j^{v_i}$, whose vertices are inside K_b, there is an edge of $\widetilde{G}_j^{v_i}$ connecting a vertex of C to a vertex of $\overline{C}$.

Lemma 6.2.11. *Let us consider an exceptional pair* (C, C') *in* $\widetilde{G}_j^{v_i}$*, where* $1 \leq i \leq b-2$ *and* $i \leq j \leq b-1$*. If* $\{w, v\} \in E(\widetilde{G}_j^{v_i})$*, then we have*

$$\mathbb{E}_C \ + \ \mathbb{E}_{C'} \ + \ \mathbf{e}_w \ + \ \mathbf{e}_v \in S_{\widetilde{G}_j^{v_i}}.$$

Proof. By Remark 6.2.10, we may assume that $V(C) \subseteq V(K_a)$ and $V(C') \subseteq V(K_b)$. The same remark implies that $C = \{u_{a-2}, u_{a-1}, u_a\}$. Let the vertices of C' be $v_{j_1}, v_{j_2}, v_{j_3}$ where $1 \leq j_1 < j_2 < j_3 \leq b$.

Case 1. $v \in V(K_a)$. We claim that $\mathbb{E}_C + \mathbf{e}_v,\ \mathbb{E}_{C'} + \mathbf{e}_w \in S_{\widetilde{G}_j^{v_i}}$. Given $\{w, v\} \in E(\widetilde{G}_j^{v_i})$, as per the construction of $\widetilde{G}_j^{v_i}$, $v = u_a$. Since $C = \{u_{a-2}, u_{a-1}, u_a\}$, we see that v is adjacent to both u_{a-2} and u_{a-1}, so $\mathbb{E}_C + \mathbf{e}_v \in S_{\widetilde{G}_j^{v_i}}$. By Remark 6.2.10, w is adjacent to a vertex of C', hence $\mathbb{E}_{C'} + \mathbf{e}_w \in S_{\widetilde{G}_j^{v_i}}$.

Case 2. $v = v_k \in V(K_b)$. We claim that $\mathbb{E}_C + \mathbf{e}_w,\ \mathbb{E}_{C'} + \mathbf{e}_v \in S_{\widetilde{G}_j^{v_i}}$. Since w is adjacent to u_a, $\mathbb{E}_C + \mathbf{e}_w \in S_{\widetilde{G}_j^{v_i}}$. Since w is adjacent to v_k, by Remarks 6.2.8 and 6.2.9, $k \geq i+1$. The same remarks imply that $j_1 \geq i,\ j_2 \geq i+1$. Hence v_k is adjacent to either v_{j_2} or v_{j_3}. This yields $\mathbb{E}_{C'} + \mathbf{e}_v \in S_{\widetilde{G}_j^{v_i}}$.

In both cases, we get the desired containment. □

For the set A as defined in Section 6.1 and the graph $\widetilde{G}_j^{v_i}$, where $1 \leq i \leq b-2$, and $i \leq j \leq b-1$, we have the following lemma.

Lemma 6.2.12. $\overline{S_{\widetilde{G}_j^{v_i}}} \subset S_{\widetilde{G}_j^{v_i}} \cup A$*, for all* $1 \leq i \leq b-2$ *and* $i \leq j \leq b-1$*.*

Proof. Let α be an arbitrary element in $\overline{S_{\widetilde{G}_j^{v_i}}}$. Due to the similar edge removal process, the proof is similar to that of Lemma 6.2.6. By similar arguments as in the proof of Lemma 6.2.6, we reduce to the case $\alpha = \beta + \gamma$, where $\beta \in S_{\widetilde{G}_j^{v_i}}$, $\gamma = \mathbb{E}_C + \mathbb{E}_{C'}$ for an exceptional pair (C, C') of $\widetilde{G}_j^{v_i}$. Furthermore, we also get that $\alpha \in A$ if $\alpha_w = 0$. Assume that $\alpha_w > 0$, then so is β_w. Hence among the edges defining the vector β, there is at least one edge of the form $\{w, v\}$. Using Lemma 6.2.11, we get that the semigroup $S_{\widetilde{G}_j^{v_i}}$ contains $\mathbb{E}_C + \mathbb{E}_{C'} + \mathbf{e}_w + \mathbf{e}_v$, hence it also contains α. □

From the above observations, $S_{\widetilde{G}_j^{v_i}} \subset S'_{\widetilde{G}_j^{v_i}} \subset \overline{S_{\widetilde{G}_j^{v_i}}} \subset S_{\widetilde{G}_j^{v_i}} \cup A$, for all $1 \leq i \leq b-2$, and $i \leq j \leq b-1$.

Proposition 6.2.13. *The edge ring* $\Bbbk[\widetilde{G}_j^{v_i}]$ *of the graph* $\widetilde{G}_j^{v_i}$ *is non-normal and satisfies* (S_2)*-condition, for all* $1 \leq i \leq b-2$ *and* $i \leq j \leq b-1$*.*

Proof. According to our construction of the graph $\widetilde{G}_j^{v_i}$, the pair of odd cycles $(\{u_{a-2}, u_{a-1}, u_a\}, \{v_{b-2}, v_{b-1}, v_b\})$ is contained in every $\widetilde{G}_j^{v_i}$ and is exceptional, for all $1 \leq i \leq b-2$, and $i \leq j \leq b-1$. Hence, $\Bbbk[\widetilde{G}_j^{v_i}]$ is always non-normal.

Let us consider an element $\alpha \in A$ such that $\alpha \in \overline{S_{\widetilde{G}_j^{v_i}}} \setminus S_{\widetilde{G}_j^{v_i}}$. The common vertex w is regular in $\widetilde{G}_j^{v_i}$, and corresponding to this regular vertex, we have $S_{\widetilde{G}_j^{v_i}} \cap F_w := \mathbb{Z}_{\geq 0}\mathcal{A}_{\widetilde{G}_j^{v_i} \setminus w}$. By a similar proof as that of Proposition 6.2.7, we can demonstrate that there exists no $\beta \in S_{\widetilde{G}_j^{v_i}} \cap F_w$, such that $\alpha + \beta \in S_{\widetilde{G}_j^{v_i}}$, and hence $S_{\widetilde{G}_j^{v_i}} = S'_{\widetilde{G}_j^{v_i}}$. □

Let the graph $\widetilde{G}_{b-1}^{v_{b-2}} := \widetilde{G}$ (Figure 6.4). We observe that, $\widetilde{G}$ is a subgraph of $G_{a,b}$ with $|V(\widetilde{G})| = |V(G_{a,b})|$ and $|E(\widetilde{G})| = a+b+2 = d+1$. By Proposition 6.2.13, we know that the edge ring $\Bbbk[\widetilde{G}]$ is non-normal and also satisfies (S_2)-condition. Therefore, we observe that $\widetilde{G}$ is the graph on d vertices with the least number of edges, $d+1$ edges, such that the edge ring is non-normal and meets (S_2)-condition. This completes the proof of a part of the statement of Theorem 6.1.1.

6.3 Addition of edges to $G_{a,b}$ breaks non-normality or (S_2)-condition

In this section, we prove that any addition of (one or more) new edges to $G_{a,b}$ either breaks the non-normality of the edge ring or violates the (S_2)-condition.

Let us construct a new graph G' on the vertex set $V(G') = V(G_{a,b})$, by introducing one or more edges to $G_{a,b}$. Since K_{a+1} and K_{b+1} are complete graphs, each of the new edges will be of the form $\{u_i, v_j\}$, for some $1 \leq i \leq a$ and $1 \leq j \leq b$. For instance, addition of a single edge $\{u_2, v_3\}$ to the graph $G_{a,b}$ is illustrated in Figure 6.5.

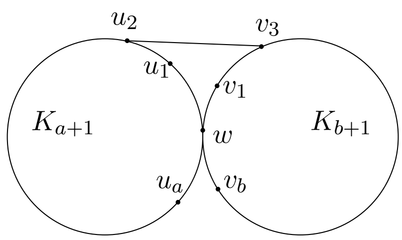

Figure 6.5: The graph G' obtained by adding edge $\{u_2, v_3\}$ to $G_{a,b}$

We can observe that for any G', all of its vertices are regular and the fundamental sets are: $\{i\}$, for some $i \in V(G')$ and $\{i, j\}$, $\forall \{i, j\} \notin E(G')$.

Proposition 6.3.1. *Let G' be a graph on the vertex set $V(G') = V(G_{a,b})$, such that G' is constructed by adding one or more new edges to $G_{a,b}$. Then for any G', the edge ring $\Bbbk[G']$ is either normal or it does not satisfy (S_2)-condition.*

Proof. Suppose we construct a graph G' by adding at least minimal number of edges $\{u_i, v_j\}$, where $1 \leq i \leq a$ and $1 \leq j \leq b$, such that we connect all the exceptional pairs of $G_{a,b}$. Thus, G' satisfies the odd cycle condition, and therefore the corresponding edge ring $\Bbbk[G']$ is normal.

Now, let us consider the case where we construct a graph G' such that $\Bbbk[G']$ is non-normal. Then, we prove that for any such G', the edge ring $\Bbbk[G']$ will not satisfy (S_2)-condition.

Suppose, we construct G' by adding new edges $\{u_i, v_j\}$ to the graph $G_{a,b}$, for some $1 \leq i \leq a$ and $1 \leq j \leq b$, such that G' consists of at least one pair of 3-cycles, $(\{u_{i_1}, u_{i_2}, u_{i_3}\}, \{v_{j_1}, v_{j_2}, v_{j_3}\})$ with either $i_k \neq i$ or $j_k \neq j$ for any $1 \leq k \leq 3$. This pair will be exceptional in G' and thus, the corresponding edge ring $\Bbbk[G']$ is non-normal.

Now, we consider any exceptional pair (C, C') of the graph G'. For any regular vertex $v \in V(G') \backslash [V(C) \cup V(C')]$ such that $v \neq w$, we observe that $G' \backslash v$ is a connected graph with the common vertex w. Let us consider the regular vertex $w \in V(G') \backslash [V(C) \cup V(C')]$. As per our construction, the graph G' contains edges of the type $\{u_i, v_j\}$, for some $1 \leq i \leq a$ and $1 \leq j \leq b$. The existence of such edges in G' ensures the connectedness of the graph $G' \backslash w$.

Thus for any regular vertex $v \in V(G') \backslash [V(C) \cup V(C')]$, we observe that the graph $G' \backslash v$ is always a connected graph. Hence both C and C' belong to the same connected components of $G' \backslash v$.

Since both K_{a+1} and K_{b+1} are complete graphs, any vertex in $V(K_{a+1})$ or $V(K_{b+1})$ is adjacent to all the other vertices of K_{a+1} and K_{b+1} respectively. Hence for all $v \in V(G')$, we have $[V(C) \cup V(C')] \cap [\{v\} \cup N_{G'}(\{v\})] \neq \emptyset$. Let us consider the fundamental set of the form $\{u_i, v_j\}$, such that $\{u_i, v_j\} \notin E(G')$. By the completeness of K_{a+1} and K_{b+1}, $\{u_i, v_j\} \cup N_{G'}(\{u_i, v_j\}) = V(G')$. Hence for any fundamental set T of G', we have

$$\Big[V(C) \cup V(C')\Big] \cap \Big[T \cup N_{G'}(T)\Big] \neq \emptyset.$$

Therefore, by Theorem 3.4.6, $\mathbb{E}_C + \mathbb{E}_{C'} \in S'_{G'}$. In particular, $S_{G'} \neq S'_{G'}$. Hence, the edge ring $\Bbbk[G']$ does not satisfy (S_2)-condition. □

6.4 Conclusions

A finite simple connected graph on d vertices with a non-normal edge ring must contain at least one exceptional pair of odd cycles. Thus the minimal graph on d vertices satisfying the above condition must be a graph consisting of two disjoint minimal odd cycles and a path (of at least length 2) connecting the two cycles. This minimal graph will have exactly $d+1$ number of edges. In Section 6.2, we proved the existence of such a minimal graph $\widetilde{G}$, which satisfies the main theorem (Theorem 6.1.1).

Moreover, for every finite simple connected graph on d vertices, the edge ring is always normal if $d \leq 6$. For this reason, we suppose that $d \geq 7$ in Theorem 6.1.1.

We have examined the graph $G_{a,b}$ in detail. From Section 6.3, we can conclude that any addition of (one or more) new edges to $G_{a,b}$ either breaks the non-normality of the edge ring or violates (S_2)-condition. Thus, we may conclude that $G_{a,b}$ is the graph on d vertices with the maximum number of edges such that, the corresponding edge ring is non-normal and satisfies (S_2)-condition. For the graph $G_{a,b}$, we have $|V(G_{a,b})| = d = a+b+1$ and $3 \leq a \leq b$. Therefore, in order to maximize the number of edges in $G_{a,b}$, we have to consider $a=3$ and $b = d-4$. That is,

$$\left|E(G_{a,b})\right| \leq \binom{4}{2} + \binom{d-3}{2} = \frac{d^2-7d+24}{2}.$$

This provides us very strong supporting evidence that $\frac{d^2-7d+24}{2}$ could be the maximal number of edges possible for a graph on d vertices such that, its edge ring is non-normal and satisfies (S_2)-condition.

Proof of Theorem 6.1.1. Let us consider the graph $G_{3,b}$ on d vertices such that $d \geq 7$. We have $|E(G_{3,b})| = \frac{d^2-7d+24}{2}$ and by Proposition 6.1.4, the edge ring $\Bbbk[G_{3,b}]$ is non-normal and satisfies (S_2)-condition.

Through the edge removal processes discussed in Section 6.2, by eliminating one edge from the graph $G_{3,b}$ per step, we can gradually build up a graph on d vertices with $d+1$ edges such that, its edge ring is non-normal and satisfies (S_2)-condition. Proposition 6.2.7 and Proposition 6.2.13 guarantee that the edge ring of each of the graphs obtained after each removal step is always non-normal and will satisfy (S_2)-condition.

Therefore we prove that for any given integers d and m such that, $d \geq 7$ and $d+1 \leq m \leq \frac{d^2-7d+24}{2}$, we can always construct a finite simple connected graph on d vertices and having m edges such that, the edge ring of the graph is non-normal and satisfies (S_2)-condition. □

Remark 6.4.1. By further automatic computations using `Macaulay2`, we believe that every non-normal edge ring of the family of graphs studied in this chapter that satisfies (S_2)-condition is Cohen–Macaulay.

Chapter 7

On a special family of cactus graphs and (S_2)-condition

A cactus graph is a connected graph in which every block is either an edge or a cycle. In this chapter, we will examine a special category of cactus graphs of diameter 4, where all the blocks are 3-cycles. Our main focus is to prove that the corresponding edge ring of this special family of graphs satisfies the (S_2)-condition. This chapter and its contents are a part of the author's ongoing collaboration with R. Dinu.

7.1 On triangular cacti

All the concepts and notations related to edge rings that are used in this chapter are defined in Chapter 3.

Let G be a simple finite connected graph. A vertex $v \in V(G)$ is called a *cutpoint* if the subgraph $G\backslash v$ of G has more connected components than that of G. From Section 3.4, recall that a vertex v is regular in G if every connected component of $G\backslash v$ contains at least one odd cycle. If a cutpoint v is regular in G, then v is called a *regular cutpoint*. We say that a connected graph without a cutpoint is *non-separable*. A maximal non-separable subgraph of G is called a *block* of the graph G.

A *cactus graph* is a connected graph in which every block is either an edge or a cycle. The cactus graph whose blocks are all n-cycles is defined as *n-cactus graph*. In this chapter, we will focus only on the 3-cactus graphs. Let us call a 3-cactus graph as the *triangular cactus*.

The 3-cycles are triangular cacti of diameter 1. The family of graphs studied in Chapter 4, consisting of 3-cycles that share a single common vertex, are triangular cacti of diameter 2. A generic illustration of a triangular cactus of diameter 3 is shown in Figure 7.1. Therefore, we observe that any triangular cactus of diameter ≤ 3 satisfies the odd cycle condition, and by Theorem 3.4.5, it has a normal Cohen–Macaulay edge ring.

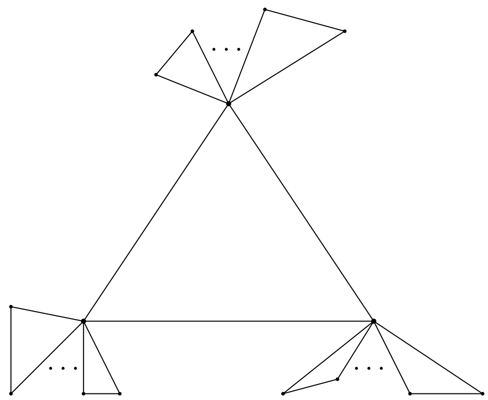

Figure 7.1: A general triangular cactus of diameter 3

We focus on the non-normal edge rings associated with triangular cacti that satisfy (S_2)-condition. A broader objective of this study is to prove the following conjecture.

Conjecture 7.1.1. The edge ring associated with a triangular cactus of diameter ≥ 4, satisfies (S_2)-condition.

In this chapter, we will exclusively study the family of triangular cacti whose diameter is 4 and prove Conjecture 7.1.1 for this specific family. Let **G** be the triangular cactus of diameter 4. For an illustration of **G** in general, see Figure 7.2.

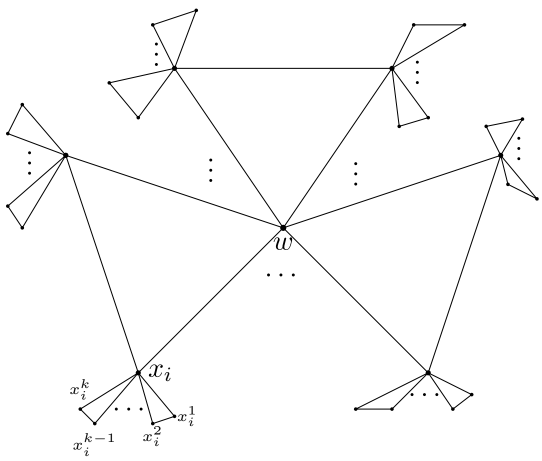

Figure 7.2: A general form of a triangular cactus of diameter 4

The main theorem of our study is as follows.

Theorem 7.1.2. *Let* $\mathbf{G}$ *be a triangular cactus with* $\operatorname{diam} \mathbf{G} = 4$*. Then, the edge ring* $\Bbbk[\mathbf{G}]$ *is non-normal and satisfies* (S_2)*-condition.*

7.2 Towards the proof of Theorem 7.1.2

This section centers on the proof of Theorem 7.1.2. Here, we will study the two types of graph $\mathbf{G}$ and collect the necessary results about them to finally prove our main theorem.

Let us consider that $|V(\mathbf{G})| = d$ and n copies of 3-cycles are attached to the vertex w in $\mathbf{G}$. Note that, w is always a cutpoint of $\mathbf{G}$. Moreover, all the non-cutpoints of $\mathbf{G}$ are regular in $\mathbf{G}$.

For graph $\mathbf{G}$, we consider $A_{\mathbf{G}} = \{\rho(e) : e \in E(\mathbf{G})\}$, $S_{\mathbf{G}} := \mathbb{Z}_{\geq 0} A_{\mathbf{G}}$ and $\overline{S_{\mathbf{G}}}$ denotes the normalization of $S_{\mathbf{G}}$. We know that the edge ring $\Bbbk[\mathbf{G}]$ is the affine semigroup ring of $S_{\mathbf{G}}$. Recall that, for any regular vertex v and fundamental set T of $\mathbf{G}$, we denote F_v and F_T as the facets of $\mathbb{Q}_{\geq 0} A_{\mathbf{G}}$ corresponding to the hyperplanes $\mathcal{H}_v$ and $\mathcal{H}_T$ respectively.

Recall the notion of exceptional pairs of a graph, explained in Section 3.4. The exceptional pairs in $\mathbf{G}$ are of the form (C, C') where $C = \{x_p, x_p^{k-1}, x_p^k\}$ and $C' = \{x_q, x_q^{k'-1}, x_q^{k'}\}$ for some $p, q \in [2n]$ such that $x_p \neq x_q$ and $\{x_p, x_q\} \notin E(\mathbf{G})$. An illustration of an exceptional pair in $\mathbf{G}$ is shown in Figure 7.3.

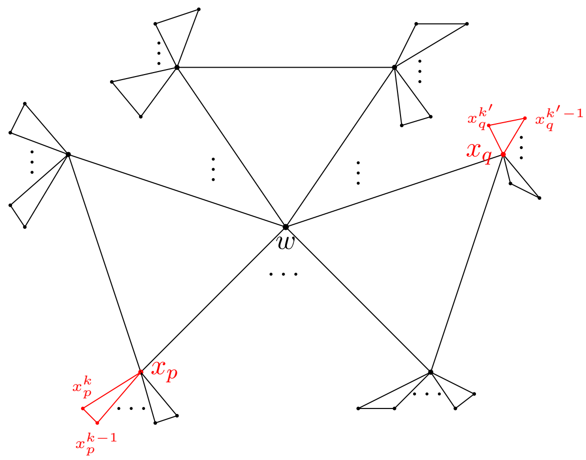

Figure 7.3: Illustration of an exceptional pair

For any odd cycle C in $\mathbf{G}$, recall that $\mathbb{E}_C := \sum_{i \in V(C)} \mathbf{e}_i$, where $\mathbf{e}_i$ represents the i^{th} canonical vector of $\mathbb{R}^d$.

Lemma 7.2.1. *Let $(C_i, C_{i'})$ and $(C_j, C_{j'})$ be any two exceptional pairs in $\mathbf{G}$. Then,*

$$\mathbb{E}_{C_i} + \mathbb{E}_{C_{i'}} + \mathbb{E}_{C_j} + \mathbb{E}_{C_{j'}} \in S_{\mathbf{G}}$$

if and only if both (C_i, C_j) and $(C_{i'}, C_{j'})$ $\Big($ or $(C_i, C_{j'})$ and $(C_{i'}, C_j)\Big)$ are not exceptional.

Proof. Let us consider $C_p = \{x_p, x_p^{k_p-1}, x_p^{k_p}\}$, for $p \in \{i, i', j, j'\}$. We know that, the pair (C_p, C_q) exceptional in $\mathbf{G}$ implies that $x_p \neq x_q$ and $\{x_p, x_q\} \notin E(\mathbf{G})$.

($\Longrightarrow$) Let us assume that, $\mathbb{E}_{C_i} + \mathbb{E}_{C_{i'}} + \mathbb{E}_{C_j} + \mathbb{E}_{C_{j'}} \in S_{\mathbf{G}}$. This implies that $\mathbb{E}_{C_i} + \mathbb{E}_{C_{i'}} + \mathbb{E}_{C_j} + \mathbb{E}_{C_{j'}}$ is a linear combination of $\rho(e)$ for some $e \in E(\mathbf{G})$.

According to the structure of the graph $\mathbf{G}$, $\{x_u^k, x_v^{k'}\} \notin E(\mathbf{G})$, for any $u \neq v \in [2n]$. Moreover, for the exceptional pairs $(C_i, C_{i'})$ and $(C_j, C_{j'})$, we know that $x_i \neq x_{i'}$, $x_j \neq x_{j'}$, $\{x_i, x_{i'}\} \notin E(\mathbf{G})$, and $\{x_j, x_{j'}\} \notin E(\mathbf{G})$. Therefore, $\mathbb{E}_{C_i} + \mathbb{E}_{C_{i'}} + \mathbb{E}_{C_j} + \mathbb{E}_{C_{j'}}$ can be expressed as a linear combination of $\rho(e)$ for some $e \in E(\mathbf{G})$ only if both of the vertices x_i and $x_{i'}$ are such that

$$x_i, x_{i'} \in \{x_j, x_{j'}\} \cup N_{\mathbf{G}}\big(\{x_j, x_{j'}\}\big).$$

This also implies $x_j, x_{j'} \in \{x_i, x_{i'}\} \cup N_{\mathbf{G}}\big(\{x_i, x_{i'}\}\big)$. That is, either both $\mathbb{E}_{C_i} + \mathbb{E}_{C_j}$, $\mathbb{E}_{C_{i'}} + \mathbb{E}_{C_{j'}} \in S_{\mathbf{G}}$ or both $\mathbb{E}_{C_i} + \mathbb{E}_{C_{j'}}$, $\mathbb{E}_{C_{i'}} + \mathbb{E}_{C_j} \in S_{\mathbf{G}}$. Therefore, from the given odd cycles C_p $(p = i, i', j, j')$, we can choose any two pairs of odd cycles that are not exceptional.

($\Longleftarrow$) Let us assume that both the pairs (C_i, C_j) and $(C_{i'}, C_{j'})$, are not exceptional. Therefore, we have one of the following cases:

- $V(C_i) \cap V(C_j) \neq \emptyset$ and $V(C_{i'}) \cap V(C_{j'}) \neq \emptyset$;
- there exists a bridge between C_i, C_j and between $C_{i'}, C_{j'}$;
- $V(C_i) \cap V(C_j) \neq \emptyset$ and there exists a bridge between $C_{i'}, C_{j'}$ or vice versa.

In any of these cases, the method of the proof remains the same and therefore, without loss of generality, we assume that C_i, C_j shares a common vertex, i.e., $x_i = x_j$ and there exists a bridge between $C_{i'}, C_{j'}$, i.e., $\{x_{i'}, x_{j'}\} \in E(\mathbf{G})$. Hence,

$$\begin{aligned} \mathbb{E}_{C_i} + \mathbb{E}_{C_{i'}} + \mathbb{E}_{C_j} + \mathbb{E}_{C_{j'}} \;=\; & (\mathbf{e}_{x_i} + \mathbf{e}_{x_i^{k_i}}) + (\mathbf{e}_{x_i} + \mathbf{e}_{x_i^{k_i-1}}) \\ & + (\mathbf{e}_{x_j} + \mathbf{e}_{x_j^{k_j-1}}) + (\mathbf{e}_{x_{i'}} + \mathbf{e}_{x_{j'}}) \\ & + (\mathbf{e}_{x_{i'}^{k_{i'}}} + \mathbf{e}_{x_{i'}^{k_{i'}-1}}) + (\mathbf{e}_{x_{j'}^{k_{j'}}} + \mathbf{e}_{x_{j'}^{k_{j'}-1}}). \end{aligned}$$

This implies, $\mathbb{E}_{C_i} + \mathbb{E}_{C_{i'}} + \mathbb{E}_{C_j} + \mathbb{E}_{C_{j'}}$ is a linear combination of $\rho(e)$ for some $e \in E(\mathbf{G})$. Thus, $\mathbb{E}_{C_i} + \mathbb{E}_{C_{i'}} + \mathbb{E}_{C_j} + \mathbb{E}_{C_{j'}} \in S_{\mathbf{G}}$. □

Lemma 7.2.2. *Let (C, C') be exceptional in $\mathbf{G}$. Then, for any $\{v, w\} \in E(\mathbf{G})$ with $v \in V(C) \cup V(C') \cup N_{\mathbf{G}}\big(V(C) \cup V(C')\big)$, we have*

$$\mathbb{E}_C + \mathbb{E}_{C'} + \mathbf{e}_v + \mathbf{e}_w \in S_{\mathbf{G}}.$$

Proof. Since (C, C') is exceptional in $\mathbf{G}$, we have $V(C) \cap V(C') = \emptyset$ and $N_{\mathbf{G}}\big(V(C)\big) \cap N_{\mathbf{G}}\big(V(C')\big) = \{w\}$ (see, Figure 7.3). Hence both $\mathbb{E}_C + \mathbf{e}_w$ and $\mathbb{E}_{C'} + \mathbf{e}_w$ can be express as a linear combination of $\rho(e)$ for some $e \in E(\mathbf{G})$. Now, without loss of generality, we assume that $v \in V(C) \cup N_{\mathbf{G}}\big(V(C)\big)$. Thus we can write $\mathbb{E}_C + \mathbf{e}_v = \sum_{e \in E(\mathbf{G})} \rho(e)$. Hence, $\mathbb{E}_C + \mathbb{E}_{C'} + \mathbf{e}_v + \mathbf{e}_w = (\mathbb{E}_C + \mathbf{e}_v) + (\mathbb{E}_{C'} + \mathbf{e}_w)$, can be expressed as a linear combination of $\rho(e)$ for some $e \in E(\mathbf{G})$. Therefore, we have $\mathbb{E}_C + \mathbb{E}_{C'} + \mathbf{e}_v + \mathbf{e}_w \in S_{\mathbf{G}}$. □

Based on the fact whether w in $\mathbf{G}$ is a regular cutpoint or not, we classify the graph $\mathbf{G}$ into two types. Let us look at these two cases in-depth in the following subsections.

7.2.1 Type 1: w is a regular cutpoint of G

Let $\mathbf{G}'$ be the triangular cactus $\mathbf{G}$ where any 3-cycle in $\mathbf{G}'$ containing the vertex w is such that at least one of its remaining vertices x_i will have at least one 3-cycle $\{x_i, x_i^{k-1}, x_i^k\}$ attached to it. This is the first type of triangular cactus $\mathbf{G}$ that we will be studying.

Let us denote the vertices x_i with no 3-cycles $\{x_i, x_i^{k-1}, x_i^k\}$ attached to them as ζ_i and let there be l such ζ_i in $\mathbf{G}'$. That is, $\zeta_i \in V(\mathbf{G}')$ has no 3-cycles attached to each of them, for all $1 \le i \le l$. Note that as per the description of $\mathbf{G}'$, for any $\{x_i, x_j\} \in E(\mathbf{G}')$, at most one of the vertices (x_i or x_j) can be a vertex ζ_i. Therefore we have, $0 \le l \le n$. An illustration of the graph $\mathbf{G}'$ is shown in Figure 7.4.

Now, let us look at the regular vertices and fundamental sets in $\mathbf{G}'$. To recollect the concept of regular vertices and fundamental sets, see Section 3.4. The vertex w and all the x_i of $\mathbf{G}'$ with at least one 3-cycle $\{x_i, x_i^{k-1}, x_i^k\}$ attached to them, are cutpoints of $\mathbf{G}'$. That is, the set of cutpoints in $\mathbf{G}'$ is $\{w\} \cup \Big(\{x_i \colon 1 \le i \le 2n\} \setminus \bigcup_{i=1}^{l} \{\zeta_i\}\Big)$. The vertex w and all the non-cutpoints of $\mathbf{G}'$ are regular in $\mathbf{G}'$.

Let us now focus on the fundamental sets in $\mathbf{G}'$. For $i \in [2n]$, let s_i number of 3-cycles, not containing the vertex w, be attached to $x_i \in V(\mathbf{G}')$. Then the fundamental set in $\mathbf{G}'$, containing the vertex w, is of the form $\{w\} \cup \bigcup_{1 \le i \le 2n} \{x_i^{k_1}, x_i^{k_2}, \ldots, x_i^{k_{s_i}}\}$, where $k_u \in [2s_i]$ and none of the vertices $x_i^{k_u}$ are adjacent to each other. Note that, for the fundamental set T with $w \in T$, and any exceptional pair (C, C') in $\mathbf{G}'$, we have $\big(T \cup N_{\mathbf{G}'}(T)\big) \cap \big(V(C) \cup V(C')\big) \neq \emptyset$.

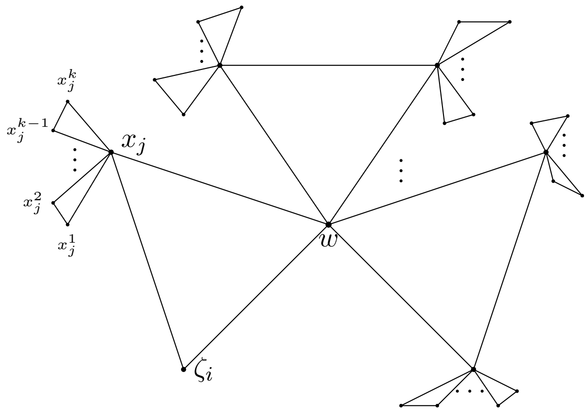

Figure 7.4: An illustration of graph $\mathbf{G}'$

For graph $\mathbf{G}'$, and for any $i, j \in [2n]$, the building blocks of any fundamental set not containing w are as follows.

(i) Suppose s is the number of 3-cycles attached to the vertex x_i that do not contain the vertex w. Then, the set $\{x_i^{k_1}, x_i^{k_2}, \ldots, x_i^{k_s}\}$, where $k_u \in [2s]$, and none of the vertices $x_i^{k_u}$ adjacent to each other, is fundamental in $\mathbf{G}'$.

(ii) Let $\{x_i, x_j\} \in E(\mathbf{G}')$, and let t be the number of 3-cycles attached to the vertex x_j that do not contain the vertex w. Then, the set $\{x_i, x_j^{k_1}, x_j^{k_2}, \ldots, x_j^{k_t}\}$, where none of the vertices $x_j^{k_v}$ are adjacent to each other for $k_v \in [2t]$, forms a fundamental set in $\mathbf{G}'$.

Note that any fundamental set T in $\mathbf{G}'$ with $w \notin T$ is either one of the listed building blocks or can be expressed as their union. Moreover, for any fundamental set T in $\mathbf{G}'$ consisting of building blocks listed in (ii), we have $w \in N_{\mathbf{G}'}(T)$.

Proposition 7.2.3. *For the triangular cactus $\mathbf{G}'$ (Figure 7.4), let $\mathfrak{T}$ be the set of all fundamental sets T in $\mathbf{G}'$ such that $w \in N_{\mathbf{G}'}(T)$ and $\big(T \cup N_{\mathbf{G}'}(T)\big) \cap \big(V(C_i) \cup V(C_i')\big) = \emptyset$, for any exceptional pair (C_i, C_i'). Then,*

$$\overline{S_{\mathbf{G}'}} \backslash S_{\mathbf{G}'} = \bigcup_{T \in \mathfrak{T}} (q_i + F_T) \cup \bigcup_i (q_i + F_w), \tag{7.1}$$

where the index i is taken over all possible exceptional pairs (C_i, C_i') of $\mathbf{G}'$, for which we define $q_i = \sum_i (\mathbb{E}_{C_i} + \mathbb{E}_{C_i'}) \notin S_{\mathbf{G}'}$.

Proof. As we have seen in Section 3.4, the normalization of edge ring $\Bbbk[\mathbf{G}']$ can be expressed as

$$\overline{S_{\mathbf{G}'}} = S_{\mathbf{G}'} \cup \mathbb{Z}_{\geq 0}\{\mathbb{E}_C + \mathbb{E}_{C'} : (C, C') \text{ are exceptional in } \mathbf{G}'\}.$$

Also note that in (7.1), we choose i in accordance with Lemma 7.2.1.

($\supset$) Let $\alpha \in q_i + F_w$ for some q_i. This implies that α does not have any contribution from any edges adjacent to w. Thus, $\alpha \in \overline{S_{\mathbf{G}'}} \backslash S_{\mathbf{G}'}$. Note that, all $T \in \mathfrak{T}$ is such that $\big(T \cup N_{\mathbf{G}}(T)\big) \cap \big(V(C_i) \cup V(C'_i)\big) = \emptyset$. Therefore, any combination of edges of the facet F_T added to q_i never belong to $S_{\mathbf{G}'}$. Hence, $(q_i + F_T) \in \overline{S_{\mathbf{G}'}} \backslash S_{\mathbf{G}'}$, for all $T \in \mathfrak{T}$.

($\subset$) Now let us prove that $\overline{S_{\mathbf{G}'}} \subset S_{\mathbf{G}'} \cup \bigcup_{T \in \mathfrak{T}} (q_i + F_T) \cup \bigcup_i (q_i + F_w)$. Let $\alpha \in \overline{S_{\mathbf{G}'}}$. We can write $\alpha = \beta + \gamma$, where $\beta \in S_{\mathbf{G}'}$ and $\gamma \in \mathbb{Z}_{\geq 0}\{\mathbb{E}_C + \mathbb{E}_{C'} : (C, C') \text{ are exceptional in } \mathbf{G}'\}$.

Case 1. Let $\alpha_w = 0$. This implies $\beta_w = 0$ and we have $\beta \in F_w$. From the structure of $\mathbf{G}'$, we can see that any $\mathbb{E}_{C_i} + \mathbb{E}_{C_{i'}} + \mathbb{E}_{C_j} + \mathbb{E}_{C_{j'}} \in S_{\mathbf{G}'}$ (in Lemma 7.2.1) is always contained in the facet F_w. Thus, we express $\alpha = q_i + F_w$, for some i.

Case 2. Let $\alpha_w > 0$. This implies $\beta_w > 0$ and indicates that among the edges defining the vector β, there must be at least one edge adjacent to w, say $\{v, w\}$. By Lemma 7.2.2, we have $\mathbb{E}_{C_i} + \mathbb{E}_{C'_i} + \mathbf{e}_v + \mathbf{e}_w \in S_{\mathbf{G}'}$ if the vertex v belongs to $\big(V(C_i) \cup V(C'_i)\big) \cup N_{\mathbf{G}'}\big(V(C_i) \cup V(C'_i)\big)$. Let us consider that $v \notin \big(V(C_i) \cup V(C'_i)\big) \cup N_{\mathbf{G}'}\big(V(C_i) \cup V(C'_i)\big)$, and therefore $\mathbb{E}_{C_i} + \mathbb{E}_{C'_i} + \mathbf{e}_v + \mathbf{e}_w \in \overline{S_{\mathbf{G}'}} \backslash S_{\mathbf{G}'}$. By observing the structure of the fundamental sets, the existence of edge $\{v, w\}$ in the formation of β implies that β belongs to some facets corresponding to the fundamental sets with building blocks of type (ii). Therefore, we see that $\{v, w\}$ is contained in some facet F_T corresponding to the fundamental set T of $\mathbf{G}'$ such that $v \in T$, $w \in N_{\mathbf{G}'}(T)$, and $\big(\{v\} \cup N_{\mathbf{G}'}(\{v\})\big) \cap \big(V(C_i) \cup V(C'_i)\big) = \emptyset$. Hence, any $\mathbb{E}_{C_i} + \mathbb{E}_{C'_i} + \mathbf{e}_v + \mathbf{e}_w \in \overline{S_{\mathbf{G}'}} \backslash S_{\mathbf{G}'}$ can be expressed as $q_i + F_T$ for some $T \in \mathfrak{T}$.

In both cases, we get the desired containment. □

7.2.2 Type 2: w is not regular in G

Let $\widetilde{\mathbf{G}}$ be our second type of triangular cactus $\mathbf{G}$ such that there exists at least one 3-cycle in $\widetilde{\mathbf{G}}$ containing w, and the remaining vertices of this cycle do not have any other 3-cycles attached to them. Moreover, $\operatorname{diam} \widetilde{\mathbf{G}} = 4$.

As earlier, we denote the vertices x_i with no 3-cycles $\{x_i, x_i^{k-1}, x_i^k\}$ attached to them as ζ_i and let there be l such ζ_i in $\widetilde{\mathbf{G}}$. Note that as per the description of $\widetilde{\mathbf{G}}$, there exists at least a pair of vertices (ζ_i, ζ_j) in $\widetilde{\mathbf{G}}$ such that $\{\zeta_i, \zeta_j\} \in E(\widetilde{\mathbf{G}})$. Since $\operatorname{diam} \widetilde{\mathbf{G}} = 4$, the number of vertices $\zeta_i \in V(\widetilde{\mathbf{G}})$ is at most $2(n-1)$. Therefore we have, $2 \leq l \leq 2(n-1)$. A generic diagram of $\widetilde{\mathbf{G}}$ is shown in Figure 7.5.

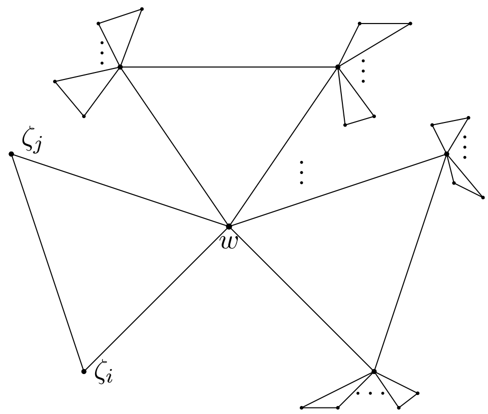

Figure 7.5: An illustration of graph $\widetilde{\mathbf{G}}$

The set of cutpoints of $\widetilde{\mathbf{G}}$ is $\{w\} \cup \left(\{x_i \colon 1 \leq i \leq 2n\} \setminus \bigcup_{i=1}^{l} \{\zeta_i\}\right)$. Since there exists at least one $\{\zeta_i, \zeta_j\} \in E(\widetilde{\mathbf{G}})$, we observe that w is not regular in $\widetilde{\mathbf{G}}$. Note that, the only regular vertices in $\widetilde{\mathbf{G}}$ are the non-cutpoints of $\widetilde{\mathbf{G}}$.

For $i \in [2n]$, let s_i number of 3-cycles, not containing the vertex w, be attached to $x_i \in V(\widetilde{\mathbf{G}})$. Then $\{w\} \cup \bigcup_{1 \leq i \leq 2n} \{x_i^{k_1}, x_i^{k_2}, \ldots, x_i^{k_{s_i}}\}$, where $k_u \in [2s_i]$ and none of the vertices $x_i^{k_u}$ adjacent to each other, is the only fundamental set in $\widetilde{\mathbf{G}}$ that contains the vertex w.

Let us consider that there are m pair of vertices $(\zeta_p, \zeta_{p'})$ in $\widetilde{\mathbf{G}}$ such that $\{\zeta_p, \zeta_{p'}\} \in E(\widetilde{\mathbf{G}})$. For such a pair of vertices $(\zeta_p, \zeta_{p'})$, we define ω_p as $\omega_p \in \{\zeta_p, \zeta_{p'}\}$, $1 \leq p \leq m$. In $\widetilde{\mathbf{G}}$, and for any $i, j \in [2n]$, the building blocks of any fundamental set not containing w, are as follows.

(a) The set $\bigcup_{p=1}^{m} \{\omega_p\}$ is fundamental in $\widetilde{\mathbf{G}}$.

(b) Suppose that s number of 3-cycles that do not contain the vertex w, are attached to x_i. Then the set $\{x_i^{k_1}, x_i^{k_2}, \ldots, x_i^{k_s}\}$, where $k_u \in [2s]$ such that none of the vertices $x_i^{k_u}$ adjacent to each other, is a fundamental set in $\widetilde{\mathbf{G}}$.

(c) Let $\{x_i, x_j\} \in E(\widetilde{\mathbf{G}})$ and t number of 3-cycles that do not contain w be attached to x_j. Then, $\bigcup_{p=1}^{m}\{\omega_p\} \cup \{x_i, x_j^{k_1}, x_j^{k_2}, \ldots, x_j^{k_t}\}$ with none of the vertices $x_j^{k_v}$, $k_v \in [2t]$, adjacent to each other, is a fundamental set in $\widetilde{\mathbf{G}}$.

Any fundamental set T in $\widetilde{\mathbf{G}}$ with $w \notin T$ is either one of the listed building blocks or can be expressed as their union. For any fundamental set T consisting of the building blocks in (a) or (c), we observe that $w \in N_{\widetilde{\mathbf{G}}}(T)$.

Proposition 7.2.4. *For the triangular cactus $\widetilde{\mathbf{G}}$, let $\mathfrak{T}$ be the set of all fundamental sets T in $\widetilde{\mathbf{G}}$ such that $w \in N_{\widetilde{\mathbf{G}}}(T)$ and $\big(T \cup N_{\widetilde{\mathbf{G}}}(T)\big) \cap \big(V(C_i) \cup V(C_i')\big) = \emptyset$, for any exceptional pair (C_i, C_i'). Then, we have*

$$\overline{S_{\widetilde{\mathbf{G}}}} \backslash S_{\widetilde{\mathbf{G}}} = \bigcup_{T \in \mathfrak{T}} (q_i + F_T), \tag{7.2}$$

where the index i is taken over all possible exceptional pair (C_i, C_i') of $\widetilde{\mathbf{G}}$, for which we define $q_i = \sum_i (\mathbb{E}_{C_i} + \mathbb{E}_{C_i'}) \notin S_{\widetilde{\mathbf{G}}}$.

Proof. For our graph $\widetilde{\mathbf{G}}$, we do not have any facet F_w since the vertex w is not regular. Therefore note that, for any $\beta \in S_{\widetilde{\mathbf{G}}}$, we have $\beta_w > 0$. Further, we can proceed with a similar proof as that of Proposition 7.2.3 and prove (7.2). □

We combine all the above results and proceed to the proof of our main theorem.

Proof of Theorem 7.1.2. Since $\operatorname{diam} \mathbf{G} = 4$, there exists at least one pair of odd cycles of the form $(\{x_p, x_p^{k-1}, x_p^{k}\},\ \{x_q, x_q^{k'-1}, x_q^{k'}\})$ in $\mathbf{G}$ such that the distance between x_p and x_q is 2. All such pairs of odd cycles are exceptional in $\mathbf{G}$. Therefore, the graph $\mathbf{G}$ does not satisfy the odd cycle condition and by Theorem 3.4.5, we conclude that the edge ring $\Bbbk[\mathbf{G}]$ is non-normal irrespective of the two types of $\mathbf{G}$.

In Sections 7.2.1 and 7.2.2, we explored the two types of graph $\mathbf{G}$. Let us assume that both $\mathbf{G}'$ and $\widetilde{\mathbf{G}}$ are triangular cacti on d vertices. We know that for a graph $\mathbf{G}$ on d vertices, all the facets F_T and F_v correspond to hyperplanes $\mathcal{H}_T$ and $\mathcal{H}_v$ respectively and therefore are of dimension $d-1$. Now, we compare (7.1) and (7.2) with the description of holes given in (2.1) and observe that, all $q_i + F_T$ and $q_i + F_w$ in both (7.1) and (7.2) correspond to the holes of dimension $d-1$. Hence by Theorem 2.2.2, we say that the edge ring $\Bbbk[\mathbf{G}]$ always satisfies (S_2)-condition. □

7.3 Conclusions

We have seen that the associated edge ring of a triangular cactus of diameter 4 is always non-normal and satisfies (S_2)-condition. Moreover, our primary theorem, Theorem 7.1.2 provides supporting pieces of evidence to Conjecture 7.1.1.

Remark 7.3.1. By further automatic computations using `Macaulay2`, we believe that every non-normal edge ring of triangular cacti satisfying (S_2)-condition is always Cohen–Macaulay. To the best of our knowledge, no triangular cactus has been identified whose edge ring does not satisfy (S_2)-condition. Hence, we expect that the edge ring of every triangular cactus is Cohen–Macaulay.

Part IV

Some graph-theoretical approach

Chapter 8

On super edge-magic total strength of some unicyclic graphs

Here, we work on some unicyclic graphs and provide shreds of evidence to conjecture that the super edge-magic total strength of a certain family of unicyclic graphs consisting of odd cycle of length n and having m edges is equal to $2m + \frac{n+3}{2}$. This chapter and all of its contents are part of the author's ongoing research [40] on super edge-magic total strength.

8.1 Introduction to the main conjecture

A graph labeling is an assignment of integers to either the vertices or edges, or both, subject to certain conditions. In this chapter, the domain will usually be the set of all vertices and edges; such labelings are called *total labelings*. A useful survey on graph labeling can be found in [14].

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. Let us assume that $|V(G)| = p$ and $|E(G)| = q$, then G is called a *(p, q)-graph*. Let $uv := \{u, v\}$, for any edge $\{u, v\} \in E(G)$. For any undefined graph theoretical terms and notations in this chapter, one may refer to [5] or [6].

An *edge-magic total labeling* of a (p, q)-graph G is a bijection

$$f\colon V(G) \cup E(G) \longrightarrow [p + q]$$

such that for all edges $uv \in E(G)$, $f(u) + f(uv) + f(v) = c(f)$, where $c(f)$ is a constant and is called the *magic constant* of f. A graph is said to be *edge-magic total* if it has an edge-magic total labeling.

The concept of *edge-magic total strength* of a graph G was introduced in [1] as the smallest magic constant over all edge-magic total labelings of G. Let us denote it by $em(G)$. Therefore, for an edge-magic total labeling f of G with magic constant $c(f)$,

$$em(G) = \min\big\{c(f)\colon f \text{ be an edge-magic total labeling of } G\big\}.$$

In this study, we explore an edge-magic total labeling f of the (p, q)-graph G, such that $f(V(G)) = [p]$. This type of labeling is defined as *super edge-magic total labeling.* If a graph G has a super edge-magic total labeling, then G is called a *super edge-magic total* graph.

A necessary and sufficient condition for a graph to be a super edge-magic total graph is stated in the following lemma.

Lemma 8.1.1 ([11, Lemma 1])**.** *A (p, q)-graph G is a super edge-magic total graph if and only if there exists a bijection $f\colon V(G) \longrightarrow [p]$, such that the set $\{f(u) + f(v)\colon uv \in E(G)\}$ is consecutive. And, f extends to a super edge-magic total labeling with magic constant $c(f) = p + q + \min\{f(u) + f(v)\colon uv \in E(G)\}$.*

Remark 8.1.2. In the above lemma, the vertex labeling f can be extended to a super edge-magic total labeling of G by defining

$$f(uv) = p + q + \min\big\{f(\tilde{u}) + f(\tilde{v})\colon \tilde{u}\tilde{v} \in E(G)\big\} - f(u) - f(v),$$

for every edge $uv \in E(G)$.

For any regular super edge-magic total graph, we have the following result.

Lemma 8.1.3 ([11, Lemma 4])**.** *Let G be an r-regular (p, q)-graph, where $r > 0$. Let f be any super edge-magic total labeling of G. Then q is odd and $c(f) = \frac{4p+q+3}{2}$, for all super edge-magic total labeling f.*

By using Lemma 8.1.3, we can derive that for an odd cycle C of length n, and any super edge-magic total labeling f of C,

$$\begin{aligned} c(f) = 2n + \min\{f(u) + f(v)\colon uv \in E(C)\} = 2n + \frac{n+3}{2} \\ \Longrightarrow \min\{f(u) + f(v)\colon uv \in E(C)\} = \frac{n+3}{2}. \end{aligned}$$

The article [9] illustrates that through the adoption of a super edge-magic total labeling for a graph, it is possible to introduce additional vertices and edges to the existing graph in a manner that ensures the resulting constructed graph retains its super edge-magic total properties.

Theorem 8.1.4 ([9, Theorem 2.4]). *Let G_p be a connected super edge-magic total (p, q)-graph with $p \geq 3$. Let f be a super edge-magic total labeling of G_p and let us consider $F_f(G_p) = \{f(u)+f(v) : uv \in E(G_p)\}$. Let $\max(F_f(G_p)) = p + t$, and for some $a \in V(G_p)$, $f(a) = t$. We construct a graph $\widetilde{G_p}$, by taking each copy of G_p and mK_1, $m \geq 1$ and connecting all the vertices of mK_1 to the vertex $a \in V(G_p)$. Then, $\widetilde{G_p}$ is also a super edge-magic total graph.*

Avadayappan–Jeyanthi–Vasuki [2] defined the *super edge-magic total strength* of a graph G as the minimum magic constant over all the super edge-magic total labelings of G, let us denote it as $sm(G)$. That is, we have

$$sm(G) = \min\big\{c(f) : f \text{ is a super edge-magic total labeling of } G\big\}.$$

Remark 8.1.5. Let G be a super edge-magic total (p, q)-graph. For any $v \in V(G)$, the number of edges adjacent to vertex v is called the *degree* of v, denoted by $\deg(v)$. The vertices that have degree 1 are called the *pendant vertices*. Let f be a super edge-magic total labeling of G, with the magic constant $c(f)$. As noted in [1], each edge's magic constants are added together to produce the following result:

$$qc(f) = \sum_{v \in V(G)} \deg(v) f(v) + \sum_{e \in E(G)} f(e). \tag{8.1}$$

Furthermore, since $\mathrm{Im}(f) = [p + q]$ and $f(V(G)) = [p]$, we can derive that $p + q + 3 \leq sm(G) \leq 3p$.

In this chapter, our attention is directed towards the family of unicyclic graphs, comprising an odd cycle C (of length n) and k_i pendant vertices attached to each vertex $i \in V(C)$.

Let us consider $G(n; k_1, \ldots, k_n)$ to be the unicyclic (p, q)-graph consisting of an odd cycle $C = \{a_1, \ldots, a_n\}$ and k_i number of pendant vertices adjacent to each of the vertex a_i, $1 \leq i \leq n$. Swaminathan and Jeyanthi [46] established a range for the super edge-magic total strength of this family of unicyclic graphs.

Theorem 8.1.6 ([46, Theorem 4]). *The family of unicyclic (p, q)-graph $G(n; k_1, \ldots, k_n)$, where $n = 2s + 1$, is a super edge-magic total graph and*

$$\begin{aligned}
&2q + 2 + \frac{1}{q}\left(m_2 + 2m_3 + \cdots + (n-1)m_n + \frac{n(n-1)}{2}\right) \\
&\leq sm\big(G(n; k_1, \ldots, k_n)\big) \\
&\leq 2(k_1 + k_3 + \cdots + k_{2s+1}) + 3(k_2 + k_4 + \cdots + k_{2s}) + 2n + s + 2,
\end{aligned}$$

where $m_1 \geq m_2 \geq \cdots \geq m_n$ *are integers such that* $\{m_1, \ldots, m_n\} = \{k_1, \ldots, k_n\}$.

Corollary 8.1.7 ([46, Corollary 4.1])**.** *For any unicyclic graph of the form* $G(n; k_1, \ldots, k_n)$ *with* $k_i = k$ *for any* $1 \leq i \leq n$*,*

$$sm(G(n; k, \ldots, k)) = 2n(k+1) + \frac{n+3}{2}.$$

The prime focus of this study is to provide supporting pieces of evidence for the following conjecture.

Conjecture 8.1.8. Let $G(n; k_1, \ldots, k_n)$ be the super edge-magic total unicyclic (p, q)-graph consisting of an odd cycle $C = \{a_1, \ldots, a_n\}$ and k_i number of pendant vertices adjacent to each a_i, $1 \leq i \leq n$. Then,

$$sm(G(n; k_1, \ldots, k_n)) = 2q + \frac{n+3}{2}.$$

Within this chapter, we conduct a detailed investigation of three distinct graphs within the family of unicyclic graphs $G(n; k_1, \ldots, k_n)$. Our exploration yields significant support for Conjecture 8.1.8.

8.2 Unicyclic graph $G_{n,k,c}$

Let $G_{n,k,c} := G(n; k, \ldots, k, k+c)$, where $1 \leq c < \frac{2n(k+1)}{n-3}$. That is, we consider that $G_{n,k,c}$ is the unicyclic graph consisting of an odd cycle $C = \{a_1, \ldots, a_n\}$, with k number of pendant vertices adjacent to each of the vertices a_i, $1 \leq i \leq n-1$ and $k+c$ number of pendant vertices adjacent to vertex a_n. For illustration, see Figure 8.1. The number of vertices and edges of the graph $G_{n,k,c}$ is $p = q = n(k+1) + c$. Let the vertex set $V(G_{n,k,c})$ be

$$V(C) \cup \{a_{i,j} \colon 1 \leq i \leq n-1, 1 \leq j \leq k\} \cup \{a_{n,j} \colon 1 \leq j \leq k+c\}$$

and let the edge set $E(G_{n,k,c})$ be

$$E(C) \cup \{a_i a_{i,j} \colon 1 \leq i \leq n-1, 1 \leq j \leq k\} \cup \{a_n a_{n,j} \colon 1 \leq j \leq k+c\}.$$

Theorem 8.2.1. *The unicyclic graph* $G_{n,k,c}$ *is a super edge-magic total graph with super edge-magic total strength given by*

$$sm(G_{n,k,c}) = 2n(k+1) + 2c + \frac{n+3}{2}.$$

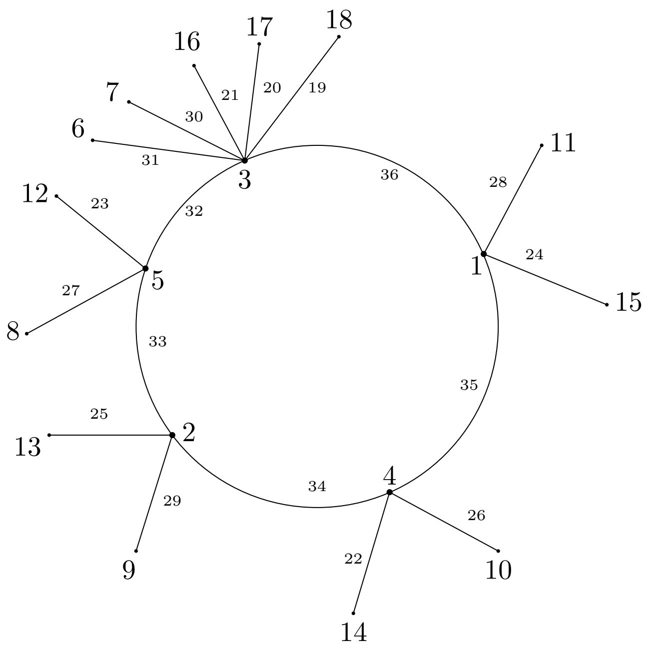

Figure 8.1: The graph $G_{5,2,3}$

Proof. By Theorem 8.1.6, the graph $G_{n,k,c} := G(n; k, \ldots, k, k + c)$, $1 \leq c < \frac{2n(k+1)}{n-3}$, is super edge-magic total. From (8.1), for any super edge-magic total labeling f of $G_{n,k,c}$, we have

$$\begin{aligned} qc(f) &= \sum_{v \in V(G_{n,k,c})} \deg(v) f(v) + \sum_{e \in E(G_{n,k,c})} f(e) \\ &= q(2q+1) + (k+1) \sum_{a_i \in V(C_n)} f(a_i) + c f(a_n). \end{aligned}$$

By assigning smaller labels to vertices with higher degrees, we get the least possible value of $c(f)$, i.e., $c(f) \geq 2q + 1 + \frac{(k+1)n(n+1)}{2q} + \frac{c}{q}$. Hence, we have

$$sm(G_{n,k,c}) \geq 2q + 1 + \frac{n(k+1)(n+1)}{2q} + \frac{c}{q}.$$

Since the super edge-magic total strength $sm(G_{n,k,c})$ is an integer, we consider the integer part of $\frac{n(k+1)(n+1)}{2q} + \frac{c}{q}$. We have

$$\frac{n+1}{2} - \left(\frac{n(k+1)(n+1)}{2q} + \frac{c}{q} \right) = \frac{(n-1)c}{2(n(k+1)+c)}.$$

Since $1 \leq c < \frac{2n(k+1)}{n-3}$, we have $(n-1)c < 2(n(k+1)+c)$. Therefore, $0 < \frac{(n-1)c}{2(n(k+1)+c)} < 1$. Hence,

$$\begin{aligned} sm(G_{n,k,c}) &\geq 2q + 1 + \frac{(n+1)}{2} \\ &= 2n(k+1) + 2c + \frac{n+3}{2}. \end{aligned}$$

That is,

$$sm(G_{n,k,c}) \geq 2n(k+1) + 2c + \frac{n+3}{2}. \tag{8.2}$$

By Theorem 8.1.4 and Theorem 8.1.6, the graph $G_{n,k,c}$ can be considered as the super edge-magic total graph constructed from the graph $G(n; k, \ldots, k)$ with super edge-magic total labeling f' of $G(n; k, \ldots, k)$ defined as follows.

For $1 \leq i \leq n$,

$$f'(a_i) = \begin{cases} \dfrac{i+1}{2} & \text{if } i \text{ is odd,} \\ \\ \dfrac{n+i+1}{2} & \text{if } i \text{ is even.} \end{cases}$$

For $1 \leq i \leq n-1;\ 1 \leq j \leq k$,

$$f'(a_{i,j}) = n(k+1) - (n-1)(j-1) - (i-1).$$

And, $f'(a_{n,k}) = n + j,\ 1 \leq j \leq k$.

Now, we consider a vertex labeling $f\colon V(G_{n,k,c}) \longrightarrow [p]$ as follows:

$$f(v) = \begin{cases} f'(v) & \text{if } v \in V(G(n; k, \ldots, k)), \\ \\ n(k+1) + j - k & \text{if } v = a_{n,j},\ \text{for } k+1 \leq j \leq k+c. \end{cases} \tag{8.3}$$

As per the labeling defined in (8.3), for any $uv \in E(G_{n,k,c})$ we observe the following.

- If $u, v \in V(G_{n,k,c})$, since f' is a super edge-magic total labeling of the graph $G(n; k, \ldots, k)$, then $\{f(u) + f(v)\} = \{f'(u) + f'(v)\}$ is a consecutive sequence with highest element $n(k+1) + \frac{n+1}{2}$.
- If $u = a_n$ and $v = a_{n,j}$, $k+1 \leq j \leq k+c$, then we observe that $\{f(u) + f(v)\} = \left\{\frac{n+1}{2} + n(k+1) + 1, \ldots, \frac{n+1}{2} + n(k+1) + c\right\}$ is a consecutive sequence.

Therefore, we see that $\{f(u)+f(v)\colon uv \in E(G_{n,k,c})\}$ is a consecutive sequence and $\min\{f(u)+f(v)\colon uv \in E(G_{n,k,c})\} = \frac{n+3}{2}$.

Thus by Lemma 8.1.1, the vertex labeling f extends to a super edge-magic total labeling of $G_{n,k,c}$ with $c(f) = 2n(k+1) + 2c + \frac{n+3}{2}$. Hence,

$$sm(G_{n,k,c}) \leq 2n(k+1) + 2c + \frac{n+3}{2}. \tag{8.4}$$

From (8.2) and (8.4), we have $sm(G_{n,k,c}) = 2n(k+1) + 2c + \frac{n+3}{2}$. □

Example 8.2.2. Super edge-magic total labeling of the graph $G_{5,2,3}$ with strength $sm(G_{5,2,3}) = 40$, is illustrated in Figure 8.1.

Example 8.2.3. Super edge-magic total labeling of graph $G_{9,3,4}$ with $sm(G_{9,3,4}) = 86$ is illustrated in Figure 8.2.

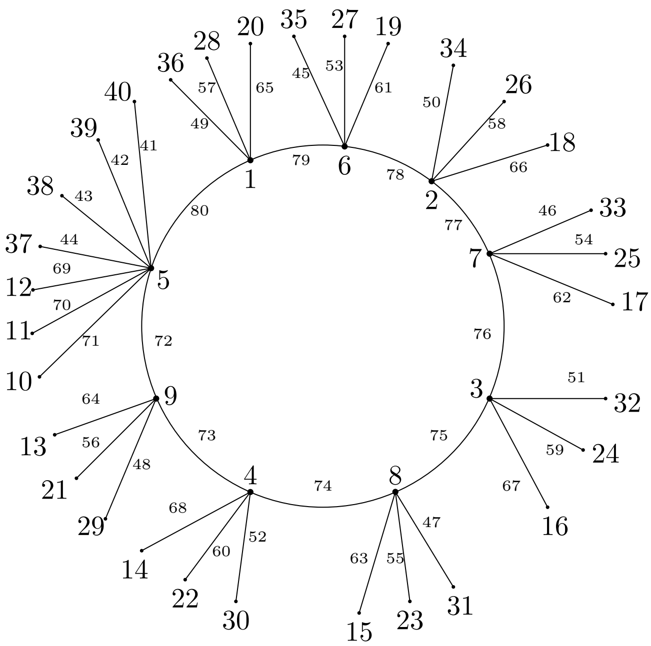

Figure 8.2: The graph $G_{9,3,4}$

8.3 Unicyclic graph $G_{n,k,-c}$

Let us consider the unicyclic graph $G_{n,k,-c} := G(n; k, \ldots, k, k-c)$, $1 \leq c \leq k$. For example, see Figure 8.3.

For $G_{n,k,-c}$, the number of vertices and edges are $p = q = n(k+1)-c$. Let the vertex set $V(G_{n,k,-c})$ be

$$V(C) \cup \{a_{i,j} \colon 1 \le i \le n-1, 1 \le j \le k\} \cup \{a_{n,j} \colon 1 \le j \le k-c\}$$

and the edge set $E(G_{n,k,-c})$ be equal to

$$E(C) \cup \{a_i a_{i,j} \colon 1 \le i \le n-1, 1 \le j \le k\} \cup \{a_n a_{n,j} \colon 1 \le j \le k-c\},$$

where $1 \le c \le k$.

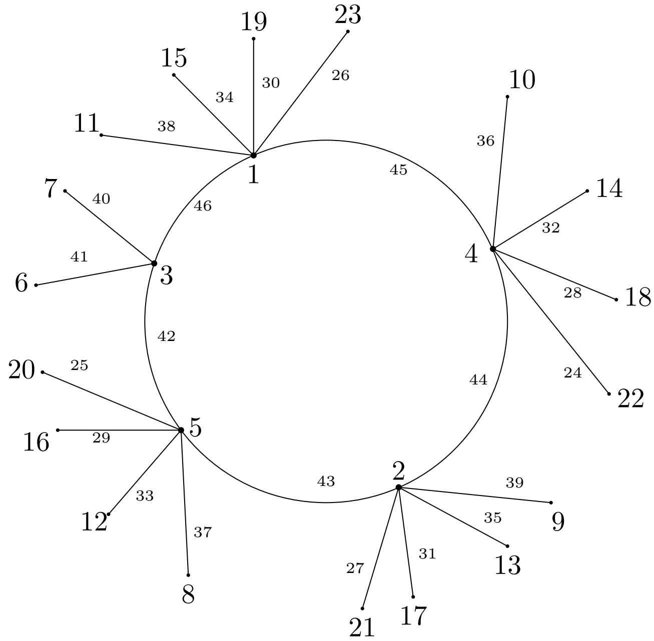

Figure 8.3: The graph $G_{5,4,-2}$

Theorem 8.3.1. *The unicyclic graph $G_{n,k,-c}$ is a super edge-magic total graph with super edge-magic total strength*

$$sm(G_{n,k,-c}) = 2n(k+1) - 2c + \frac{n+3}{2}.$$

Proof. By Theorem 8.1.6, the graph $G_{n,k,-c}$ is super edge-magic total and the lower bound of its super edge-magic total strength is:

$$sm(G_{n,k,-c}) \ge 2q + 2 + \frac{1}{q}\left(\frac{k(n-1)}{2} + \frac{n(n-1)}{2} - c(n-1)\right).$$

Hence we have

$$
\begin{aligned}
&sm(G_{n,k,-c}) \\
&\geq 2q + 2 + \frac{1}{q}\left(\frac{k(n-1)}{2} + \frac{n(n-1)}{2} - c(n-1)\right) \\
&= 2n(k+1) - 2c + 2 + \frac{1}{n(k+1)-c}\left(\frac{n(n-1)(k+1)}{2} - c(n-1)\right) \\
&= 2n(k+1) - 2c + 2 + \frac{n-1}{2}\left(\frac{n(k+1)-2c}{n(k+1)-c}\right).
\end{aligned}
$$

Since $sm(G_{n,k,-c})$ is an integer, we will consider the integer part of $\frac{n-1}{2}\left(\frac{n(k+1)-2c}{n(k+1)-c}\right)$. We have $\frac{n-1}{2} - \frac{n-1}{2}\left(\frac{n(k+1)-2c}{n(k+1)-c}\right) = \frac{(n-1)c}{2(n(k+1)-c)}$. Since $c \leq k$, we observe that $(n-1)c < 2(n(k+1)-c)$. Therefore,

$$
0 < \frac{n-1}{2} - \frac{n-1}{2}\left(\frac{n(k+1)-2c}{n(k+1)-c}\right) < 1.
$$

Hence for $G_{n,k,-c}$, we have

$$
\begin{aligned}
sm(G_{n,k,-c}) &\geq 2n(k+1) - 2c + 2 + \frac{n-1}{2} \\
&= 2n(k+1) - 2c + \frac{n+3}{2}.
\end{aligned}
$$

That is,

$$
sm(G_{n,k,-c}) \geq 2n(k+1) - 2c + \frac{n+3}{2}. \tag{8.5}
$$

Now, we define a vertex labeling $f\colon V(G_{n,k,-c}) \longrightarrow [p]$ as follows:

For $1 \leq i \leq n$,

$$
\begin{aligned}
&f(a_i) = \begin{cases} \dfrac{i+1}{2} & \text{if } i \text{ is odd,} \\[2ex] \dfrac{n+i+1}{2} & \text{if } i \text{ is even.} \end{cases} \\
&f(a_{i,j}) = n(k+2) - c - (n-1)j - i,\ 1 \leq i \leq n-1,\ 1 \leq j \leq k. \\
&f(a_{n,j}) = n + j,\ 1 \leq j \leq k - c.
\end{aligned} \tag{8.6}
$$

As per the labeling defined in (8.6), for any $uv \in E(G_{n,k,-c})$ we observe the following.

- If $u, v \in V(C_n)$ then, $\{f(u) + f(v)\} = \left\{1 + \frac{n+1}{2}, \ldots, n + \frac{n+1}{2}\right\}$ is a consecutive sequence.

- If $u = a_n$ and $v = a_{n,j}$, $1 \leq j \leq k-c$, then we have $\{f(u)+f(v)\} = \left\{n + \frac{n+3}{2}, \ldots, n + k - c + \frac{n+1}{2}\right\}$, a consecutive sequence.
- If $u = a_i$ and $v = a_{i,j}$, for $1 \leq i \leq n-1$; $1 \leq j \leq k$, then we have $\{f(u) + f(v)\} = \left\{n + k - c + \frac{n+3}{2}, \ldots, n(k+1) - c + 2\right\}$, which is a consecutive sequence.

Thus we observe that $\{f(u) + f(v)\colon uv \in E(G_{n,k,-c})\}$ is a consecutive sequence with $\min\{f(u) + f(v)\colon uv \in E(G_{n,k,-c})\} = \frac{n+3}{2}$. Therefore by Lemma 8.1.1, the vertex labeling f extends to a super edge-magic total labeling of $G_{n,k,-c}$ with a magic constant $c(f) = 2n(k+1) - 2c + \frac{n+3}{2}$. Hence,

$$sm(G_{n,k,-c}) \leq 2n(k+1) - 2c + \frac{n+3}{2}. \tag{8.7}$$

From (8.5) and (8.7), we have $sm(G_{n,k,-c}) = 2n(k+1) - 2c + \frac{n+3}{2}$. □

Example 8.3.2. Super edge-magic total labeling of the graph $G_{5,4,-2}$ with super edge-magic total strength $sm(G_{5,4,-2}) = 50$, is illustrated in Figure 8.3.

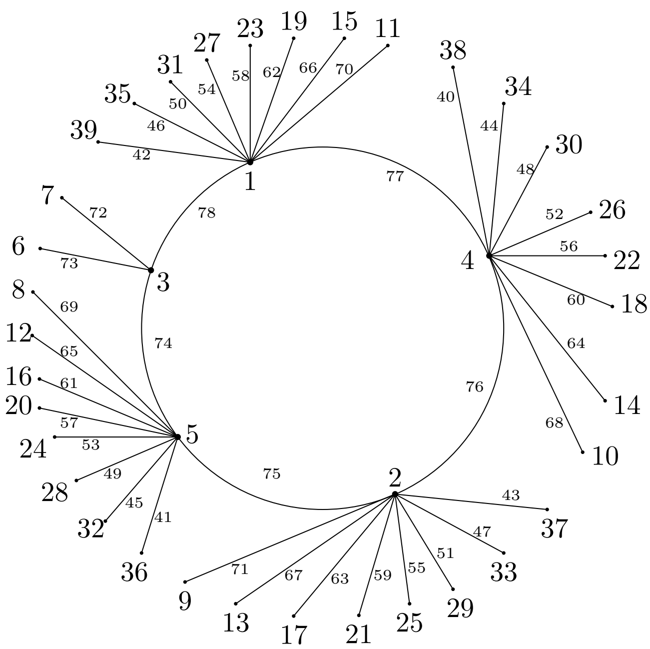

Figure 8.4: The graph $G_{5,8,-6}$

Example 8.3.3. Super edge-magic total labeling of the graph $G_{5,8,-6}$ with super edge-magic total strength $sm(G_{5,8,-6}) = 82$, is illustrated in Figure 8.4.

8.4 Unicyclic graph $G(n; k, r)$

Let $G(n; k, r)$ be the unicyclic graph $G(n, k_1, \ldots, k_n)$ with $k_i = k$, if $i \neq r, n-r$ and $k_r = k_{n-r} = k+1$ for any odd number r, $1 \leq r < n$. For an illustration, see Figure 8.5.

Let $p = q = n(k+1) + 2$, be the number of vertices and edges of $G(n; k, r)$. Let the vertex set $V(G(n; k, r))$ be

$$V(C) \cup \{a_{i,j} : 1 \leq i \leq n, 1 \leq j \leq k\} \cup \{a_{r,k+1}, a_{n-r,k+1}\},$$

and the edge set $E(G(n; k, r))$ be

$$E(C) \cup \{a_i a_{i,j} : 1 \leq i \leq n, 1 \leq j \leq k\} \cup \{a_i a_{i,k+1} : i \in \{r, n-r\}\}.$$

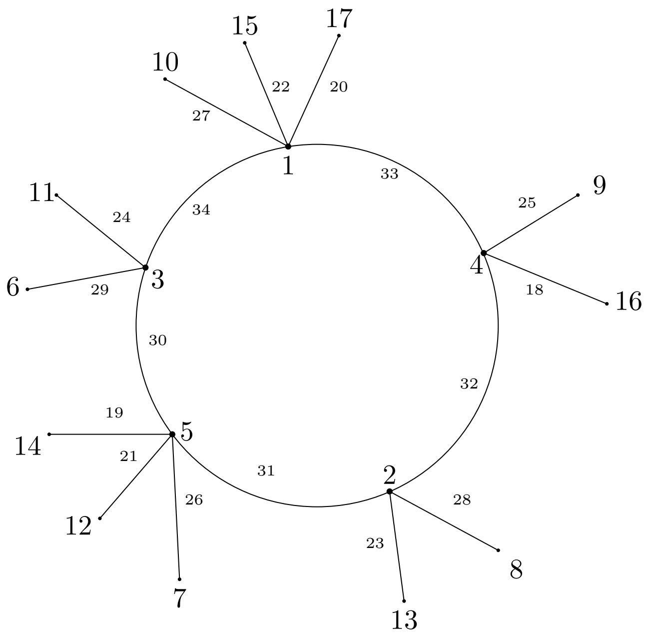

Figure 8.5: The graph $G(5; 2, 1)$

Theorem 8.4.1. *The unicyclic graph $G(n; k, r)$, where r is any odd number such that $1 \leq r < n$, admits a super edge-magic total labeling and has a super edge-magic total strength*

$$sm(G(n; k, r)) = 2n(k+1) + 4 + \frac{n+3}{2}.$$

Proof. By Theorem 8.1.6, the unicyclic graph $G(n;k,r)$ is a super edge-magic total graph with

$$\begin{aligned} &sm(G(n;k,r)) \\ &\geq 2q+2+\frac{1}{q}\left((k+1)+2k+\cdots+(n-1)k+\frac{n(n-1)}{2}\right) \\ &= 2n(k+1)+6+\frac{1}{n(k+1)+2}\left(\frac{nk(n-1)}{2}+\frac{n(n-1)}{2}+1\right) \\ &= 2n(k+1)+6+\frac{n-1}{2}\left(\frac{n(k+1)}{n(k+1)+2}\right)+\frac{1}{n(k+1)+2}\ . \end{aligned}$$

That is, we have

$$sm(G(n;k,r)) \geq 2n(k+1)+6+\frac{n-1}{2}\left(\frac{n(k+1)}{n(k+1)+2}\right)+\frac{1}{n(k+1)+2}\ .$$

We know that $sm(G(n;k,r))$ is an integer and we see that

$$\begin{aligned} &\frac{n-1}{2}-\left(\frac{n-1}{2}\left(\frac{n(k+1)}{n(k+1)+2}\right)+\frac{1}{n(k+1)+2}\right) \\ &= \frac{n-1}{2}\left(\frac{2}{n(k+1)+2}\right)-\frac{1}{n(k+1)+2} \\ &= \frac{n-2}{n(k+1)+2}\ < 1. \end{aligned}$$

Hence, we observe that the integer part of $\frac{n-1}{2}\left(\frac{n(k+1)}{n(k+1)+2}\right)+\frac{1}{n(k+1)+2}$ is $\frac{n-1}{2}$ and we have

$$sm(G(n;k,r)) \geq 2n(k+1)+6+\frac{n-1}{2} = 2n(k+1)+4+\frac{n+3}{2}.$$

Therefore, we can express

$$sm(G(n;k,r)) \geq 2n(k+1)+4+\frac{n+3}{2}. \tag{8.8}$$

Now, if we prove that there exists a super edge-magic total labeling f of $G(n;k,r)$ with magic constant $c(f)=2n(k+1)+4+\frac{n+3}{2}$, then our proof is complete.

Let us define a vertex labeling $f\colon V(G(n;k,r)) \longrightarrow [p]$ as follows.

For $1 \leq i \leq n$,

$$f(a_i)=\begin{cases} \dfrac{i+1}{2} & \text{if } i \text{ is odd,} \\ \\ \dfrac{n+i+1}{2} & \text{if } i \text{ is even.} \end{cases} \tag{8.9}$$

And,

$$
\begin{aligned}
&f(a_{i,j}) = n(k-j+2) - 2f(a_i) + 2,\ 1 \le i \le n,\ 1 \le j \le k-1,\\
&f(a_{i,k}) = \begin{cases} 2n+2-2f(a_i) & \text{if } 1 \le f(a_i) \le \frac{n+1}{2},\\ n(k+2)+4-2f(a_i) & \text{if } \frac{n+3}{2} \le f(a_i) \le n - f(a_r),\\ n(k+2)+2-2f(a_i) & \text{if } n - f(a_r) + 1 \le f(a_i) \le n, \end{cases}\\
&f(a_{r,k+1}) = n(k+1) + 2,\\
&f(a_{n-r,k+1}) = nk + r + 3.
\end{aligned}
\tag{8.10}
$$

As per the above labeling, for $uv \in E(G(n;k,r))$, we observe that:

- For $u, v \in V(C)$, $\{f(u) + f(v)\} = \{1 + \frac{n+1}{2}, \ldots, n + \frac{n+1}{2}\}$ is a consecutive sequence.
- Let us consider $u = a_i$ and $v = a_{i,j}$, for any $1 \le i \le n$, and $1 \le j \le k-1$. Then the set

  $$\{f(u)+f(v)\} = \{n(k-j+2)-f(a_i)+2\colon 1 \le i \le n,\ 1 \le j \le k-1\}$$

  is a consecutive sequence with minimal element $2n+2$ and maximal element $n(k+1)+1$.
- Let $u = a_i$ and $v = a_{i,k}$, $1 \le i \le n$.
  - If $1 \le f(a_i) \le \frac{n+1}{2}$, then we observe that

    $$\{f(a_i) + f(a_{i,k})\} = \{n + \frac{n+3}{2}, \ldots, 2n+1\},$$

    is a consecutive sequence.
  - If $\frac{n+3}{2} \le f(a_i) \le n - f(a_r)$, then $\{f(a_i) + f(a_{i,k})\}$ is consecutive and equals $\{n(k+1)+4+f(a_r), \ldots, n(k+1)+4+\frac{n-3}{2}\}$.
  - If $n - f(a_r) + 1 \le f(a_i) \le n$, then we see that the set $\{f(a_i) + f(a_{i,k})\} = \{n(k+1)+2, \ldots, n(k+1)+1+f(a_r)\}$, is consecutive.
- For $u = a_r$ and $v = a_{r,k+1}$, $f(u) + f(v) = n(k+1) + 2 + f(a_r)$.
- If $u = a_{n-r}$ and $v = a_{n-r,k+1}$, then $f(u) + f(v) = n(k+1) + 3 + f(a_r)$.

Therefore, we observe that $\{f(u) + f(v)\colon uv \in E(G(n;k,r))\}$ is a consecutive sequence whose minimum element is $\frac{n+3}{2}$.

Hence by Lemma 8.1.1, the vertex labeling f extends to a super edge-magic total labeling of $G(n;k,r)$ with $c(f) = 2n(k+1)+4+\frac{n+3}{2}$. Hence,

$$sm(G(n;k,r)) \leq 2n(k+1)+4+\frac{n+3}{2}. \tag{8.11}$$

From (8.8) and (8.11), we have

$$2n(k+1)+4+\frac{n+3}{2} \leq sm(G(n;k,r)) \leq 2n(k+1)+4+\frac{n+3}{2}.$$

This implies, $sm(G(n;k,r)) = 2n(k+1)+4+\frac{n+3}{2}$. □

Example 8.4.2. A super edge-magic total labeling of $G(5;2,1)$ with super edge-magic total strength $sm(G(5;2,1)) = 38$ is illustrated in Figure 8.5.

Example 8.4.3. Super edge-magic total labeling of $G(5;2,3)$ with super edge-magic total strength $sm(G(5;2,3)) = 38$ is illustrated in Figure 8.6.

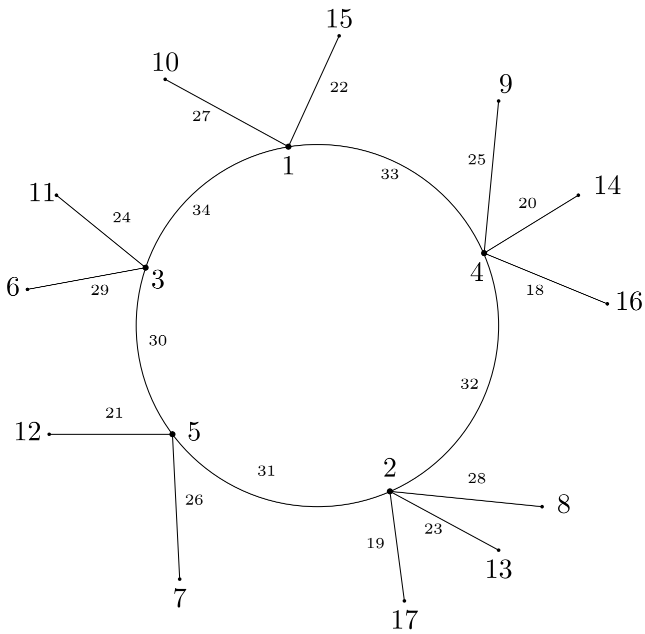

Figure 8.6: The graph $G(5;2,3)$

8.5 Conclusions

In this chapter, we determine the super edge-magic total strength of three variations of $G(n; k_1, \ldots, k_n)$, a certain family of unicyclic (p, q)-graphs. All three of them have super edge-magic total strength equal to $2q + \frac{n+3}{2}$. These results can be considered as the preliminary steps to provide evidence in proving the Conjecture 8.1.8.

Bibliography

[1] S. Avadayappan, P. Jeyanthi, and R. Vasuki, Magic strength of a graph, *Indian J. Pure Appl. Math.* **31** (7) (2000), 873–883.

[2] S. Avadayappan, P. Jeyanthi, and R. Vasuki, Super magic strength of a graph, *Indian J. Pure Appl. Math.* **32** (11) (2001), 1621–1630.

[3] J.B. Babujee and N. Rao, Edge-magic trees, *Indian J. Pure Appl. Math.* **33** (2002), 1837–1840.

[4] K. Bhaskara, A. Higashitani, and N. Shibu Deepthi, h-vectors of edge rings of odd-cycle compositions, preprint, arXiv:2311.13573 (2023).

[5] B. Bollobás, "Graph theory: An introductory course", Graduate Texts in Mathematics **63**, Springer (1979).

[6] J. A. Bondy and U. S. R. Murty, "Graph theory", Graduate Texts in Mathematics **244**, Springer (2008).

[7] A. Borzì and A. D'Alì, Graded algebras with cyclotomic Hilbert series, *J. Pure Appl. Algebra* **225** (2021), 106764, 9 pp.

[8] W. Bruns and J. Herzog, "Cohen–Macaulay rings" Revised edition, Cambridge University Press (1998).

[9] Darmaji, S. Wahyudi, Rinurwati, and S.W. Saputro, On the construction of super edge-magic total graphs, *Electron. J. Graph Theory Appl.* **10** (1) (2022), 301–309.

[10] H. Enomoto, A.S. Llado, T. Nakamigawa, and G. Ringel, Super edge-magic graphs, *SUT J. Math.* **34** (1998), 105–109.

[11] R. M. Figueroa-Centeno, R. Ichishima, and F. A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.* **231** (2001) 153–168.

[12] R. M. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle, Magical coronations of graphs, *Australas. J. Combin.* **26** (2002), 199–208.

[13] R. M. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle, On super edge-magic graphs, *Ars Combin.* **64** (2002), 81–95.

[14] J. A. Gallian, A dynamic survey of graph labeling, Twenty-fifth edition, *Electron. J. Combin.* (2022), #DS6.

[15] S. Goto, R. Takahashi, and N. Taniguchi, Almost Gorenstein rings – towards a theory of higher dimension, *J. Pure Appl. Algebra* **219** (2015), 2666–2712.

[16] S. Goto and K. Watanabe, On Graded rings II ($\mathbb{Z}^n$-graded rings), *Tokyo J. Math.* **1** (1978), 237–261.

[17] J. Herzog and T. Hibi, “Monomial ideals”, Graduate Texts in Mathematics **260**, Springer, New York (2010).

[18] J. Herzog, T. Hibi, and H. Ohsugi, “Binomial ideals”, Graduate Texts in Mathematics **279**, Springer, Cham (2018).

[19] T. Hibi, “Algebraic combinatorics on convex polytopes”, Carslaw Publications (1992).

[20] T. Hibi and L. Katthän, Edge rings satisfying Serre’s condition (R_1), *Proc. Amer. Math. Soc.* **142** (2014), 2537–2541.

[21] A. Higashitani, Almost Gorenstein homogeneous rings and their h-vectors, *J. Algebra* **456** (2016), 190–206.

[22] A. Higashitani and K. Kimura, A necessary condition for an edge ring to satisfy Serre’s condition (S_2), *Advanced Studies in Pure Mathematics* **77** (2018), 121–128.

[23] A. Higashitani and K. Matsushita, Levelness versus almost Gorensteinness of edge rings of complete multipartite graphs, *Comm. Algebra* **50** (6) (2022), 2637–2652.

[24] A. Higashitani and N. Shibu Deepthi, The h-vectors of the edge rings of a special family of graphs, *Comm. Algebra* **51** (12) (2023), 5287–5296.

[25] A. Higashitani and K. Yanagawa, Non-level semi-standard graded Cohen–Macaulay domain with h-vector (h_0, h_1, h_2), *J. Pure Appl. Algebra* **222** (2018), 191–201.

[26] M. Hochster, Rings of invariants of tori, Cohen–Macaulay rings generated by monomials, and polytopes, *Ann. of Math.* **96** (2) (1972), 318–337.

[27] T. Kálmán and A. Postnikov, Root polytopes, Root polytopes, Tutte polynomials, and a duality theorem for bipartite graphs, *Proc. Lond. Math. Soc.* **114** (3) (2017), 561–588.

[28] L. Katthän, Non-normal affine monoid algebras, *Manuscr. Math.* **146** (2015), 223–233.

[29] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970), 451–461.

[30] H. Matsumura, "Commutative ring theory", Cambridge University Press (1986).

[31] N. Matsuoka and S. Murai, Uniformly Cohen–Macaulay simplicial complexes, *J. Algebra* **455** (2016), 14–31.

[32] E. Miller and B. Sturmfels, "Combinatorial commutative algebra", Graduate Texts in Mathematics **227**, Springer (2005).

[33] A.A.G. Ngurah and E. T. Baskoro, On magic and antimagic total labelings of generalized Petersen graph, *Util. Math.* **63** (2003), 97–107.

[34] H. Ohsugi and T. Hibi, Normal polytopes arising from finite graphs, *J. Algebra* **207** (1998), 409–426.

[35] H. Ohsugi and T. Hibi, Compressed polytopes, initial ideals and complete multipartite graphs, *Illinois J. Math.* **44** (2) (2000), 391–406.

[36] G. Ringel and A. Llado, Another tree conjecture, *Bull. ICA* **18** (1996), 83–85.

[37] J. Sadláčk, "Problem 27" in Theory of graphs and its applications (Smolenice, 1963), Publ. House Czechoslovak Acad. Sci., Prague (1964), 163–164.

[38] U. Schäfer and P. Schenzel, Dualizing complexes of affine semigroup rings, *Trans. Amer. Math. Soc.* **322** (2) (1990), 561–582.

[39] N. Shibu Deepthi, Non-normal edge rings satisfying (S_2)-condition, *Acta Math. Vietnam.* (in press).

[40] N. Shibu Deepthi, Super edge-magic total strength of some unicyclic graphs, preprint, arXiv:2212.05329 (2022).

[41] A. Simis, W. V. Vasconcelos, and R. H. Villarreal, The integral closure of subrings associated to graphs, *J. Algebra* **199** (1998), 281–289.

[42] S. Slamin, M. Bača, Y. Lin, M. Miller, and R. Simanjuntak, Edge-magic total labeling of wheels, fans, and friendship graphs, *Bull. ICA*, **35** (2002), 89–98.

[43] R. P. Stanley, Hilbert functions of graded algebras, *Adv. Math.* **28** (1978), 57–83.

[44] R. P. Stanley, "Combinatorics and commutative algebra", Second edition, Birkhäuser (1995).

[45] B. Sturmfels, "Gröbner bases and convex polytopes", University Lecture Series Vol. 8, American Mathematics Society, Providence, RI (1995).

[46] V. Swaminathan and P. Jeyanthi, Super edge-magic strength of fire crackers, banana trees and unicyclic graphs, *Discrete Math.* **306** (2006), 1624–1636.

[47] N. V. Trung and L. T. Hoa, Affine semigroups and Cohen–Macaulay rings generated by monomials, *Trans. Amer. Math. Soc.* **298** (1986), 145–167.

[48] R. H. Villarreal, Normality of subrings generated by square free monomials, *J. Pure Appl. Algebra* **113** (1996), 91–106.

[49] R. H. Villarreal, "Monomial algebras", Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL (2015).

[50] W.D. Wallis, "Magic Graphs", Birkhäuser, Boston (2001).

[51] Z. Wang and D. Lu, The edge rings of compact graphs, preprint, arXiv:2309.07587 (2023).

[52] K. Yanagawa, Castelnuovo's Lemma and h-vectors of Cohen–Macaulay homogeneous domains, *J. Pure Appl. Algebra* **105** (1995), 107–116.