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# THE EQUIVARIANT SPAN OF THE UNIT SPHERES IN REPRESENTATION SPACES

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## 1. Introduction

Let $G$ be a finite group and $M$ be a smooth $G$-manifold. We define $\mathrm{Span}_G(M)$ to be the largest integer $k$ such that $M$ has $k$ linearly independent smooth $G$-vector fields. Let $V$ be an orthogonal $G$-representation space and let $S(V)$ denote the unit sphere in $V$. In the case where $G$ acts freely on $S(V)$, $\mathrm{Span}_G(S(V))$ $(=\mathrm{Span}(S(V)/G))$ has been studied by Becker [6], Iwata [13], Sjerve [23] and Yoshida [29]. In this paper, we consider $\mathrm{Span}_G(S(V))$ when $G$ does not act freely on $S(V)$. Our main results are Theorems 1.1 and 1.2, which are generalizations of Theorems 2.1 and 2.2 in [6] respectively. Our method is due to Becker [6].

Let $H$ be a subgroup of $G$, then we write $H<G$.

**Theorem 1.1.** *Let $G$ be a finite group and let $V$, $W$ be unitary $G$-representation spaces. Suppose that*
(i) $\dim_C V^H = \dim_C W^H$ *for all* $H<G$,
(ii) *For each* $H<G$, $\dim_R V^H \geqq 2k$ *if* $V^H \neq \{0\}$.
*Then* $\mathrm{Span}_G(S(V)) \geqq k-1$ *if and only if* $\mathrm{Span}_G(S(W)) \geqq k-1$.

Let $\xi$ and $\eta$ be orthogonal $G$-vector bundles over a compact $G$-space. Denote by $S(\xi)$ (resp. $S(\eta)$) the unit sphere bundle of $\xi$ (resp. $\eta$). Then $S(\xi)$ and $S(\eta)$ are said to be *G-fiber homotopy equivalent* if there are fiber-preserving $G$-maps:

$$f\colon S(\xi) \to S(\eta),\quad f'\colon S(\eta) \to S(\xi)$$

such that $f\circ f'$ and $f'\circ f$ are fiber-preserving $G$-homotopic to the identity ([6], [19]).

Let $\boldsymbol{R}P^{k-1}$ denote the $(k-1)$-dimensional real projective space with trivial $G$-action and let $\eta_k$ denote the non-trivial line bundle over $\boldsymbol{R}P^{k-1}$ with trivial $G$-action.

**Theorem 1.2.** *Let $G$ be a finite group and let $V$ be an orthogonal $G$-representation space. Then we have the following*:

(i) *Suppose that* $\mathrm{Span}_G(S(V)) \geqq k-1$. *Then there are an integer t and a G-fiber homotopy equivalence*

$$f\colon S((\eta_k \otimes \underline{\underline{V}}) \oplus \underline{\underline{\boldsymbol{R}}}^t) \to S(\underline{\underline{V}} \oplus \underline{\underline{\boldsymbol{R}}}^t).$$

*Moreover we suppose that* $\dim_{\boldsymbol{R}} V^H \geqq k+1$ *if* $V^H \neq \{0\}$ *for each* $H < G$. *Then there is a G-fiber homotopy equivalence*

$$f\colon S(\eta_k \otimes \underline{\underline{V}}) \to S(\underline{\underline{V}}).$$

(ii) *Suppose that* $\dim_{\boldsymbol{R}} V^H \geqq 2k$ *if* $V^H \neq \{0\}$ *for each* $H < G$ *and there is a G-fiber homotopy equivalence*

$$f\colon S(\eta_k \otimes \underline{\underline{V}}) \to S(\underline{\underline{V}}).$$

*Then* $\mathrm{Span}_G(S(V)) \geqq k-1$.

*Here* $\underline{\underline{V}}$ *denotes the trivial G-vector bundle* $\boldsymbol{R}P^{k-1} \times V \to \boldsymbol{R}P^{k-1}$.

Throughout this paper $G$ will be a finite group.

The paper is organized as follows:

In § 2, we discuss some preliminary results. In § 3, we consider equivariant duality, reducibility and coreducibility. In § 4, we consider stunted projective spaces with linear $G$-actions. In §§ 5 and 6, we state an equivariant version of the theorem of James. In § 7, we prove Theorem 1.1. In § 8, we prove Theorem 1.2. In § 9, we give an example.

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## 2. Preliminary results

First we shall fix some notations. Let $X$ and $Y$ be $G$-spaces. Let $A$ be a $G$-subspace of $X$ and let $\alpha\colon A \to Y$ be a $G$-map. Denote by $F((X, A), Y; \alpha)$ the space of all maps $f\colon X \to Y$ such that $f|A = \alpha$ in the compact open topology. $F((X, A), Y; \alpha)$ is a $G$-space with the following $G$-action: if $f\colon X \to Y$ and $g \in G$, we put

$$(g \cdot f)(x) = g(f(g^{-1}x)).$$

For $H < G$, $X^H$ denotes the $H$-fixed point set in $X$. The set $F((X, A), Y; \alpha)^G$ is just the set of $G$-maps $f\colon X \to Y$ such that $f|A = \alpha$. Denote by $[(X, A), Y; \alpha]^G$ the set of $G$-homotopy classes rel $A$ of $G$-maps $f\colon X \to Y$ such that $f|A = \alpha$. If $A = \phi$, we write $F(X, Y)$ (resp. $[X, Y]^G$) instead of $F((X, A), Y; \alpha)$ (resp. $[(X, A), Y; \alpha]^G$), for simplicity. If $X$, $Y$ are $G$-spaces with base points, then we denote the set of $G$-homotopy classes relative to the base points of pointed $G$-maps from $X$ to $Y$ by $[X, Y]_0^G$. The base points are $G$-fixed points as usual. For $H < G$, $(H)$ denotes the conjugacy class of $H$ in $G$. Denote by $G_x$ the isotropy group at $x \in X$ and we put

$$\mathrm{Iso}(X) = \{(G_x) \mid x \in X\} .$$

For a space $Z$, we define $\mathrm{conn}(Z)$ to be the largest integer $n$ such that $Z$ is $n$-connected. In particular, when $Z$ is not path-connected (resp. $Z=\phi$), we put $\mathrm{conn}(Z)=-1$ (resp. $\mathrm{conn}(Z)=\infty$).

The following two lemmas are easily seen by the definition of $G$-complexes (see Bredon [8] and Waner [26]).

**Lemma 2.1.** *Let $f\colon X\to Y$ be a $G$-map of $G$-spaces such that $f^H=f\mid X^H$: $X^H\to Y^H$ is an $n_H$-equivalence for each $H<G$. Let $(K, L)$ be a pair of $G$-complexes and $\alpha\colon L\to X$ be a $G$-map. Then*

$$f_*\colon [(K, L), X; \alpha]^G \to [(K, L), Y; f\circ\alpha]^G$$

*is surjective if* $\dim(K^H-L)\leqq n_H$ *and bijective if* $\dim(K^H-L)\leqq n_H-1$ *for each* $(H)\in\mathrm{Iso}(K-L)$.

**Lemma 2.2.** *Let $(K, L)$ be a pair of $G$-complexes and $X$ be a $G$-space. Let $\alpha\colon L\to X$ be a $G$-map. Then the $G$-fixed point morphism*

$$\phi_G\colon [(K, L), X; \alpha]^G \to [(K^G, L^G), X^G; \alpha^G]$$

*is surjective if* $\dim(K^H-L\cup K^G)\leqq\mathrm{conn}(X^H)+1$ *and bijective if* $\dim(K^H-L\cup K^G)\leqq\mathrm{conn}(X^H)$ *for each* $(H)\in\mathrm{Iso}(K-L\cup K^G)$.

DEFINITION 2.3. Let $X$ be a $G$-space. Then $X$ is said to be *$G$-path-connected* if and only if $\mathrm{conn}(X^H)\geqq 0$ for all $H<G$.

Let $X$ and $Y$ be $G$-spaces. We recall that the join $X*Y$ is the space obtained from the union of $X$, $Y$ and $X\times Y\times[0, 1]$ by identifying

$$(x, y, 0) = x , \quad (x, y, 1) = y \qquad \text{for } x\in X, y\in Y .$$

We generally omit to write in the identification map, so that the image of $(x, y, t)$ in $X*Y$ is denoted by the same expression. A canonical $G$-action on $X*Y$ is given by $g\cdot(x, y, t)=(gx, gy, t)$. Let $V$ be an orthogonal $G$-representation space. We see that

$$(X*Y)^H = X^H*Y^H$$

and

$$\mathrm{conn}((X*S(V))^H) = \mathrm{conn}(X^H)+\dim_{\boldsymbol{R}} V^H$$

for $H<G$. Let $i_{S(V)}\colon S(V)\to X*S(V)$ be an inclusion map defined by $i_{S(V)}(v)=(-, v, 1)$. We have the following theorem (cf. [18; Theorem 3.6], [20]):

**Theorem 2.4.** *Let $K$ be a $G$-complex and $X$ be a $G$-space. Let $V$ be* *an orthogonal G-representation space. Assume that* $\mathrm{conn}(X^H)\geqq 0$ *for each* $(H)\in \mathrm{Iso}(K)$. *Then the suspension map*

$$\tau_*^V\colon [K, X]^G \to [(K*S(V), S(V)), X*S(V); i_{S(V)}]^G$$

*is surjective if* $\dim K^H \leqq n_H$ *and bijective if* $\dim K^H \leqq n_H - 1$ *for each* $(H)\in \mathrm{Iso}(K)$, *where*

$$n_H = \min_{L<H} \begin{cases} 2\,\mathrm{conn}\,(X^H)+1 & \text{if } H=L \text{ and } V^H \neq \{0\}\,, \\ \mathrm{conn}\,(X^L) & \text{if } V^H \neq V^L\,, \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Let $D(V)$ denote the unit disk in $V$. We define a $G$-map

$$\lambda\colon X \to F((D(V), S(V)), X*S(V); i_{S(V)})$$

by $\lambda(x)(tv)=(x, v, t)$ for $x\in X$, $v\in S(V)$, $t\in[0, 1]$. Consider the following commutative diagram:

$$\begin{array}{ccc} & \overset{\tau_*^V}{\nearrow} & [(K*S(V), S(V)), X*S(V); i_{S(V)}]^G \\ [K, X]^G & & \downarrow \varphi \\ & \underset{\lambda_*}{\searrow} & [K, F((D(V), S(V)), X*S(V); i_{S(V)})]^G\,, \end{array}$$

where $\varphi$ is the exponential correspondence given by

$$\varphi(f)(k)(tv) = f(k, v, t) \qquad \text{for } k\in K,\ v\in S(V),\ t\in[0, 1]\,.$$

As is easily seen, $\varphi$ is bijective. Using Lemma 2.2, we see that

$$\lambda^H\colon X^H \to F((D(V), S(V)), X*S(V); i_{S(V)})^H$$

is an $n_H$-equivalence for each $(H)\in \mathrm{Iso}(K)$ by the same argument as in the proof of Theorem 3.6 in [18]. We are now in a position to apply Lemma 2.1.
q.e.d.

## 3. Equivariant duality, reducibility and coreducibility

In this section, we recall the definitions of equivariant duality, reducibility and coreducibility (see [18] and [26]) and consider an equivariant version of Atiyah's duality theorem. Let $X$ and $Y$ be pointed $G$-spaces. The reduced join $X\wedge Y$ has a natural $G$-action induced from the diagonal action on $X\times Y$. For an orthogonal $G$-representation space $V$, $\Sigma^V$ denotes the one-point compactification of $V$ and $\Sigma^V X=\Sigma^V \wedge X$ is called $\Sigma^V$-*suspension* of $X$. We remark that $\Sigma^V$ is a pointed finite $G$-complex ([12]).

DEFINITION 3.1. Let $X$ and $X^*$ be $G$-path-connected pointed finite $G$-

complexes. Let $U$ be an orthogonal $G$-representation space. Then a pointed $G$-map

$$\mu\colon \Sigma^U \to X \wedge X^*$$

is said to be a *(U-)duality G-map* if $\mu^H\colon \Sigma^{U^H} \to X^H \wedge X^{*H}$ is a duality map in the usual sense ([6], [24]) for each $H<G$.

DEFINITION 3.2. Let $X$ be a $G$-path-connected pointed finite $G$-complex and $V$ be an orthogonal $G$-representation space.

(i) A pointed $G$-map $f\colon \Sigma^V \to X$ is said to be a *(V-)reduction G-map* if $f^H\colon \Sigma^{V^H} \to X^H$ is a reduction map in the usual sense ([3]) for each $H<G$, and then $X$ is called *G- (V-)reducible.*

(ii) A pointed $G$-map $f\colon X \to \Sigma^V$ is said to be a *(V-)coreduction G-map* if $f^H\colon X^H \to \Sigma^{V^H}$ is a coreduction map in the usual sense ([3]) for each $H<G$, and then $X$ is called *G- (V-)coreducible.*

Let $M$ be a path-connected closed smooth manifold with trivial $G$-action. Let $\xi$ be a smooth $G$-vector bundle over $M$. The fibers $\xi_x$ for $x \in M$ are orthogonal $G$-representation spaces. Since $M$ is path-connected, $\xi_x$ does not depend on the choice of $x \in M$. So we put $V=\xi_x$. Assume that $V^G \neq \{0\}$. Then $T(\xi)$ is a $G$-path-connected pointed finite $G$-complex ([12]), where $T(\xi)$ denotes the Thom space of $\xi$.

**Proposition 3.3.** *If $T(\xi)$ is G-V-coreducible, then there is a G-fiber homotopy equivalence $f\colon S(\xi \oplus \underline{\boldsymbol{R}}^1) \to S(\underline{V} \oplus \underline{\boldsymbol{R}}^1)$. Conversely, if there is a G-fiber homotopy equivalence $f\colon S(\xi) \to S(\underline{V})$, then $T(\xi)$ is G-V-coreducible.*

Using Equivariant Dold Theorem (Kawakubo [19; Theorem 2.1]) and Equivariant J.H.C. Whitehead Theorem (Bredon [8; Chap. II Corollary (5.5)]), the proof is almost parallel to that of Proposition 2.8 in [3]. So we omit it.

Let $\omega$, $\xi_1$ and $\xi_2$ be smooth $G$-vector bundles over $M$. We put $V=\omega_x$, $W_1=(\xi_1)_x$ and $W_2=(\xi_2)_x$ for $x \in M$. Assume that $V^G \neq \{0\}$, $W_1^G \neq \{0\}$ and $W_2^G \neq \{0\}$. Then $T(\omega)$, $T(\xi_1)$ and $T(\xi_2)$ are $G$-path-connected pointed finite $G$-complexes.

**Lemma 3.4.** *If there are a reduction G-map $\alpha\colon \Sigma^V \to T(\omega)$ and a coreduction G-map $\beta\colon T(\xi_1 \oplus \xi_2) \to \Sigma^{W_1 \oplus W_2}$, then there is a duality G-map*

$$\mu\colon \Sigma^{W_1 \oplus W_2 \oplus V} \to T(\xi_1) \wedge T(\xi_2 \oplus \omega)\,.$$

Using Equivariant J.H.C. Whitehead Theorem ([8]), the proof is quite similar to that of (13.2) in [6]. So we omit it.

## 4. Linear actions on stunted projective spaces

Let $V$ be an orthogonal $G$-representation space and $\varepsilon_{\boldsymbol{R}}$ be the non-trivial orthogonal 1-dimensional $\boldsymbol{Z}_2$-representation space. Then $\varepsilon_{\boldsymbol{R}}\otimes V$ is an orthogonal $(\boldsymbol{Z}_2\times G)$-representation space.

DEFINITION 4.1. (i) $\boldsymbol{R}P(V)=S(\varepsilon_{\boldsymbol{R}}\otimes V)/(\boldsymbol{Z}_2\times \{e\})$,
(ii) For $m\geqq k$, $P_k(V\oplus\boldsymbol{R}^m)=\boldsymbol{R}P(V\oplus\boldsymbol{R}^m)/\boldsymbol{R}P(V\oplus\boldsymbol{R}^{m-k})$.

Then $P_k(V\oplus\boldsymbol{R}^m)$ is a pointed finite $G$-complex ([12]). We see that, if $m>k$, then for $H<G$

$$P_k(V\oplus\boldsymbol{R}^m)^H = P_k(V^H\oplus\boldsymbol{R}^m)\,,$$
$$\dim P_k(V\oplus\boldsymbol{R}^m)^H = \dim_{\boldsymbol{R}} V^H+m-1$$

and

$$\mathrm{conn}(P_k(V\oplus\boldsymbol{R}^m)^H) = \dim_{\boldsymbol{R}} V^H+m-k-1\,.$$

In particular, if $m>k$, then $P_k(V\oplus\boldsymbol{R}^m)$ is $G$-path-connected. Atiyah [3] identifies the Thom space of a multiple of $\eta_k$ as a stunted projective space. As $G$-spaces this identification takes the form

$$T(\eta_k\otimes(\underline{\underline{V}}\oplus\underline{\underline{\boldsymbol{R}}}^{m-k})) = P_k(V\oplus\boldsymbol{R}^m)\,.$$

Let $a_k(\boldsymbol{R})$ $(k>0)$ be the integer defined by [4; § 5]. We recall that the group $\tilde{J}(\boldsymbol{R}P^{k-1})$ is cyclic of order $a_k(\boldsymbol{R})$ ([1], [2]). We remark that $a_k(\boldsymbol{R})\geqq k$ for $k>0$.

**Lemma 4.2.** *Let $m$, $n$ and $k$ be integers such that $m\equiv 0 \bmod a_k(\boldsymbol{R})$, $n\equiv k \bmod a_k(\boldsymbol{R})$ and $n>m\geqq 2k\geqq 4$. Let $U$ be an arbitrary orthogonal $G$-representation space. Then we have the following:*

(i) *If $\Sigma^U P_k(V\oplus\boldsymbol{R}^m)$ is $G$-$U\oplus V\oplus\boldsymbol{R}^{m-1}$-reducible, then there is a duality $G$-map*

$$\mu_1\colon \Sigma^{\boldsymbol{R}^{m-1}U\oplus V\oplus\boldsymbol{R}^{n-k}} \to P_k(\boldsymbol{R}^m)\wedge\Sigma^U P_k(V\oplus\boldsymbol{R}^n)\,,$$

(ii) *If $\Sigma^U P_k(V\oplus\boldsymbol{R}^n)$ is $G$-$U\oplus V\oplus\boldsymbol{R}^{n-k}$-coreducible, then there is a duality $G$-map*

$$\mu_2\colon \Sigma^{U\oplus V\oplus\boldsymbol{R}^{m-1}\oplus\boldsymbol{R}^{n-k}} \to \Sigma^U P_k(V\oplus\boldsymbol{R}^m)\wedge P_k(\boldsymbol{R}^n)\,.$$

Proof. We remark that

$$T(\underline{\underline{U}}\oplus(\eta_k\otimes(\underline{\underline{V}}\oplus\boldsymbol{R}^{m-k}))) = \Sigma^U P_k(V\oplus\boldsymbol{R}^m)\,,$$
$$T(\underline{\underline{U}}\oplus(\eta_k\otimes(\underline{\underline{V}}\oplus\underline{\underline{\boldsymbol{R}}}^{n-k}))) = \Sigma^U P_k(V\oplus\boldsymbol{R}^n)\,.$$

First we show (i). By assumption, there is a reduction $G$-map

$$\alpha\colon \Sigma^{U\oplus V\oplus\boldsymbol{R}^{m-1}} \to T(\underline{\underline{U}}\oplus(\eta_k\otimes(\underline{\underline{V}}\oplus\boldsymbol{R}^{m-k})))\,.$$

Set

$$\omega = \underline{\underline{U}}\oplus(\eta_k\otimes(\underline{\underline{V}}\oplus\underline{\underline{\boldsymbol{R}}}^{m-k}))\,, \quad \xi_1 = \eta_k\otimes\underline{\underline{\boldsymbol{R}}}^{m-k}\,,$$
$$\xi_2 = \eta_k\otimes\underline{\underline{\boldsymbol{R}}}^{n-m}\,.$$

Since $\xi_1\oplus\xi_2$ is trivial, there is a coreduction ($G$-)map

$$\beta\colon T(\xi_1\oplus\xi_2)\to\Sigma^{\boldsymbol{R}^{n-k}}.$$

Applying Lemma 3.4 to $\alpha$, $\beta$, $\omega$, $\xi_1$ and $\xi_2$, we have a duality $G$-map $\mu_1$.

Next we show (ii). By assumption, there is a coreduction $G$-map

$$\beta\colon T(\underline{\underline{U}}\oplus(\eta_k\otimes(\underline{\underline{V}}\oplus\underline{\underline{\boldsymbol{R}}}^{n-k})))\to\Sigma^{U\oplus V\oplus\boldsymbol{R}^{n-k}}\,.$$

Since $m\equiv 0 \bmod a_k(\boldsymbol{R})$ and $m\geqq 2k$, there is a reduction ($G$-)map

$$\alpha\colon \Sigma^{\boldsymbol{R}^{m-1}}\to T(\eta_k\otimes\underline{\underline{\boldsymbol{R}}}^{m-k})\,.$$

Set

$$\omega = \eta_k\otimes\underline{\underline{\boldsymbol{R}}}^{m-k}\,, \quad \xi_1 = \underline{\underline{U}}\oplus(\eta_k\otimes(\underline{\underline{V}}\oplus\underline{\underline{\boldsymbol{R}}}^{m-k}))\,,$$
$$\xi_2 = \eta_k\otimes\underline{\underline{\boldsymbol{R}}}^{n-m}\,.$$

Applying Lemma 3.4, we have a duality $G$-map $\mu_2$. q.e.d.

**Lemma 4.3.** *Let $m$ and $k$ be integers such that $m>k>0$. Let $V$ be an orthogonal $G$-representation space. Assume that $P_k(V\oplus\boldsymbol{R}^m)$ is either $G$-$V\oplus\boldsymbol{R}^{m-1}$-reducible or $G$-$V\oplus\boldsymbol{R}^{m-k}$-coreducible. Then we have*

$$\dim_{\boldsymbol{R}} V^K - \dim_{\boldsymbol{R}} V^H \geqq k$$

*if $V^K\neq V^H$ for $K<H<G$.*

Proof. Let $K<H<G$ such that $V^K\neq V^H$. First we assume that $P_k(V\oplus\boldsymbol{R}^m)$ is $G$-$V\oplus\boldsymbol{R}^{m-1}$-reducible. Then, by definition, $P_k(V^H\oplus\boldsymbol{R}^m)$ and $P_k(V^K\oplus\boldsymbol{R}^m)$ are reducible. It follows from Atiyah [3; Theorem 6.2] that $\dim_{\boldsymbol{R}} V^H+m\equiv 0 \bmod a_k(\boldsymbol{R})$ and $\dim_{\boldsymbol{R}} V^K+m\equiv 0 \bmod a_k(\boldsymbol{R})$. Thus we see that $\dim_{\boldsymbol{R}} V^K-\dim_{\boldsymbol{R}} V^H\equiv 0 \bmod a_k(\boldsymbol{R})$. Now we have

$$\dim_{\boldsymbol{R}} V^K-\dim_{\boldsymbol{R}} V^H\geqq a_k(\boldsymbol{R})\geqq k\,.$$

Next we assume that $P_k(V\oplus\boldsymbol{R}^m)$ is $G$-$V\oplus\boldsymbol{R}^{m-k}$-coreducible. Then $P_k(V^H\oplus\boldsymbol{R}^m)$ and $P_k(V^K\oplus\boldsymbol{R}^m)$ are coreducible. By Atiyah [3; Proposition 2.8], we have

$$J(\eta_k\otimes(\underline{\underline{V}}^H\oplus\underline{\underline{\boldsymbol{R}}}^{m-k})-(\underline{\underline{V}}^H\oplus\underline{\underline{\boldsymbol{R}}}^{m-k}))=0 \quad \text{in } \tilde{J}(\boldsymbol{R}P^{k-1})\,,$$
$$J(\eta_k\otimes(\underline{\underline{V}}^K\oplus\underline{\underline{\boldsymbol{R}}}^{m-k})-(\underline{\underline{V}}^K\oplus\underline{\underline{\boldsymbol{R}}}^{m-k}))=0 \quad \text{in } \tilde{J}(\boldsymbol{R}P^{k-1})\,.$$

Thus we obtain that $\dim_{\boldsymbol{R}} V^K-\dim_{\boldsymbol{R}} V^H\equiv 0 \bmod a_k(\boldsymbol{R})$. Now we see that

$$\dim_{\boldsymbol{R}} V^K - \dim_{\boldsymbol{R}} V^H \geqq a_k(\boldsymbol{R}) \geqq k . \qquad \text{q.e.d.}$$

**Proposition 4.4.** *Let $m$, $n$ and $k$ be integers such that $m\equiv 0 \bmod a_k(\boldsymbol{R})$, $n\equiv k \bmod a_k(\boldsymbol{R})$ and $n>m\geqq 2k\geqq 4$. Let $V$ be an orthogonal $G$-representation space. Then the following two conditions are equivalent:*
(i) *$P_k(V\oplus\boldsymbol{R}^m)$ is $G$-$V\oplus\boldsymbol{R}^{m-1}$-reducible,*
(ii) *$P_k(V\oplus\boldsymbol{R}^n)$ is $G$-$V\oplus\boldsymbol{R}^{n-k}$-coreducible.*

Proof. First we show that (i) implies (ii). By Lemma 4.2, there is a duality $G$-map

$$\mu_1\colon \Sigma^{\boldsymbol{R}^{m-1}\oplus V\oplus \boldsymbol{R}^{n-k}} \to P_k(\boldsymbol{R}^m)\wedge P_k(V\oplus\boldsymbol{R}^n) .$$

We put $U=V\oplus\boldsymbol{R}^1$. For $s>0$, we define a homomorphism

$$\begin{aligned}\bar{\Gamma}_s(\mu_1)\colon & [\Sigma^{sU} P_k(V\oplus\boldsymbol{R}^n),\ \Sigma^{sU}\Sigma^{V\oplus\boldsymbol{R}^{n-k}}]_0^G \\ & \to [\Sigma^{sU}\Sigma^{\boldsymbol{R}^{m-1}\oplus V\oplus\boldsymbol{R}^{n-k}},\ \Sigma^{sU}P_k(\boldsymbol{R}^m)\wedge\Sigma^{V\oplus\boldsymbol{R}^{n-k}}]_0^G\end{aligned}$$

by the following: if $f\colon \Sigma^{sU}P_k(V\oplus\boldsymbol{R}^n)\to\Sigma^{sU}\Sigma^{V\oplus\boldsymbol{R}^{n-k}}$ is a pointed $G$-map, then $\bar{\Gamma}_s(\mu_1)([f])$ is represented by the composition

$$\begin{aligned}&\Sigma^{sU}\Sigma^{\boldsymbol{R}^{m-1}\oplus V\oplus\boldsymbol{R}^{n-k}} \xrightarrow{1\wedge\mu_1} \Sigma^{sU}P_k(\boldsymbol{R}^m)\wedge P_k(V\oplus\boldsymbol{R}^n) \xrightarrow{T_1} \\ &P_k(\boldsymbol{R}^m)\wedge\Sigma^{sU}P_k(V\oplus\boldsymbol{R}^n) \xrightarrow{1\wedge f} P_k(\boldsymbol{R}^m)\wedge\Sigma^{sU}\Sigma^{V\oplus\boldsymbol{R}^{n-k}} \xrightarrow{T_2} \Sigma^{sU}P_k(\boldsymbol{R}^m)\wedge\Sigma^{V\oplus\boldsymbol{R}^{n-k}},\end{aligned}$$

where $T_1$ and $T_2$ are the switching maps. Then we have the following:

**Assertion 4.4.1.** *If $s>\dim_{\boldsymbol{R}} V+m+n+1$, then $\bar{\Gamma}_s(\mu_1)$ is an isomorphism.*

The proof is quite similar to that of Assertion 4.1.1 in [18]. So we omit it.

On the other hand, the standard identification

$$\nu_1\colon \Sigma^{\boldsymbol{R}^{m-1}\oplus V\oplus\boldsymbol{R}^{n-k}} \to \Sigma^{\boldsymbol{R}^{m-1}}\wedge\Sigma^{V\oplus\boldsymbol{R}^{n-k}}$$

is a duality $G$-map. We define a homomorphism

$$\Gamma_s(\nu_1)\colon [\Sigma^{sU}\Sigma^{\boldsymbol{R}^{m-1}},\ \Sigma^{sU}P_k(\boldsymbol{R}^m)]_0^G \to [\Sigma^{sU}\Sigma^{\boldsymbol{R}^{m-1}\oplus V\oplus\boldsymbol{R}^{n-k}},\ \Sigma^{sU}P_k(\boldsymbol{R}^m)\wedge\Sigma^{V\oplus\boldsymbol{R}^{n-k}}]_0^G$$

by the following: if $f\colon \Sigma^{sU}\Sigma^{\boldsymbol{R}^{m-1}}\to\Sigma^{sU}P_k(\boldsymbol{R}^m)$ is a pointed $G$-map, then $\Gamma_s(\nu_1)([f])=[f']$, where $f'$ is the composition

$$\Sigma^{sU}\Sigma^{\boldsymbol{R}^{m-1}\oplus V\oplus\boldsymbol{R}^{n-k}} \xrightarrow{1\wedge\nu_1} \Sigma^{sU}\Sigma^{\boldsymbol{R}^{m-1}}\wedge\Sigma^{V\oplus\boldsymbol{R}^{n-k}} \xrightarrow{f\wedge 1} \Sigma^{sU}P_k(\boldsymbol{R}^m)\wedge\Sigma^{V\oplus\boldsymbol{R}^{n-k}} .$$

For $s>\dim_{\boldsymbol{R}} V+m+n+1$, we put

$$\begin{aligned}D_s(\nu_1, \mu_1) = \bar{\Gamma}_s(\mu_1)^{-1}\circ\Gamma_s(\nu_1)\colon & [\Sigma^{sU}\Sigma^{\boldsymbol{R}^{m-1}},\ \Sigma^{sU}P_k(\boldsymbol{R}^m)]_0^G \\ & \to [\Sigma^{sU}P_k(V\oplus\boldsymbol{R}^n),\ \Sigma^{sU}\Sigma^{V\oplus\boldsymbol{R}^{n-k}}]_0^G .\end{aligned}$$

Since $m\equiv 0 \bmod a_k(\boldsymbol{R})$ and $m\geqq 2k$, there is a reduction ($G$-)map $f_1$: $\Sigma^{\boldsymbol{R}^{m-1}}\to P_k(\boldsymbol{R}^m)$. Let $f_2$: $\Sigma^{sU}P_k(V\oplus\boldsymbol{R}^n)\to\Sigma^{sU}\Sigma^{V\oplus\boldsymbol{R}^{n-k}}$ be a pointed $G$-map such that $D_s(\nu_1, \mu_1)([1\wedge f_1])=[f_2]$. As is easily seen, $f_2$ is a coreduction $G$-map. Here we consider the suspension map

$$\sigma_*^{sU}: [P_k(V\oplus\boldsymbol{R}^n), \Sigma^{V\oplus\boldsymbol{R}^{n-k}}]_0^G \to [\Sigma^{sU}P_k(V\oplus\boldsymbol{R}^n), \Sigma^{sU}\Sigma^{V\oplus\boldsymbol{R}^{n-k}}]_0^G .$$

Let $K<H<G$ such that $(sU)^H\neq(sU)^K$. Since $U=V\oplus\boldsymbol{R}^1$, we see that $V^H\neq V^K$. Applying Lemma 4.3, we have

$$\begin{cases}\dim(P_k(V\oplus\boldsymbol{R}^n)^H) = \dim_{\boldsymbol{R}} V^H+n-1\,,\\ 2\operatorname{conn}((\Sigma^{V\oplus\boldsymbol{R}^{n-k}})^H)+1 = 2(\dim_{\boldsymbol{R}} V^H+n-k-1)+1\geqq\dim_{\boldsymbol{R}} V^H+n-1\,,\\ \operatorname{conn}((\Sigma^{V\oplus\boldsymbol{R}^{n-k}})^K) = \dim_{\boldsymbol{R}} V^K+n-k-1\geqq\dim_{\boldsymbol{R}} V^H+n-1\,.\end{cases}$$

By the suspension theorem [18; Theorem 3.6], we see that $\sigma_*^{sU}$ is surjective. Let $f_3$: $P_k(V\oplus\boldsymbol{R}^n)\to\Sigma^{V\oplus\boldsymbol{R}^{n-k}}$ be a pointed $G$-map such that $\sigma_*^{sU}([f_3])=[f_2]$. Then it is easy to see that $f_3$ is also a coreduction $G$-map. That is, $P_k(V\oplus\boldsymbol{R}^n)$ is $G$-$V\oplus\boldsymbol{R}^{n-k}$-coreducible.

Similarly, using $\mu_2$ in Lemma 4.2, we see that (ii) implies (i). q.e.d.

## 5. An equivariant version of the theorem of James

First we fix some notations. Let $V_k(V)$ denote the Stiefel manifold of orthogonal $k$-frames in an orthogonal $G$-representation space $V$ with $G$-action defined by

$$g\cdot(v_1, \cdots, v_k) = (gv_1, \cdots, gv_k)\,.$$

Then $V_k(V)$ is a smooth $G$-manifold. If $\dim_{\boldsymbol{R}} V^H\geqq k$ for some $H<G$, then we see that

$$V_k(V)^H = V_k(V^H)$$

and

$$\operatorname{conn}(V_k(V)^H) = \dim_{\boldsymbol{R}} V^H-k-1\,.$$

Let

$$q_k\colon V_k(V)\to S(V)$$

send $(v_1, \cdots, v_k)$ to $v_k$. We remark that $q_k$: $V_k(V)\to S(V)$ is a smooth $G$-fiber bundle in the sense of Bierstone [7]. We remark the following:

**Lemma 5.1.** $\mathrm{Span}_G(S(V))\geqq k-1$ *if and only if* $q_k$: $V_k(V)\to S(V)$ *has a smooth $G$-cross-section.*

Let $m>k>0$. There is a well-known mapping

$$\tau_k\colon P_k(V\oplus \boldsymbol{R}^m)\to V_k(V\oplus \boldsymbol{R}^m)$$

by

$$\tau_k([x])=(e_{n+m-k+1}-2(e_{n+m-k+1}, x)x, \cdots, e_{n+m}-2(e_{n+m}, x)x)\,,$$

where $n=\dim_{\boldsymbol{R}} V$ and $e_i$ denotes the $i$-th unit vector in $V\oplus \boldsymbol{R}^m$. We see that $\tau_k$ is a $G$-map and for $H<G$

$$\tau_k^H\colon P_k(V\oplus \boldsymbol{R}^m)^H\to V_k(V\oplus \boldsymbol{R}^m)^H$$

is a $2(\dim_{\boldsymbol{R}} V^H+m-k)$-equivalence (see James [16; Lemma 8.1]). We remark that $\tau_1\colon P_1(V\oplus \boldsymbol{R}^m)\to S(V\oplus \boldsymbol{R}^m)$ $(=V_1(V\oplus \boldsymbol{R}^m))$ is a $G$-homeomorphism. Let

$$p\colon S(V\oplus \boldsymbol{R}^m)\to P_k(V\oplus \boldsymbol{R}^m)$$

and

$$\pi'\colon P_k(V\oplus \boldsymbol{R}^m)\to P_1(V\oplus \boldsymbol{R}^m)$$

be the natural projection and the collapsing map respectively. For $S(V\oplus \boldsymbol{R}^m)$, we choose a base point $x_0\in S(\boldsymbol{R}^{m-k})(\subset S(V\oplus \boldsymbol{R}^{m-k})\subset S(V\oplus \boldsymbol{R}^m))$. There is a pointed $G$-map $u\colon P_1(V\oplus \boldsymbol{R}^m)\to S(V\oplus \boldsymbol{R}^m)$ such that $u$ and $\tau_1$ are $G$-homotopic. We put

$$\pi=u\circ\pi'\colon P_k(V\oplus \boldsymbol{R}^m)\to S(V\oplus \boldsymbol{R}^m)\,.$$

Then $p$ and $\pi$ are pointed $G$-maps.

**Lemma 5.2.** *Let $m>k>0$. Let $f\colon S(V\oplus \boldsymbol{R}^m)\to P_k(V\oplus \boldsymbol{R}^m)$ be a pointed $G$-map. Then $f$ is a reduction $G$-map if and only if the composition*

$$S(V\oplus \boldsymbol{R}^m)^H\xrightarrow{f^H} P_k(V\oplus \boldsymbol{R}^m)^H\xrightarrow{\pi^H} S(V\oplus \boldsymbol{R}^m)^H$$

*is an ordinary homotopy equivalence (i.e. has degree $\pm1$) for each $(H)\in$* $\mathrm{Iso}(S(V\oplus \boldsymbol{R}^m))$.

The proof is easy.

A $G$-homeomorphism

$$h\colon S(V)*S(\boldsymbol{R}^m)\to S(V\oplus \boldsymbol{R}^m)$$

is given by $h(x, y, t)=(x\cdot\cos(\pi t/2), y\cdot\sin(\pi t/2))$. In [14], James defined the intrinsic map

$$\mu\colon V_k(V)*V_k(\boldsymbol{R}^m)\to V_k(V\oplus \boldsymbol{R}^m)\,.$$

We see that $\mu$ is a $G$-map and the following diagram commutes:

$$\begin{array}{ccc} V_k(V)*V_k(\boldsymbol{R}^m) & \xrightarrow{\mu} & V_k(V\oplus \boldsymbol{R}^m) \\ \downarrow{\scriptstyle q_k*q_k} & & \downarrow{\scriptstyle q_k} \\ S(V)*S(\boldsymbol{R}^m) & \xrightarrow{h} & S(V\oplus \boldsymbol{R}^m)\,. \end{array}$$

Now we prove the following theorem, which is a generalization of Proposition 11.5 in [6] (see also Theorem 8.2 in [16]):

**Theorem 5.3.** *Let $m$ and $k$ be integers such that $m\equiv 0 \bmod a_k(\boldsymbol{R})$ and $m\geqq 2k\geqq 4$. If* $\mathrm{Span}_G(S(V))\geqq k-1$, *then $P_k(V\oplus\boldsymbol{R}^m)$ is $G$-$V\oplus\boldsymbol{R}^{m-1}$-reducible.*

Proof. Since $m\equiv 0 \bmod a_k(\boldsymbol{R})$ and $m\geqq 2k$, there is a reduction ($G$-)map $\rho$: $S(\boldsymbol{R}^m)\to P_k(\boldsymbol{R}^m)$. It follows from Lemma 5.1 that there is a $G$-cross-section of $q_k$

$$\Delta\colon S(V)\to V_k(V).$$

Then we define a $G$-map

$$\gamma\colon S(V\oplus\boldsymbol{R}^m)\to V_k(V\oplus\boldsymbol{R}^m)$$

by the composition

$$S(V\oplus\boldsymbol{R}^m)\xrightarrow{h^{-1}} S(V)*S(\boldsymbol{R}^m)\xrightarrow{\Delta*\rho} V_k(V)*P_k(\boldsymbol{R}^m)\xrightarrow{1*\tau_k}$$
$$V_k(V)*V_k(\boldsymbol{R}^m)\xrightarrow{\mu} V_k(V\oplus\boldsymbol{R}^m).$$

Consider a map

$$\tau_{k*}\colon [S(V\oplus\boldsymbol{R}^m), P_k(V\oplus\boldsymbol{R}^m)]^G\to[S(V\oplus\boldsymbol{R}^m), V_k(V\oplus\boldsymbol{R}^m)]^G.$$

Since $\tau_k^H\colon P_k(V\oplus\boldsymbol{R}^m)^H\to V_k(V\oplus\boldsymbol{R}^m)^H$ is a $2(\dim_{\boldsymbol{R}} V^H+m-k)$-equivalence for each $H<G$, it follows from Lemma 2.1 that $\tau_{k*}$ is bijective. Moreover we see that

$$[S(V\oplus\boldsymbol{R}^m), P_k(V\oplus\boldsymbol{R}^m)]^G\cong[S(V\oplus\boldsymbol{R}^m), P_k(V\oplus\boldsymbol{R}^m)]_0^G.$$

Hence there is a pointed $G$-map

$$\lambda\colon S(V\oplus\boldsymbol{R}^m)\to P_k(V\oplus\boldsymbol{R}^m)$$

such that $\tau_{k*}([\lambda])=[\gamma]$. As is easily seen, the composition

$$S(V\oplus\boldsymbol{R}^m)^H\xrightarrow{\lambda^H} P_k(V\oplus\boldsymbol{R}^m)^H\xrightarrow{\pi^H} S(V\oplus\boldsymbol{R}^m)^H$$

is an ordinary homotopy equivalence for each $H<G$. By Lemma 5.2, $\lambda$ is a reduction $G$-map. That is, $P_k(V\oplus\boldsymbol{R}^m)$ is $G$-$V\oplus\boldsymbol{R}^{m-1}$-reducible. q.e.d.

## 6. A converse of Theorem 5.3

Let $m$ and $k$ be integers such that $m\equiv 0 \bmod a_k(\boldsymbol{R})$ and $m\geqq 2k\geqq 4$. Let $\kappa$: $S(\boldsymbol{R}^m)\to V_k(\boldsymbol{R}^m)$ be a 1-section of $q_k$. That is, the composition $S(\boldsymbol{R}^m)\xrightarrow{\kappa} V_k(\boldsymbol{R}^m)\xrightarrow{q_k} S(\boldsymbol{R}^m)$ has degree 1. For $n>k$, we define

$$\theta_\kappa\colon V_k(\boldsymbol{R}^n)*S(\boldsymbol{R}^m)\to V_k(\boldsymbol{R}^{n+m})$$

by the composition

$$V_k(\boldsymbol{R}^n)*S(\boldsymbol{R}^m)\xrightarrow{1*\kappa}V_k(\boldsymbol{R}^n)*V_k(\boldsymbol{R}^m)\xrightarrow{\mu}V_k(\boldsymbol{R}^{n+m}),$$

where $\mu$ is the intrinsic map (see §5). By Theorem 3.1 in [15], $\theta_\kappa$ is a $(2n-2k+m-1)$-equivalence. The following Theorem is a converse of Theorem 5.3.

**Theorem 6.1.** *Let $m$ and $k$ be integers such that $m\equiv 0 \bmod a_k(\boldsymbol{R})$ and $m\geqq 2k\geqq 4$. Let $V$ be an orthogonal $G$-representation space. Assume that*
(i) *For each $H<G$, $\dim_{\boldsymbol{R}} V^H\geqq 2k$ if $V^H\neq\{0\}$,*
(ii) *$P_k(V\oplus\boldsymbol{R}^m)$ is $G$-$V\oplus\boldsymbol{R}^{m-1}$-reducible.*
*Then* $\mathrm{Span}_G(S(V))\geqq k-1$.

Proof. First we show the following Assertion 6.1.1.

**Assertion 6.1.1.** *There is a $G$-map*

$$\gamma_0\colon S(V\oplus\boldsymbol{R}^m)\to V_k(V\oplus\boldsymbol{R}^m)$$

*such that $\gamma_0$ satisfies the following:*
(6.1.2) $\gamma_0(S(\boldsymbol{R}^m))\subset V_k(\boldsymbol{R}^m)(\subset V_k(V\oplus\boldsymbol{R}^m))$,
(6.1.3) *the composition*

$$S(\boldsymbol{R}^m)\xrightarrow{\gamma_0|S(\boldsymbol{R}^m)}V_k(\boldsymbol{R}^m)\xrightarrow{q_k}S(\boldsymbol{R}^m)$$

has degree 1,
(6.1.4) *the composition*

$$S(V\oplus\boldsymbol{R}^m)^H\xrightarrow{\gamma_0^H}V_k(V\oplus\boldsymbol{R}^m)^H\xrightarrow{q_k^H}S(V\oplus\boldsymbol{R}^m)^H$$

*has degree 1 for each $H<G$.*

Proof of Assertion 6.1.1. By assumption, we have a reduction $G$-map

$$\lambda'\colon S(V\oplus\boldsymbol{R}^m)\to P_k(V\oplus\boldsymbol{R}^m).$$

Let $\pi\colon P_k(V\oplus\boldsymbol{R}^m)\to S(V\oplus\boldsymbol{R}^m)$ be the pointed $G$-map as in §5. We put

$$\lambda=\lambda'\circ(\pi\circ\lambda')\colon S(V\oplus\boldsymbol{R}^m)\to P_k(V\oplus\boldsymbol{R}^m).$$

Then $\lambda$ is also a reduction $G$-map such that

$$\deg(\pi\circ\lambda)^H=1\qquad\text{for all } H<G.$$

We put

$$\gamma_1=\tau_k\circ\lambda\colon S(V\oplus\boldsymbol{R}^m)\to V_k(V\oplus\boldsymbol{R}^m).$$

We consider a $G$-map

$$\gamma_2 = \gamma_1 | S(\boldsymbol{R}^m) \colon S(\boldsymbol{R}^m) \to V_k(V \oplus \boldsymbol{R}^m).$$

First we assume that $V^G \neq \{0\}$. Since $m \equiv 0 \bmod a_k(\boldsymbol{R})$ and $m \geqq 2k$, there is a ($G$-)cross-section of $q_k$

$$\Delta \colon S(\boldsymbol{R}^m) \to V_k(\boldsymbol{R}^m) \subset V_k(V \oplus \boldsymbol{R}^m).$$

Since $\operatorname{conn}(V_k(V \oplus \boldsymbol{R}^m)^G) \geqq \dim S(\boldsymbol{R}^m)$, $\gamma_2$ and $\Delta$ are $G$-homotopic. Remark that $(S(V \oplus \boldsymbol{R}^m), S(\boldsymbol{R}^m))$ has the $G$-homotopy extension property. We have a $G$-map

$$\gamma_0 \colon S(V \oplus \boldsymbol{R}^m) \to V_k(V \oplus \boldsymbol{R}^m)$$

such that $\gamma_0$ and $\gamma_1$ are $G$-homotopic and $\gamma_0 | S(\boldsymbol{R}^m) = \Delta$. As is easily seen, $\gamma_0$ satisfies our required properties.

Next we assume that $V^G = \{0\}$. In this case, $\gamma_2 = \gamma_1^G \colon S(\boldsymbol{R}^m) \to V_k(\boldsymbol{R}^m)$ is a 1-section of $q_k$. Therefore we put $\gamma_0 = \gamma_1$.

This completes the proof of Assertion 6.1.1.

We put $\gamma_3 = \gamma_0 | S(\boldsymbol{R}^m) \colon S(\boldsymbol{R}^m) \to V_k(\boldsymbol{R}^m) (\subset V_k(V \oplus \boldsymbol{R}^m))$. Consider a map

$$\begin{aligned}\theta_{\gamma_3 *} \colon & [(S(V \oplus \boldsymbol{R}^m), S(\boldsymbol{R}^m)), V_k(V) * S(\boldsymbol{R}^m); i_{S(\boldsymbol{R}^m)}]^G \\ & \to [(S(V \oplus \boldsymbol{R}^m), S(\boldsymbol{R}^m)), V_k(V \oplus \boldsymbol{R}^m); \gamma_3]^G.\end{aligned}$$

Since $\gamma_3$ is a 1-section, $\theta_{\gamma_3}^H \colon V_k(V)^H * S(\boldsymbol{R}^m) \to V_k(V \oplus \boldsymbol{R}^m)^H$ is a $(2 \dim_{\boldsymbol{R}} V^H - 2k + m - 1)$-equivalence for each $(H) \in \operatorname{Iso}(S(V \oplus \boldsymbol{R}^m) - S(\boldsymbol{R}^m))$. Applying Lemma 2.1, $\theta_{\gamma_3 *}$ is surjective. Therefore we have a $G$-map

$$\gamma_4 \colon S(V \oplus \boldsymbol{R}^m) \to V_k(V) * S(\boldsymbol{R}^m)$$

such that $\theta_{\gamma_3 *}([\gamma_4]) = [\gamma_0]$ and $\gamma_4 | S(\boldsymbol{R}^m) = i_{S(\boldsymbol{R}^m)}$. As is easily seen, the composition

$$S(V)^H * S(\boldsymbol{R}^m) \xrightarrow{h^H} S(V \oplus \boldsymbol{R}^m)^H \xrightarrow{\gamma_4^H} V_k(V)^H * S(\boldsymbol{R}^m) \xrightarrow{q_k^H * 1} S(V)^H * S(\boldsymbol{R}^m)$$

has degree 1 for each $H < G$, where $h$ is as in § 5. Consider the following suspension map

$$\tau_*^{\boldsymbol{R}^m} \colon [S(V), V_k(V)]^G \to [(S(V) * S(\boldsymbol{R}^m), S(\boldsymbol{R}^m)), V_k(V) * S(\boldsymbol{R}^m); i_{S(\boldsymbol{R}^m)}]^G.$$

Since $\dim S(V)^H \leqq 2 \operatorname{conn}(V_k(V^H)) + 1$ for each $(H) \in \operatorname{Iso}(S(V))$, it follows from Theorem 2.4 that $\tau_*^{\boldsymbol{R}^m}$ is surjective. Then we have a $G$-map

$$\gamma_5 \colon S(V) \to V_k(V)$$

such that $\tau_*^{\boldsymbol{R}^m}([\gamma_5]) = [\gamma_4 \circ h]$. As is easily seen, the composition

$$S(V)^H \xrightarrow{\gamma_5^H} V_k(V)^H \xrightarrow{q_k^H} S(V)^H$$

has degree 1 for each $(H)\in \mathrm{Iso}(S(V))$. Let $K<H<G$ such that $V^K \neq V^H$. Using Lemma 4.3, we have $\dim_{\boldsymbol{R}} V^K - \dim_{\boldsymbol{R}} V^H \geqq k \geqq 2$. Thus it follows from Rubinsztein [22; Theorem 8.4] that $q_k \circ \gamma_5$ is $G$-homotopic to the identity. Since $q_k$: $V_k(V) \to S(V)$ is a smooth $G$-fiber bundle in the sense of Bierstone [7], $q_k$ has the smooth $G$-homotopy lifting property. Using Wasserman [27; Corollary 1.12], we see that $q_k$ has a smooth $G$-cross-section. Now, by Lemma 5.1, we have $\mathrm{Span}_G(S(V)) \geqq k-1$. q.e.d.

## 7. Proof of Theorem 1.1

Let $V$ and $W$ be unitary $G$-representation spaces such that $\dim_{\boldsymbol{C}} V^H = \dim_{\boldsymbol{C}} W^H$ for all $H<G$. By Lee-Wasserman [21; Proposition 3.17], there are direct sum decompositions

$$\begin{cases} V = V_1 \oplus V_2 \oplus \cdots \oplus V_r\,, \\ W = W_1 \oplus W_2 \oplus \cdots \oplus W_r \end{cases}$$

such that $V_i$ and $W_i$ $(1 \leqq i \leqq r)$ are irreducible unitary $G$-representation spaces and $V_i$ is conjugate to $W_i$ by a field automorphism of $\boldsymbol{C}$ for $1 \leqq i \leqq r$. That is, there are integers $n(i)(1 \leqq i \leqq r)$ such that $(n(i), |G|)=1$ and $W_i = \psi^{n(i)}(V_i)$ for $1 \leqq i \leqq r$, where $\psi^s$ denotes the equivariant $s$-th Adams operation and $|G|$ denotes the order of $G$. Since $\psi^{s+|G|} = \psi^s$, we may assume that $n(i)$ $(1 \leqq i \leqq r)$ are odd integers. Let $\varepsilon_{\boldsymbol{C}}$ be the non-trivial unitary 1-dimensional $\boldsymbol{Z}_2$-representation space. Then $\varepsilon_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} V$ and $\varepsilon_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} W$ are unitary $(Z_2 \times G)$-representation spaces and

$$\begin{cases} \varepsilon_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} V = (\varepsilon_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} V_1) \oplus (\varepsilon_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} V_2) \oplus \cdots \oplus (\varepsilon_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} V_r)\,, \\ \varepsilon_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} W = (\varepsilon_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} W_1) \oplus (\varepsilon_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} W_2) \oplus \cdots \oplus (\varepsilon_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} W_r) \end{cases}$$

are decompositions of $\varepsilon_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} V$ and $\varepsilon_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} W$ into direct sums of irreducible unitary $(\boldsymbol{Z}_2 \times G)$-representation spaces respectively. Since $n(i)$ $(1 \leqq i \leqq r)$ are odd, there are integers $\bar{n}(i)$ $(1 \leqq i \leqq r)$ such that $(\bar{n}(i), 2|G|)=1$ and $n(i)\cdot \bar{n}(i) \equiv 1 \bmod |2G|$. Then we have

$$\begin{cases} \varepsilon_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} V_i = \psi^{\bar{n}(i)}(\varepsilon_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} W_i) & \text{for } 1 \leqq i \leqq r\,, \\ \varepsilon_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} W_i = \psi^{n(i)}(\varepsilon_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} V_i) & \text{for } 1 \leqq i \leqq r\,. \end{cases}$$

The following lemma is due to Tornehave [25] (see also [11]).

**Lemma 7.1.** *There are* $(\boldsymbol{Z}_2 \times G)$*-maps*

$$\begin{cases}\varphi_i\colon S(\mathcal{E}_C \underset{C}{\otimes} V_i) \to S(\mathcal{E}_C \underset{C}{\otimes} W_i)\,,\\ \psi_i\colon S(\mathcal{E}_C \underset{C}{\otimes} W_i) \to S(\mathcal{E}_C \underset{C}{\otimes} V_i)\end{cases}$$

*for* $1 \leqq i \leqq r$ *such that*

$$\deg \varphi_i^K = n(i)^{d_i(K)} \quad \textit{and} \quad \deg \psi_i^K = \bar{n}(i)^{d_i(K)}$$

*for each* $K < \boldsymbol{Z}_2 \times G$, *where* $d_i(K) = \dim_C (\mathcal{E}_C \underset{C}{\otimes} V_i)^K$ $(= \dim_C (\mathcal{E}_C \underset{C}{\otimes} W_i)^K)$.

We put

$$\begin{cases}\varphi = \varphi_1 * \cdots * \varphi_r\colon S(\mathcal{E}_C \underset{C}{\otimes} V) \to S(\mathcal{E}_C \underset{C}{\otimes} W)\,,\\ \psi = \psi_1 * \cdots * \psi_r\colon S(\mathcal{E}_C \underset{C}{\otimes} W) \to S(\mathcal{E}_C \underset{C}{\otimes} V)\,.\end{cases} \tag{7.2}$$

Then, for each $K < \boldsymbol{Z}_2 \times G$, we have

$$\deg(\psi \circ \varphi)^K \equiv 1 \bmod 2|G| \quad \text{and} \quad \deg(\varphi \circ \psi)^K \equiv 1 \bmod 2|G|\,.$$

Let $U$ be a unitary $G$-representation space and $m \geqq 2$. We define a homomorphism

$$\Psi\colon [\Sigma^{U \oplus \boldsymbol{R}^{m-1}}, \Sigma^{U \oplus \boldsymbol{R}^{m-1}}]_0^G \to \prod_{(H) \in \mathrm{Iso}(\Sigma^{U \oplus \boldsymbol{R}^{m-1}})} \boldsymbol{Z}$$

by the following: if $f\colon \Sigma^{U \oplus \boldsymbol{R}^{m-1}} \to \Sigma^{U \oplus \boldsymbol{R}^{m-1}}$ is a pointed $G$-map, then $\Psi([f]) = \prod_{(H) \in \mathrm{Iso}(\Sigma^{U \oplus \boldsymbol{R}^{m-1}})} \deg f^H$ (for details see Rubinsztein [22]). By the same argument as in tom Dieck [10; Proposition 1.2.3], we have the following:

**Lemma 7.3.** *Let* $x \in \prod_{(H) \in \mathrm{Iso}(\Sigma^{U \oplus \boldsymbol{R}^{m-1}})} \boldsymbol{Z}$ *be an arbitrary element. Then* $|G|x \in \mathrm{Im}\, \Psi$.

**Proposition 7.4.** *Let* $m > k \geqq 2$. *Let $V$ and $W$ be unitary $G$-representation spaces such that* $\dim_C V^H = \dim_C W^H$ *for all* $H < G$. *Then the following two conditions are equivalent*:

(i) *There is a reduction G-map*

$$f\colon \Sigma^{V \oplus \boldsymbol{R}^{m-1}} \to P_k(V \oplus \boldsymbol{R}^m)\,,$$

(ii) *There is a reduction G-map*

$$g\colon \Sigma^{W \oplus \boldsymbol{R}^{m-1}} \to P_k(W \oplus \boldsymbol{R}^m)\,.$$

Proof. It suffices to show that (i) implies (ii). Let

$$\begin{cases}\varphi\colon S(\mathcal{E}_C \underset{C}{\otimes} V) \to S(\mathcal{E}_C \underset{C}{\otimes} W)\,,\\ \psi\colon S(W) \to S(V)\end{cases}$$

be a $(\boldsymbol{Z}_2\times G)$-map and a $G(\subset\boldsymbol{Z}_2\times G)$-map as in (7.2) respectively. We put a $(\boldsymbol{Z}_2\times G)$-map

$$\varphi_1=\varphi*1_{S(\varepsilon_{\boldsymbol{R}}\otimes\boldsymbol{R}^m)}\colon S((\varepsilon_{\boldsymbol{C}}\underset{\boldsymbol{C}}{\otimes}V)\oplus(\varepsilon_{\boldsymbol{R}}\otimes\boldsymbol{R}^m))\to S((\varepsilon_{\boldsymbol{C}}\underset{\boldsymbol{C}}{\otimes}W)\oplus(\varepsilon_{\boldsymbol{R}}\otimes\boldsymbol{R}^m))$$

and a pointed $G$-map

$$\psi_1=\psi*1_{S(\boldsymbol{R}^m)}\colon S(W\oplus\boldsymbol{R}^m)\to S(V\oplus\boldsymbol{R}^m)\,.$$

Remark that $\varphi_1$ induces a pointed $G$-map

$$\varphi_2\colon P_k(V\oplus\boldsymbol{R}^m)\to P_k(W\oplus\boldsymbol{R}^m)$$

such that the following diagram commutes:

$$\begin{array}{ccc} S((\varepsilon_{\boldsymbol{C}}\underset{\boldsymbol{C}}{\otimes}V)\oplus(\varepsilon_{\boldsymbol{R}}\otimes\boldsymbol{R}^m)) & \xrightarrow{\varphi_1} & S((\varepsilon_{\boldsymbol{C}}\underset{\boldsymbol{C}}{\otimes}W)\oplus(\varepsilon_{\boldsymbol{R}}\otimes\boldsymbol{R}^m)) \\ \downarrow p_1 & & \downarrow p_2 \\ P_k(V\oplus\boldsymbol{R}^m) & \xrightarrow{\varphi_2} & P_k(W\oplus\boldsymbol{R}^m)\,, \end{array}$$

where $p_1$ and $p_2$ are the natural projections as in § 5. We define a pointed $G$-map

$$g_1\colon \Sigma^{W\oplus\boldsymbol{R}^{m-1}}\to P_k(W\oplus\boldsymbol{R}^m)$$

by the composition

$$\Sigma^{W\oplus\boldsymbol{R}^{m-1}}\xrightarrow{d_2} S(W\oplus\boldsymbol{R}^m)\xrightarrow{\psi_1} S(V\oplus\boldsymbol{R}^m)\xrightarrow{d_1}$$
$$\Sigma^{V\oplus\boldsymbol{R}^{m-1}}\xrightarrow{f} P_k(V\oplus\boldsymbol{R}^m)\xrightarrow{\varphi_2} P_k(W\oplus\boldsymbol{R}^m)\,,$$

where $d_1$ and $d_2$ are pointed $G$-homeomorphisms. Let $\pi_1\colon P_k(V\oplus\boldsymbol{R}^m)\to S(V\oplus\boldsymbol{R}^m)$ and $\pi_2\colon P_k(W\oplus\boldsymbol{R}^m)\to S(W\oplus\boldsymbol{R}^m)$ be the natural collapsing maps as in § 5. Let

$$g_2\colon \Sigma^{W\oplus\boldsymbol{R}^{m-1}}\to\Sigma^{W\oplus\boldsymbol{R}^{m-1}}$$

be a $G$-map defined by the composition

$$\Sigma^{W\oplus\boldsymbol{R}^{m-1}}\xrightarrow{g_1} P_k(W\oplus\boldsymbol{R}^m)\xrightarrow{\pi_2} S(W\oplus\boldsymbol{R}^m)\xrightarrow{d_2^{-1}}\Sigma^{W\oplus\boldsymbol{R}^{m-1}}\,.$$

Then it is easy to see that

$$\deg g_2^H\equiv\deg(d_1\circ\pi_1\circ f)^H \mod 2|G| \qquad \text{for each } H<G\,.$$

Since $f$ is a reduction $G$-map, we remark that $\deg(d_1\circ\pi_1\circ f)^H=\pm1$ for each $H<G$. Let $a(H)$ be an integer such that

$$\deg g_2^H=\deg(d_1\circ\pi_1\circ f)^H+2a(H)|G|$$

for each $(H)\in \mathrm{Iso}(\Sigma^{W\oplus \boldsymbol{R}^{m-1}})$. By Lemma 7.3, there is a pointed $G$-map

$$g_3\colon \Sigma^{W\oplus \boldsymbol{R}^{m-1}} \to \Sigma^{W\oplus \boldsymbol{R}^{m-1}}$$

such that $\deg g_3^H = a(H)|G|$ for each $(H)\in \mathrm{Iso}(\Sigma^{W\oplus \boldsymbol{R}^{m-1}})$. We define a pointed $G$-map

$$g_4\colon \Sigma^{W\oplus \boldsymbol{R}^{m-1}} \to P_k(W\oplus \boldsymbol{R}^m)$$

by the composition

$$\Sigma^{W\oplus \boldsymbol{R}^{m-1}} \xrightarrow{g_3} \Sigma^{W\oplus \boldsymbol{R}^{m-1}} \xrightarrow{d_2} S(W\oplus \boldsymbol{R}^m) \xrightarrow{p_2} P_k(W\oplus \boldsymbol{R}^m)\,.$$

Then we see that the composition

$$(\Sigma^{W\oplus \boldsymbol{R}^{m-1}})^H \xrightarrow{g_4^H} P_k(W\oplus \boldsymbol{R}^m)^H \xrightarrow{\pi_2^H} S(W\oplus \boldsymbol{R}^m)^H \xrightarrow{(d_2^{-1})^H} (\Sigma^{W\oplus \boldsymbol{R}^{m-1}})^H$$

has degree $2a(H)|G|$ for each $(H)\in \mathrm{Iso}(\Sigma^{W\oplus \boldsymbol{R}^{m-1}})$. Since $m\geqq 2$, pointed $G$-homotopy classes of pointed $G$-maps from $\Sigma^{W\oplus \boldsymbol{R}^{m-1}}$ to $P_k(W\oplus \boldsymbol{R}^m)$ form a group. Then we put

$$g = g_1 - g_4\colon \Sigma^{W\oplus \boldsymbol{R}^{m-1}} \to P_k(W\oplus \boldsymbol{R}^m)\,.$$

It is easy to see that the composition

$$(\Sigma^{W\oplus \boldsymbol{R}^{m-1}})^H \xrightarrow{g^H} P_k(W\oplus \boldsymbol{R}^m)^H \xrightarrow{\pi_2^H} S(W\oplus \boldsymbol{R}^m)^H \xrightarrow{(d_2^{-1})^H} (\Sigma^{W\oplus \boldsymbol{R}^{m-1}})^H$$

has $\deg(d_1\circ\pi_1\circ f)^H$ $(=\pm 1)$ for each $(H)\in \mathrm{Iso}(\Sigma^{W\oplus \boldsymbol{R}^{m-1}})$. It follows from Lemma 5.2 that $g$ is a reduction $G$-map. q.e.d.

Proof of Theorem 1.1. We may assume that $k\geqq 2$. Let $m$ be an integer such that $m\equiv 0 \bmod a_k(\boldsymbol{R})$ and $m\geqq 2k$. If $\mathrm{Span}_G(S(V))\geqq k-1$, it follows from Theorem 5.3 that $P_k(V\oplus \boldsymbol{R}^m)$ is $G$-$V\oplus \boldsymbol{R}^{m-1}$-reducible. According to Proposition 7.4, $P_k(W\oplus \boldsymbol{R}^m)$ is $G$-$W\oplus \boldsymbol{R}^{m-1}$-reducible. By Theorem 6.1, $\mathrm{Span}_G(S(W))\geqq k-1$.

The converse is quite similar. q.e.d.

## 8. Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

**Lemma 8.1.** *Let $U$ be an orthogonal $G$-representation space such that $\dim_{\boldsymbol{R}} U^H \geqq k+1$ if $U^H \neq \{0\}$ for each $H<G$. Assume that there are an integer $m$ and a $G$-fiber homotopy equivalence*

$$f\colon S((\eta_k\otimes \underline{U})\oplus \underline{\boldsymbol{R}}^m) \to S(\underline{U}\oplus \underline{\boldsymbol{R}}^m)\,.$$

*Then we have a G-fiber homotopy equivalence*

$$\bar{f}\colon S(\eta_k\otimes\underline{U}) \to S(\underline{U})\,.$$

Proof. First we show that the following Assertion 8.1.1.

**Assertion 8.1.1.** *There are an integer $n$ ($\geqq m$) and a G-map*

$$f_1\colon S((\eta_k\otimes\underline{U})\oplus\underline{\boldsymbol{R}}^n) \to S(U\oplus\boldsymbol{R}^n)$$

*such that a restriction*

$$f_1\,|\,S((\eta_k\otimes\underline{U})\oplus\underline{\boldsymbol{R}}^n)_x\colon S((\eta_k\otimes\underline{U})\oplus\underline{\boldsymbol{R}}^n)_x \to S(U\oplus\boldsymbol{R}^n)$$

*for $x\in\boldsymbol{R}P^{k-1}$ is a G-homotopy equivalence and a restriction $f_1\,|\,S(\underline{\boldsymbol{R}}^n)$ is the natural projection $S(\underline{\boldsymbol{R}}^n)\to S(\boldsymbol{R}^n)\subset S(U\oplus\boldsymbol{R}^n)$.*

Proof of Assertion 8.1.1. We put $f_2=p_1\circ f\colon S((\eta_k\otimes\underline{U})\oplus\underline{\boldsymbol{R}}^m)\to S(U\oplus\boldsymbol{R}^m)$, where $p_1\colon S(\underline{U}\oplus\underline{\boldsymbol{R}}^m)\to S(U\oplus\boldsymbol{R}^m)$ is the natural projection.

Suppose first that $U^G\neq\{0\}$. By assumption, we see that $\mathrm{conn}(S(U\oplus\boldsymbol{R}^m)^G)\geqq\dim S(\underline{\boldsymbol{R}}^m)$. Then $f_2\,|\,S(\underline{\boldsymbol{R}}^m)\colon S(\underline{\boldsymbol{R}}^m)\to S(U\oplus\boldsymbol{R}^m)$ and the natural projection $p_2\colon S(\underline{\boldsymbol{R}}^m)\to S(\boldsymbol{R}^m)\subset S(U\oplus\boldsymbol{R}^m)$ are $G$-homotopic. Since $(S((\eta_k\otimes\underline{U})\oplus\underline{\boldsymbol{R}}^m), S(\underline{\boldsymbol{R}}^m))$ has the $G$-homotopy extension property, we have a $G$-map

$$f_1\colon S((\eta_k\otimes\underline{U})\oplus\underline{\boldsymbol{R}}^m) \to S(U\oplus\boldsymbol{R}^m)$$

such that $f_1$ and $f_2$ are $G$-homotopic and $f_1\,|\,S(\underline{\boldsymbol{R}}^m)=p_2$. We put $n=m$. It is easy to see that $f_1$ has our required properties.

Suppose second that $U^G=\{0\}$. Remark that $f_2^G\colon S(\underline{\boldsymbol{R}}^m)\to S(\boldsymbol{R}^m)$ is a map such that $(f_2^G)_x\colon S(\boldsymbol{R}^m)\to S(\boldsymbol{R}^m)$ is a homotopy equivalence for $x\in\boldsymbol{R}P^{k-1}$. It is well-known that there is a map $h\colon S(\underline{\boldsymbol{R}}^{m'})\to S(\boldsymbol{R}^{m'})$ such that $f_2^G\tilde{*}h\colon S(\underline{\boldsymbol{R}}^{m+m'})\to S(\boldsymbol{R}^{m+m'})$ is homotopic to the natural projection $p_3\colon S(\underline{\boldsymbol{R}}^{m+m'})\to S(\boldsymbol{R}^{m+m'})$, where $\tilde{*}$ denotes the fiberwise join. We put

$$f_3=f_2\tilde{*}h\colon S((\eta_k\otimes\underline{U})\oplus\underline{\boldsymbol{R}}^{m+m'}) \to S(U\oplus\boldsymbol{R}^{m+m'})\,.$$

Then $f_3\,|\,S(\underline{\boldsymbol{R}}^{m+m'})=f_3^G$ is ($G$-)homotopic to $p_3$. By the same argument as in the case when $U^G\neq\{0\}$, we have a $G$-map

$$f_1\colon S((\eta_k\otimes\underline{U})\oplus\underline{\boldsymbol{R}}^{m+m'}) \to S(U\oplus\boldsymbol{R}^{m+m'})$$

such that $f_1$ is $G$-homotopic to $f_3$ and $f_1\,|\,S(\underline{\boldsymbol{R}}^{m+m'})=p_3$. We put $n=m+m'$. Then $f_1$ has our required properties.

This completes the proof of Assertion 8.1.1.

We see that $f_1$ induces a $G$-map

$$f_4\colon S(\eta_k\otimes\underline{U})*S(\boldsymbol{R}^n) \to S(U)*S(\boldsymbol{R}^n)$$

such that the following diagram commutes:

$$\begin{array}{ccc} S(\eta_k\otimes\underline{\underline{U}})\tilde{*}S(\underline{\underline{\boldsymbol{R}}}^n) & \overset{f_1}{\searrow} & \\ \downarrow q & & S(U)*S(\boldsymbol{R}^n)\,, \\ S(\eta_k\otimes\underline{\underline{U}})*S(\boldsymbol{R}^n) & \underset{f_4}{\nearrow} & \end{array}$$

where $q$ is the natural projection. Then $f_4|S(\boldsymbol{R}^n)=i_{S(\boldsymbol{R}^n)}\colon S(\boldsymbol{R}^n)\rightarrow S(U)*S(\boldsymbol{R}^n)$. For each $(H)\in\mathrm{Iso}(S(\eta_k\otimes\underline{\underline{U}}))$ $(=\mathrm{Iso}(S(U)))$, we see that $\dim S(\eta_k\otimes\underline{\underline{U}})^H\leqq 2\,\mathrm{conn}(S(U)^H)+1$. It follows from Theorem 2.4 that we obtain a $G$-map

$$f_5\colon S(\eta_k\otimes\underline{\underline{U}})\rightarrow S(U)$$

such that $f_5*1_{S(\boldsymbol{R}^n)}$ is $G$-homotopic to $f_4$. By Equivariant Dold Theorem ([19]), it is easy to see that

$$\bar{f}=p_4\times f_5\colon S(\eta_k\otimes\underline{\underline{U}})\rightarrow \boldsymbol{R}P^{k-1}\times S(U)$$

gives a $G$-fiber homotopy equivalence, where $p_4\colon S(\eta_k\otimes\underline{\underline{U}})\rightarrow\boldsymbol{R}P^{k-1}$ is the natural projection. q.e.d.

Proof of Theorem 1.2. We may assume that $k\geqq 2$. Let $m$ and $n$ be integers such that $m\equiv 0 \bmod a_k(\boldsymbol{R})$, $n\equiv k \bmod a_k(\boldsymbol{R})$ and $n>m\geqq 2k$.

First we show (i). By Theorem 5.3, $P_k(V\oplus\boldsymbol{R}^m)$ is $G$-$V\oplus\boldsymbol{R}^{m-1}$-reducible. Applying Proposition 4.4, $P_k(V\oplus\boldsymbol{R}^n)$ is $G$-$V\oplus\boldsymbol{R}^{n-k}$-coreducible. It follows from Proposition 3.3 that we have a $G$-fiber homotopy equivalence

$$f_1\colon S((\eta_k\otimes(\underline{\underline{V}}\oplus\underline{\underline{\boldsymbol{R}}}^{n-k}))\oplus\underline{\underline{\boldsymbol{R}}}^1)\rightarrow S(\underline{\underline{V}}\oplus\underline{\underline{\boldsymbol{R}}}^{n-k}\oplus\underline{\underline{\boldsymbol{R}}}^1)\,.$$

Since $n\equiv k \bmod a_k(\boldsymbol{R})$ and $n>2k$, we have a $G$-fiber homotopy equivalence

$$f_2\colon S((\eta_k\otimes\underline{\underline{V}})\oplus\underline{\underline{\boldsymbol{R}}}^{n-k+1})\rightarrow S(\underline{\underline{V}}\oplus\underline{\underline{\boldsymbol{R}}}^{n-k+1})\,.$$

The first result follows. The second result follows from Lemma 8.1

Next we show (ii). Since $n\equiv k \bmod a_k(\boldsymbol{R})$ and $n>2k$, we have a $G$-fiber homotopy equivalence

$$f_3\colon S(\eta_k\otimes(\underline{\underline{V}}\oplus\underline{\underline{\boldsymbol{R}}}^{n-k}))\rightarrow S(\underline{\underline{V}}\oplus\underline{\underline{\boldsymbol{R}}}^{n-k})\,.$$

By Proposition 3.3, $P_k(V\oplus\boldsymbol{R}^n)$ is $G$-$V\oplus\boldsymbol{R}^{n-k}$-coreducible. Applying Proposition 4.4, $P_k(V\oplus\boldsymbol{R}^m)$ is $G$-$V\oplus\boldsymbol{R}^{m-1}$-reducible. It follows from Theorem 6.1 that $\mathrm{Span}_G(S(V))\geqq k-1$. q.e.d.

## 9. An example

Let $G$ be a metacyclic group

$$\{a,\, b\,|\,a^m=b^q=e,\; bab^{-1}=a^r\}\;,$$

where $m$ is a positive odd integer, $q$ is an odd prime integer, $(r-1, m)=1$ and $r$ is a primitive $q$-th root of $1 \bmod m$. Let $\boldsymbol{Z}_m=\langle a\rangle<G$ and let $t^h(h\in\boldsymbol{Z})$ be the unitary 1-dimensional $\boldsymbol{Z}_m$-representation space with $a$ acting on $\boldsymbol{C}^1$ as multiplication with $\exp(2\pi h\sqrt{-1}/m)$. Let $T_h$ denote the induced representation space $\mathrm{Ind}\,^G_{\boldsymbol{Z}_m}(t^h)$ of the $\boldsymbol{Z}_m$-representation space $t^h$. Then $T_h$ is a unitary $q$-dimensional $G$-representation space (for details see [9; §47] or [17]). We put

$$V_n = T_{h_1}\oplus T_{h_2}\oplus\cdots\oplus T_{h_n},$$

where $(h_i, m)=1$ for $1\leqq i\leqq n$.

EXAMPLE 9.1. If $n\geqq 9$, then $\mathrm{Span}_G(S(V_n))=\rho(2n, \boldsymbol{R})-1$.
Here $\rho(s, \boldsymbol{R})$ denotes the largest integer $k$ such that $s\equiv 0 \bmod a_k(\boldsymbol{R})$ ([1]).

Proof of Example 9.1. Since $\dim_{\boldsymbol{R}} V_n=2nq$ and $q$ is odd, $\mathrm{Span}(S(V_n))=\rho(2nq, \boldsymbol{R})-1=\rho(2n, \boldsymbol{R})-1$. Thus we have

$$\text{(9.1.1)} \qquad \mathrm{Span}_G(S(V_n))\leqq \mathrm{Span}(S(V_n)) = \rho(2n, \boldsymbol{R})-1.$$

By Becker [6; Theorems 1.1 and 2.2], there is a $\boldsymbol{Z}_m$-fiber homotopy equivalence

$$f_1\colon S(\eta_{\rho(2n,\boldsymbol{R})}\underset{\boldsymbol{R}}{\otimes}\underline{\underline{nt}}) \to S(\underline{\underline{nt}}).$$

By the same argument as in [5; II. Proposition 2.2], we have a $G$-fiber homotopy equivalence

$$f_2\colon S(\eta_{\rho(2n,\boldsymbol{R})}\underset{\boldsymbol{R}}{\otimes}\underline{\underline{nT_1}}) \to S(\underline{\underline{nT_1}}).$$

Since $n\geqq 9$, we see that $\dim_{\boldsymbol{R}} nT_1^H\geqq 2\rho(2n, \boldsymbol{R})$ if $nT_1^H\neq\{0\}$ for each $H<G$. Applying Theorem 1.2, we have $\mathrm{Span}_G(S(nT_1))\geqq\rho(2n, \boldsymbol{R})-1$. It is easy to see that $\dim_{\boldsymbol{C}} V_n^H=\dim_{\boldsymbol{C}} nT_1^H$ for all $H<G$. Thus it follows from Theorem 1.1 that we have

$$\text{(9.1.2)} \qquad \mathrm{Span}_G(S(V_n))\geqq\rho(2n, \boldsymbol{R})-1.$$

Combining (9.1.1) and (9.1.2), we have $\mathrm{Span}_G(S(V_n))=\rho(2n, \boldsymbol{R})-1$. q.e.d.

Added in proof. Professor P. May kindly informed me that Dr. U. Namboodiri has obtained similar results [30].

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## References

[1] J.F. Adams: *Vector fields on spheres,* Ann. of Math. **75** (1962), 603–632.
[2] ———: *On the groups J(X)-II,* Topology **3** (1965), 137–171.
[3] M.F. Atiyah: *Thom complexes,* Proc. London Math. Soc. **11** (1961), 291–310.

[4] ———, R. Bott and A. Shapiro: *Clifford modules*, Topology **3** (1964), Suppl. 1, 3–38.

[5] ——— and D.O. Tall: *Group representations, λ-rings and the J-homomorphism*, Topology **8** (1969), 253–297.

[6] J.C. Becker: *The Span of spherical space forms*, Amer. J. Math. **94** (1972), 991–1026.

[7] E. Bierstone: *The equivariant covering homotopy property for differentiable G-fiber bundles*, J. Differential Geom. **8** (1973), 615–622.

[8] G.E. Bredon: Equivariant cohomology theories, Lecture Notes in Math. 34, Springer-Verlag, 1967.

[9] C.W. Curtis and I. Reiner: Representation theory of finite groups and associative algebras, Interscience, 1962.

[10] T. tom Dieck: Transformation groups and representation theory, Lecture Notes in Math. 766, Springer-Verlag, 1979.

[11] ——— and T. Petrie: *Geometric modules over the Burnside ring*, Invent. Math. **47** (1978), 273–287.

[12] S. Illman: *Smooth equivariant triangulations of G-manifolds for G a finite group*, Math. Ann. **233** (1978), 199–220.

[13] K. Iwata: *Span of lens spaces*, Proc. Amer. Math. Soc. **26** (1970), 687–688.

[14] I.M. James: *The intrinsic join: A study of the homotopy groups of Stiefel manifolds*, Proc. London Math. Soc. **8** (1958), 507–535.

[15] ———: *Cross-sections of Stiefel manifolds*, Proc. London Math. Soc. **8** (1958), 536–547.

[16] ———: *Spaces associated with Stiefel manifolds*, Proc. London Math. Soc. **9** (1959), 115–140.

[17] S. Kakutani: *On the groups $J_{Z_{m,q}}(*)$*, Osaka J. Math. **17** (1980), 513–524.

[18] ——— and M. Murayama: *Equivariant duality, reducibility and coreducibility*, preprint 1981.

[19] K. Kawakubo: *Compact Lie group actions and fiber homotopy type*, J. Math. Soc. Japan **33** (1981), 295–321.

[20] K. Komiya and M. Morimoto: *Equivariant desuspension of G-maps*, Osaka J. Math. **18** (1981), 525–532.

[21] C.N. Lee and A.G. Wasserman: On the groups $JO(G)$, Memoirs of Amer. Math. Soc. 159 (1975).

[22] R.L. Rubinsztein: On the equivariant homotopy of spheres, Dissertationes Math. 134 (1976).

[23] D. Sjerve: *Vector bundles over orbit manifolds*, Trans. Amer. Math. Soc. **138** (1969), 97–106.

[24] E.H. Spanier: *Function spaces and duality*, Ann. of Math. **70** (1959), 338–378.

[25] J. Tornehave: *Equivariant maps of spheres with conjugate orthogonal actions*, Aarhus Universitet preprint series No. 24 (1976/77).

[26] S. Waner: *Equivariant homotopy theory and Milnor's theorem*, Trans. Amer. Math. Soc. **258** (1980), 351–368.

[27] A.G. Wasserman: *Equivariant differential topology*, Topology **8** (1969), 127–150.

[28] K. Wirthmüller: *Equivariant S-duality*, Arch. Math. **26** (1975), 427–431.

[29] T. Yoshida: *A remark on vector fields on lens spaces*, J. Sci. Hiroshima Univ. Ser. A-I **31** (1967), 13–15.
[30] U. Namboodiri: *Equivariant vector fields on spheres*, Dr. Phil. Thesis, Univ. of Chicago, 1982.

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