<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>The equivariant span of the unit spheres in representation spaces</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Kakutani, Shin-ichiro</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 20(2) P.439-P.460</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1983</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/9625">https://doi.org/10.18910/9625</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/9625</td>
</tr>
<tr>
<td><strong>rights</strong></td>
<td></td>
</tr>
</tbody>
</table>
THE EQUIVARIANT SPAN OF THE UNIT SPHERES IN REPRESENTATION SPACES

SHIN-ICHIRO KAKUTANI

(Received September 21, 1981)

1. Introduction

Let $G$ be a finite group and $M$ be a smooth $G$-manifold. We define $\text{Span}_G(M)$ to be the largest integer $k$ such that $M$ has $k$ linearly independent smooth $G$-vector fields. Let $V$ be an orthogonal $G$-representation space and let $S(V)$ denote the unit sphere in $V$. In the case where $G$ acts freely on $S(V)$, $\text{Span}_G(S(V)) (= \text{Span}(S(V)/G))$ has been studied by Becker [6], Iwata [13], Sjerve [23] and Yoshida [29]. In this paper, we consider $\text{Span}_G(S(V))$ when $G$ does not act freely on $S(V)$. Our main results are Theorems 1.1 and 1.2, which are generalizations of Theorems 2.1 and 2.2 in [6] respectively. Our method is due to Becker [6].

Let $H$ be a subgroup of $G$, then we write $H < G$.

**Theorem 1.1.** Let $G$ be a finite group and let $V, W$ be unitary $G$-representation spaces. Suppose that
(i) $\dim_H V = \dim_H W$ for all $H < G$,
(ii) For each $H < G$, $\dim_H V^H \geq 2k$ if $V^H \neq \{0\}$.
Then $\text{Span}_G(S(V)) \geq k-1$ if and only if $\text{Span}_G(S(W)) \geq k-1$.

Let $\xi$ and $\eta$ be orthogonal $G$-vector bundles over a compact $G$-space. Denote by $S(\xi)$ (resp. $S(\eta)$) the unit sphere bundle of $\xi$ (resp. $\eta$). Then $S(\xi)$ and $S(\eta)$ are said to be $G$-fiber homotopy equivalent if there are fiber-preserving $G$-maps:
$$ f: S(\xi) \to S(\eta), \quad f': S(\eta) \to S(\xi) $$
such that $f \circ f'$ and $f' \circ f$ are fiber-preserving $G$-homotopic to the identity ([6], [19]).

Let $\mathbb{R}P^{k-1}$ denote the $(k-1)$-dimensional real projective space with trivial $G$-action and let $\eta_k$ denote the non-trivial line bundle over $\mathbb{R}P^{k-1}$ with trivial $G$-action.

**Theorem 1.2.** Let $G$ be a finite group and let $V$ be an orthogonal $G$-representation space. Then we have the following:
4. Preliminary results

First we shall fix some notations. Let $X$ and $Y$ be $G$-spaces. Let $A$ be a $G$-subspace of $X$ and let $\alpha: A \to Y$ be a $G$-map. Denote by $F((X, A), Y; \alpha)$ the space of all maps $f: X \to Y$ such that $f|_A = \alpha$ in the compact open topology. $F((X, A), Y; \alpha)$ is a $G$-space with the following $G$-action: if $f: X \to Y$ and $g \in G$, we put

$$(g \cdot f)(x) = g(f(g^{-1}x)).$$

For $H < G$, $X^H$ denotes the $H$-fixed point set in $X$. The set $F((X, A), Y; \alpha)^G$ is just the set of $G$-maps $f: X \to Y$ such that $f|_A = \alpha$. Denote by $[(X, A), Y; \alpha]^G$ the set of $G$-homotopy classes rel $A$ of $G$-maps $f: X \to Y$ such that $f|_A = \alpha$. If $A = \phi$, we write $F(X, Y)$ (resp. $[X, Y]^G$) instead of $F((X, A), Y; \alpha)$ (resp. $[(X, A), Y; \alpha]^G$), for simplicity. If $X, Y$ are $G$-spaces with base points, then we denote the set of $G$-homotopy classes relative to the base points of pointed $G$-maps from $X$ to $Y$ by $[X, Y]_G^G$. The base points are $G$-fixed points as usual. For $H < G$, $(H)$ denotes the conjugacy class of $H$ in $G$. Denote by $G_x$ the isotropy group at $x \in X$ and we put
For a space $Z$, we define $\text{conn}(Z)$ to be the largest integer $n$ such that $Z$ is $n$-connected. In particular, when $Z$ is not path-connected (resp. $Z=\phi$), we put $\text{conn}(Z)=-1$ (resp. $\text{conn}(Z)=\infty$).

The following two lemmas are easily seen by the definition of $G$-complexes (see Bredon [8] and Waner [26]).

**Lemma 2.1.** Let $f: X\to Y$ be a $G$-map of $G$-spaces such that $f^H=f|X^H$: $X^H\to Y^H$ is an $n_H$-equivalence for each $H< G$. Let $(K, L)$ be a pair of $G$-complexes and $\alpha: L\to X$ be a $G$-map. Then

$$f^*_\alpha: [(K, L), X; \alpha]_G \to [(K, L), Y; f\circ \alpha]_G$$

is surjective if $\dim(K^H-L)\leq n_H$ and bijective if $\dim(K^H-L)\leq n_H-1$ for each $(H)\in \text{Iso}(K-L)$.

**Lemma 2.2.** Let $(K, L)$ be a pair of $G$-complexes and $X$ be a $G$-space. Let $\alpha: L\to X$ be a $G$-map. Then the $G$-fixed point morphism

$$\phi_\alpha: [(K, L), X; \alpha]_G \to [(K^G, L^G), X^G; \alpha^G]$$

is surjective if $\dim(K^H-L\cup K^G)\leq \text{conn}(X^H)+1$ and bijective if $\dim(K^H-L\cup K^G)\leq \text{conn}(X^H)$ for each $(H)\in \text{Iso}(K-L\cup K^G)$.

**Definition 2.3.** Let $X$ be a $G$-space. Then $X$ is said to be $G$-path-connected if and only if $\text{conn}(X^H)\geq 0$ for all $H< G$.

Let $X$ and $Y$ be $G$-spaces. We recall that the join $X*Y$ is the space obtained from the union of $X$, $Y$ and $X\times Y\times [0,1]$ by identifying

$$(x, y, 0) = x, \quad (x, y, 1) = y \quad \text{ for } x\in X, y\in Y.$$  

We generally omit to write in the identification map, so that the image of $(x, y, t)$ in $X*Y$ is denoted by the same expression. A canonical $G$-action on $X*Y$ is given by $g*(x, y, t)=(gx, gy, t)$. Let $V$ be an orthogonal $G$-representation space. We see that

$$(X*Y)^H = X^H*Y^H$$

and

$$\text{conn}((X*S(V))^H) = \text{conn}(X^H)+\dim R V^H$$

for $H< G$. Let $i_{S(V)}: S(V)\to X*S(V)$ be an inclusion map defined by $i_{S(V)}(v)=(-, v, 1)$. We have the following theorem (cf. [18; Theorem 3.6], [20]):

**Theorem 2.4.** Let $K$ be a $G$-complex and $X$ be a $G$-space. Let $V$ be
an orthogonal $G$-representation space. Assume that $\text{conn}(X^H) \geq 0$ for each $(H) \in \text{Iso}(K)$. Then the suspension map

$$
\tau^G_{*} : [K, X]^G \rightarrow [(K \ast S(V), S(V)), X \ast S(V); i_{s(V)}]^G
$$

is surjective if $\dim K^H \leq n^H$ and bijective if $\dim K^H \leq n^H - 1$ for each $(H) \in \text{Iso}(K)$, where

$$
n^H = \min \begin{cases} 
2 \text{conn}(X^H) + 1 & \text{if } H = L \text{ and } V^H \neq \{0\}, \\
\text{conn}(X^L) & \text{if } V^H = V^L, \\
\infty & \text{otherwise}.
\end{cases}
$$

Proof. Let $D(V)$ denote the unit disk in $V$. We define a $G$-map

$$
\lambda : X \rightarrow F((D(V), S(V)), X \ast S(V); i_{s(V)})
$$

by $\lambda(x)(tv) = (x, v, t)$ for $x \in X$, $v \in S(V)$, $t \in [0, 1]$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
[K, X]^G & \xrightarrow{\tau^G_{*}} & [(K \ast S(V), S(V)), X \ast S(V); i_{s(V)}]^G \\
\downarrow \phi & & \downarrow \\
[K, F((D(V), S(V)), X \ast S(V); i_{s(V)})]^G
\end{array}
$$

where $\phi$ is the exponential correspondence given by

$$
\phi(f)(k)(tv) = f(k, v, t) \quad \text{for } k \in K, v \in S(V), t \in [0, 1].
$$

As is easily seen, $\phi$ is bijective. Using Lemma 2.2, we see that

$$
\lambda^H : X^H \rightarrow F((D(V), S(V)), X \ast S(V); i_{s(V)})^H
$$

is an $n^H$-equivalence for each $(H) \in \text{Iso}(K)$ by the same argument as in the proof of Theorem 3.6 in [18]. We are now in a position to apply Lemma 2.1.

q.e.d.

3. Equivariant duality, reducibility and coreducibility

In this section, we recall the definitions of equivariant duality, reducibility and coreducibility (see [18] and [26]) and consider an equivariant version of Atiyah's duality theorem. Let $X$ and $Y$ be pointed $G$-spaces. The reduced join $X \land Y$ has a natural $G$-action induced from the diagonal action on $X \times Y$. For an orthogonal $G$-representation space $V$, $\Sigma^V$ denotes the one-point compactification of $V$ and $\Sigma^V X = \Sigma^V \land X$ is called $\Sigma^V$-suspension of $X$. We remark that $\Sigma^V$ is a pointed finite $G$-complex ([12]).

**Definition 3.1.** Let $X$ and $X^*$ be $G$-path-connected pointed finite $G$-
Let $U$ be an orthogonal $G$-representation space. Then a pointed $G$-map

$$\mu: \Sigma^U \to X \wedge X^*$$

is said to be a $(U)$-duality $G$-map if $\mu^H: \Sigma^H \to X^H \wedge X^{*H}$ is a duality map in the usual sense ([6], [24]) for each $H<G$.

**Definition 3.2.** Let $X$ be a $G$-path-connected pointed finite $G$-complex and $V$ be an orthogonal $G$-representation space.

(i) A pointed $G$-map $f: \Sigma^V \to X$ is said to be a $(V)$-reduction $G$-map if $f^H: \Sigma^V \to X^H$ is a reduction map in the usual sense ([3]) for each $H<G$, and then $X$ is called $G$- $(V)$-reducible.

(ii) A pointed $G$-map $f: X \to \Sigma^V$ is said to be a $(V)$-coreduction $G$-map if $f^H: X^H \to \Sigma^V$ is a coreduction map in the usual sense ([3]) for each $H<G$, and then $X$ is called $G$- $(V)$-coreducible.

Let $M$ be a path-connected closed smooth manifold with trivial $G$-action. Let $\xi$ be a smooth $G$-vector bundle over $M$. The fibers $\xi_x$ for $x \in M$ are orthogonal $G$-representation spaces. Since $M$ is path-connected, $\xi_x$ does not depend on the choice of $x \in M$. So we put $V = \xi_x$. Assume that $V^G \neq \{0\}$. Then $T(\xi)$ is a $G$-path-connected pointed finite $G$-complex ([12]), where $T(\xi)$ denotes the Thom space of $\xi$.

**Proposition 3.3.** If $T(\xi)$ is $G$-$V$-coreducible, then there is a $G$-fiber homotopy equivalence $f: S(\xi \oplus B^1) \to S(V \oplus B^1)$. Conversely, if there is a $G$-fiber homotopy equivalence $f: S(\xi) \to S(V)$, then $T(\xi)$ is $G$-$V$-coreducible.

Using Equivariant Dold Theorem (Kawakubo [19; Theorem 2.1]) and Equivariant J.H.C. Whitehead Theorem (Bredon [8; Chap. II Corollary (5.5)]), the proof is almost parallel to that of Proposition 2.8 in [3]. So we omit it.

Let $\omega$, $\xi_1$ and $\xi_2$ be smooth $G$-vector bundles over $M$. We put $V = \omega_x$, $W_1 = (\xi_1)_x$ and $W_2 = (\xi_2)_x$ for $x \in M$. Assume that $V^G \neq \{0\}$, $W_1^G \neq \{0\}$ and $W_2^G \neq \{0\}$. Then $T(\omega)$, $T(\xi_1)$ and $T(\xi_2)$ are $G$-path-connected pointed finite $G$-complexes.

**Lemma 3.4.** If there are a reduction $G$-map $\alpha: \Sigma^V \to T(\omega)$ and a coreduction $G$-map $\beta: T(\xi_1 \oplus \xi_2) \to \Sigma^{W_1 \oplus W_2}$, then there is a duality $G$-map

$$\mu: \Sigma^{W_1 \oplus W_2} \to T(\xi_1) \wedge T(\xi_2 \oplus \omega).$$

Using Equivariant J.H.C. Whitehead Theorem ([8]), the proof is quite similar to that of (13.2) in [6]. So we omit it.
4. Linear actions on stunted projective spaces

Let $V$ be an orthogonal $G$-representation space and $\varepsilon_R$ be the non-trivial orthogonal 1-dimensional $Z_2$-representation space. Then $\varepsilon_R \otimes V$ is an orthogonal $(Z_2 \times G)$-representation space.

**Definition 4.1.**
(i) $R^P(V) = \{\varepsilon_R \otimes V\} / (Z_2 \times \{e\})$,
(ii) For $m \geq k$, $P_R(V \oplus R^m) = R^P(V \oplus R^m) / R^P(V \oplus R^{m-k})$.

Then $P_R(V \oplus R^m)$ is a pointed finite $G$-complex ([12]). We see that, if $m > k$, then for $H < G$

$$P_R(V \oplus R^m)_H = P_R(V^H \oplus R^m),$$

$$\dim P_R(V \oplus R^m)_H = \dim_R V^H + m - 1$$

and

$$\text{conn}(P_R(V \oplus R^m)_H) = \dim_R V^H + m - k - 1.$$  

In particular, if $m > k$, then $P_R(V \oplus R^m)$ is $G$-path-connected. Atiyah [3] identifies the Thom space of a multiple of $\eta_k$ as a stunted projective space. As $G$-spaces this identification takes the form

$$T(\eta_k \otimes (V \oplus R^{m-1})) = P_R(V \oplus R^m).$$

Let $a_k(R) (k > 0)$ be the integer defined by [4; § 5]. We recall that the group $J(R)$ is cyclic of order $a_k(R) ([1], [2])$. We remark that $a_k(R) \geq k$ for $k > 0$.

**Lemma 4.2.** Let $m$, $n$ and $k$ be integers such that $m \equiv 0 \mod a_k(R)$, $n \equiv k \mod a_k(R)$ and $n > m \geq 2k \geq 4$. Let $U$ be an arbitrary orthogonal $G$-representation space. Then we have the following:

(i) If $\Sigma^m P_R(V \oplus R^m)$ is $G-U \oplus V \oplus R^{m-1}$-reducible, then there is a duality $G$-map

$$\mu_1: \Sigma R^{m-1} \oplus V \oplus R^n \rightarrow P_R(R^m) \setminus \Sigma^m P_R(V \oplus R^m),$$

(ii) If $\Sigma^m P_R(V \oplus R^m)$ is $G-U \oplus V \oplus R^{m-1}$-coreducible, then there is a duality $G$-map

$$\mu_2: \Sigma U \oplus V \oplus R^{m-1} \oplus R^n \rightarrow \Sigma^m P_R(V \oplus R^m) \setminus P_R(R^n).$$

**Proof.** We remark that

$$T(U \oplus (\eta_k \otimes (V \oplus R^{m-k}))) = \Sigma^m P_R(V \oplus R^m),$$

$$T(U \oplus (\eta_k \otimes (V \oplus R^{m-k}))) = \Sigma^m P_R(V \oplus R^n).$$

First we show (i). By assumption, there is a reduction $G$-map

$$\alpha: \Sigma^m U \oplus V \oplus R^{m-1} \rightarrow T(U \oplus (\eta_k \otimes (V \oplus R^{m-k}))).$$
EQUIVARIANT SPAN OF THE UNIT SPHERES

Set

\[ \omega = U \oplus (\eta_k \otimes (V \oplus R^{m-k})) \], \hspace{1em} \xi_1 = \eta_k \otimes R^{m-k}, \hspace{1em} \xi_2 = \eta_k \otimes R^{n-m}. \]

Since \( \xi_1 \oplus \xi_2 \) is trivial, there is a coreduction \((G-)\)map

\[ \beta: T(\xi_1 \oplus \xi_2) \to \Sigma R^{n-k}. \]

Applying Lemma 3.4 to \( \alpha, \beta, \omega, \xi_1 \) and \( \xi_2 \), we have a duality \( G \)-map \( \mu_1 \).

Next we show (ii). By assumption, there is a coreduction \( G \)-map

\[ \beta: T(U \oplus (\eta_k \otimes (V \oplus R^{m-k}))) \to \Sigma U \oplus V \oplus R^{n-k}. \]

Since \( m \equiv 0 \pmod{a(R)} \) and \( m \geq 2k \), there is a reduction \((G-)\)map

\[ \alpha: \Sigma R^{n-1} \to T(\eta_k \otimes R^{m-k}). \]

Set

\[ \omega = \eta_k \otimes R^{m-k}, \hspace{1em} \xi_1 = U \oplus (\eta_k \otimes (V \oplus R^{m-k})), \hspace{1em} \xi_2 = \eta_k \otimes R^{n-m}. \]

Applying Lemma 3.4, we have a duality \( G \)-map \( \mu_2 \).

**Lemma 4.3.** Let \( m \) and \( k \) be integers such that \( m \geq k > 0 \). Let \( V \) be an orthogonal \( G \)-representation space. Assume that \( P_h(V \oplus R^m) \) is either \( G-V \oplus R^{m-1} \)-reducible or \( G-V \oplus R^{m-k} \)-coreducible. Then we have

\[ \dim_R V^K - \dim_R V^H \geq k \]

if \( V^K \not= V^H \) for \( K < H < G \).

**Proof.** Let \( K < H < G \) such that \( V^K \not= V^H \). First we assume that \( P_h(V \oplus R^m) \) is \( G-V \oplus R^{m-1} \)-reducible. Then, by definition, \( P_h(V^H \oplus R^m) \) and \( P_h(V^K \oplus R^m) \) are reducible. It follows from Atiyah [3; Theorem 6.2] that \( \dim_R V^H \equiv m \equiv 0 \pmod{a_h(R)} \) and \( \dim_R V^K \equiv m \equiv 0 \pmod{a_h(R)} \). Thus we see that \( \dim_R V^K - \dim_R V^H \equiv 0 \pmod{a_h(R)} \). Now we have

\[ \dim_R V^K - \dim_R V^H \geq 0 \]

Next we assume that \( P_h(V \oplus R^m) \) is \( G-V \oplus R^{m-k} \)-coreducible. Then \( P_h(V^H \oplus R^m) \) and \( P_h(V^K \oplus R^m) \) are coreducible. By Atiyah [3; Proposition 2.8], we have

\[ J(\eta_k \otimes (V^H \oplus R^{m-k})) - (V^H \oplus R^{m-k}) = 0 \]

in \( \hat{J}(R^{P_{-1}}) \),

\[ J(\eta_k \otimes (V^K \oplus R^{m-k})) - (V^K \oplus R^{m-k}) = 0 \]

in \( \hat{J}(R^{P_{-1}}) \).

Thus we obtain that \( \dim_R V^K - \dim_R V^H \equiv 0 \pmod{a_h(R)} \). Now we see that
Proposition 4.4. Let $m$, $n$ and $k$ be integers such that $m \equiv 0 \mod a_k(R)$, $n \equiv k \mod a_k(R)$ and $n > m^2 k^4$. Let $V$ be an orthogonal $G$-representation space. Then the following two conditions are equivalent:

(i) $P_k(V \oplus R^m)$ is $G$-V@R$^m$-reducible,

(ii) $P_k(V \oplus R^n)$ is $G$-V@R$^n$-coreducible.

Proof. First we show that (i) implies (ii). By Lemma 4.2, there is a duality $G$-map

$$\mu_1: \Sigma R^{m-1} \oplus V \oplus R^m \to P_k(R^m) \wedge P_k(V \oplus R^n).$$

We put $U = V \oplus R$. For $s > 0$, we define a homomorphism

$$\Gamma_s(\mu_1): [\Sigma^s P_k(V \oplus R^n), \Sigma^s V \oplus R^m]_0 \to [\Sigma^s \Sigma R^{m-1} \oplus V \oplus R^m, \Sigma^s P_k(R^m) \wedge \Sigma V \oplus R^m]_0$$

by the following: if $\cdot: \Sigma^s P_k(V \oplus R^n) \to \Sigma^s \Sigma^s V \oplus R^m$ is a pointed $G$-map, then $\Gamma_s(\mu_1)(\cdot)$ is represented by the composition

$$\begin{align*}
\Sigma^s P_k(R^m) \wedge \Sigma^s P_k(V \oplus R^n) \xrightarrow{1 / \mu_1} \Sigma^s P_k(R^m) \wedge P_k(V \oplus R^n) \xrightarrow{T_1} P_k(R^m) \wedge \Sigma^s \Sigma V \oplus R^m \xrightarrow{1 / f} P_k(R^m) \wedge \Sigma^s V \oplus R^m \wedge \Sigma V \oplus R^m,
\end{align*}$$

where $T_1$ and $T_2$ are the switching maps. Then we have the following:

Assertion 4.4.1. If $s > \dim R V + m + n + 1$, then $\Gamma_s(\mu_1)$ is an isomorphism.

The proof is quite similar to that of Assertion 4.1.1 in [18]. So we omit it.

On the other hand, the standard identification

$$\nu_1: \Sigma R^{m-1} \oplus V \oplus R^m \to \Sigma \Sigma V \oplus R^m$$

is a duality $G$-map. We define a homomorphism

$$\Gamma_s(\nu_1): [\Sigma^s \Sigma R^{m-1}, \Sigma^s P_k(R^m)]_0 \to [\Sigma^s \Sigma R^{m-1} \oplus V \oplus R^m, \Sigma^s P_k(R^m) \wedge \Sigma V \oplus R^m]_0$$

by the following: if $\cdot: \Sigma^s \Sigma R^{m-1} \to \Sigma^s P_k(R^m)$ is a pointed $G$-map, then $\Gamma_s(\nu_1)(\cdot) = [\cdot]$, where $\cdot$ is the composition

$$\begin{align*}
\Sigma^s \Sigma R^{m-1} \oplus V \oplus R^m \xrightarrow{1 / \nu_1} \Sigma^s \Sigma R^{m-1} \wedge \Sigma V \oplus R^m \wedge \Sigma V \oplus R^m \xrightarrow{1 / f} \Sigma^s P_k(R^m) \wedge \Sigma V \oplus R^m.
\end{align*}$$

For $s > \dim R V + m + n + 1$, we put

$$D_s(\nu_1, \mu_1) = \Gamma_s(\mu_1)^{-1} \circ \Gamma_s(\nu_1): [\Sigma^s \Sigma R^{m-1}, \Sigma^s P_k(R^m)]_0 \to [\Sigma^s P_k(V \oplus R^n), \Sigma^s \Sigma V \oplus R^m]_0.$$
Since $m \equiv 0 \mod a_k(R)$ and $m \geq 2k$, there is a reduction $(G)$-map $f_1: \Sigma R^{*k} \to P_k(R^n)$. Let $f_2: \Sigma^G P_k(V \oplus R^n) \to \Sigma^G \Sigma^G R^{*k}$ be a pointed $G$-map such that $D_i(v, \mu_v)([1 \wedge f_1]) = [f_2]$. As is easily seen, $f_2$ is a coreduction $G$-map. Here we consider the suspension map

$$\sigma^G_{*}: [P_k(V \oplus R^n), \Sigma^G \Sigma^G R^{*k}] \to [\Sigma^G P_k(V \oplus R^n), \Sigma^G \Sigma^G R^{*k}]^G.$$ 

Let $K < H < G$ such that $(sU)^H = (sU)^K$. Since $U = V \oplus R^1$, we see that $V^H = V^K$. Applying Lemma 4.3, we have

$$\begin{cases}
\dim (P_k(V \oplus R^n)^H) = \dim R V^H + n - 1, \\
2 \operatorname{conn} ((\Sigma^G R^{*k})^H) + 1 = 2(\dim R V^H + n - k - 1) + 1 \geq \dim R V^H + n - 1, \\
\operatorname{conn} ((\Sigma^G R^{*k})^H) = \dim R V^K + n - k - 1 \geq \dim R V^H + n - 1.
\end{cases}$$

By the suspension theorem [18; Theorem 3.6], we see that $\sigma^G_{*}$ is surjective. Let $f_3: P_k(V \oplus R^n) \to \Sigma^G \Sigma^G R^{*k}$ be a pointed $G$-map such that $\sigma^G_{*}([f_3]) = [f_3]$. Then it is easy to see that $f_3$ is also a coreduction $G$-map. That is, $P_k(V \oplus R^n)$ is $G$-$V \oplus R^{*k}$-coreducible. Similarly, using $\mu_2$ in Lemma 4.2, we see that (ii) implies (i). q.e.d.

5. An equivariant version of the theorem of James

First we fix some notations. Let $V_k(V)$ denote the Stiefel manifold of orthogonal $k$-frames in an orthogonal $G$-representation space $V$ with $G$-action defined by

$$g \cdot (v_1, \ldots, v_k) = (gv_1, \ldots, g v_k).$$

Then $V_k(V)$ is a smooth $G$-manifold. If $\dim R V^H \geq k$ for some $H \leq G$, then we see that

$$V_k(V)^H = V_k(V^H)$$

and

$$\operatorname{conn}(V_k(V)^H) = \dim R V^H - k - 1.$$

Let

$$q_k: V_k(V) \to S(V)$$

send $(v_1, \ldots, v_k)$ to $v_k$. We remark that $q_k: V_k(V) \to S(V)$ is a smooth $G$-fiber bundle in the sense of Bierstone [7]. We remark the following:

**Lemma 5.1.** $\operatorname{Span}(S(V)) \geq k - 1$ if and only if $q_k: V_k(V) \to S(V)$ has a smooth $G$-cross-section.

Let $m > k > 0$. There is a well-known mapping
τₖ: Pₖ(V ⊕ ℜⁿ) → Vₖ(V ⊕ ℜᵐ)

by

\[ \tauₖ([x]) = (e_{n+m-k-2}x, e_{n+m-k+1}x, \ldots, e_{n+m}x) \]

where \( n = \text{dim}_R V \) and \( e_i \) denotes the \( i \)-th unit vector in \( V ⊕ ℜ^n \). We see that \( \tauₖ \) is a \( G \)-map and for \( H < G \)

\[ \tauₖ^H: Pₖ(V ⊕ ℜ^n)^H → Vₖ(V ⊕ ℜ^n)^H \]

is a 2(\( \text{dim}_R V + m - k \))-equivalence (see James [16; Lemma 8.1]). We remark that \( τ_i: P_i(V ⊕ ℜ^n) → S(V ⊕ ℜ^n) (= V_i(V ⊕ ℜ^n)) \) is a \( G \)-homeomorphism. Let

\[ p: S(V ⊕ ℜ^n) → Pₖ(V ⊕ ℜ^n) \]

and

\[ π': Pₖ(V ⊕ ℜ^n) → P_i(V ⊕ ℜ^n) \]

be the natural projection and the collapsing map respectively. For \( S(V ⊕ ℜ^n) \), we choose a base point \( x_0 ∈ S(R^{n+1}) ⊂ S(V ⊕ ℜ^n) \). There is a pointed \( G \)-map \( u: P_i(V ⊕ ℜ^n) → S(V ⊕ ℜ^n) \) such that \( u \) and \( τ_i \) are \( G \)-homotopic. We put

\[ π = u \circ π': Pₖ(V ⊕ ℜ^n) → S(V ⊕ ℜ^n) \]

Then \( p \) and \( π \) are pointed \( G \)-maps.

**Lemma 5.2.** Let \( m ≫ k ≫ 0 \). Let \( f: S(V ⊕ ℜ^n) → Pₖ(V ⊕ ℜ^n) \) be a pointed \( G \)-map. Then \( f \) is a reduction \( G \)-map if and only if the composition

\[ S(V ⊕ ℜ^n)^H \xrightarrow{f^H} Pₖ(V ⊕ ℜ^n)^H \xrightarrow{π^H} S(V ⊕ ℜ^n)^H \]

is an ordinary homotopy equivalence (i.e. has degree \( \pm 1 \)) for each \( (H) ∈ \text{Iso}(S(V ⊕ ℜ^n)) \).

The proof is easy.

A \( G \)-homeomorphism

\[ h: S(V) * S(ℜ^n) → S(V ⊕ ℜ^n) \]

is given by \( h(x, y, t) = (x \cdot \cos(πt/2), y \cdot \sin(πt/2)) \). In [14], James defined the intrinsic map

\[ μ: Vₖ(V) * Vₖ(ℜ^n) → Vₖ(V ⊕ ℜ^n) \]

We see that \( μ \) is a \( G \)-map and the following diagram commutes:

\[ \begin{CD}
Vₖ(V) * Vₖ(ℜ^n) @>{μ}>> Vₖ(V ⊕ ℜ^n) \\
@VV{qₖ}V \downarrow{qₖ}
S(V) * S(ℜ^n) @>{h}>> S(V ⊕ ℜ^n) \end{CD} \]
Now we prove the following theorem, which is a generalization of Proposition 11.5 in [6] (see also Theorem 8.2 in [16]):

**Theorem 5.3.** Let $m$ and $k$ be integers such that $m \equiv 0 \mod a_k(R)$ and $m \geq 2k \geq 4$. If $\text{Span}_G(S(V)) \geq k - 1$, then $P_k(V \oplus R^m)$ is $G \cdot V \oplus R^{m-1}$-reducible.

Proof. Since $m \equiv 0 \mod a_k(R)$ and $m \geq 2k$, there is a reduction $(G,\pi)$-map $\rho: S(R^m) \rightarrow P_k(R^m)$. It follows from Lemma 5.1 that there is a $G$-cross-section of $q_k$

$$\Delta: S(V) \rightarrow V_k(V).$$

Then we define a $G$-map

$$\gamma: S(V \oplus R^m) \rightarrow V_k(V \oplus R^m)$$

by the composition

$$S(V \oplus R^m) \xrightarrow{h^{-1}} S(V) \ast S(R^m) \xrightarrow{\Delta \ast \rho} V_k(V) \ast P_k(R^m) \xrightarrow{\mu} V_k(V \oplus R^m).$$

Consider a map

$$\tau_{k^*}: [S(V \oplus R^m), P_k(V \oplus R^m)]^G \rightarrow [S(V \oplus R^m), V_k(V \oplus R^m)]^G.$$

Since $\tau_{k^*}^H: P_k(V \oplus R^m)^H \rightarrow V_k(V \oplus R^m)^H$ is a $2(\dim_R V^H + m - k)$-equivalence for each $H < G$, it follows from Lemma 2.1 that $\tau_{k^*}$ is bijective. Moreover we see that

$$[S(V \oplus R^m), P_k(V \oplus R^m)]^G \cong [S(V \oplus R^m), P_k(V \oplus R^m)]^G.$$

Hence there is a pointed $G$-map

$$\lambda: S(V \oplus R^m) \rightarrow P_k(V \oplus R^m)$$

such that $\tau_{k^*}([\lambda]) = [\gamma]$. As is easily seen, the composition

$$S(V \oplus R^m)^H \xrightarrow{\lambda^H} P_k(V \oplus R^m)^H \xrightarrow{\pi^H} S(V \oplus R^m)^H$$

is an ordinary homotopy equivalence for each $H < G$. By Lemma 5.2, $\lambda$ is a reduction $G$-map. That is, $P_k(V \oplus R^m)$ is $G \cdot V \oplus R^{m-1}$-reducible. q.e.d.

6. A converse of Theorem 5.3

Let $m$ and $k$ be integers such that $m \equiv 0 \mod a_k(R)$ and $m \geq 2k \geq 4$. Let $\kappa: S(R^m) \rightarrow V_k(R^m)$ be a 1-section of $q_k$. That is, the composition

$$S(R^m) \xrightarrow{\kappa} V_k(R^m) \xrightarrow{q_k} S(R^m)$$

has degree $1$. For $n > k$, we define
\[ \theta_\varepsilon : V_k(R^n) \otimes S(R^m) \to V_k(R^{n+m}) \]

by the composition

\[ V_k(R^n) \otimes S(R^m) \xrightarrow{1 \ast \kappa} V_k(R^n) \otimes V_k(R^m) \xrightarrow{\mu} V_k(R^{n+m}) , \]

where \( \mu \) is the intrinsic map (see §5). By Theorem 3.1 in [15], \( \theta_\varepsilon \) is a \((2n-2k+m-1)\)-equivalence. The following Theorem is a converse of Theorem 5.3.

**Theorem 6.1.** Let \( m \) and \( k \) be integers such that \( m \equiv 0 \mod a_k(R) \) and \( m \geq 2k \geq 4 \). Let \( V \) be an orthogonal \( G \)-representation space. Assume that

(i) For each \( H \lhd G \), \( \dim_R V^H \geq 2k \) if \( V^H \neq \{0\} \),

(ii) \( P_s(V \oplus R^m) \) is \( G \)-\( V \oplus R^{m-1} \)-reducible.

Then \( \text{Span}_G(S(V)) \geq k-1 \).

**Proof.** First we show the following Assertion 6.1.1.

**Assertion 6.1.1.** There is a \( G \)-map

\[ \gamma_0 : S(V \oplus R^m) \to V_k(V \oplus R^m) \]

such that \( \gamma_0 \) satisfies the following:

(6.1.2) \( \gamma_0(S(R^m)) \subset V_k(R^n) \subset V_k(V \oplus R^m) \),

(6.1.3) the composition

\[ S(R^m) \xrightarrow{\gamma_0} V_k(R^n) \xrightarrow{q_k} S(R^m) \]

has degree 1,

(6.1.4) the composition

\[ S(V \oplus R^m)^H \xrightarrow{\gamma_0^H} V_k(V \oplus R^m)^H \xrightarrow{q_k^H} S(V \oplus R^m)^H \]

has degree 1 for each \( H \lhd G \).

**Proof of Assertion 6.1.1.** By assumption, we have a reduction \( G \)-map

\[ \lambda' : S(V \oplus R^m) \to P_s(V \oplus R^m) . \]

Let \( \pi : P_s(V \oplus R^m) \to S(V \oplus R^m) \) be the pointed \( G \)-map as in §5. We put

\[ \lambda = \lambda' \circ (\pi \circ \lambda') : S(V \oplus R^m) \to P_s(V \oplus R^m) . \]

Then \( \lambda \) is also a reduction \( G \)-map such that

\[ \deg(\pi \circ \lambda)^H = 1 \quad \text{for all } H \lhd G . \]

We put

\[ \gamma_1 = \tau_k \circ \lambda : S(V \oplus R^m) \to V_k(V \oplus R^m) . \]
We consider a $G$-map

$$\gamma_2 = \gamma_1|S(R^m): S(R^m) \to V\delta(V \oplus R^n).$$

First we assume that $V^G \neq \{0\}$. Since $m \equiv 0 \mod a_i(R)$ and $m \geq 2k$, there is a $(G)$-cross-section of $q_k$

$$\Delta: S(R^m) \to V\delta(V \oplus R^n) \subset V\delta(V \oplus R^n).$$

Since $\text{conn}(V\delta(V \oplus R^n)^G) \geq \dim S(R^m)$, $\gamma_2$ and $\Delta$ are $G$-homotopic. Remark that $(S(V \oplus R^n), S(R^m))$ has the $G$-homotopy extension property. We have a $G$-map

$$\gamma_0: S(V \oplus R^n) \to V\delta(V \oplus R^n)$$

such that $\gamma_0$ and $\gamma_1$ are $G$-homotopic and $\gamma_0|S(R^m) = \Delta$. As is easily seen, $\gamma_0$ satisfies our required properties.

Next we assume that $V^G = \{0\}$. In this case, $\gamma_2 = \gamma_1: S(R^m) \to V\delta(V \oplus R^n)$ is a 1-section of $q_k$. Therefore we put $\gamma_0 = \gamma_1$.

This completes the proof of Assertion 6.1.1.

We put $\gamma_3 = \gamma_0|S(R^m): S(R^m) \to V\delta(V \oplus R^n) \subset V\delta(V \oplus R^n)$. Consider a map

$$\theta_{\gamma_3^*: [S(V \oplus R^n), S(R^m), V\delta(V) \ast S(R^m); i_{S(R^m)}]^G} \rightarrow [S(V \oplus R^n), S(R^m), V\delta(V) \ast S(R^m); \gamma_3]^G.$$  

Since $\gamma_3$ is a 1-section, $\theta_{\gamma_3^*: V\delta(V) \ast S(R^m) \to V\delta(V \oplus R^n)^H}$ is a $(2 \dim_R V^H - 2k + m - 1)$-equivalence for each $(H) \in \text{Iso}(S(V \oplus R^n) - S(R^m))$. Applying Lemma 2.1, $\theta_{\gamma_3^*}$ is surjective. Therefore we have a $G$-map

$$\gamma_4: S(V \oplus R^n) \to V\delta(V) \ast S(R^m)$$

such that $\theta_{\gamma_4^*: [\gamma_4]} = [\gamma_0]$ and $\gamma_4|S(R^m) = i_{S(R^m)}$. As is easily seen, the composition

$$S(V)^H \ast S(R^m) \xrightarrow{h^H} S(V \oplus R^n)^H \xrightarrow{\gamma_4^H} V\delta(V) \ast S(R^m) \xrightarrow{q_k^H \ast 1} S(V)^H \ast S(R^m)$$

has degree 1 for each $H \leq G$, where $h$ is as in §5. Consider the following suspension map

$$\tau_{R^m}: [S(V), V\delta(V)]^G \rightarrow [(S(V) \ast S(R^m), S(R^m)), V\delta(V) \ast S(R^m); i_{S(R^m)}]^G.$$

Since $\dim S(V)^H \leq 2 \cdot \text{conn}(V\delta(V))^H + 1$ for each $(H) \in \text{Iso}(S(V))$, it follows from Theorem 2.4 that $\tau_{R^m}$ is surjective. Then we have a $G$-map

$$\gamma_5: S(V) \to V\delta(V)$$

such that $\tau_{R^m}^G([\gamma_5]) = [\gamma_4 \circ h]$. As is easily seen, the composition
\[ S(V)^H \xrightarrow{\gamma_s^H} V_s(V)^H \xrightarrow{q_s^H} S(V)^H \]

has degree 1 for each \((H) \in \text{Iso}(S(V))\). Let \(K < H < G\) such that \(V^K \neq V^H\). Using Lemma 4.3, we have \(\dim_R V^K - \dim_R V^H \geq k \geq 2\). Thus it follows from Rubinsztein [22; Theorem 8.4] that \(q_s \circ \gamma_s\) is \(G\)-homotopic to the identity. Since \(q_s: V_s(V) \rightarrow S(V)\) is a smooth \(G\)-fiber bundle in the sense of Bierstone [7], \(q_s\) has the smooth \(G\)-homotopy lifting property. Using Wasserman [27; Corollary 1.12], we see that \(q_s\) has a smooth \(G\)-cross-section. Now, by Lemma 5.1, we have \(\text{Span}_c(S(V)) \geq k - 1\).

q.e.d.

7. Proof of Theorem 1.1

Let \(V\) and \(W\) be unitary \(G\)-representation spaces such that \(\dim_c V^H = \dim_c W^H\) for all \(H < G\). By Lee-Wasserman [21; Proposition 3.17], there are direct sum decompositions

\[
\begin{align*}
V &= V_1 \oplus V_2 \oplus \cdots \oplus V_r, \\
W &= W_1 \oplus W_2 \oplus \cdots \oplus W_r
\end{align*}
\]

such that \(V_i\) and \(W_i\) (\(1 \leq i \leq r\)) are irreducible unitary \(G\)-representation spaces and \(V_i\) is conjugate to \(W_i\) by a field automorphism of \(C\) for \(1 \leq i \leq r\). That is, there are integers \(n(i)(1 \leq i \leq r)\) such that \((n(i), |G|) = 1\) and \(W_i = \psi^{n(i)}(V_i)\) for \(1 \leq i \leq r\), where \(\psi^t\) denotes the equivariant \(s\)-th Adams operation and \(|G|\) denotes the order of \(G\). Since \(\psi^{t+|G|} = \psi^t\), we may assume that \(n(i)(1 \leq i \leq r)\) are odd integers. Let \(\varepsilon_c\) be the non-trivial unitary 1-dimensional \(\mathbb{Z}_2\)-representation space. Then \(\varepsilon_c \otimes V\) and \(\varepsilon_c \otimes W\) are unitary \((\mathbb{Z}_2 \times G)\)-representation spaces and

\[
\begin{align*}
\varepsilon_c \otimes V &= (\varepsilon_c \otimes V_1) \oplus (\varepsilon_c \otimes V_2) \oplus \cdots \oplus (\varepsilon_c \otimes V_r), \\
\varepsilon_c \otimes W &= (\varepsilon_c \otimes W_1) \oplus (\varepsilon_c \otimes W_2) \oplus \cdots \oplus (\varepsilon_c \otimes W_r)
\end{align*}
\]

are decompositions of \(\varepsilon_c \otimes V\) and \(\varepsilon_c \otimes W\) into direct sums of irreducible unitary \((\mathbb{Z}_2 \times G)\)-representation spaces respectively. Since \(n(i)(1 \leq i \leq r)\) are odd, there are integers \(\bar{n}(i)(1 \leq i \leq r)\) such that \((\bar{n}(i), 2|G|) = 1\) and \(n(i) \cdot \bar{n}(i) \equiv 1 \text{ mod } |2G|\). Then we have

\[
\begin{align*}
(\varepsilon_c \otimes V_i) &= \psi^{\bar{n}(i)}(\varepsilon_c \otimes W_i) & \text{ for } 1 \leq i \leq r, \\
(\varepsilon_c \otimes W_i) &= \psi^{n(i)}(\varepsilon_c \otimes V_i) & \text{ for } 1 \leq i \leq r.
\end{align*}
\]

The following lemma is due to Torneh have [25] (see also [11]).

Lemma 7.1. There are \((\mathbb{Z}_2 \times G)\)-maps
\[
\begin{align*}
\{ \varphi_i &: S(\mathcal{C}_i \otimes V_i) \to S(\mathcal{C}_i \otimes W_i), \\
\psi_i &: S(\mathcal{C}_i \otimes W_i) \to S(\mathcal{C}_i \otimes V_i)
\end{align*}
\]

for \(1 \leq i \leq r\) such that 
\[
\text{deg} \, \varphi_i^K = n(i)^d_i(K) \quad \text{and} \quad \text{deg} \, \psi_i^K = n(i)^d_i(K)
\]

for each \(K < \mathbb{Z}_2 \times G\), where \(d_i(K) = \dim \mathcal{C}(\mathcal{C}_i \otimes V_i)^K\) \((= \dim \mathcal{C}(\mathcal{C}_i \otimes W_i)^K)\).

We put
\[
(7.2)
\]

\[
(\varphi = \varphi_1 \ast \cdots \ast \varphi_r) \colon S(\mathcal{C} \otimes V) \to S(\mathcal{C} \otimes W), \quad (\psi = \psi_1 \ast \cdots \ast \psi_r) \colon S(\mathcal{C} \otimes W) \to S(\mathcal{C} \otimes V).
\]

Then, for each \(K < \mathbb{Z}_2 \times G\), we have
\[
\text{deg} (\psi \circ \varphi)^K \equiv 1 \mod 2|G| \quad \text{and} \quad \text{deg} (\varphi \circ \psi)^K \equiv 1 \mod 2|G|.
\]

Let \(U\) be a unitary \(G\)-representation space and \(m \geq 2\). We define a homomorphism
\[
\Psi : [\Sigma^U \otimes R^{m-1}, \Sigma^U \otimes R^{m-1}] \to \prod_{(H) \in \text{Iso}(\Sigma^U \otimes R^{m-1})} \mathbb{Z}
\]

by the following: if \(f : \Sigma^U \otimes R^{m-1} \to \Sigma^U \otimes R^{m-1}\) is a pointed \(G\)-map, then \(\Psi([f]) = \prod_{(H) \in \text{Iso}(\Sigma^U \otimes R^{m-1})} \text{deg} f^H\) (for details see Rubinsztein [22]). By the same argument as in tom Dieck [10; Proposition 1.2.3], we have the following:

**Lemma 7.3.** Let \(x \in \prod_{(H) \in \text{Iso}(\Sigma^U \otimes R^{m-1})} \mathbb{Z}\) be an arbitrary element. Then \(|G| \cdot x \in \text{Im} \, \Psi\).

**Proposition 7.4.** Let \(m > k \geq 2\). Let \(V\) and \(W\) be unitary \(G\)-representation spaces such that \(\dim \mathcal{C} V^H = \dim \mathcal{C} W^H\) for all \(H < G\). Then the following two conditions are equivalent:

(i) There is a reduction \(G\)-map
\[
f : \Sigma^V \otimes R^{m-1} \to P_k(V \oplus R^n),
\]

(ii) There is a reduction \(G\)-map
\[
g : \Sigma^W \otimes R^{m-1} \to P_k(W \oplus R^n).
\]

**Proof.** It suffices to show that (i) implies (ii). Let
\[
(\varphi : S(\mathcal{C} \otimes V) \to S(\mathcal{C} \otimes W),
\psi : S(W) \to S(V)
\]

Then, for each \(K < \mathbb{Z}_2 \times G\), we have
\[
\text{deg} (\psi \circ \varphi)^K \equiv 1 \mod 2|G| \quad \text{and} \quad \text{deg} (\varphi \circ \psi)^K \equiv 1 \mod 2|G|.
\]
be a \((\mathbb{Z}_2 \times G)\)-map and a \(G(\subset \mathbb{Z}_2 \times G)\)-map as in (7.2) respectively. We put a \((\mathbb{Z}_2 \times G)\)-map
\[
\varphi_1 = \varphi^*1_{S(\mathcal{C} \otimes \mathbb{R}^m)}: S((\mathcal{C} \otimes V) \oplus (\mathcal{R} \otimes \mathbb{R}^m)) \to S((\mathcal{C} \otimes W) \oplus (\mathcal{R} \otimes \mathbb{R}^m))
\]
and a pointed \(G\)-map
\[
\psi_1 = \psi^*1_{S(\mathbb{R}^m)}: S(W \oplus \mathbb{R}^m) \to S(V \oplus \mathbb{R}^m).
\]
Remark that \(\varphi_1\) induces a pointed \(G\)-map
\[
\varphi_2: P_{\text{A}}(V \oplus \mathbb{R}^m) \to P_{\text{A}}(W \oplus \mathbb{R}^m)
\]
such that the following diagram commutes:
\[
\begin{array}{ccc}
S((\mathcal{C} \otimes V) \oplus (\mathcal{R} \otimes \mathbb{R}^m)) & \xrightarrow{\varphi_1} & S((\mathcal{C} \otimes W) \oplus (\mathcal{R} \otimes \mathbb{R}^m)) \\
\downarrow{p_1} & & \downarrow{p_2} \\
P_{\text{A}}(V \oplus \mathbb{R}^m) & \xrightarrow{\varphi_2} & P_{\text{A}}(W \oplus \mathbb{R}^m),
\end{array}
\]
where \(p_1\) and \(p_2\) are the natural projections as in § 5. We define a pointed \(G\)-map
\[
g_1: \Sigma^{W \oplus \mathbb{R}^m-1} \to P_{\text{A}}(W \oplus \mathbb{R}^m)
\]
by the composition
\[
\Sigma^{W \oplus \mathbb{R}^m-1} \xrightarrow{d_2} S(W \oplus \mathbb{R}^m) \xrightarrow{\psi_1} S(V \oplus \mathbb{R}^m) \xrightarrow{d_1} \Sigma^{V \oplus \mathbb{R}^m-1} \xrightarrow{f} P_{\text{A}}(V \oplus \mathbb{R}^m) \xrightarrow{\varphi_2} P_{\text{A}}(W \oplus \mathbb{R}^m),
\]
where \(d_1\) and \(d_2\) are pointed \(G\)-homeomorphisms. Let \(\pi_1: P_{\text{A}}(V \oplus \mathbb{R}^m) \to S(V \oplus \mathbb{R}^m)\) and \(\pi_2: P_{\text{A}}(W \oplus \mathbb{R}^m) \to S(W \oplus \mathbb{R}^m)\) be the natural collapsing maps as in § 5. Let
\[
g_2: \Sigma^{W \oplus \mathbb{R}^m-1} \to \Sigma^{W \oplus \mathbb{R}^m-1}
\]
be a \(G\)-map defined by the composition
\[
\Sigma^{W \oplus \mathbb{R}^m-1} \xrightarrow{g_1} P_{\text{A}}(W \oplus \mathbb{R}^m) \xrightarrow{\pi_2} S(W \oplus \mathbb{R}^m) \xrightarrow{d_2^{-1}} \Sigma^{W \oplus \mathbb{R}^m-1}.
\]
Then it is easy to see that
\[
\deg g_2^H \equiv \deg (d_1 \circ \pi_1 \circ f)^H \mod 2 |G| \quad \text{for each } H < G.
\]
Since \(f\) is a reduction \(G\)-map, we remark that \(\deg (d_1 \circ \pi_1 \circ f)^H = \pm 1\) for each \(H < G\). Let \(a(H)\) be an integer such that
\[
\deg g_2^H = \deg (d_1 \circ \pi_1 \circ f)^H + 2a(H) |G|
\]
for each \((H) \in \text{Iso}(\Sigma^W \oplus R^{m-1})\). By Lemma 7.3, there is a pointed \(G\)-map
\[
g_3: \Sigma^W \oplus R^{m-1} \to \Sigma^W \oplus R^{m-1}
\]
such that \(\deg g_3^H = a(H)|G|\) for each \((H) \in \text{Iso}(\Sigma^W \oplus R^{m-1})\). We define a pointed \(G\)-map
\[
g_4: \Sigma^W \oplus R^{m-1} \to P_h(W \oplus R^m)
\]
by the composition
\[
\begin{array}{c}
\Sigma^W \oplus R^{m-1} \xrightarrow{g_3} \Sigma^W \oplus R^{m-1} \\
\quad \quad \downarrow d_2 \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \downarrow
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
Then we have a $G$-fiber homotopy equivalence

$$f: S(\eta_k \otimes U) \to S(U).$$

**Proof.** First we show that the following Assertion 8.1.1.

**Assertion 8.1.1.** There are an integer $n (\geq m)$ and a $G$-map

$$f_1: S((\eta_k \otimes U) \oplus \mathbb{R}^n) \to S(U \oplus \mathbb{R}^n)$$

such that a restriction

$$f_1|S((\eta_k \otimes U) \oplus \mathbb{R}^n)_x: S((\eta_k \otimes U) \oplus \mathbb{R}^n)_x \to S(U \oplus \mathbb{R}^n)$$

for $x \in \mathbb{R}^{P^{k-1}}$ is a $G$-homotopy equivalence and a restriction $f_1|S(\mathbb{R}^n)$ is the natural projection $S(\mathbb{R}^n) \to S(\mathbb{R}^n) \subset S(U \oplus \mathbb{R}^n)$.

**Proof of Assertion 8.1.1.** We put $f_2 = p_1 \circ f: S((\eta_k \otimes U) \oplus \mathbb{R}^n) \to S(U \oplus \mathbb{R}^n)$, where $p_1: S(U \oplus \mathbb{R}^n) \to S(U \oplus \mathbb{R}^n)$ is the natural projection.

Suppose first that $U^G \neq \{0\}$. By assumption, we see that $\mathrm{conn}(S(\mathbb{R}^m)^G) \geq \dim S(\mathbb{R}^n)$. Then $f_2|S(\mathbb{R}^n): S(\mathbb{R}^n) \to S(U \oplus \mathbb{R}^n)$ and the natural projection $p_2: S(\mathbb{R}^n) \to S(\mathbb{R}^n) \subset S(U \oplus \mathbb{R}^n)$ are $G$-homotopic. Since $(S((\eta_k \otimes U) \oplus \mathbb{R}^n), S(\mathbb{R}^n))$ has the $G$-homotopy extension property, we have a $G$-map

$$f_1: S((\eta_k \otimes U) \oplus \mathbb{R}^n) \to S(U \oplus \mathbb{R}^n)$$

such that $f_1$ and $f_2$ are $G$-homotopic and $f_1|S(\mathbb{R}^n) = p_2$. We put $n = m$. It is easy to see that $f_1$ has our required properties.

Suppose second that $U^G = \{0\}$. Remark that $f_2^G: S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is a map such that $(f_2^G)_x: S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is a homotopy equivalence for $x \in \mathbb{R}^{P^{k-1}}$. It is well-known that there is a map $h: S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ such that $f_2^G \bar{h}: S(\mathbb{R}^{n+m}) \to S(\mathbb{R}^{n+m})$ is homotopic to the natural projection $p_3: S(\mathbb{R}^{n+m}) \to S(\mathbb{R}^{n+m})$, where $\bar{\cdot}$ denotes the fiberwise join. We put

$$f_3 = f_2^G \bar{h}: S((\eta_k \otimes U) \oplus \mathbb{R}^{n+m}) \to S(U \oplus \mathbb{R}^{n+m}).$$

Then $f_3|S(\mathbb{R}^{n+m}) = f_2^G$ is $(G)$-homotopic to $p_3$. By the same argument as in the case when $U^G \neq \{0\}$, we have a $G$-map

$$f_1: S((\eta_k \otimes U) \oplus \mathbb{R}^{n+m}) \to S(U \oplus \mathbb{R}^{n+m})$$

such that $f_1$ is $G$-homotopic to $f_3$ and $f_1|S(\mathbb{R}^{n+m}) = p_3$. We put $n = m + m'$. Then $f_1$ has our required properties.

This completes the proof of Assertion 8.1.1.

We see that $f_1$ induces a $G$-map

$$f_4: S(\eta_k \otimes U) \ast S(\mathbb{R}^n) \to S(U) \ast S(\mathbb{R}^n)$$
such that the following diagram commutes:

\[
\begin{array}{ccc}
S(\eta_k \otimes U) \times S(R^*) & \xrightarrow{f_1} & S(U) \times S(R^*) \\
q \downarrow & & \downarrow \\
S(\eta_k \otimes U) \times S(R^*) & \xrightarrow{f_i} & S(U) \times S(R^*)
\end{array}
\]

where \( q \) is the natural projection. Then \( f_1 \mid S(R^*) = i_{S(R^*)} : S(R^*) \to S(U) \times S(R^*) \).

For each \((H) \subseteq \text{Iso}(S(\eta_k \otimes U))\) \(= \text{Iso}(S(U))\), we see that \( \dim S(\eta_k \otimes U)^H \leq 2 \text{ conn}(S(U)^H) + 1 \). It follows from Theorem 2.4 that we obtain a \( G \)-map

\[ f_5 : S(\eta_k \otimes U) \to S(U) \]

such that \( f_5 \circ 1_{S(R^*)} \) is \( G \)-homotopic to \( f_i \). By Equivariant Dold Theorem ([19]), it is easy to see that

\[ f_5 = p_i \times f_5 : S(\eta_k \otimes U) \to R^{p_i-1} \times S(U) \]

gives a \( G \)-fiber homotopy equivalence, where \( p_i : S(\eta_k \otimes U) \to R^{p_i-1} \) is the natural projection.

Proof of Theorem 1.2. We may assume that \( k \geq 2 \). Let \( m \) and \( n \) be integers such that \( m \equiv 0 \mod a_k(R) \), \( n \equiv k \mod a_k(R) \) and \( n > m \geq 2k \).

First we show (i). By Theorem 5.3, \( P_\delta(V \oplus R^n) \) is \( G-V \oplus R^{*k-1} \)-reducible. Applying Proposition 4.4, \( P_\delta(V \oplus R^n) \) is \( G-V \oplus R^{*k-1} \)-coreducible. It follows from Proposition 3.3 that we have a \( G \)-fiber homotopy equivalence

\[ f_1 : S((V \oplus R^{*k}) \oplus R^*) \to S(V \oplus R^{*k-1} \oplus R^*) \]

Since \( n \equiv k \mod a_k(R) \) and \( n > 2k \), we have a \( G \)-fiber homotopy equivalence

\[ f_2 : S((V \oplus R^{*k}) \oplus R^{*k+1}) \to S(V \oplus R^{*k+1}) \]

The first result follows. The second result follows from Lemma 8.1

Next we show (ii). Since \( n \equiv k \mod a_k(R) \) and \( n > 2k \), we have a \( G \)-fiber homotopy equivalence

\[ f_2 : S((V \oplus R^{*k}) \oplus R^{*k+1}) \to S(V \oplus R^{*k+1}) \]

By Proposition 3.3, \( P_\delta(V \oplus R^n) \) is \( G-V \oplus R^{*k} \)-coreducible. Applying Proposition 4.4, \( P_\delta(V \oplus R^n) \) is \( G-V \oplus R^{*k} \)-reducible. It follows from Theorem 6.1 that \( \text{Span}_e(S(V)) \geq k-1 \).

q.e.d.

9. An example

Let \( G \) be a metacyclic group

\[ \{a, b \mid a^n = b^2 = e, bab^{-1} = a' \} \]
where \( m \) is a positive odd integer, \( q \) is an odd prime integer, \((r - 1, m) = 1\) and \( r \) is a primitive \( q \)-th root of 1 mod \( m \). Let \( \mathbb{Z}_m = \langle a \rangle < G \) and let \( t^h (h \in \mathbb{Z}) \) be the unitary 1-dimensional \( \mathbb{Z}_m \)-representation space with \( a \) acting on \( C^n \) as multiplication with \( \exp(2\pi h\sqrt{-1}/m) \). Let \( T_h \) denote the induced representation space \( \text{Ind}_{\mathbb{Z}_m}^G(t^h) \) of the \( \mathbb{Z}_m \)-representation space \( t^h \). Then \( T_h \) is a unitary \( q \)-dimensional \( G \)-representation space (for details see [9; §47] or [17]). We put

\[
V_n = T_{h_1} \oplus T_{h_2} \oplus \cdots \oplus T_{h_n},
\]

where \((h_i, m) = 1\) for \( 1 \leq i \leq n \).

**Example 9.1.** If \( n \geq 9 \), then \( \text{Span}_G(S(V_n)) = \rho(2n, \mathbb{R}) - 1 \).

Here \( \rho(s, \mathbb{R}) \) denotes the largest integer \( k \) such that \( s \equiv 0 \mod a_k(\mathbb{R}) \) ([1]).

Proof of Example 9.1. Since \( \dim_{\mathbb{R}} V_n = 2nq \) and \( q \) is odd, \( \text{Span}(S(V_n)) = \rho(2nq, \mathbb{R}) - 1 = \rho(2n, \mathbb{R}) - 1 \). Thus we have

\[
(9.1.1) \quad \text{Span}_G(S(V_n)) = \rho(2n, \mathbb{R}) - 1.
\]

By Becker [6; Theorems 1.1 and 2.2], there is a \( \mathbb{Z}_m \)-fiber homotopy equivalence

\[
f_1: S(\eta_{\rho(2n, \mathbb{R})} \otimes n\mathbb{R}) \rightarrow S(n\mathbb{R}).
\]

By the same argument as in [5; II. Proposition 2.2], we have a \( G \)-fiber homotopy equivalence

\[
f_2: S(\eta_{\rho(2n, \mathbb{R})} \otimes nT_1) \rightarrow S(nT_1).
\]

Since \( n \geq 9 \), we see that \( \dim_{\mathbb{R}} nT_1^n \geq 2 \rho(2n, \mathbb{R}) \) if \( nT_1^n \neq \{0\} \) for each \( H < G \). Applying Theorem 1.2, we have \( \text{Span}_G(S(nT_1)) \supseteq \rho(2n, \mathbb{R}) - 1 \). It is easy to see that \( \text{dim}_C V_n^n = \text{dim}_C nT_1^n \) for all \( H < G \). Thus it follows from Theorem 1.1 that we have

\[
(9.1.2) \quad \text{Span}_G(S(V_n)) \supseteq \rho(2n, \mathbb{R}) - 1.
\]

Combining (9.1.1) and (9.1.2), we have \( \text{Span}_G(S(V_n)) = \rho(2n, \mathbb{R}) - 1 \). q.e.d.

Added in proof. Professor P. May kindly informed me that Dr. U. Namboodiri has obtained similar results [30].

**References**

EQUIVARIANT SPAN OF THE UNIT SPHERES


Department of Mathematics
Faculty of Science
Kochi University
Kochi, 780 Japan