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SUPPLEMENTARY REMARKS ON CATEGORIES OF INDECOMPOSABLE MODULES

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In the previous papers [3], [4], we have defined a full sub-category $\mathfrak{A}$ in the category $\mathfrak{M}_R$ of modules over a ring R, whose objects consist of injective modules or directsums of completely indecomposable modules.

Making use of those ideas, in this short note, we shall give a proof of Z. Papp's theorem in [9] as an application of [3], Theorem 1 and generalize Theorems 4 and 7 in [4] to cases of semi-T-nilpotent system and quasi-projective module, respectively. Especially, we shall show that if R is a right perfect ring, then every quasi-projective module is a directsum of completely indecomposable modules and the Krull-Remak-Schmidt's theorem is valid for those direct decompositions.

In this note, we always assume that the ring R has the identity and every module is an unitary R-module. We shall use the same notations and definitions in [3], [4] and [5] for categories, those in [1] and [8] for semi-perfect modules and those in [5] for quasi-projective modules.

1. Papp's theorem

We shall give an application of [3], Theorem 1.

Theorem 1 ([9], Z. Papp). *Let R be a ring. If every* (*right*) *R-injective module is a directsum of indecomposable modules, then R is* (*right*) *noetherian.*

Proof. It is known by [2], Proposition 4.1 that R is noetherian if and only if any directsum of injective modules is also injective. Let $\mathfrak{A}$ be the full sub-category of all injective R-modules in the category of right R-modules and $\mathfrak{J}$ the Jacobson radical of $\mathfrak{A}$. Then $\mathfrak{A}/\mathfrak{J}$ is a completely reducible C_3-abelian category by the assumption and [3], Theorem 1. Let $\{Q_i\}_1^\infty$ be a family of injective modules, and E an injective hull of $\sum \oplus Q_i$. From the assumption $E = \sum \oplus E_i$, where E_j's are (completely) indecomposable. Hence, $\sum \oplus E_j = \sum \oplus Q_i$ in $\mathfrak{A}/\mathfrak{J}$ by [3], Theorem 1. Therefore, $E \approx \sum \oplus Q_i$, which means that $\sum \oplus Q_i$ is injective. Hence, R is noetherian.

2. Exchange property

Let M be a directsum of completely indecomposable modules M_α; $M=\sum\oplus M_\alpha$. We have defined the ($\aleph_0$-) *exchange property in* M for a direct summand N of M in [4]. Namely, we have $M=N\oplus\sum_I\oplus T'_\alpha$ for any decomposition $M=\sum_I\oplus T_\alpha$ (with Card $I\leqslant\aleph_0$), where $T'_\alpha\subseteq T_\alpha$ for all $\alpha\in I$.

Let $M=N\oplus N'$. If N has the exchange property in M, then N and N' are directsums of indecomposable modules.

Now we assume $M=\sum_I\oplus M_\alpha$. A family $\{M_\beta\}_J$ ($J\subseteq I$) is called a *semi-T-nilpotent system* with respect to the radical of $[M_\beta, M'_\beta]_R$ if the following condition is satisfied. J is a finite or empty set or if J is otherwise, for any subfamily $\{M_{\beta_i}\}$ with $\beta_i\in J$ and $\beta_i\neq\beta_j$ if $i\neq i$ and any set of non isomorphisms $f_i:M_{\beta_i}\to M_{\beta_{i+1}}$, there exists a natural number n such that $f_nf_{n-1}f\cdots f_1(m)=0$ for $m\in M_{\beta_1}$, where n may depend on m, (cf. [5]). Then we have a generalization of [4], Theorem 4 as follows;

Theorem 2. *Let $M=\sum_I\oplus \dot{M}_\alpha$ with M_α completely indecomposable and $M=N_1\oplus N_2$. If the dense submodule of N_1[1] is a directsum of indecomposable modules which are a semi-T-nilpotent system with respect to the radical, then N_i has the exchange property in M for $i=1, 2$.*

Proof. We first note that $N_1=\sum\oplus M'_\beta$ by the assumption and [7], Corollary to Theorem 1. Furthermore, since the ideal $\mathfrak{J}$ of $S_{N_1}=[N_1, N_1]_R$ defined in [3], §3 is equal to the Jacobson radical of S_{N_1} by [7], Theorem 1. Hence, we have from the first part of the proof of [4], Theorem 4 that N_2 has the exchange property. Let $M=\sum_K\oplus T_\beta$ with any Card K. We shall use the same notation in [4]. If we consider the category $\mathfrak{A}/\mathfrak{J}$ in [4], then $\bar{M}=\sum\oplus\bar{T}_\beta=\bar{N}_1\oplus\bar{N}_2$ in $\mathfrak{A}/\mathfrak{J}$ by [4], Theorem 1. Since $\mathfrak{A}/\mathfrak{J}$ is a completely reducible C_3-abelian category, $\bar{M}=\bar{N}_1\oplus(\sum\oplus\bar{T}'_\beta)$, where $\bar{T}_\beta=\bar{T}'_\beta\oplus\bar{T}''_\beta$ and we may assume that T'_β and T''_β are in $\mathfrak{A}$ and submodules in T_β by [4], Proposition 2. Since $\sum_K\oplus\bar{T}''_\beta\approx\bar{N}_1$, $N_1\approx\sum\oplus T''_\beta$ by the assumption and [7], Theorem 1. Let p be a homomorphism of M to $\sum_K\oplus T''_\beta$ such that $\bar{p}$ is a projection of $\bar{M}$ to $\sum_K\oplus\bar{T}''_\beta$ with Ker $\bar{p}=\sum_K\oplus\bar{T}'_\beta$. Then p splits. Put $L=\mathrm{Ker}\,p$, then $T_\beta=T''_\beta\oplus T^*_\beta$, where $T^*_\beta=T_\beta\cap L$. Since $L=\sum\oplus T^*_\beta$ and $\bar{L}=\mathrm{Ker}\,\bar{p}$, $\sum_K\oplus\bar{T}^*_\beta=\sum_K\oplus\bar{T}'_\beta$. Hence, $\bar{M}=\bar{N}_1\oplus(\sum\oplus\bar{T}'_\beta)=\sum_K\oplus\bar{T}''_\beta\oplus\sum_K\oplus\bar{T}'_\beta$. Therefore, Ker $\bar{p}\cap\bar{N}_1=\bar{0}$ and $\bar{p}(\bar{N}_1)=\bar{p}(\bar{M})=\sum_K\oplus\bar{T}''_\beta$, which implies $p|N_1$ is isomorphic. Hence, $M=N_1\oplus\mathrm{Ker}\,p=N_1\oplus$

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1) See [4], §1 for the definition.

$\sum_K \oplus T^*_\beta$.

Corollary. *Let M be as above and N a direct summand of M. Then the following statements are equivalent.*

1) *Every direct summand of N has the $\aleph_0$-exchange property in M.*
2) *Every direct summand of N has the exchange property in M.*
3) $N=\sum_J \oplus N_\beta; N_\beta \approx M_{\pi(\beta)}$ *and* $\{N_\beta\}$ *is a semi-T-nilpotent system with respect to the radical of* $[N_\gamma, N_\delta]_R$.

Proof. 1)→3). Let N be a direct summand of M and $N=\sum_J \oplus N'_\beta$, $N'_\beta \approx M_\beta$ with Card $J \leqslant$ Card I. We first note that every direct summand P of N has the $\aleph_0$-exchange property in N. Let $M=N\oplus Q$ and $N=P_1\oplus P_2=\sum_{i=1}^{\infty}\oplus T_i$. Then $M=P_1\oplus(P_2\oplus Q)=\sum\oplus T_i\oplus Q$. Since P_1 has the $\aleph_0$-exchange property in M, $M=P_1\oplus\sum\oplus T'_i\oplus Q'$, where $T'_i\subseteq T_i$ and $Q'\subseteq Q$. Hence, $N=P_1\oplus\sum\oplus T'_i \oplus P_1\cap Q'$ and $P_1\cap Q'\subseteq P_1\cap Q=(0)$. Now put $P_1=\sum_{J_0}\oplus N_\gamma$ for any $J_0\subseteq I$ with Card $J_0\leqslant\aleph_0$. Then $\{N_\gamma\}_{J_0}$ is a semi-T-nilpotent system by [7], Theorem 1. Hence, $\{N_\gamma\}_J$ is a semi-T-nilpotent system. 3)→2). Since the ideal $\mathfrak{J}$ of $[N, N]_R$ defined in [3] is the Jacobson radical by [7], Theorem 1, every direct summand of N is a directsum of indecomoposable modules and has the exchange property in M by Theorem 2. 2)→1). It is clear.

Lemma 1. *Let M be as above. We assume that* $M=N_1\oplus N_2=N'_1\oplus N'_2$. *If* N_1 *has the exchange property in M and there exists an automorphism f of M such that* $f(N_i)=N'_i$ *for* $i=1, 2$ *then* N'_1 *has the exchange property in M.*

Proof. It is clear.

Lemma 2. *Let M,* N_1 *and* N_2 *be as above. We assume* $N_i=\sum_{\alpha\in J_i}\oplus M_{i\alpha}$, *Card* J_i *are infinite and* $M_{i\alpha}$*'s are indecomposable modules for* $i=1, 2$. *Let* $\{f_i\}_1^\infty$, $\{g_i\}_1^\infty$, *be sets of non-isomorphic homomorphisms of* $M_{1\alpha_i}$ *to* $M_{2\alpha_i}$ *and* $M_{2\alpha_i}$ *to* $M_{1\alpha_{i+1}}$, *respectively. Furthermore, we assume that* N_1 *has the* $\aleph_0$*-exchange property. Then for any m in* $M_{1\alpha_1}$ *there exists n such that* $g_n f_n g_{n-1} f_{n-1}\cdots g_1 f_1(m)=0$.

Proof. We shall make use of the same argument in [3], Lemma 9. Put $M'_{1\alpha_i}=\{m_i+f_i(m_i)\,|\,m_i\in M_{1\alpha_i}\}$ and $M'_{2\alpha_i}=\{m_i+g_i(m_i)\,|\,m_i\in M_{2\alpha_i}\}$. Then $M=M'_{1\alpha_1}\oplus M_{2\alpha_1}\oplus M'_{1\alpha_2}\oplus M_{2\alpha_2}\oplus\cdots\oplus M_{10}\oplus M_{20}=M_{1\alpha_1}\oplus M'_{2\alpha_1}\oplus M_{1\alpha_1}\oplus M'_{2\alpha_2}\oplus\cdots\oplus M_{10}\oplus M_{20}$, wherre $M_{i0}=\sum_{J_i-\{1,2\cdots\}}\oplus M_{i\alpha}$. Since $T=M'_{1\alpha_1}\oplus M'_{1\alpha_2}\oplus\cdots\oplus M_{10}\approx N_1$, T has the $\aleph_0$-exchange property in M by Lemma 1. Hence, $M=T\oplus M^*_{1\alpha_1}+M'^*_{2\alpha_1}\oplus M^*_{1\alpha_2}\oplus M'^*_{2\alpha_2}\oplus\cdots\oplus M^*_{20}$, where $M^*_{1\alpha_j}=0$ or $M_{1\alpha_j}$ ($M'^*_{2\alpha_j}=0$ or $M'_{2\alpha_j}$). In this case we can use the same argument in [3], Lemma 9.

From Lemma 2 we have

Proposition 1. *Let $M=\sum\oplus M_\alpha$ with M_α completely indecomposable. We assume that $M=N_1\oplus N_2$ and $N_i=\sum_\gamma\sum_{\beta\in J_\gamma}\oplus M^{(i)}_{\gamma\beta}$, where $M^{(i)}_{\gamma\beta}\approx M^{(i)}_{\gamma\beta'}$ and $M^{(i)}_{\gamma'\beta'}\not\approx M^{(i)}_{\gamma'\beta''}$ if $\gamma\neq\gamma'$, where $M^{(i)}_{\alpha\beta}$'s are indecomposable. We further assume Card $J^{(2)}_\gamma\geqslant$ Card $J^{(1)}_\gamma$ for all Card $J^{(2)}_\gamma$ which is smaller than or equal to $\aleph_0$. Then N_1 has the ($\aleph_0$–) exchange property if and only if $\{M^{(1)}_{\gamma\beta}\}$ is a semi-T-nilpotent system with respect to the radical.*

Now we take the category $\mathfrak{A}$ of all R-modules which is a directsum of some completely indecomposable modules. Let M be an object in $\mathfrak{A}$. We call M having *the exchange property in* $\mathfrak{A}$ if M has the exchange property in P for any object P in $\mathfrak{A}$ which contains M as a direct summand.

Corollary 2. *Let $\mathfrak{A}$ be the above. Then we have the following equivalent statements for $M=\sum_I\oplus M_\alpha$ in $\mathfrak{A}$.*

1) *M has the exchange property in $\mathfrak{A}$.*

2) *$\{M_\alpha\}_I$ is a semi-T-nilpotent system with respect to the radical, where M_α's are completely indecomposable.*

Proof. 2)→1). It is clear from Corollary to Theorem 2. 1)→2). Let $M=\sum_\alpha\sum_{I_\alpha\in\beta}\oplus M_{\alpha\beta}$, $M_{\alpha\beta}\approx M_{\alpha\beta'}$ and $M_{\alpha\beta}\not\approx M_{\alpha',\beta'}$ if $\alpha\neq\alpha'$. Put $P=\sum_1^\infty\oplus P_n$, $P_n=M$. Since M has the exchange property in P by the assumption, $\{M_\alpha\}$ is a semi-T-nilpotent system by Proposition 1.

Finally, we shall consider a special case. Let Z be the ring of integers (or Z may be a Dedekind domain) and $\{P_i\}_I$ a family of primes. Let M be a directsum of any copies of Z/P^{n_i}, where i runs over a sub-set of I and n_i's are integers. Then $M=\sum_{i\in I}\oplus M_{P_i}$, where $M_{P_i}=\sum\oplus Z/P_i^{n_i}$. In this case, every submodule N of M is a directsum of N_P, where $N_P=N\cap M_P$. Hence, a direct summand N of M has the exchange property in M if and only if N_P has the exchange property in M_P for each P.

Corollary 3. *Let Z, M_P and M be as above. We assume $M=N_1\oplus N_2$ and $N_i=\sum\oplus M^{(i)}_{P_j}$; $M^{(i)}_{P_j}\approx Z/P^{n_j}_{P_j}$. Then N_1 has the exchange property in M if and only if either $\{M^{(1)}_{P_j}\}$ or $\{M^{(2)}_{P_j}\}_j$ is a semi-T-nilpotent system with respect to the radical for every P_j.*

Proof. It is clear from Lemma 2 and [3], Lemma 12.

Remark. If $P_1\neq P_2$, then $\sum_{n=1}^\infty\oplus Z/P_1^n$ has the exchange property in $P=\sum_1^\infty\oplus Z/P_1^n\oplus\sum_1^\infty\oplus Z/P_2^n$ from the above remark. However, $\{Z/P_i^n\}_n$ are not semi-T-nilpotent systems for $i=1$, 2. Hence, M does not have the exchange property in $\mathfrak{A}$.

3. Quasi-projective modules

First, we consider projective modules of a special type.

Lemma 3. *Let P and Q be projective R-modules such that $J(P)$ and $J(Q)$ are small in P and Q, respectively. Then $[P/J(P), Q/J(Q)]_{R/J(R)}=0$ if and only if $[Q/J(Q), P/J(P)]_{R/J(R)}=0$, where $J(*)$ is the Jacobson radical of $(*)$.*

Proof. Put $T=P\oplus Q$. Then $J(T)$ is a unique maximal one among small submodules in T. We assume $[P, Q]_R=[P, J(Q)]_R$ and f an element in $[Q, P]_R$. We put $f_T=\begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix}$ in $S_T=[T, T]_R$. Since $[P, Q]_R f=[P, J(Q)]_R f\subseteq[Q, J(Q)]_R\subseteq J(S_Q)$ by [4], Proposition 1. Hence, $S_T f_T$ is in $J(S_T)$. Therefore, $f_T(T)\subseteq J(P)\oplus J(Q)$. Hence, $f(Q)\subseteq J(P)$ and $[Q, P]_R=[Q, J(P)]_R$. It is clear that $[P, J(Q)]_R=[P, Q]_R$ if and only if $[P/J(P), Q/J(Q)]_R=0$, since P is projective.

Proposition 2. *Let P and Q be as above. We further assume that P is completely indecomposable, then the following are equivalent.*

1) *P is isomorphic to a direct summand of Q.*
2) *$P/J(P)$ is isomorphic to a sub-module of $Q/J(Q)$.*

Proof. It is clear, since $J(P)$ is a unique maximal sub-module in P by [4], Theorem 5.

Changing slightly the proofs in [10], Lemma 1 and [5], Proposition 1, we have

Lemma 4. *Let M be a quasi-projective, then $J(S_M)=\{f\mid\in S_M, f(M)$ is small in $M\}$. Furthermore, $J(M)$ is small if and only if $[M, J(M)]_R=J(S_M)$, where, $S_M=[M, M]_R$.*

We note that a quasi-projective module with projective cover is nothing but a factor module of projective module P with respect to a small R-sub-module K in P which is a S_P-module by [6], Propositions 2.1 and 2.2. Furthermore, if we take the ring of column summable matrices, we know Proposition 2.4 in [6] is valid for a directsum of infinite components, (cf. [3], § 3).

Proposition 3. *Let M be a quasi-projective. We assume that M has projective cover P. Then $S_M\approx S_P/A$ and $P/J(P)\approx M/J(M)$, where A is an ideal contained in $J(S_P)$. Furthermore, $J(P)$ is small in P if and only if $J(M)$ is small in M.*

Proof. We have the exact sequence $0\to[P, K]_R\to S_P\to[P, M]_R\to 0$ from an exact sequence $0\to K\to P\overset{\nu}{\to}M\to 0$. $A=[P, K]_R$ is a two-sided ideal by [6], Proposition 2.2. Let f be in $[P, M]_R$. Since P is projective, we have g in S_P such that $\nu g=f$. Hence, $f(K)=\nu g(K)\subseteq\nu(K)=0$. Therefore, $[P, M]_R=S_M$. Since $K\subseteq J(P)$, $J(M)\approx J(P)/K$ and $P/J(P)\approx M/J(M)$. Furthermore, $A\subseteq J(S_P)$ by Lemma 4. The last part is clear.

Lemma 5. *Let $\{M_\alpha\}_I$ be a family of quasi-projective modules and I an infinite set. We assume $M=\sum_I\oplus M_\alpha$ is quasi-projective. Then $J(M)$ is small in M if and only if $J(M_\alpha)$ is small in M_α for all $\alpha\in I$ and $\{M_\alpha\}_I$ is a semi-T-nilpotent system with respect to the radical of $[M_\alpha, M_\beta]_R$.*

Proof. We can make use of the same argument in [5], Theorem 3 from Lemma 4.

Theorem 3. *Let M be a quasi-projective module with projective cover P. Then P is semi-perfect if and only if* 1) *$M=\sum_I\oplus M_\alpha$; M_α's are completely indecomposable R-modules,* 2) *$\{M_\alpha\}_I$ is a semi-T-nilpotent system with respect to the Jacobson radical of $[M_\alpha, M_\beta]_R$ and* 3) *M_α has a projective cover for all $\alpha\in I$. In this case any direct decomposition of $M/J(M)$ is lifted to M.*

Proof. We assume P is semi-perfect. Then 1) is clear from [6], Proposition 2.4 and the above remark. 2) is clear from Proposition 3 and Lemma 4. 3) is clear from [6], Proposition 2.4. Conversely, we assume 1), 2) and 3). Let P_α be a projective cover of M_α via ν_α and $Q=\sum\oplus P_\alpha$. We have an exact sequence $0\to K\to P\xrightarrow{\nu}M\to 0$ with K small. Hence, we have $f\in[Q, P]_R$ and $g\in[P, Q]_R$ such that $fg=I_P$ and $\nu'=\nu g$, where $\nu=\sum\oplus\nu_\alpha$. Since ν and ν' induce natural isomorphisms $P/J(P)\approx M/J(M)\approx Q/J(Q)$, g is isomorphic. Furthermore, P_α is semi-perfect from Proposition 3 and [4], Theorem 5. We know from 2) and Lemma 4 that $J(M)$ is small in M. Hence, $J(P)$ is small in P by Proposition 3. Therefore, P is semi-perfect by [8], Theorem 5.2. The last part is clear from Proposition 3, [6], Proposition 2.4 and [8], Theorem 4.3.

Corollary. *If R is a right perfect (resp, semi-perfect) ring, then every (resp. finitely generated) quasi-projective module is a directsum of completely indecomposable modules and the Krull-Remak-Schmidt's theorem is valid for those decompositions.*

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