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Rigidity theorems on lattices of topologies on vector spaces

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Abstract

We consider a lattice isomorphism between lattices of topologies on vector spaces over topological fields. We show that if the isomorphism preserves the lattices of vector topologies, then the map is induced by a composition of a semilinear isomorphism and a translation. As a corollary, the distribution of vector topologies in the lattice of topologies determines the structure of the topological field and that of the vector space.

We also consider a lattice isomorphism between lattices of vector topologies which preserves the sets of Hausdorff vector topologies. If such an isomorphism exists, the coefficient fields are algebraically isomorphic, and the vector spaces have the same dimension.

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Chapter 1

Introduction

The notion of topology is undoubtedly one of the most important concept in mathematics. It defines the convergence in a given space and is used not only to solve problems but also to formulate problems. One method of studying a space is to consider the partially ordered set of all topologies defined on the space with the inclusion order. The inclusion order controls some topological properties. For example, a topology which is finer than a Hausdorff topology is always Hausdorff, and a topology which is coarser than a compact topology is always compact. The partially ordered set of topologies with the inclusion order forms an algebraic structure called lattice structure, which is one of the generalizations of Boolean algebra. G. Birkhoff studied in [5], topologies by comparing with respect to the inclusion order. Since then, the lattice structure of topologies has been intensively studied. For example, one important problem was whether each topology in a lattice of topologies has a complement like Boolean algebra. This problem involved many mathematicians and was solved in [14]. A survey paper [10] is a good reference on lattice of topologies.

When a given set has some mathematical structure, it is often the case that the set may have several natural topologies. For example, the space of all continuous real valued functions can have several norms, which define different topologies. The partially ordered set consisting of all natural topologies on a fixed mathematical object with the inclusion order may form an interesting lattice structure. For an algebraic system, the term natural topology means all operations of the algebra are continuous with respect to the topology. For example, it is known that for groups, rings, and vector spaces, the set of natural topologies have lattice structures. It is natural to consider the relation between the lattice structures and the algebraic structures of the base set. For example, it is recently shown in [9], that the lattice of group topologies on every nilpotent group satisfies the semimodular property, which is a weakened condition of distributive property. We refer readers to [3] as a survey paper on the lattices of topologies on algebraic systems. This thesis is also related to this theme, mainly on vector spaces. In this case, we fix a topological field and a vector space over the field. Then we consider a partially ordered set with the inclusion order, consisting of all topologies on the vector space by which the addition and the scalar multiple are continuous. In other words, with these topologies, the vector space becomes a topological vector spaces. The main results of this thesis are rigidity results on these lattice structures on vector spaces. We see the results in detail in the next section.

1.1 Main results

For a set X, we denote by $\Sigma(X)$, the lattice of topologies on X with the inclusion order. For a vector space X over a topological field K, we denote by $\tau_K(X)$, the lattice of all vector topologies on X. Namely, $\tau_K(X)$ consists of topologies on X such that the addition and the scalar multiple are continuous with respect to them.

The following are main results of this thesis.

Theorem A (Theorem 3.2.5). *Let K, L be Hausdorff topological fields, X be a vector space over K with its dimension is bigger than one, and Y be a vector space over L. Then, for each lattice isomorphism $\Phi : \Sigma(X) \to \Sigma(Y)$ that maps $\tau_K(X)$ to $\tau_L(Y)$, there exists a unique triple (ψ, ϕ, y_0) consists of an isomorphism $\psi : K \to L$ between topological fields, a ψ-semilinear isomorphism $\phi : X \to Y$, and a point y_0 of Y such that*

- *if the cardinality $|X|$ is infinite, Φ is $(\phi + y_0)_*$, and*
- *if the cardinality $|X|$ is finite, Φ is either $(\phi + y_0)_*$ or $C_Y \circ (\phi + y_0)_*$,*

where $(\phi + y_0)_* : \Sigma(X) \to \Sigma(Y)$ and $C_Y : \Sigma(Y) \to \Sigma(Y)$ are maps between the lattices of topologies defined below.

Theorem A states roughly speaking, how vector topologies are in the lattice of topologies is unique up to isomorphism class of topological fields and vector structures.

Theorem A is a vector space analogue to a result of J. Hartmanis [8]. His result is on the group $\mathrm{Aut}(\Sigma(X))$ of lattice automorphisms from $\Sigma(X)$ to itself.

Theorem (Hartmanis). *Let X be a set.*

- *If $|X|$ is 1,2, or infinite, then $\mathrm{Aut}(\Sigma(X))$ is isomorphic to the symmetric group of X.*
- *If $|X|$ is finite and more than 2, then $\mathrm{Aut}(\Sigma(X))$ is isomorphic to the direct product of the symmetric group of X and the two-element group $\mathbb{Z}/2\mathbb{Z}$.*

Let us explain this result. A bijection $f : X \to Y$ between two sets X and Y induces a lattice isomorphism f_* between $\Sigma(X)$ and $\Sigma(Y)$ by

$$f_*(T) = \{V \subset Y \mid f^{-1}(V) \in T\},\ T \in \Sigma(X).$$

When $X = Y$, this induces a group homomorphism from the symmetric group of X to $\mathrm{Aut}(\Sigma(X))$. When the cardinality $|X|$ is finite, for a topology T on X, a family $C_X(T)$ defined by

$$\{X \setminus U \mid U \in T\}$$

is also a topology on X since taking an infinite union is reduced to taking a finite union. This induces another group homomorphism C_X from $\mathbb{Z}/2\mathbb{Z}$ to $\mathrm{Aut}(\Sigma(X))$. The above result shows that every lattice automorphism comes from these two types. We extend the theorem of Hartmanis to the case when $X \neq Y$ by slightly modifying his original proof in [8].

Theorem (Theorem 3.1.1)**.** *Let X, Y be two non-empty sets and Φ be a lattice isomorphism from $\Sigma(X)$ to $\Sigma(Y)$. We have a unique bijection $\phi : X \to Y$ such that*

- *if $|X|$ is 1,2, or infinite, then $\Phi = \phi_*$, and*
- *if $|X|$ is finite more than 2, then either $\Phi = \phi_*$ or $\Phi = C_Y \circ \phi_*$.*

This result is an analogue to Theorem A, and we use it to prove Theorem A.

Clearly, the restriction of the map Φ in Theorem A to $\tau_K(X)$ induces a lattice isomorphism between $\tau_K(X)$ and $\tau_L(Y)$. Thus a natural question is whether we can weaken the assumption of Theorem A to the existence of a lattice isomorphism between the lattices of vector topologies. This question is negatively answered by an example of $X = \mathbb{Q}^2, Y = \mathbb{R}^2$ (Example 3.2.7). However, if we also consider sets $\tau_K^H(X), \tau_L^H(Y)$ consisting of Hausdorff vector topologies on X and Y, respectively, we obtain the next similar result to Theorem A.

Theorem B (Theorem 3.2.8)**.** *Let K, L be Hausdorff topological fields, X be a vector space over K with its dimension is bigger than two, and Y be a vector space over L. If there is a lattice isomorphism $\Phi : \tau_K(X) \to \tau_L(Y)$ that maps $\tau_K^H(X)$ to $\tau_L^H(Y)$, then the fields K and L are isomorphic algebraically, and the dimensions of X and Y are the same.*

There is an example of topological fields and vector spaces such that they satisfies the assumption of Theorem B but any field isomorphism is not continuous (Example 3.2.9).

Key ingredients of proofs of Theorem A and Theorem B are the "fundamental theorem of affine geometry" (Theorem 2.3.1) and the "fundamental theorem of projective geometry" (Theorem 2.3.2). These fundamental theorems are classical and have been generalized in various way (See [11, 13] for examples). Let us give a brief explanation of basic ones. It is well-known that every vector space has a structure of an affine space and has an associated projective space. These fundamental theorems assert that these obtained geometric spaces are unique for the vector space: if we have a bijection that preserves parallel lines between two affine spaces (subspaces between projective spaces, respectively) constructed from two vector spaces, then the bijection is induced by an isomorphism between the vector spaces. In [11], the bijective assumption is dropped on the map between spaces, and in [13], the fundamental theorem is considered not in vector spaces but in tori. Since these fundamental theorems use vector subspaces whereas Theorem A and Theorem B are concerned with vector topologies, to accomplish the proofs, we use a bridge $(\mathfrak{S}, \mathfrak{T})$ (see Definition 3.2.1) between vector subspaces and topologies, which is an antitone Galois connection. Here, $\mathfrak{S}$ assigns the intersection $\bigcup_{0 \in U \in T} U$ of all open neighborhoods of the zero for each vector topology T, and $\mathfrak{T}$ maps a vector subspace S to a vector topology consisting of all S-invariant open subsets of the strongest vector topology.

1.2 Structure of this thesis

This thesis consists of three chapters. In Chapter 2, we prepare notations and recall basic results. More precisely, in Section 2.1, we explain preliminaries on lattices, especially on lattice of topologies on a fixed set. In Section 2.2, we focus on properties of lattices of vector topologies on a fixed vector space. In Section 2.3, we review the fundamental theorem of affine geometry and the fundamental theorem of projective geometry. This section ends with a proof of the fundamental theorem of projective geometry.

The purpose of Chapter 3 is to prove main theorems. In Section 3.1, we give a proof of modified result due to J. Hartmanis (Theorem 3.1.1). In Section 3.2, we prove Theorem A (Theorem 3.2.5) and Theorem B (Theorem 3.2.8). We also see some examples, which show the sharpness of our main results.

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Chapter 2

Preliminaries

In this chapter, we prepare notations and see properties on lattices, lattice of topologies, topological vector spaces and fundamental theorems of affine, projective geometries.

2.1 Preliminaries on lattices

A *lattice* $(L, \leq)$ is a partially ordered set (abbreviated to poset) such that for each pair of two elements $x, y \in L$, there exist a supremum (least upper bound) and an infimum (greatest lower bound). Namely, a supremum s (an infimum i, respectively) satisfies

- $x, y \leq s$ ($i \leq x, y$, respectively,)
- if $s' \in L$ satisfies $x, y \leq s'$, then $s \leq s'$ (if $i' \in L$ satisfies $i' \leq x, y$, then $i' \leq i$, respectively.)

By the second condition and the antisymmetric law, these supremum and infimum uniquely exist for each pair $(x, y) \in L \times L$. Thus we denoted by $x \vee y, x \wedge y$, the supremum and the infimum of (x, y) and call them *join* and *meet* of x, y, respectively. From another perspective, each lattice $(L, \leq)$ has an algebraic structure $(L, \vee, \wedge)$ consisting of two binary operations $\vee, \wedge : L \times L \to L$. They satisfies the following three laws:

(commutative law) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ for $x, y \in L$,

(associative law) $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ for $x, y, z \in L$,

(absorption law) $(x \vee y) \wedge x = x$ and $(x \wedge y) \vee x = x$ for $x, y \in L$.

Conversely, if we have an algebraic structure $(L, \vee, \wedge)$ that satisfies these three laws, we recover a lattice $(L, \leq)$ by defining a binary relation $\leq$ as

$$x \leq y \Leftrightarrow x \vee y = y.$$

Therefore, we can study lattices both by algebraic manners and by order theoretic manners.
A lattice $(L, \leq)$ is called *complete* if not only two-elements pair but also any subset S of L has a supremum $\bigvee S$ and an infimum $\bigwedge S$ in L. Thus a complete lattice has a top element and a bottom element as an infimum and a supremum of the empty set.
From the algebraic view point, it is natural to define a map ϕ between two lattices $(L_1, \leq_1)$ and $(L_2, \leq_2)$ as *lattice homomorphism* if ϕ preserves the join and the meet. Namely, $\phi : L_1 \to L_2$ satisfies

$$\phi(x \vee_1 y) = \phi(x) \vee_2 \phi(y),$$
$$\phi(x \wedge_1 y) = \phi(x) \wedge_2 \phi(y),$$

where $\vee_1, \vee_2$ are the join operations of L_1, L_2, respectively and $\wedge_1, \wedge_2$ are the meet operations of L_1, L_2, respectively. Moreover, a map ϕ between two complete lattices $(L_1, \leq_1), (L_2, \leq_2)$ is called *complete lattice homomorphism* if for any subset S of L_1, the map ϕ preserves the supremum and the infimum of S. If a lattice homomorphism ϕ has an inverse lattice homomorphism, we call ϕ *lattice isomorphism.* It is easy to show that a map ϕ is a lattice isomorphism if and only if ϕ is a bijective order preserving map. Furthermore, a lattice isomorphism is always a complete lattice homomorphism.
The followings are classical examples of lattices.

Example 2.1.1. Let $\mathbb{Z}$ be the set of integers with the standard order $\leq$. The poset $(\mathbb{Z}, \leq)$ is a lattice, where the join and the meet of $x, y \in \mathbb{Z}$ is the least common multiple and the greatest common divisor of x, y, respectively.

Example 2.1.2. Let X be a vector space over a field K. We denoted by $\sigma_K(X)$, the set of all K-vector subspaces. Then the poset $(\sigma_K(X), \subset)$ is a complete lattice. In fact, for a family of subsets $\{S_\lambda\}_{\lambda \in \Lambda}$, the subspace generated by all elements of S_λ and $\bigcap_{\lambda \in \Lambda} S_\lambda$ are the join and the meet of the family, respectively, where we consider $\bigcap_{\lambda \in \Lambda} S_\lambda$ as X if Λ is empty.

Definition 2.1.3. Let X be a non-empty set. We denote by $\Sigma(X)$, the partially ordered set consisting of all topologies on X with the inclusion order $\subset$. Here every element of $\Sigma(X)$ is a family of subsets of X such that it has the empty set and X, and that it is closed under taking a finite intersection and an infinite union.

For a family $\{T_\lambda\}_{\lambda \in \Lambda}$ of topologies on X, the topology generated by $\bigcup_{\lambda \in \Lambda} T_\lambda$ and $\bigcap_{\lambda \in \Lambda} T_\lambda$ are the supremum and the infimum of the family with respect to the order $\subset$. Thus $(\Sigma(X), \subset)$ is a complete lattice, called *lattice of topologies* on X.

Let $f : X \to Y$ be a map between two sets X, Y. Then f induces two maps $f_* : \Sigma(X) \to \Sigma(Y)$ and $f^* : \Sigma(Y) \to \Sigma(X)$ defined by

$$f_*(T) = \{V \subset Y \mid f^{-1}(V) \in T\},\ T \in \Sigma(X),$$
$$f^*(T') = \{f^{-1}(V) \subset X \mid V \in T'\},\ T' \in \Sigma(Y).$$

By definition, f_* and f^* preserve the order $\subset$. Thus every bijection induces lattice isomorphisms. Due to J. Hartmanis [8, Theorem 4], which we see in detail in Section 3.1, these induced isomorphisms are a major part of lattice isomorphisms between lattices of topologies. The other type of isomorphisms between lattices of topologies occurs when X is a finite set. We define a map C_X from the power of power set of X to itself by

$$C_X(T) = \{X \setminus U \mid U \in T\}.$$

We call C_X *complement map* of X. Every infinitely many union of subsets of X is equal to finitely many union of subsets of X since X is finite. Thus $C_X(T)$ is a topology on X if T is a topology. The map C_X is clearly order preserving involution, which implies that $C_X : \Sigma(X) \to \Sigma(X)$ is a lattice isomorphism (automorphism).

Let S be a subset of a poset $(\mathbb{P}, \leq)$. Then if exists, the supremum (the infimum, respectively) of the lower bound (upper bound, respectively) of S is equal to the infimum (the supremum, respectively) of S:

$$\inf S = \sup\{x \in \mathbb{P} \mid \forall s \in S \ x \leq s\},$$
$$(\sup S = \inf\{x \in \mathbb{P} \mid \forall s \in S \ s \leq x\}, \text{ respectively.})$$

These equalities are established by checking the definitions of the infimum and the supremum. Thus we obtain the following lemma stating that we do not have to express an infimum or a supremum of S explicitly to show that a given poset is a complete lattice.

Lemma 2.1.4. *Let $(\mathbb{P}, \leq)$ be a poset. If there exists a supremum (an infimum, respectively) for any subset S of $\mathbb{P}$, then the poset $(\mathbb{P}, \leq)$ is a complete lattice.*

Let L be a lattice with the bottom element 0. An element a of L is called *atom* or *point* if a is the next minimum element to 0. Namely, a is not 0, and if an element x satisfies $0 \leq x \leq a$, then $x = 0$ or $x = a$. A lattice is called *atomic* if every element of the lattice is expressed as a supremum of a family of atoms.

Definition 2.1.5. Let $\mathfrak{p}$ be the set of all atoms. A function called *type* $t : \mathfrak{p} \times \mathfrak{p} \to \mathbb{N}$ is defined by

$$t(a_1, a_2) = \#\{a \in \mathfrak{p} \mid a \subset a_1 \vee a_2\}.$$

That is, the function $t(\cdot, \cdot)$ counts the number of atoms which are weaker or equal to the join of atoms.

When a set X has more than one point, each atom of the lattice of topologies $\Sigma(X)$ is of form

$$a(D) = \{\emptyset, D, X\},$$

where D is a proper subset of X. Thus when X has more or equal to two points, the cardinality of atoms $\mathfrak{p}_X$ of $\Sigma(X)$ is $2^{|X|}-2$ if X is a finite set, and $2^{|X|}$ if X is an infinite set. It is known that every lattice of topologies is atomic. In fact, for a topology T of X, we set a subset $\mathcal{A}$ of atoms by

$$\{a(D) \mid \emptyset \subsetneq D \subsetneq X, D \in T\}.$$

Then the supremum $\bigvee \mathcal{A}$ is equal to T. For a given set X whose cardinality is more or equal to two, we define three subsets of the set of atoms $\mathfrak{p}_X$ by

$$\begin{aligned}\mathfrak{n}_X &= \{a(\{x\}) \in \mathfrak{p}_X \mid x \in X\},\\ \mathfrak{m}_X &= \{a(X \setminus \{x\}) \in \mathfrak{p}_X \mid x \in X\},\\ \mathfrak{l}_X &= \mathfrak{p}_X \setminus (\mathfrak{n}_X \cup \mathfrak{m}_X).\end{aligned}$$

We abbreviate $a(\{x\}), a(X \setminus \{x\})$ to $a(x), a(x^c)$, respectively. Let us observe the type function with respect to the decomposition $\mathfrak{p}_X = \mathfrak{n}_X \sqcup \mathfrak{m}_X \sqcup \mathfrak{l}_X$ when $|X| \geq 3$. First, for distinct atoms $a(D_1), a(D_2)$, the join $a(D_1) \vee a(D_2)$ is of form

$$\{\emptyset, D_1 \cap D_2, D_1, D_2, D_1 \cup D_2, X\}.$$

Thus $t(a(D_1), a(D_2))$ is at most 4. Let $a(x_1), a(x_2)$ be two distinct elements from $\mathfrak{n}_X$. Then since $\{x_1\} \cap \{x_2\} = \emptyset$, the type $t(a(x_1), a(x_2))$ is 3. A similar argument shows the type is 3 for two distinct atoms from $\mathfrak{m}_X$. Let $a(x_1), a({x_2}^c)$ be distinct atoms from $\mathfrak{n}_X, \mathfrak{m}_X$, respectively. Then $\{x_1\} \cap X \setminus \{x_2\}$ is $\{x_1\}$ or $\emptyset$, and $\{x_1\} \cup X \setminus \{x_2\}$ is $X \setminus \{x_2\}$ or X. Thus $t(a(x_1), a({x_2}^c))$ is 2. For atoms $a(x_1)$ from $\mathfrak{n}_X$ and $a(D)$ from $\mathfrak{l}_X$, since $\{x_1\} \cap D$ is $\{x_1\}$ or $\emptyset$, the type $t(a(x_1), a(D))$ is 2 or 3. A similar argument shows that for $a({x_1}^c) \in \mathfrak{m}_X, a(D) \in \mathfrak{l}_X$, the type $t(a({x_1}^c), a(D))$ is 2 or 3. For an atom $a(D)$ from $\mathfrak{l}_X$, we take elements x from D and x' from $X \setminus D$. Since $a(D)$ does not belong to $\mathfrak{n}_X \cup \mathfrak{m}_X$, we have $\{x\} \subsetneq D$ and $\{x'\} \subsetneq X \setminus D$. We set D' as the set $\{x, x'\}$. Then $\emptyset, D \cap D', D, D', D \cup D', X$ are all distinct. Therefore, for each $a(D) \in \mathfrak{l}_X$, there exists an atom $a(D')$ with $t(a(D), a(D')) = 4$. In summary, we obtain the following:

Proposition 2.1.6. *Let p, q be distinct atoms of $\Sigma(X)$. Then we have*

(1) if $p, q \in \mathfrak{n}_X$ or if $p, q \in \mathfrak{m}_X$, then the type $t(p, q) = 3$,

(2) if $p \in \mathfrak{n}_X, q \in \mathfrak{m}_X$, then the type $t(p, q) = 2$,

(3) if $p \in \mathfrak{n}_X \cup \mathfrak{m}_X, q \in \mathfrak{l}_X$, then the type $t(p, q)$ is 2 or 3, and

(4) for each $p \in \mathfrak{l}_X$, there exists $q \in \mathfrak{l}_X$ such that the type $t(p, q) = 4$.

2.2 Preliminaries on topological vector spaces

A topology T on a commutative field K is called *field topology* if the three field operations

$$\begin{aligned}
\text{(additive operation): } & K \times K \ni (\alpha, \beta) \mapsto \alpha + \beta \in K, \\
\text{(multiple operation): } & K \times K \ni (\alpha, \beta) \mapsto \alpha\beta \in K \text{ and} \\
\text{(inverse operation): } & K \setminus \{0\} \ni \alpha \mapsto \alpha^{-1} \in K \setminus \{0\}
\end{aligned}$$

are continuous, where we endow $K \times K$ with the product topology and $K \setminus \{0\}$ with the relative topology of T. We denote by T_K, a field topology of K. A field endowed with a field topology is called *topological field*.

Important examples of topological fields come from the notion of *valued field*. Recall that a function $\nu : K \to \mathbb{R}_{\geq 0}$ is a *valuation* if

$$\begin{aligned}
&\nu(\alpha) = 0 \Leftrightarrow \alpha = 0, \\
&\nu(\alpha\beta) = \nu(\alpha)\nu(\beta) \text{ and} \\
&\nu(\alpha + \beta) \leq \nu(\alpha) + \nu(\beta)
\end{aligned}$$

hold for $\alpha, \beta \in K$. A *valued field* is a pair (K, ν) consisting of a field K and a valuation ν. A canonical metric d_ν is defined on the valued field by $d_\nu(\alpha, \beta) = \nu(\alpha - \beta)$. This metric endows the field K a Hausdorff field topology. We call a valued field *complete* if the canonical metric space (K, d_ν) is a complete metric space. It is known that for a given valued field (K, ν), the metric completion $\hat{K}$ has a field structure and that the valuation ν is extended to $\hat{K}$, denoted by $\hat{\nu}$. Thus we obtain a complete valued field $(\hat{K}, \hat{\nu})$.

Example 2.2.1. The field of real numbers $\mathbb{R}$ with the standard absolute value $|\cdot|$ is a valued field.
Another example is the field of p-adic numbers $\mathbb{Q}_p$ for a fixed prime number p. This is obtained by a completion of the valued field $(\mathbb{Q}, |\cdot|_p)$, where $\mathbb{Q}$ is the field of rational numbers and $|\cdot|_p$ is a valuation defined by

$$\left|\frac{a}{b}p^n\right|_p = p^{-n}, \ |0|_p = 0$$

for an integer n and for non-zero integers $a, b \in \mathbb{Z}$ which are prime to p.

Let K be a topological field and U be a non-empty proper open subset of K. Namely, $\emptyset \subsetneq U \subsetneq K$ holds. Since translations are homeomorphism, K has a proper open neighborhood U' of zero. For a non-zero element $\alpha \in K$, the multiple map defined by

$$K \ni x \mapsto \alpha x \in K$$

is a homeomorphism and fixes zero. Thus for each non-zero element, by mapping U', we obtain an open neighborhood of zero to which the element does not belong. Now, let x, y be two distinct elements from K. There exists an open neighborhood U' of zero such that $x - y \notin U'$. By the continuity of the addition at $(0,0)$, we have an open neighborhood V of zero such that $V + V \subset U'$. Then $x - V, y + V$ are disjoint neighborhoods of x and y, respectively. Therefore we obtain a known result:

Proposition 2.2.2. *A non-Hausdorff topological field is an indiscrete space.*

Let X be a vector space over a topological field K. A topology on X is called *vector topology* or *compatible* if the linear operations:

$$\begin{aligned} \text{(addition)}: X \times X \ni (x, y) \mapsto x + y \in X \text{ and} \\ \text{(scalar multiple)}: K \times X \ni (\alpha, x) \mapsto \alpha * x \in X \end{aligned}$$

are continuous. A *topological vector space* is a vector space endowed with a vector topology.

Definition 2.2.3. Let X be a vector space over a topological field K. We denote by $\tau_K(X)$, the set of all vector topologies on X.

We recall one of a generalization of a linear map called semilinear map. A map $\phi : X \to Y$ between a vector space X over K and a vector space Y over L is called *ψ-semilinear* if $\psi : K \to L$ is an isomorphism between the fields, and ϕ satisfies

$$\begin{aligned} \phi(x + x') =& \phi(x) + \phi(x') \text{ for } x, x' \in X, \\ \phi(\alpha * x) =& \psi(\alpha) * \phi(x) \text{ for } \alpha \in K, x \in X. \end{aligned}$$

We sometimes abbreviate ψ-semilinear to semilinear. We call ϕ *semilinear isomorphism* if the semilinear ϕ is a bijection. In particular, when $K = L$ and $\psi = \mathrm{id}_K$, these conditions are the same as those of linear maps. When X is not 0-dimensional, assume that a map $\phi : X \to Y$ is both ψ_1-semilinear and ψ_2-semilinear. For a fixed non-zero element $x_0 \in X$, we have $\psi_1(\alpha)\phi(x_0) = \phi(\alpha x_0) = \psi_2(\alpha)\phi(x_0)$. Thus each semilinear isomorphism has a unique associated field isomorphism.

We next see that semilinear maps and linear maps induce maps between $\tau_K(X)$ and $\tau_L(Y)$.

Proposition 2.2.4. *Let X be a vector space over a topological field K and Y be a vector space over a topological field L. Then a semilinear map $\phi : X \to Y$ induces a map $\phi^* : \tau_L(Y) \to \tau_K(X)$ if the associated field isomorphism $\psi : K \to L$ for ϕ is continuous. The map ϕ_* is a map between $\tau_K(X) \to \tau_L(Y)$ if ϕ is surjective and $\psi : K \to L$ is an open surjective map.*

Proof. Let T' be a vector topology of Y. We show that $\phi^*(T')$ is a vector topology of X. Fix $x_1, x_2 \in X$ and $U \in \phi^*(T')$ as an open neighborhood of $x_1 + x_2$. By definition, we have $U = \phi^{-1}(V)$ for some $V \in T'$. Then $\phi(x_1) + \phi(x_2) \in V$, and there are $V_1, V_2 \in T'$

such that $\phi(x_1) \in V_1, \phi(x_2) \in V_2$ with $V_1 + V_2 \subset V$ since T' is a vector topology. Now $\phi^{-1}(V_1), \phi^{-1}(V_2) \in \phi^*(T')$ are open neighborhoods of x_1, x_2, respectively such that $\phi^{-1}(V_1) + \phi^{-1}(V_2) \subset U$.
Fix $\alpha \in K, x \in X$ and $U = \phi^{-1}(V) \in \phi^*(T')$ as an open neighborhood of $\alpha * x$. Then $\psi(\alpha) * \phi(x) \in V \in T'$, and there are $O \in T_L, V' \in T'$ such that $O * V' \subset V$ since T' is a vector topology. By the continuity of ψ, we have $\psi^{-1}(O) \in T_K, \phi^{-1}(V') \in \phi^*(T')$ are open neighborhoods of α, x, respectively such that $\psi^{-1}(O) * \phi^{-1}(V') \subset U$. Therefore $\phi^*(T')$ is a vector topology.

Next we assume that $\phi : X \to Y$ is surjective and the associated map $\psi : K \to L$ is an open surjective map. Let T be a vector topology on X, and we show that $\phi_*(T)$ is a vector topology on Y. Note that the equality

$$\phi^{-1}(\phi(U)) = \bigcup_{x \in \mathrm{Ker}(\phi)} (U + x)$$

holds. Thus $\phi : (X, T) \to (Y, \phi_*(T))$ is an open map. Fix $y_1, y_2 \in Y$ and V as an open neighborhood of $y_1 + y_2$. Since ϕ is surjective, we take $x_1, x_2 \in X$ such that $\phi(x_1) = y_1, \phi(x_2) = y_2$. By definition, $\phi^{-1}(V)$ is an open neighborhood of $x_1 + x_2$ in T. Thus there are $x_1 \in U_1, x_2 \in U_2 \in T$ such that $U_1 + U_2 \subset \phi^{-1}(V)$ since T is a vector topology. Then $\phi(U_1), \phi(U_2)$ are open neighborhoods of y_1, y_2 such that $\phi(U_1) + \phi(U_2) \subset V$.
Fix $\beta \in L, y \in Y$ and V as an open neighborhood of $\beta * y$. Since ψ, ϕ are surjective, we have $\alpha \in K, x \in X$ such that $\psi(\alpha) = \beta, y = \phi(x)$. The set $\phi^{-1}(V)$ is an open neighborhood of $\alpha * x$, and thus there are open neighborhoods $O \in T_K, U \in T$ such that $O * U \subset \phi^{-1}(V)$. Since ψ is an open map, $\psi(O), \phi(U)$ are neighborhoods of β, y, respectively such that $\psi(O) * \phi(U) \subset V$. Therefore $\phi_*(T)$ is a vector topology. □

When $K = L$ as a topological field, since the identity map is open and surjective, we obtain the next corollary.

Corollary 2.2.5. *Let X, Y be vector spaces over a topological field K. A linear map $\phi : X \to Y$ induces a map $\phi^* : \tau_K(Y) \to \tau_K(X)$. If ϕ is surjective, then ϕ induces $\phi_* : \tau_K(X) \to \tau_K(Y)$.*

Let $\{T_\lambda\}_{\lambda \in \Lambda}$ be a family of vector topologies on X. We denote by $T_{\sup}$, the topology whose subbase is $\bigcup_{\lambda \in \Lambda} T_\lambda$. We see that $T_{\sup}$ is a supremum of the family $\{T_\lambda\}_{\lambda \in \Lambda}$ in $\tau_K(X)$. Since $T_{\sup}$ is a supremum in $\Sigma(X)$, it suffices to show that $T_{\sup} \in \tau_K(X)$. Now for each $\lambda \in \Lambda$, since $T_\lambda \in \tau_K(X)$ and $T_\lambda \subset T_{\sup}$, the two linear operations are continuous:

$$(X \times X, T_{\sup} \times T_{\sup}) \ni (x, y) \mapsto x + y \in (X, T_\lambda),$$
$$(K \times X, T_K \times T_{\sup}) \ni (\alpha, x) \mapsto \alpha * x \in (X, T_\lambda),$$

where $T_{\sup} \times T_{\sup}, T_K \times T_{\sup}$ are the product topologies. This implies that $T_{\sup}$ is a vector topology. More precisely, let $U \in T_{\sup}$ be a non-empty open subset. Then there is a finite number of open subsets $U_1, U_2, \dots, U_n$ such that $U_i \in T_{\lambda_i}$ and $U = \bigcap_{i=1}^n U_i$. Thus the preimages of U by the addition and the scalar multiple are the intersections of the preimages of U_i. The above continuities imply that the preimage of U_i are in $T_{\text{sup}} \times T_{\text{sup}}$, $T_K \times T_{\text{sup}}$. Therefore, the preimage of U is in $T_{\text{sup}} \times T_{\text{sup}}$ and $T_K \times T_{\text{sup}}$. Thus $(\tau_K(X), \subset)$ is a complete lattice. In particular, $\tau_K(X)$ has a top element. We denote by $T_K^{\max}(X)$, the top element (strongest topology) of $(\tau_K(X), \subset)$.

Since the condition of a map ϕ between topological spaces (X, T) and (Y, T') being continuous is equivalent to $\phi^*(T') \subset T$, by Corollary 2.2.5, we obtain the following:

Corollary 2.2.6. *A linear map $\phi : X \to Y$ is continuous with respect to $T^{\max}(X)$ and any vector topology on Y.*

Proposition 2.2.7. *Let X be a non-zero vector space over K. The top element $T_K^{\max}(X)$ is Hausdorff if and only if K is a Hausdorff topological field. Moreover, if K is not a Hausdorff space, then $T_K^{\max}(X)$ is an indiscrete topology.*

Proof. Assume that K is not a Hausdorff space and there exists proper open subset $\emptyset \subsetneq U \subsetneq X$ in $T^{\max}(X)$. Then by a translation, we can assume that U is an open neighborhood of zero such that an element $x_0 \notin U$. We define a map f by

$$f : K \ni \alpha \mapsto \alpha * x_0 \in X.$$

Then by the continuity of f, the preimage $f^{-1}(U)$ is a proper open subset of K. On the other hand, by Proposition 2.2.2, K is an indiscrete space. Thus $f^{-1}(U) = K$, which is contradiction. Therefore when K is not Hausdorff, $T^{\max}(X)$ is an indiscrete topology. In particular, $T^{\max}(X)$ is not a Hausdorff space.
Next we assume that K is a Hausdorff topological field. In the topological vector space X, maps

$$X \times X \ni (x, y) \mapsto x - y \in X \text{ and}$$
$$X \times X \ni (x, y) \mapsto x + y \in X$$

are continuous. Therefore, to prove that the space is Hausdorff, it suffices to show that for a non-zero element, there is an open neighborhood of zero such that the element does not belong to it. For a non-zero element $x_0 \in X$, we fix a decomposition of X into $X' \bigoplus Kx_0$ for some subspace X' and the subspace Kx_0 generated by x_0. We define a g map by

$$g : X \ni x = x' + \alpha * x_0 \mapsto \alpha \in K, x' \in X'.$$

Namely, g is a map that assigns component of x_0 with respect to the decomposition. Since the linear map $g : (X, T^{\max}(X)) \to (K, T_K)$ is continuous, the preimages of disjoint open neighborhoods O_1, O_2 of $0, 1 \in K$ are open with respect to $T^{\max}(X)$. Thus we can separate zero and x_0 in $(X, T^{\max}(X))$. Therefore $T^{\max}(X)$ is Hausdorff. □

Remark 2.2.8. By Proposition 2.2.7, $\tau_K(X)$ is a one point set if K is not a Hausdorff space. Thus we assume that K is a Hausdorff space for the rest of this thesis.

Definition 2.2.9. We denote by $\tau_K^H(X)$, the subset of $\tau_K(X)$ consisting of Hausdorff vector topologies.

Under the assumption of Remark 2.2.8, $T^{\max}(X)$ is a Hausdorff topology. That is, $T^{\max}(X) \in \tau_K^H(X)$ holds. Next proposition states furthermore, if X admits only one Hausdorff vector topology, then the lattice of vector topology is understood as the lattice of subspaces. For a proof of Proposition 2.2.10, see [1] for example.

Proposition 2.2.10. *Let X be a vector space over a topological field K. If X admits only one Hausdorff vector topology, then the lattice of vector topology $(\tau_K(X), \subset)$ is isomorphic to $(\sigma_K(X), \supset)$ by $\mathfrak{S}$ and $\mathfrak{T}$ (see Definition 3.2.1 for the definition of $\mathfrak{S}, \mathfrak{T}$).*

Remark 2.2.11. Although $\tau_K(X)$ is a subposet of $\Sigma(X)$, the lattice $\tau_K(X)$ is not a sublattice of $\Sigma(X)$ in general. More precisely, the meet operations of $\Sigma(X)$ and $\tau_K(X)$ do not coincide in general.

For the rest of this section, we see in particular, results on the lattice of vector topologies on a vector space over a valued field. We need them to explain examples.

Next result due to A. Tikhonov states that a finite-dimensional vector space over a complete valued field admits only one Hausdorff vector topology (see [6]).

Proposition 2.2.12. *Let X be a finite-dimensional vector space over a complete valued field K. Then $T_K^{\max}(X)$ is the only Hausdorff vector topology on X.*

By combining Proposition 2.2.10 and Proposition 2.2.12, we obtain the next proposition.

Proposition 2.2.13. *In the situation of Proposition 2.2.12, the lattice of vector topologies $\tau_K(X)$ is isomorphic to the lattice of subspaces $\sigma_K(X)$.*

For the case when the coefficient valued field is not complete, the next result is shown in [1].

Proposition 2.2.14. *Let X be a finite-dimensional vector space over a valued field (K, ν) whose completion $(\hat{K}, \hat{\nu})$ is a locally compact and not a discrete space. Then the lattice of vector topologies $(\tau_K(X), \subset)$ is isomorphic to the lattice of subspaces $(\sigma_{\hat{K}}(\hat{X}), \supset)$ by*

$$\hat{\mathfrak{S}} : \tau_K(X) \ni T \mapsto \bigcap_{0 \in U \in T} \overline{I(U)} \in \sigma_{\hat{K}}(\hat{X}),$$

$$\hat{\mathfrak{T}} : \sigma_{\hat{K}}(\hat{X}) \ni S \mapsto I^*(\mathfrak{T}_X(S)) \in \tau_K(X),$$

where $\hat{X}$ is a $\hat{K}$-vector space obtained by $\hat{K} \bigotimes_K X$, the map $I : X \to \hat{X}$ is defined by $x \mapsto 1 \otimes x$ and we take the closure with respect to $T^{\max}(\hat{X})$.

Proposition 2.2.15. *In the situation of Proposition 2.2.14, a vector topology T is Hausdorff if and only if $\hat{\mathfrak{S}}(T) \cap I(X) = \{0\}$.*

2.3 Geometries from vector spaces

In this section, we recall the fundamental theorem of affine geometry and the fundamental theorem of projective geometry.

For a vector space X, we say two subsets A_1, A_2 of X *parallel lines* if there exist two points $x_1, x_2 \in X$ and a 1-dimensional subspace l such that $A_i = x_i + l$ for $i = 1, 2$. Then the fundamental theorem of affine geometry is stated as follow.

Theorem 2.3.1 (Fundamental theorem of affine geometry)**.** *Let X be a vector space over K with $\dim_K(X) \geq 2$ and Y be a vector space over L. If a bijection $\phi : X \to Y$ maps parallel lines of X to parallel lines of Y, then ϕ is a composition of a semilinear map and a translation.*

In [12], an elementary proof is given for $X = Y$ case. This proof is easily modified to the case of $X \neq Y$.

Theorem 2.3.2 (Fundamental theorem of projective geometry)**.** *Let X be a vector space over K with $\dim_K(X) \geq 3$ and Y be a vector space over L. Let $\Phi : \sigma_K(X) \to \sigma_L(Y)$ be a lattice isomorphism. Then there exists a semilinear map $\phi : X \to Y$ such that $\Phi(S) = \phi(S)$ for all $S \in \sigma_K(X)$.*

The fundamental theorem of projective geometry holds for the sublattice consisting of finite-dimensional subspaces. Namely, let $\sigma_K^{<\infty}(X)$ denote the sublattice of $\sigma_K(X)$ consisting of finite-dimensional subspaces of X. Then the next statement holds.

Theorem 2.3.3. *Let X, Y, K, L satisfy the same assumption in Theorem 2.3.2. Let Φ be a lattice isomorphism from $\sigma_K^{<\infty}(X)$ to $\sigma_L^{<\infty}(Y)$. Then there exists a field isomorphism $\psi : K \to L$ and a ψ-semilinear isomorphism $\phi : X \to Y$ such that $\Phi(S) = \phi(S)$ for all $S \in \sigma_K^{<\infty}(X)$.*

An easy consequence of Theorem 2.3.3 is every lattice isomorphism between the lattices of finite-dimensional subspaces extends to the lattices of all subspaces.

A proof of fundamental theorem of projective geometry is given in [4]. We end this section by showing that the proof according to [4] also holds in the situation of Theorem 2.3.3. For a vector $x \in X$, we denote by Kx, a subspace of X generated by x.

Proof. The proof is divided into several claims.

Claim 1. *For non-zero vectors $x \in X, y \in Y$ such that $\Phi(Kx) = Ly$ and for a vector $x' \in X$ with $Kx \neq Kx'$, there exists a unique $y' \in Y$ such that*

$$\begin{cases} \Phi(Kx') = Ly', \\ \Phi(K(x - x')) = L(y - y'). \end{cases}$$

When $x' = 0$, then Claim 1 is valid by $y' = 0$. Next, we consider when $x' \neq 0$. From $K(x - x') \subset Kx + Kx'$, we deduce that

$$\Phi(K(x - x')) \subset \Phi(Kx + Kx') = \Phi(Kx) + \Phi(Kx').$$

The assumption $Kx \neq Kx'$ implies $x \neq x'$. Thus $\Phi(K(x-x'))$ is 1-dimensional. Thus there exists a non-zero $y_0 \in Y$ with $\Phi(K(x-x')) = Ly_0$. Since $y_0 \in \Phi(Kx) + \Phi(Kx')$ and $\Phi(Kx) = Ly$ hold, there exist $y_1 \in \Phi(Kx')$ and $l \in L$ such that $y_0 = ly - y_1$. Since $K(x-x') \not\subset Kx'$, we have $Ly_0 \not\subset \Phi(Kx')$ and thus, $l \neq 0$. We define y' as $l^{-1}y_1$. Then since we assumed that $x' \neq 0$ and $Kx \neq Kx'$, we have $Ly' = Ly_1 \subset \Phi(Kx')$ implies that $Ly' = \Phi(Kx')$. Furthermore,

$$\Phi(K(x-x')) = Ly_0 = L(l^{-1}y_0) = L(y-y'),$$

and thus y' meets all our requirements.

Assume now that y'' also satisfies $\Phi(Kx) = Ly''$ and $\Phi(K(x-x')) = L(y-y'')$, which implies that

$$Ly' = Ly'' \text{ and } L(y-y') = L(y-y'').$$

Consequently, there exist numbers $l_1, l_2 \in L$ such that

$$y' = l_1 y'' \text{ and } l_2(y-y'') = y - y' = y - l_1 y''.$$

Thus we obtain $(l_2 - 1)y = (l_2 - l_1)y''$, which implies that $\Phi(Kx') = Ly'' \subset Ly = \Phi(Kx)$ if $l_2 \neq l_1$. This is contradiction since we assumed $Kx \neq Kx'$ and $x' \neq 0$. Therefore $l_2 = l_1$ and $l_2 = 1$ hold since $y \neq 0$, which implies $y' = y''$. This completes the proof of Claim 1.

From Claim 1, for fixed non-zero vectors $x \in X, y \in Y$ such that $\Phi(Kx) = Ly$, we define a function $h(x, y : x')$ from $(X \setminus Kx) \cup \{0\}$ to Y by which x' is sent to y' such that

$$\begin{cases} \Phi(Kx') = Ly', \\ \Phi(K(x-x')) = L(y-y'). \end{cases}$$

Claim 2. *For non-zero vectors $x, x' \in X$ such that $Kx \neq Kx'$ and $y, y' \in Y$ such that $\Phi(Kx) = Ly$ and $\Phi(Kx') = Ly'$, we have*

$$h(x, y : x') = y' \Leftrightarrow h(x', y' : x) = y.$$

It is clear that the equation $\Phi(K(x-x')) = L(y-y')$ is equivalent to $\Phi(K(x'-x)) = L(y'-y)$, and thus by definition of h, Claim 2 holds.

Claim 3. *If x_1, x_2, x_3 are three independent vectors in X, then*

$$K(x_2 - x_3) = (Kx_2 + Kx_3) \cap (K(x_1 - x_2) + K(x_1 - x_3)).$$

It is obvious that

$$K(x_2 - x_3) \subset (Kx_2 + Kx_3) \cap (K(x_1 - x_2) + K(x_1 - x_3)) =: J.$$

Conversely, let $j \in J$ be of the form

$$j = ax_2 + bx_3 = d(x_1 - x_2) + e(x_1 - x_3), \ a, b, d, e \in K.$$

Since x_1, x_2, x_3 are independent, we have $a = -d, b = -e, d + e = 0$. Hence $j = a(x_2 - x_3)$ belongs to $K(x_2 - x_3)$. This proves Claim 3.

Claim 4. *If x_1, x_2, x_3 are independent vectors in X and $y_1 \in Y$ satisfies $\Phi(Kx_1) = Ly_1$, then $h(x_1, y_1 : x_2) = y_2$ and $h(x_1, y_1 : x_3) = y_3$ implies $h(x_2, y_2 : x_3) = y_3$ for $y_2, y_3 \in Y$.*

By hypothesis, the following equations are valid:

$$\begin{cases} \Phi(Kx_1) = Ly_1, & \Phi(K(x_1 - x_2)) = L(y_1 - y_2), \\ \Phi(Kx_2) = Ly_2, & \Phi(K(x_1 - x_3)) = L(y_1 - y_3), \\ \Phi(Kx_3) = Ly_3. \end{cases}$$

The independence of three vectors y_1, y_2, y_3 is a consequence of the independence of x_1, x_2, x_3.
Hence we may apply Claim 3, both $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$. Consequently, we obtain

$$\begin{aligned} \Phi(K(x_2 - x_3)) &= \Phi((Kx_2 + Kx_3) \cap (K(x_1 - x_2) + K(x_1 - x_3)) \\ &= (Ly_2 + Ly_3) \cap (L(y_1 - y_2) + L(y_1 - y_3)) \\ &= L(y_2 - y_3). \end{aligned}$$

Now the validity of the three equations

$$\Phi(Kx_2) = Ly_2,\ \Phi(Kx_3) = Ly_3,\ \Phi(K(x_2 - x_3)) = L(y_2 - y_3)$$

implies that $h(x_2, y_2 : x_3) = y_3$.

Claim 5. *If x_1, x_2, x_3 are three independent vectors in X, then*

$$\begin{cases} K(x_1 - x_2 - x_3) = (K(x_1 - x_2) + Kx_3) \cap (K(x_1 - x_3) + Kx_2), \\ K(x_2 + x_3) = (Kx_2 + Kx_3) \cap (K(x_1 - x_2 - x_3) + Kx_1). \end{cases}$$

Claim 5 follows from similar arguments in the proof of Claim 3.

Claim 6. *Let $x_1 \in X$ and $y_1 \in Y$ be non-zero vectors satisfying $\Phi(Kx_1) = Ly_1$. For $x_2, x_3 \in X$, if $Kx_1 \cap (Kx_2 + Kx_3) = 0$ holds, then h preserves the addition. Namely we have*

$$h(x_1, y_1 : x_2 + x_3) = h(x_1, y_1 : x_2) + h(x_1, y_1 : x_3).$$

(Case 1:Three vectors x_1, x_2, x_3 are independent.)
In this case, we have $Kx_1 \neq Kx_2, Kx_1 \neq Kx_3$ and $Kx_1 \neq K(x_2 + x_3)$ from the independence of x_1, x_2, x_3. We put

$$y_2 \coloneqq h(x_1, y_1 : x_2),\ y_3 \coloneqq h(x_1, y_1 : x_3).$$

The independence of y_1, y_2, y_3 follows from that of x_1, x_2, x_3. By using Claim 5 and definition of y_2, y_3, we find that

$$\begin{aligned} \Phi(K(x_1 - x_2 - x_3)) &= \Phi((K(x_1 - x_2) + Kx_3) \cap (K(x_1 - x_3) + Kx_2)) \\ &= (L(y_1 - y_2) + Ly_3) \cap (L(y_1 - y_3) + Ly_2) \\ &= L(y_1 - y_2 - y_3), \\ \Phi(K(x_2 + x_3)) &= \Phi((Kx_2 + Kx_3) \cap (K(x_1 - x_2 - x_3) + Kx_1)) \\ &= (Ly_2 + Ly_3) \cap (L(y_1 - y_2 - y_3) + Ly_1) \\ &= L(y_2 + y_3), \end{aligned}$$

where we use the first equality to show the second equality. The above two equalities imply that $h(x_1, y_1 : x_2 + x_3) = h(x_1, y_1 : x_2) + h(x_1, y_1 : x_3)$.

(Case 2:Three vectors x_1, x_2, x_3 are not independent.)
Our assumptions $Kx_1 \neq 0$ and $Kx_1 \cap (Kx_2 + Kx_3) = 0$ imply that $Kx_2 \subset Kx_3$ or $Kx_3 \subset Kx_2$. Moreover, if $Kx_2 = 0$ or $Kx_3 = 0$, then our claim is trivial. Thus we only treat the case when $Kx_2 = Kx_3 \neq 0$ and $Kx_1 \neq Kx_2$. Since $\dim_K X \geq 3$, we can take a non-zero vector $x \in X$ such that x_1, x_2, x are independent. We divide two cases.

(1. $x_2 + x_3 \neq 0$ case) By using Case 1, we obtain

$$\begin{aligned} h(x_1, y_1 : x) + h(x_1, y_1 : x_2 + x_3) &= h(x_1, y_1 : x + x_2 + x_3) \\ &= h(x_1, y_1 : x + x_2) + h(x_1, y_1 : x_3) \\ &= h(x_1, y_1 : x) + h(x_1, y_1 : x_2) + h(x_1, y_1 : x_3) \end{aligned}$$

Thus we obtain $h(x_1, y_1 : x_2 + x_3) = h(x_1, y_1 : x_2) + h(x_1, y_1 : x_3)$.

(2. $x_2 + x_3 = 0$ case) A similar argument in Case 1 holds. Take w so that x_1, x_2, w are independent. Then

$$\begin{aligned} h(x_1, y_1 : w) + h(x_1, y_1 : x_2) + h(x_1, y_1 : x_3) &= h(x_1, y_1 : w + x_2) + h(x_1, y_1 : x_3) \\ &= h(x_1, y_1 : w + x_2 + x_3) \\ &= h(x_1, y_1 : w). \end{aligned}$$

Thus $h(x_1, y_1 : x_2) + h(x_1, y_1 : x_3) = 0 = h(x_1, y_1 : x_2 + x_3)$. This completes the proof of Claim 6.

Now, since $\dim_K X \geq 3$, we fix three independent vectors $x_1, x_2, x_3 \in X$ for the rest of the proof of Theorem 2.3.3. Take $y_1 \in Y$ such that $\Phi(Kx_1) = Ly_1$. Let $y_2, y_3 \in Y$ be

$$\begin{cases} y_2 = h(x_1, y_1 : x_2), \\ y_3 = h(x_1, y_1 : x_3). \end{cases}$$

Then by Claim 2 and Claim 4, we have

Claim 7.

$$\begin{cases} h(x_2, y_2 : x_1) &= y_1, \\ h(x_2, y_2 : x_3) &= y_3, \\ h(x_3, y_3 : x_1) &= y_1, \\ h(x_3, y_3 : x_2) &= y_2. \end{cases}$$

Claim 8. *Let i, j be two distinct numbers of 1, 2 or 3. For a vector $x \in X$ with $Kx_i \neq Kx$ and $Kx_j \neq Kx$, then $h(x_i, y_i : x) = h(x_j, y_j : x)$.*

(Case 1:Three vectors x_i, x_j, x are independent.)
By Claim 7, we have $h(x_i, y_i : x_j) = y_j$. Set y as $h(x_i, y_i : x) = y$. Then by Claim 4, we have $h(x_j, y_j : x) = y = h(x_i, y_i : x)$.

(Case 2:Three vectors x_i, x_j, x are not independent.)
In this case, $Kx \subset Kx_i + Kx_j$. Let x_k be the last element of $\{x_1, x_2, x_3\}$, $x_k \neq x_i, x_k \neq$

x_j. If $x = 0$, the claim is trivial since $h(x_i, y_i : x) = 0 = h(x_j, y_j : x)$. If $x \neq 0$, since x_i, x_k, x are independent, by Case 1, we have

$$h(x_i, y_i : x) = h(x_k, y_k : x).$$

Again since x_j, x_k, x are independent, by Case 1, we have

$$h(x_j, y_j : x) = h(x_k, y_k : x).$$

Therefore, we obtain the desired equality $h(x_i, y_i : x) = h(x_j, y_j : x)$ in Claim 8.

We can define a map $\phi : X \to Y$ by

$$\phi(x) = y \text{ if and only if } Kx_i \neq Kx, h(x_i, y_i : x) = y \text{ for some } i.$$

By Claim 8, the choice of x_i is not dependent in the definition of ϕ. By definition of ϕ, we have $h(x_i, y_i : x) = \phi(x)$ for some $x_i \in \{x_1, x_2, x_3\}$. Thus $\Phi(Kx) = L\phi(x)$ holds.

Claim 9. *ϕ is an isomorphism from the additive group X onto the additive group Y.*

First, we show that the map ϕ preserves the addition: $\phi(x + x') = \phi(x) + \phi(x')$ for $x, x' \in X$. Since x_1, x_2, x_3 are independent, there is $x_i \in \{x_1, x_2, x_3\}$ such that $Kx_i \cap (Kx + Kx') = 0$, and thus by Claim 6, we obtain

$$h(x_i, y_i : x + x') = h(x_i, y_i : x) + h(x_i, y_i : x').$$

Now since $K(x + x') \neq Kx_i, Kx \neq Kx_i, Kx' \neq Kx_i$, by definition of ϕ, we have $\phi(x + x') = \phi(x) + \phi(x')$.

Suppose $\phi(x) = 0$ for a vector $x \in X$. Then $\Phi(Kx) = L\phi(x)$ implies $\Phi(Kx) = 0$. Since Φ is one-to-one and $\Phi(0) = 0$, the subspace Kx is 0-dimensional, which implies that $x = 0$. Thus ϕ is injective.

Next, take an arbitrary non-zero element y_0 from Y. Since y_1, y_2, y_3 are independent, we can take $y_i \in \{y_1, y_2, y_3\}$ such that $Ly_i \neq Ly_0$. Then $L(y_i + y_0) \neq 0$ and $Ly_0 \neq 0$. Thus there are non-zero vectors $x, x' \in X$ such that

$$\begin{cases} \Phi(Kx) = L(y_i + y_0), \\ \Phi(Kx') = Ly_0. \end{cases}$$

Then there is $\alpha \in K$ such that $x_i + \alpha x' \in Kx$. This is proved as follows: Since $\Phi(Kx) = L(y_i + y_0) \subset Ly_i + Ly_0 = \Phi(Kx_i + Kx')$, we have $Kx \subset Kx_i + Kx'$, and thus x is represented as $x = \beta x_i + \gamma x'$ for some $\beta, \gamma \in K$. If β were zero, then $x \in Kx'$, which implies that $L(y_i + y_0) \subset Ly_0$. This is contradiction since $Ly_i \neq Ly_0$. Thus we obtain $x_i + \alpha x' \in Kx$ by putting α as $\beta^{-1}\gamma$. Now we show that $x_0 \coloneqq \alpha x'$ is sent to y_0 by ϕ. Since $\phi(x_0) \in \Phi(Kx') = Ly_0$ holds, $\phi(x_0) - y_0 \in Ly_0$. Next, since $x_i + x_0 \in Kx$, we have $L(\phi(x_i + x_0)) \subset \Phi(Kx) = L(y_i + y_0)$. Thus $\phi(x_0) - y_0 = (\phi(x_i) + \phi(x_0)) - (y_i + y_0) \in L(y_i + y_0)$. Therefore, the element $\phi(x_0) - y_0$ belongs to the intersection $Ly_0 \cap L(y_i + y_0) = 0$, which implies that $\phi : X \to Y$ is surjective, and we conclude that $\phi : X \to Y$ is a group isomorphism.

Claim 10. $\Phi(S) = \phi(S)$ *for every finite-dimensional subspace* S *of* X.

For every $x \in S$, we have $\phi(x) \in L\phi(x) = \Phi(Kx) \subset \Phi(S)$. Thus $\phi(S) \subset \Phi(S)$ holds. Take $y \in \Phi(S)$. Since ϕ is surjective, there exists $x \in X$ such that $\phi(x) = y$. Then $\Phi(Kx) = L\phi(x) = Ly \subset \Phi(S)$ implies that $x \in S$. Therefore we have $y = \phi(x) \in \phi(S)$.

For a non-zero vector $x \in X$ and $k \in K$, since $\Phi(Kx) = L\phi(x)$, we denote by $s(x, k)$, a unique $l \in L$ such that $\phi(kx) = l\phi(x)$.

Claim 11. *For non-zero vectors* $x, x' \in X$ *and* $k \in K$, *we have* $s(x, k) = s(x', k)$.

First, we assume that $Kx \neq Kx'$. Then we have

$$\begin{aligned} s(x+x', k)\phi(x) + s(x+x', k)\phi(x') &= s(x+x', k)\phi(x+x') \\ &= \phi(k(x+x')) \\ &= \phi(kx) + \phi(kx') \\ &= s(x, k)\phi(x) + s(x', k)\phi(x'). \end{aligned}$$

Since $\phi(x)$ and $\phi(x')$ are independent, $s(x, k) = s(x+x', k) = s(x', k)$ holds. Next, in case $Kx = Kx'$, since $\dim_K X \geq 2$, there exists a non-zero vector $x'' \in X$ such that $Kx'' \neq Kx = Kx'$. By the same argument, we obtain $s(x, k) = s(x'', k) = s(x', k)$.

By Claim 11, we can define a map $\psi : K \to L$ by which $k \in K$ is sent to l such that $\phi(kx) = \psi(l)\phi(x)$ holds for any non-zero vector $x \in X$.

We finish the proof of Theorem 2.3.3 by showing that $\psi : K \to L$ is a field isomorphism.

(surjectivity) Let x_0 be a non-zero vector of X. For a fixed $l \in L$, since ϕ is surjective, there exists $x \in X$ such that $\phi(x) = l\phi(x_0)$. By Claim 10, $\phi(x) \in L\phi(x_0) = \Phi(Kx_0) = \phi(Kx_0)$ holds. Since ϕ is injective, x belongs to Kx_0, and thus there exists $k \in K$ such that $x = kx_0$. Then by definition, $\phi(x) = \psi(k)\phi(x_0)$, which implies $\psi(k) = l$. Therefore ψ is surjective.

(homomorphism) Let $k_1, k_2 \in K$. By definition of ψ and by Claim 9, we have

$$\begin{aligned} (\psi(k_1) + \psi(k_2))\phi(x_0) &= \phi(k_1x_0) + \phi(k_2x_0) \\ &= \phi((k_1 + k_2)x_0) \\ &= \psi(k_1 + k_2)\phi(x_0). \end{aligned}$$

Thus, ψ preserves the addition: $\psi(k_1) + \psi(k_2) = \psi(k_1 + k_2)$.
Next, for every $k_1, k_2 \in K$, we have

$$\begin{aligned} (\psi(k_1)\psi(k_2))\phi(x_0) &= \phi(k_1(k_2x_0)) \\ &= \phi((k_1k_2)x_0) \\ &= \psi(k_1k_2)\phi(x_0). \end{aligned}$$

Thus ψ preserves the multiplication: $\psi(k_1k_2) = \psi(k_1)\psi(k_2)$.
Next, $\psi(1)\phi(x_0) = \phi(1x_0) = \phi(x_0)$ implies that $\psi(1) = 1$.

(injectivity) Let $k \in K$ be mapped to zero by ψ. Then $\phi(kx_0) = \psi(k)\phi(x_0) = 0$. Since ϕ is injective, we obtain $kx_0 = 0$ and thus $k = 0$. □

Chapter 3

Rigidity of lattices of vector topologies

3.1 An extension of a theorem of J. Hartmanis

In this section, we extend a result due to J. Hartmanis.

Theorem 3.1.1 (J. Hartmanis [8]). *Let X, Y be non-empty sets and $\Phi : \Sigma(X) \to \Sigma(Y)$ be a lattice isomorphism. Then, there exists a unique bijection $\phi : X \to Y$ such that*

- *if the cardinality $|X|$ is one, two or infinite, then $\Phi = \phi_*$, and*
- *if the cardinality $|X|$ is finite more than two, then either $\Phi = \phi_*$ or $\Phi = C_Y \circ \phi_*$ holds.*

Remark 3.1.2. The original statement of Theorem 3.1.1 in [8] is on the group of lattice automorphisms of $\Sigma(X)$. Namely, it deals with the case when $X = Y$.

The following is a slight modified proof of Theorem 3.1.1 from [8].

Proof. Let $\mathfrak{p}_X$ be the set of atoms of $\Sigma(X)$ and $\mathfrak{p}_Y$ be that of $\Sigma(Y)$. Since a lattice isomorphism is an order preserving bijection, the restriction of Φ to the set $\mathfrak{p}_X$ of atoms is a bijection between $\mathfrak{p}_X$ and $\mathfrak{p}_Y$. In particular, the cardinalities of atoms are equal. Thus X is a finite set if and only if Y is a finite set. Moreover, the cardinality of X is equal to Y if X is a finite set.

($|X| = 1, 2$ case) When $|X| = 1$, the set Y is also a one point set, and thus the claim is clear. When $|X| = 2$, let x_1, x_2 and y_1, y_2 be points of X and Y, respectively. Since $\Phi(a(x_i))$ is an atom of form $a(y_1)$ or $a(y_2)$, a map $\phi : X \to Y$ is defined so that $\Phi(a(x_i)) = a(\phi(x_i))$ holds for $i = 1, 2$. This map is a unique map satisfying our claim when $|X| = 2$.

($|X| \geq 3$ case) In this case, we decompose the sets of atoms $\mathfrak{p}_X$ and $\mathfrak{p}_Y$ into $\mathfrak{n}_X \sqcup \mathfrak{m}_X$ and $\mathfrak{n}_Y \sqcup \mathfrak{m}_Y$, respectively. We show that either

(i) $\Phi(\mathfrak{n}_X) = \mathfrak{n}_Y$ and $\Phi(\mathfrak{m}_X) = \mathfrak{m}_Y$, or

(ii) $\Phi(\mathfrak{n}_X) = \mathfrak{m}_Y$ and $\Phi(\mathfrak{m}_X) = \mathfrak{n}_Y$

holds. Let p be an atom from $\mathfrak{n}_X \sqcup \mathfrak{m}_X$. Assume that Φ sends p to $\mathfrak{l}_Y$. By (4) of Proposition 2.1.6 and by surjectivity of Φ, there exists an atom $a \in \mathfrak{p}_X$ such that $t(\Phi(p), \Phi(a)) = 4$. It is clear that the lattice isomorphism Φ preserves the type, and thus $t(p, a) = 4$. This contradicts to (3) of Proposition 2.1.6. Therefore $\Phi(\mathfrak{n}_X \cup \mathfrak{m}_X) \subset \mathfrak{n}_Y \cup \mathfrak{m}_Y$. By the same argument for Φ^{-1} implies this inequality is the equality. Next, we assume that there exist two atoms p, q from $\mathfrak{n}_X$ such that $\Phi(p) \in \mathfrak{n}_Y$ and $\Phi(q) \in \mathfrak{m}_Y$. By (2) of Proposition 2.1.6, the type $t(\Phi(p), \Phi(q))$ is 2. On the other hand, $t(p, q)$ is 3 by (1) of Proposition 2.1.6, which contradicts to Φ preserves the type. By the same argument, we obtain that Φ does not send atoms from $\mathfrak{m}_X$ to both $\mathfrak{n}_Y$ and $\mathfrak{m}_Y$. Therefore we obtain the above (i) or (ii) holds.

We consider the case (i) holds. Since $\Phi(\mathfrak{n}_X) = \mathfrak{n}_Y$ holds, we define a map $\phi : X \to Y$ so that $\Phi(a(x)) = a(\phi(x))$ holds for each $x \in X$. This map is a bijection since the restriction $\Phi \restriction_{\mathfrak{n}_X} : \mathfrak{n}_X \to \mathfrak{n}_Y$ is a bijection. Now we fix a proper subset D of X and take a proper subset D' of Y such that $\Phi(a(D)) = a(D')$. Then the inequality $a(D) \subset \bigvee_{x \in D} a(x)$ implies that

$$\begin{aligned}\Phi(a(D)) \subset & \bigvee_{x \in D} \Phi(a(x)) \\ = & \bigvee_{x \in D} a(\phi(x)) \\ = & \{\emptyset, A, X \mid A \subset \phi(D)\}.\end{aligned}$$

Thus we have $D' \subset \phi(D)$. By the same argument for Φ^{-1} and ϕ^{-1}, the equality $D' = \phi(D)$ holds, which implies that Φ coincides with ϕ_* at all atoms. Since $\Sigma(X)$ is atomic lattice, we have $\Phi = \phi_*$.

Next, we assume that (ii) holds. Since Φ is a complete lattice isomorphism, we have $\Phi(\bigvee \mathfrak{n}_X) = \bigvee \Phi(\mathfrak{n}_X) = \bigvee \mathfrak{m}_Y$. The supremum $\bigvee \mathfrak{n}_X$ is the discrete topology, which is the top element in $\Sigma(X)$. Thus Φ sends $\bigvee \mathfrak{n}_X$ to the top element in $\Sigma(Y)$, namely the discrete topology of Y. On the other hand, the topology $\bigvee \mathfrak{m}_Y$ is a cofinite topology, that is, it consists of the sets $Y \setminus F$ for finite subsets F of Y. Thus (ii) holds only if Y is a finite set. Therefore the complement map $C_Y : \Sigma(Y) \to \Sigma(Y)$ is a lattice isomorphism, and we can apply the proof of the case (i) for $C_Y \circ \Phi$.

We end the proof by showing the uniqueness the bijection when $|X| \geq 3$. Let $\phi_1, \phi_2 : X \to Y$ be two bijections such that ϕ_{i*} or $C_Y \circ \phi_{i*}$ is equal to Φ for $i = 1, 2$. Since an induced map by a bijection ϕ satisfies $\phi_*(a(D)) = a(\phi(D))$ for a proper subset $D \subset Y$, the induced map ϕ_* preserves the cardinality of the proper subsets. On the other hand, the complement map sends an atom $a(D)$ to $a(Y \setminus D)$. Thus C_Y does not coincide with any induced maps when $|X| \geq 3$. Thus only $\phi_{1*} = \phi_{2*}$ or $C_Y \circ \phi_{1*} = C_Y \circ \phi_{2*}$ occurs, which implies that $\phi_1 = \phi_2$ since $\phi_{i*}(a(x)) = a(\phi_i(x))$ hold for $i = 1, 2$. $\square$

3.2 Proof of Theorems

In this last section, we prove the main theorems: Theorem 3.2.5 and Theorem 3.2.8. Since our proofs use fundamental theorems of affine and projective geometries, we connect topologies and subspaces as follows.

Definition 3.2.1. Let X be a vector space over K. We define maps $\mathfrak{S}_X, \mathfrak{T}_X$ between $\tau_K(X)$ and $\sigma_K(X)$ by

$$\mathfrak{S}_X : \tau_K(X) \ni T \mapsto \bigcap_{0 \in U \in T} U \in \sigma_K(X),$$
$$\mathfrak{T}_X : \sigma_K(X) \ni S \mapsto \{U \in T^{\max}(X) \mid U = U + S\} \in \tau_K(X).$$

We abbreviate $\mathfrak{S}_X, \mathfrak{T}_X$ to $\mathfrak{S}, \mathfrak{T}$, respectively if there is no danger of confusion.

Proposition 3.2.2. *Let X be a vector space over a topological field K. Then $\mathfrak{S}(T)$ is a subspace. $\mathfrak{T}(S)$ is a vector topology.*

Proof. By definition, zero belongs to $\mathfrak{S}(T)$. Let x_1, x_2 are elements of $\mathfrak{S}(T)$. Then for an arbitrary open neighborhood $U \in T$ of zero, by the continuity of the addition at $(0, 0)$, we have an open neighborhood V of zero such that $V + V \subset U$. By definition, x_1, x_2 belong to V, which implies that $x_1 + x_2 \in U$. Thus $x_1 + x_2$ is in $\mathfrak{S}(T)$. Next we take $\alpha \in K$ and $x \in \mathfrak{S}(T)$. For an arbitrary open neighborhood $U \in T$ of zero, since the scalar multiple is continuous at $(\alpha, 0) \in K \times X$, there are open neighborhoods $O \in T_K$ of α and $V \in T$ of zero such that $O * V \subset U$. By definition, we have $x \in V$ and $\alpha * x \in U$. Thus $\alpha * x \in \mathfrak{S}(T)$. Therefore $\mathfrak{S}(T)$ is a subspace of X.

For a subspace S, let $\pi_S : X \to X/S$ be the natural quotient map. Then $\mathfrak{T}(S)$ is equal to ${\pi_S}^* \circ \pi_{S*}(T^{\max}(X))$. Thus by Corollary 2.2.5, $\mathfrak{T}(S)$ is a vector topology. □

We see several properties of $\mathfrak{S}, \mathfrak{T}$.

Lemma 3.2.3. *(1) The composition $\mathfrak{S} \circ \mathfrak{T}$ is an identity map of $\sigma_K(X)$.*

(2) For a vector topology T on X, the composition satisfies $T \subset \mathfrak{T} \circ \mathfrak{S}(T)$.

(3) For a vector topology T on X, the subspace $\mathfrak{S}(T)$ is $\{0\}$ if and only if T is Hausdorff.

Proof. **(1)** Let U be an open neighborhood of zero with respect to $\mathfrak{T}(S)$. By definition, $U = U + S$ holds. Thus $S \subset U$ holds since zero belongs to U. Next we assume that there exists an element x from $(\bigcap_{0 \in U \in \mathfrak{T}(S)} U) \setminus S$. Let $\pi : X \to X/S$ be the natural quotient map. Since $\pi(x) \neq 0$ and $T^{\max}(X/S)$ is Hausdorff, there are disjoint open neighborhoods V_1, V_2 of $\pi(x), \pi(0)$. Then $\pi^{-1}(V_2)$ is an open set with respect to $\pi^*(T^{\max}(X/S))$ and $T^{\max}(X)$. Since $\mathrm{Ker}\pi$ is S, the set $\pi^{-1}(V_2)$ is S-invariant. Thus $0 \in \pi^{-1}(V_2) \in \mathfrak{T}(S)$ holds, and by definition, $x \in \pi^{-1}(V_2)$. This is a contradiction.

(2) Since translations are homeomorphism with respect to each vector topology, it suffices to show that every open neighborhood U of zero with respect to T is in $\mathfrak{T} \circ \mathfrak{S}(T)$.

Since $\mathfrak{S}(T)$ has zero, $U \subset U + \mathfrak{S}(T)$ holds. For the other inclusion, let u, s be elements from U and $\mathfrak{S}(T)$, respectively. By the continuity of the addition at $(u, 0)$ with respect to T, we have open neighborhoods $u \in U', 0 \in V$ such that $U' + V \subset U$. By definition of $\mathfrak{S}(T)$, the element s is in V, and thus $u + s \in U$. Thus the equality $U = U + \mathfrak{S}(T)$ holds. Therefore by definition, $U \in \mathfrak{T} \circ \mathfrak{S}(T)$.

(3) When T is a Hausdorff topology, each non-zero element x has disjoint open neighborhoods $x \in U_1, 0 \in U_2$. Then x does not belong to $\mathfrak{S}(T)$ since $x \notin U_2$, which implies that $\mathfrak{S}(T) = \{0\}$. Assume that $\mathfrak{S}(T) = \{0\}$ holds. Then for a non-zero element x, by definition of $\mathfrak{S}(T)$, there exists an open neighborhood of zero to which x does not belong. By the same argument in the proof of Proposition 2.2.7, T is a Hausdorff topology. □

Remark 3.2.4. Recall that a pair of maps (f, g) between two posets $(\mathbb{P}, \leq_{\mathbb{P}})$ and $(\mathbb{Q}, \leq_{\mathbb{Q}})$ is called *antitone Galois connection* if $f : \mathbb{P} \to \mathbb{Q}$ and $g : \mathbb{Q} \to \mathbb{P}$ invert the orders and satisfies

$$q \leq_{\mathbb{Q}} f(p) \Leftrightarrow p \leq_{\mathbb{P}} g(q)$$

for $p \in \mathbb{P}, q \in \mathbb{Q}$. The pair of maps $(\mathfrak{S}_X, \mathfrak{T}_X)$ is an example of Galois connection. In fact, by definitions of $\mathfrak{S}_X$ and $\mathfrak{T}_X$, they invert the inclusion order. By Lemma 3.2.3, $S \subset \mathfrak{S}_X(T)$ holds if and only if $T \subset \mathfrak{T}_X(S)$ for $S \in \sigma_K(X)$ and $T \in \tau_K(X)$. Thus the pair is a Galois connection.

Theorem 3.2.5. *Let X be a vector space over a topological field K with $\dim_K(X) \geq 2$ and Y be a vector space over a topological field L. For each lattice isomorphism $\Phi : \Sigma(X) \to \Sigma(Y)$ such that $\Phi(\tau_K(X)) = \tau_L(Y)$, there exists a unique triple (ψ, ϕ, y_0) consists of an isomorphism $\psi : K \to L$ between topological fields, ψ-semilinear isomorphism $\phi : X \to Y$ and a point $y_0 \in Y$ satisfying*

- *if $|X|$ is infinite, then $\Phi = (\phi + y_0)_*$ holds, and*
- *if $|X|$ is finite, then either $\Phi = (\phi + y_0)_*$ or $\Phi = C_Y \circ (\phi + y_0)_*$ is true.*

Proof. By Theorem 3.1.1, there exists a bijection $\phi : X \to Y$ such that $\Phi = \phi_{0*}$ or $\Phi = C_Y \circ \phi_{0*}$ holds. We prove that ϕ_0 is a composition of a semilinear isomorphism and a translation map.

($\Phi = \phi_{0*}$case) We set y_0 as $\phi_0(0)$ and ϕ as $\phi_0 - y_0$ so that $\phi(0) = 0$. Since the translation map $Y \ni y \mapsto y - y_0 \in Y$ is homeomorphism with respect to each vector topology, ϕ_* also a lattice isomorphism between $\tau_K(X)$ and $\tau_L(Y)$.
Let S be a subspace of X and $x \in X$ be a point. We put T as $\mathfrak{T}_X(S)$, and then we have $\mathfrak{S}_X(T) = S$ by Lemma 3.2.3. Since a translation is a homeomorphism with respect to

T, the equality $x+S=\bigcap_{x\in U\in T} U$ holds. By taking the image by ϕ, we have

$$\begin{aligned}\phi(x+S) &= \bigcap_{x\in U\in T} \phi(U)\\ &= \bigcap_{\phi(x)\in V\in \phi_*(T)} V\\ &= \phi(x) + \bigcap_{0\in V\in\phi_*(T)} V.\end{aligned}$$

The last equality holds by the same argument used to show $x+S=\bigcap_{x\in U\in T} U$. Thus $\phi(x+S)=\phi(x)+\mathfrak{S}_Y(\phi_*(T))$ holds. In particular, $\phi(S)=\mathfrak{S}_Y(\phi_*(T))$ holds, which implies that ϕ maps subspaces of X to subspaces of Y. By the same argument for ϕ^{-1}, the map ϕ induces a lattice isomorphism between $\sigma_K(X)$ and $\sigma_L(Y)$. Thus $\phi(S)$ is 1-dimensional if and only if S is 1-dimensional. Therefore ϕ sends parallel lines of X to those of Y, and we can apply the fundamental theorem of affine geometry (Theorem 2.3.1). Namely, ϕ is a ψ-semilinear isomorphism for a field isomorphism ψ since $\phi(0)=0$. Now we show that $\psi : K\to L$ is homeomorphism. We fix a non-zero element $x_0\in X$ and denote by $Kx_0, L\phi(x_0)$, the subspaces generated by $x_0,\phi(x_0)$, respectively. Then the fields K, L are identified with 1-dimensional subspaces $Kx_0, L\phi(x_0)$ by

$$\begin{aligned}K\ni\alpha\mapsto\alpha * x_0\in Kx_0,\\ L\ni\beta\mapsto\beta*\phi(x_0)\in L\phi(x_0),\end{aligned}$$

where we endow $Kx_0, L\phi(x_0)$ with the relative topologies of $T^{\max}(X), T^{\max}(Y)$, respectively. By Corollary 2.2.6, linear maps

$$\begin{aligned}(X,T^{\max}(X))\ni x=x'+\alpha x_0\mapsto\alpha\in(K,T_K),\\ (Y,T^{\max}(Y))\ni y=y'+\beta\phi(x_0)\mapsto\beta\in(L,T_Y)\end{aligned}$$

are continuous, where $x'\in X'$ and $y'\in Y'$ are components with respect to fixed direct product decompositions $X=X'\bigoplus Kx_0, Y=Y'\bigoplus L\phi(x_0)$. Thus the restrictions to $Kx_0, L\phi(x_0)$ are continuous. Combining with the continuity of scalar multiples, we deduce that these identifications are homeomorphism. Since $\phi_*(T^{\max}(X))=T^{\max}(Y)$, the restriction $\phi\restriction_{Kx_0} Kx_0\to L\phi(x_0)$ is a homeomorphism, which is equal to $\psi : K\to L$ under the identifications.

($\Phi = C_Y\circ\phi_{0*}$case) By Theorem 3.1.1, this case occurs when X and Y are finite sets. Thus L is a finite field, and Y is a finite-dimensional space. Since we assume that topological fields are Hausdorff, L has the discrete topology. By Proposition 2.2.10, for each vector topology T on Y, there exists a subspace S such that $T=\{V\in T^{\max}(Y)\mid V=V+S\}$. Moreover, since Y is a finite set, $T^{\max}(Y)$ is discrete topology. Thus T is generated by the set

$$\{y+S\mid y\in Y\},$$

which implies that $C_Y(T) = T$ since $Y \setminus (y+S) = \bigcup_{y' \neq y}(y'+S)$. Therefore $C_Y \circ \Phi = \phi_* : \Sigma(X) \to \Sigma(Y)$ is a lattice isomorphism which preserves the lattices of vector topologies. Hence the same argument in ($\Phi = \phi_{0*}$case) can apply for $C_Y \circ \Phi$.

(uniqueness of the triple) Let (ψ_1, ϕ_1, y_1) and (ψ_2, ϕ_2, y_2) be triples satisfying the claim of Theorem 3.2.5. By the same argument in the proof of Theorem 3.1.1, $(\phi_1 + y_1)_* = (\phi_2 + y_2)_*$ and $\phi_1 + y_1 = \phi_2 + y_2$ hold. Thus by substituting zero, we have $y_1 = y_2$ and $\phi_1 = \phi_2$. We fix non-zero element $x_0 \in X$. Then for every $\alpha \in K$, the equality $\phi_1 = \phi_2$ implies

$$\psi_1(\alpha) * \phi_1(x_0) = \phi_1(\alpha * x_0) = \phi_2(\alpha * x_0) = \psi_2(\alpha) * \phi_2(x_0).$$

Since $\phi_1(x_0) = \phi_2(x_0) \neq 0$, we obtain $\psi_1 = \psi_2$. □

As J. Hartmanis studied in [8], we study the group of lattice automorphisms of lattice of topologies when $X = Y$ in Theorem 3.2.5. Let $\mathrm{Aut}(\Sigma(X), \tau_K(X))$ be the subgroup of lattice automorphisms of lattice of topologies consisting of maps preserving $\tau_K(X)$. Namely, the group is defined as

$$\{\Phi \in \mathrm{Aut}(\Sigma(X)) \mid \Phi(\tau_K(X)) = \tau_K(X)\}.$$

Let $\Gamma L_h(X)$ denotes the group of semilinear automorphisms whose associated field isomorphism is a homeomorphism of K. Then we obtain the following corollary of Theorem 3.2.5.

Corollary 3.2.6. *Let X be a vector space over a topological field K with* $\dim_K(X) \geq 2$.

- *If X is an infinite set, the following map μ is a group isomorphism:*

$$X \rtimes \Gamma L_h(X) \ni (x, \phi) \mapsto (\phi + x)_* \in \mathrm{Aut}(\Sigma(X), \tau_K(X)).$$

- *If X is a finite set, the following map μ is a group isomorphism:*

$$(X \rtimes \Gamma L_h(X)) \times \mathbb{Z}/2\mathbb{Z} \ni (x, \phi, \epsilon) \mapsto {C_X}^{\epsilon} \circ (\phi + x)_* \in \mathrm{Aut}(\Sigma(X), \tau_K(X)),$$

  where ${C_X}^0 = \mathrm{id}_{\Sigma(X)}$ *and* ${C_X}^1 = C_X$.

Here the operation of the semidirect product $X \rtimes \Gamma L_h(X)$ is defined by

$$(x_1, \phi_1) \cdot (x_2, \phi_2) = (x_1 + \phi_1(x_2), \phi_1 \circ \phi_2).$$

Proof. From Theorem 3.2.5, the homomorphism μ is bijection. We show that μ is a group homomorphism. For two bijection maps $f_1, f_2 : X \to X$, by definition, $(f_1)_* \circ (f_2)_* = (f_1 \circ f_2)_*$ holds. Thus μ is a group homomorphism if X is infinite. Moreover, by definition, for a bijection map f and a topology $T \in \Sigma(X)$, we have

$$\begin{aligned} C_X \circ f_*(T) &= \{X \setminus V \mid V \in f_*(T)\} \\ &= \{X \setminus f(U) \mid U \in T\} \\ &= \{f(X \setminus U) \mid U \in T\} \\ &= f_* \circ C_X(T). \end{aligned}$$

Thus C_X and f_* commutes, which implies that μ is a group homomorphism if X is finite. □

Under the assumption of Theorem 3.2.5, the restriction of Φ to the lattice $\tau_K(X)$ is indeed a lattice isomorphism between $\tau_K(X)$ and $\tau_L(Y)$. Next example shows however, we cannot weaken the assumption to the existence of a lattice isomorphism between the lattices of vector topologies.

Example 3.2.7. Let K be the field of rational numbers $\mathbb{Q}$ and L be the field of real numbers $\mathbb{R}$ with the standard absolute value. They are clearly not isomorphic as fields. Let X be K^2 and Y be L^2. Then by Proposition 2.2.13, the lattices of vector topologies are both isomorphic to the lattice $\sigma_{\mathbb{R}}(\mathbb{R}^2)$ of subspaces of $\mathbb{R}^2$. Thus $\tau_K(X)$ and $\tau_L(Y)$ are isomorphic.

In the above example, although the lattices of vector topologies are isomorphic, lattice isomorphisms ignore the Hausdorff vector topologies. In fact, by Proposition 2.2.15, $\tau_K(X)$ has continuum many Hausdorff topologies, whereas $\tau_L(Y)$ has only one Hausdorff topology by Proposition 2.2.12. If we put an assumption on lattice isomorphisms so that they preserves Hausdorff vector topologies, we obtain a similar result to Theorem 3.2.5.

Theorem 3.2.8. *Let X be a vector space over a topological field K with $\dim_K(X) \geq 3$ and Y be a vector space over a topological field L. If there exists a lattice isomorphism $\Phi : \tau_K(X) \to \tau_L(Y)$ such that $\Phi(\tau_K^H(X)) = \tau_L^H(Y)$, then K and L are isomorphic as fields algebraically and $\dim_K(X) = \dim_L(Y)$.*

Proof. We define two maps F, G by

$$\begin{aligned} F :& \sigma_K(X) \xrightarrow{\mathfrak{T}_X} \tau_K(X) \xrightarrow{\Phi} \tau_L(Y) \xrightarrow{\mathfrak{S}_Y} \sigma_L(Y), \\ G :& \sigma_L(Y) \xrightarrow{\mathfrak{T}_Y} \tau_L(Y) \xrightarrow{\Phi^{-1}} \tau_K(X) \xrightarrow{\mathfrak{S}_X} \sigma_K(X). \end{aligned}$$

Since $\mathfrak{S}, \mathfrak{T}$ invert the inclusion order and Φ, Φ^{-1} preserve the order, F, G preserve the order. Moreover, by applying (2) of Lemma 3.2.3 for $T = \Phi \circ \mathfrak{T}_X(S)$, the inequality $T \subset \mathfrak{T}_Y \circ \mathfrak{S}_Y(T)$ holds. Since $\mathfrak{S}_X \circ \Phi^{-1}$ inverts the order, by (1) of Lemma 3.2.3, we have

$$F \circ G(S) \subset \mathfrak{S}_X \circ \Phi^{-1}(T) = \mathfrak{S}_X \circ \mathfrak{T}_X(S) = S \tag{*}$$

The same argument shows that $G \circ F(S') \subset S'$ holds for $S' \in \sigma_K(X)$. We prove that F, G are lattice isomorphisms between the lattices of finite-dimensional subspaces $\sigma_K^{<\infty}(X)$ and $\sigma_L^{<\infty}(Y)$. More precisely, for non-positive integer d, we denote by $\sigma_K^d(X), \sigma_L^d(Y)$, the set of d-dimensional subspaces of X, Y, respectively. By induction with respect to d, we prove that the restriction $F \restriction_{\sigma_K^d(X)} : \sigma_K^d(X) \to \sigma_L^d(Y)$ is a bijection and $G \restriction_{\sigma_L^d(Y)} : \sigma_L^d(Y) \to \sigma_K^d(X)$ is the inverse map.

(Base case) By (3) of Lemma 3.2.3, the 0-dimensional subspace of X is sent by $\mathfrak{T}_X$, to Hausdorff topology $T = \mathfrak{T}_X(\{0\})$. Since Φ preserves Hausdorff vector topology, $\Phi(T)$ is a Hausdorff topology, and $\mathfrak{S}_Y$ sends $\Phi(T)$ to 0-dimensional subspace by (3) of

Lemma 3.2.3. Thus F sends the 0-dimensional subspace to the 0-dimensional subspace. The same argument holds for G. Thus $d = 0$ case holds.

(Induction step) We assume that $d = 0, 1, \ldots, d'$ cases hold and prove the case of $d = d'+1$. Assume that there is a subspace $S \in \sigma_K^{d'+1}(X)$ such that $d_S = \dim_L(F(S)) \leq d'$. By the same argument in $d = 0$ case, $d_S \neq 0$ since $\dim_K(S) \geq 1$. Thus there are distinct d_S-dimensional subspaces S_1, S_2 of S. Since F preserves the order, we have $F(S_1), F(S_2) \subset F(S)$. By the hypothesis of induction, the dimensions of $F(S_1), F(S_2)$ are the same, and thus $F(S_1) = F(S_2)$, which is a contradiction since $F \restriction_{\sigma_K^{d_S}(X)}$ is a bijection. By combining the same argument for G, we have $F \restriction_{\sigma_K^{d'+1}(X)}$ and $G \restriction_{\sigma_L^{d'+1}(Y)}$ do not decrease the dimension of subspaces. Next assume that there are two distinct $d'+1$-dimensional subspaces S_1, S_2 such that $F(S_1) = F(S_2)$ holds. Note that the above argument implies the dimension of $F(S_1) = F(S_2)$ is more or equal to $d'+1$. By (*), we have $G \circ F(S_1) = G \circ F(S_2)$ is a subspace of $S_1 \cap S_2$. Since S_1 and S_2 are distinct, the dimension of $S_1 \cap S_2$ denoted by d_S, is less or equal to d'. The same argument in $d = 0$ case shows that $d_S \neq 0$. Thus we can take two distinct d_S-dimensional subspaces S_1', S_2' of $F(S_1) = F(S_2)$. By sending by G, we have $G(S_1'), G(S_2') \subset S_1 \cap S_2$. This inclusion is actually equality since their dimensions are equal to d_S by the induction hypothesis, which is a contradiction since $G \restriction_{\sigma_L^{d_S}(Y)}$ is injective. By the same argument for G, we obtain that $F \restriction_{\sigma_K^{d'+1}(X)}$ and $G \restriction_{\sigma_L^{d'+1}(Y)}$ are injection. Lastly, assume that there is a subspace $S \in \sigma_K^{d'+1}(X)$ such that the dimension of $F(S)$ is greater than $d' + 1$. Then we can take two distinct $d' + 1$-dimensional subspaces S_1', S_2' of $F(S)$. Then by (*), we have $G(S_1'), G(S_2') \subset G \circ F(S) \subset S$. Since $G \restriction_{\sigma_L^{d'+1}(Y)}$ does not decrease the dimension, $G(S_1'), G(S_2')$ are both $d' + 1$-dimensional subspaces of S, which means that they coincide. This is a contradiction since $G \restriction_{\sigma_L^{d'+1}(Y)}$ is injective. The same argument for G shows that restrictions $F \restriction_{\sigma_K^{d'+1}(X)}$ and $G \restriction_{\sigma_L^{d'+1}(Y)}$ does not increase the dimensions of subspaces. Therefore $F \circ G(S') = S'$ and $G \circ F(S) = S$ holds for $S' \in \sigma_L^{d'+1}(Y)$ and $S \in \sigma_K^{d'+1}(X)$ since they are the same dimensional subspaces. This completes the $d = d' + 1$ case.

By Theorem 2.3.3, we can apply the fundamental theorem of projective geometry, and obtain the result of Theorem 3.2.8. □

Next example shows that in the situation of Theorem 3.2.8, we cannot conclude that the coefficient fields are isomorphic as topological fields.

Example 3.2.9. We fix a prime integer p, and let K be the field of complex p-adic numbers $\mathbb{C}_p$, that is, K is the completion of the algebraic closure of the field of p-adic numbers $\mathbb{Q}_p$. Then the valuation $|\cdot|_p$ of $\mathbb{Q}_p$ is extended to $\mathbb{C}_p$. Set L as the field of complex numbers $\mathbb{C}$ with the standard absolute value. Since $\mathbb{C}_p$ and $\mathbb{C}$ are both complete, by Proposition 2.2.13, lattices of vector topologies are isomorphic to lattices of its subspaces:

$$\tau_K(K^3) \cong \sigma_K(K^3),\ \tau_L(L^3) \cong \sigma_L(L^3).$$

Moreover, it is known that by using the axiom of choice, $\mathbb{C}_p$ and $\mathbb{C}$ are isomorphic as fields. Thus the lattices of subspaces are isomorphic:

$$\sigma_K(K^3) \cong \sigma_L(L^3).$$

Therefore we obtain a lattice isomorphism $\Phi : \tau_K(K^3) \to \tau_L(L^3)$. By Proposition 2.2.12, Hausdorff vector topologies of $\tau_K(K^3)$ and $\tau_L(L^3)$ are only $T^{\max}(K^3)$ and $T^{\max}(L^3)$, respectively. Thus Hausdorff vector topologies are mapped by any lattice isomorphism. Therefore Φ satisfies the assumption of Theorem 3.2.8. However the sequence of integers $\{p^n\}_{n=1}^{\infty}$ converges to zero in $\mathbb{C}_p$ whereas it does not converge in $\mathbb{C}$, which implies that K and L are not homeomorphic by any field isomorphisms.

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