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Osaka University

Doctoral Thesis

Studies on ℓ-adic Galois polylogarithms

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Under the supervision of Prof. Hiroaki Nakamura

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List of the author's papers cited in this thesis

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3. H. Nakamura, D. Shiraishi, *Landen's trilogarithm functional equation and ℓ-adic Galois multiple polylogarithms*, to appear in "Low Dimensional Topology and Number Theory" Springer Proceedings in Mathematics & Statistics.

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Chapter 1

Introduction

The geometric étale fundamental groupoid of $\mathbb{P}^1\backslash\{0,1,\infty\}$ with one or more rational base points has the natural action of the absolute Galois group of the base field. When the base field is a number field, the Galois action is faithful as a corollary of Belyi's theorem and is described by a non-commutative 1-cocycle called the Galois associator or called the Ihara associator. Thus, the problem is to investigate the behavior of the Galois associator to reveal in detail the structure of the absolute Galois group of a number field. The Grothendieck-Teichmüller theory was originated by V. Drinfeld and Y. Ihara in the 1980s with one of the goals to solve the problem. This theory corresponds to the Galois side (i.e., the étale analog) of the theory of the KZ (Knizhnik–Zamolodchikov) differential equation due to Drinfeld.

To analyze the behavior of the Galois associator, techniques of the non-commutative profinite combinatorial group theory are often used. Around the end of the 1980s, G. W. Anderson and Y. Ihara introduced the étale analog of the beta function (and the gamma function), and its arithmetic properties were clarified by Anderson, Coleman, and Ihara-Kaneko-Yukinari. The Anderson-Ihara's étale beta function is defined by capturing the Galois associator associated with the path along the $(0,1)$ real interval on $\mathbb{P}^1\backslash\{0,1,\infty\}$ as a power series in two variables through the profinite Fox's free derivative. The ℓ-adic Taylor coefficients of the power series are essentially described by the ℓ-adic Soulé character discovered by C. Soulé. The ℓ-adic Soulé character is the ℓ-adic étale analog of the Riemann zeta value and is closely related to the arithmetic of higher circular ℓ-units.

Against this background, the ℓ-adic Galois polylogarithm, the main subject of this paper, is an ℓ-adic special function on the absolute Galois group introduced by Z. Wojtkowiak around 1990 as the ℓ-adic étale analog of the classical complex polylogarithm. The ℓ-adic Galois polylogarithm is described by the generalized ℓ-adic Soulé character (or called the ℓ-adic Galois polylogarithmic character) introduced by H. Nakamura and Z. Wojtkowiak, and is defined as an ℓ-adic Magnus expansion coefficient (i.e., ℓ-adic iterated integral) of the Galois associator associated with a path from the standard tangential base point to a rational base point z on $\mathbb{P}^1\backslash\{0,1,\infty\}$. This definition is the ℓ-adic étale analog of the fact that the complex polylogarithm is a coefficient of the appropriate term in the KZ associator (i.e., a fundamental solution of the KZ differential equation). The KZ associator is the generating function of the complex iterated integral along a path on the three-punctured Riemann sphere $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$. From this perspective, the analog of the logarithmic function (i.e., the first order polylogarithm) is understood to be the ℓ-adic Kummer 1-cocycle along the ℓ-th power roots of z.

This paper discusses (1) arithmetic–geometric properties (functional equations) and (2) number-theoretic properties (relations with triple symbols) of ℓ-adic Galois polylogarithms. The background of the research on (1) and (2) and the specifics of this paper are as follows.

(1) arithmetic–geometric properties (functional equations):
The study of complex polylogarithm functional equations originated with the work of L. Euler and J. Landen around the end of the 18th century. Modern studies have been conducted since around 1990 by Zagier, Goncharov, Wojtkowiak, Beilinson-Deligne, Gangl, and others, and in particular, Zagier established an algebraic sufficient condition, called the tensor criterion, for the general functional equation of the complex polylogarithm to be valid. Furthermore, the complex polylogarithm has been generalized to the complex multiple polylogarithm, and numerous functional equations of them are known to date.

The polylogarithm has monodromy around the punctures $z = 0, 1, \infty$. Each of its functional equations depends on a system of paths on $\mathbb{P}^1\backslash\{0, 1, \infty\}$ that define the main polylogarithm terms and is also based on its own underlying geometry. In this sense, functional equations of polylogarithms are one of the arithmetic-geometric phenomena performed by the polylogarithms based on a subtle geometrical balance of the paths on $\mathbb{P}^1\backslash\{0, 1, \infty\}$.

On the other hand, functional equations of the ℓ-adic Galois polylogarithm were studied by Nakamura-Wojtkowiak, and several typical examples (Euler type, Abel type, inversion formula, and distribution formula) were obtained. In Nakamura-Wojtkowiak's previous work, they established a topological condition called the homotopy criterion by reinterpreting Zagier's tensor criterion in terms of fundamental groups and also gave the tensor-homotopy criterion and specific computational algorithm similar to the complex case to derive the functional equation of the ℓ-adic Galois polylogarithm.

In this paper, following Nakamura-Wojtkowiak's previous works, we derive (i) Landen-type 3-terms and (ii) Spence-Kummer-type 9-terms functional equations of the ℓ-adic Galois trilogarithm, and (iii) a generalization of Oi-Ueno type functional equation of the ℓ-adic Galois multiple polylogarithm. The details of (i), (ii), and (iii) are as follows.

For (i) and (iii): Using algebraic relations (chain rules) of ℓ-adic Galois associators arising from the S_3 symmetry of $\mathbb{P}^1\backslash\{0, 1, \infty\}$, we derive Landen's 3-terms functional equation of the ℓ-adic Galois trilogarithm and a generalization of Oi-Ueno's functional equation of the ℓ-adic Galois multiple polylogarithm. For the former functional equation (i), no dilogarithm terms appear in the conventional Landen's 3-terms functional equation of the complex trilogarithm, while the functional equation (i) contains a single ℓ-adic Galois dilogarithm term. This ℓ-adic Galois specific term, named error term by Nakamura-Wojtokowiak, is due to a certain nonlinearity inherent in the ℓ-adic Galois associator. In previous studies, no higher-order terms appeared in error terms in concrete functional equations, and this study is the first to confirm such a nontrivial phenomenon. The latter functional equation (iii) is a generalization of the Oi-Ueno functional equation of the ℓ-adic Galois multiple polylogarithm, which has two aspects: a duality in terms of indexes and a reflection in terms of variables. By a specialization of (iii)

when $z \to 1$, the duality formula of ℓ-adic Galois multiple zeta values is derived.
For (ii): The Spence-Kummer's unctional equation of the trilogarithm is a two-variable functional equation consisting of nine trilogarithms, which has the geometry of $V_{\text{non-fano}}$ the complement to the non-Fano arrangement as its underlying geometry. The space $V_{\text{non-fano}}$ is a 2-dimensional algebraic variety, that is the complement to a divisor of the moduli space $M_{0,5}$ of projective lines with ordered five points. The symmetry of $V_{\text{non-fano}}$ defines nine morphisms from $V_{\text{non-fano}}$ to $\mathbb{P}^1\backslash\{0,1,\infty\}$, and by using the nine morphisms, a system consisting of nine proper paths on $\mathbb{P}^1\backslash\{0,1,\infty\}$ is defined. By analyzing in detail the chain rule of ℓ-adic Galois associators arising from the path system, we derive the Spence-Kummer's functional equation of the ℓ-adic Galois trilogarithm. In order to compute the error term, it is necessary to specify the exact ℓ-adic Magnus expansion coefficients of the 2-variable Galois associator associated with a path on $V_{\text{non-fano}}$. This calculation is completed by a precise study of the fundamental group of $V_{\text{non-fano}}$ using the Galois theory of $M_{0,5}$ and the computational algebra system Magma. That is, in this paper, we focus on the finite étale Galois covering space V_{B_3} (the complement to the B_3 Coxeter arrangement) on $M_{0,5}$ and the natural open immersion from V_{B_3} to $V_{\text{non-fano}}$. Then, the fundamental group of $V_{\text{non-fano}}$ is realized as a subquotient of the well-known fundamental group of $M_{0,5}$ (i.e., the quotient group of the pure braid group P_4 by its center). Thus, by describing the above 2-variable ℓ-adic Galois associator in terms of pure braids, a precise analysis of it is possible.
The above proofs of (i), (ii), and (iii) are all algebraic. Therefore, by replacing the ℓ-adic Galois associator with the complex KZ associator and reinterpreting the proof as the complex version, we obtain a new algebraic proof of the complex (multiple) polylogarithm functional equation.

(2) number-theoretic properties (relationships with triple symbols):
Establishing the connection between the ℓ-adic Galois polylogarithm and other number-theoretic objects is a problem in obtaining applications to number theory. For this purpose, we consider the reduction mod ℓ of the ℓ-adic Galois polylogarithm. This is called the mod ℓ Galois polylogarithm. In this paper, following the previous work of Hikaru Hirano and Masanori Morishita on "Arithmetic topology in Ihara theory", we consider the relationship between the mod ℓ Galois polylogarithm and an arithmetic object called the triple ℓ-th power residue symbol that controls the decomposition law of a prime ideal in the certain Heisenberg extension over a number field. As a result, we interpret the triple ℓ-th power residue symbol as a special value of the mod ℓ Galois polylogarithm for the case $\ell = 2, 3$ where the triple ℓ-th power residue symbol is well-defined. In doing so, we use a rational point of a certain Diophantus equation to determine an appropriate rational point of $\mathbb{P}^1\backslash\{0,1,\infty\}$, and consider the special value of the the mod ℓ Galois polylogarithm at the rational point. This result is a reinterpretation of Hirano-Morishita's work in terms of ℓ-adic Galois polylogarithms. Since the triple ℓ-th power residue symbol is described by a mod ℓ Milnor invariant for a prime ideal, which is an arithmetic analog of the Milnor invariant of links, the above result of this paper gives a description of the mod ℓ Milnor invariant by the special value of the mod ℓ Galois polylogarithm. As an arithmetic application of this result, we discuss the relationship between the reciprocity law of the triple ℓ-th power residue symbol and the Euler-type functional equation of the ℓ-adic Galois dilogarithm.

The works in this paper expand the specifics in the ℓ-adic étale iterated integral theory originated by Wojtkowiak, and should contribute to the future development of the arithmetic of higher circular ℓ-units and the Grothendieck-Teichmüller theory.

Chapter 2

Preliminaries on fundamental groups

In this section, we prepare some notations concerning fundamental groups. We shall write Set $\left(\text{resp. Set}^{\text{fin}}\right)$ for the category of sets (resp. the category of finite sets).

2.1 Topological fundamental groups

For a connected topological manifold M and two base points $*, *'$ of M, we shall write

$$\pi_1^{\text{top}}(M; *, *')$$

for the set of homotopy classes of topological paths on M from $*$ to $*'$. For $\gamma_1 \in \pi_1^{\text{top}}(M; *, *')$ and $\gamma_2 \in \pi_1^{\text{top}}(M; *', *'')$, the composite

$$\gamma_1 \cdot \gamma_2 := \gamma_1 \gamma_2 \in \pi_1^{\text{top}}(M; *, *'') \tag{2.1.1}$$

is defined in such a way that paths are composed sequentially start ing with the left. Moreover, we shall write

$$\pi_1^{\text{top}}(M, *) := \pi_1^{\text{top}}(M; *, *)$$

for the topological fundamental group of M with respect to the binary operation determined by the above path composition.

The set $\pi_1^{\text{top}}(M; *, *')$ is understood in terms of categories as follows. We shall write

$$\text{Cov}_M \ \left(\text{resp. Cov}_M^{\text{fin}}\right)$$

for the infinite Galois category of covering spaces over M (resp. the Galois category of finite covering spaces over M). By taking the fibre $* \times_{M,f} N$ for each covering space $[f : N \to M] \in \text{Obj}\,(\text{Cov}_M)$, we obtain the fibre functor

$$F_* : \text{Cov}_M \to \text{Set}.$$

Then $\pi_1^{\text{top}}(M; *, *')$ is identified with the set of natural isomorphisms from F_* to $F_{*'}$:

$$\pi_1^{\text{top}}(M; *, *') \simeq \text{Isom}_{[\text{Cov}_M, \text{Set}]}\left(F_*, F_{*'}\right),$$

where $[\text{Cov}_M, \text{Set}]$ is the functor category from Cov_M to Set.

Consider the restriction $F_*^{\mathrm{fin}} : \mathrm{Cov}_M^{\mathrm{fin}} \to \mathrm{Set}^{\mathrm{fin}}$ of F_* to $\mathrm{Cov}_M^{\mathrm{fin}}$. Let $\widehat{\pi_1^{\mathrm{top}}(M;*,*')}$ the profinite completion of $\pi_1^{\mathrm{top}}(M;*,*')$. Then the profinite set $\widehat{\pi_1^{\mathrm{top}}(M;*,*')}$ is identified with the set of natural isomorphisms from F_*^{fin} to $F_{*'}^{\mathrm{fin}}$:

$$\widehat{\pi_1^{\mathrm{top}}(M;*,*')} \simeq \mathrm{Isom}_{[\mathrm{Cov}_M^{\mathrm{fin}},\mathrm{Set}^{\mathrm{fin}}]}\left(F_*^{\mathrm{fin}}, F_{*'}^{\mathrm{fin}}\right).$$

2.2 Geometric étale fundamental groups

Let ℓ be a prime number and K a subfield of $\mathbb{C}$ with the algebraic closure $\overline{K}$. For a connected algebraic variety V over $\overline{K}$ and a base point $*$ of V, we shall write

$$\text{ÉtCov}_V^{\mathrm{fin}} \ \left(\text{resp. } \text{ÉtCov}_V^{\ell\text{-fin}}\right)$$

for the Galois category of finite étale covering spaces over V (resp. the Galois category of finite étale covering spaces over V whose Galois closure has ℓ-th power degree). By taking the fibre $* \times_{V,f} W$ for each covering space $[f : W \to V] \in \mathrm{Obj}\left(\text{ÉtCov}_V^{\mathrm{fin}}\right)$, we obtain the fibre functor

$$F_*^{\text{fin-ét}} : \text{ÉtCov}_V^{\mathrm{fin}} \to \mathrm{Set}^{\mathrm{fin}}$$

and its restriction $F_*^{\ell\text{-ét}} : \text{ÉtCov}_V^{\ell\text{-fin}} \to \mathrm{Set}^{\mathrm{fin}}$. For two base points $*, *'$ of V, we shall write

$$\pi_1^{\text{ét}}(V;*,*') := \mathrm{Isom}_{[\text{ÉtCov}_V^{\mathrm{fin}},\mathrm{Set}^{\mathrm{fin}}]}\left(F_*^{\text{fin-ét}}, F_{*'}^{\text{fin-ét}}\right)$$

$\left(\text{resp. } \pi_1^{\ell\text{-ét}}(V;*,*') := \mathrm{Isom}_{[\text{ÉtCov}_V^{\ell\text{-fin}},\mathrm{Set}^{\mathrm{fin}}]}\left(F_*^{\ell\text{-ét}}, F_{*'}^{\ell\text{-ét}}\right)\right)$ for the profinite set of étale paths on V from $*$ to $*'$ (resp. the pro-ℓ-finite set of pro-ℓ étale paths on V from $*$ to $*'$). The composite of pro-ℓ étale paths is also defined in the same way as in (2.1.1). Moreover, we shall write

$$\pi_1^{\text{ét}}(V,*) := \pi_1^{\text{ét}}(V;*,*), \quad \left(\text{resp. } \pi_1^{\ell\text{-ét}}(V,*) := \pi_1^{\ell\text{-ét}}(V;*,*)\right)$$

for the geometric étale fundamental group of V (resp. the pro-ℓ geometric étale fundamental group of V) with respect to the binary operation induced by the path composition.

Let $V^{\mathrm{an}} := V(\mathbb{C})$ be the associated analytic space of V via the inclusion $\overline{K} \hookrightarrow \mathbb{C}$. Then we have the following sequence of categories

$$\text{ÉtCov}_V^{\ell\text{-fin}} \hookrightarrow \text{ÉtCov}_V^{\mathrm{fin}} \simeq \mathrm{Cov}_{V^{\mathrm{an}}}^{\mathrm{fin}} \hookrightarrow \mathrm{Cov}_{V^{\mathrm{an}}}.$$

The middle categorical equivalence is determined by the functor of taking associated analytic space. When two points $*, *'$ on V are regarded as points of V^{an} by $\overline{K} \hookrightarrow \mathbb{C}$, there is a canonical comparison map

$$\pi_1^{\mathrm{top}}\left(V^{\mathrm{an}};*,*'\right) \to \widehat{\pi_1^{\mathrm{top}}(V^{\mathrm{an}};*,*')} \simeq \pi_1^{\text{ét}}(V;*,*') \twoheadrightarrow \pi_1^{\ell\text{-ét}}(V;*,*'). \tag{2.2.1}$$

The comparison map (2.2.1) induces an isomorphism between $\pi_1^{\text{ét}}(V;*,*')$ (resp. $\pi_1^{\ell\text{-ét}}(V;*,*')$) and the profinite completion of $\pi_1^{\mathrm{top}}(V^{\mathrm{an}};*,*')$ (resp. the pro-ℓ completion of $\pi_1^{\mathrm{top}}(V^{\mathrm{an}};*,*')$), and we identify them by the isomorphism.

Let $G_K = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K with respect to the inclusion $K \hookrightarrow \overline{K}$. If $*$ and $*'$ are K-rational (possibly, tangential) base points of V, the profinite set $\pi_1^{\text{ét}}(V; *, *')$ has the natural Galois action defined by

$$G_K \to \mathrm{Aut}\left(\pi_1^{\text{ét}}(V; *, *')\right), \quad \sigma \mapsto [\gamma \mapsto s_*(\sigma) \cdot \gamma \cdot s_{*'}(\sigma)^{-1}]$$

where $s_* : G_K \to \pi_1^{\text{ét}}(V, *)$ is the canonical homomorphism induced by $*$.

2.3 Basic examples and the path set up

We shall recall the fundamental groups of $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ and the moduli space $\mathcal{M}_{0,5}$. Let

$$\mathbb{P}^1_{\overline{K}} \backslash \{0, 1, \infty\} = \mathrm{Spec}\left(\overline{K}\left[t, \frac{1}{t(t-1)}\right]\right) \tag{2.3.1}$$

be the projective line minus three points over $\overline{K}$. Then the associated analytic space of $\mathbb{P}^1_{\overline{K}} \backslash \{0, 1, \infty\}$ via $\overline{K} \hookrightarrow \mathbb{C}$ is the three punctured riemann sphere $\mathbb{P}^1(\mathbb{C}) \backslash \{0, 1, \infty\}$. Let

$$\overrightarrow{01} : \mathrm{Spec}\left(K((t))\right) \to \mathbb{P}^1_{\overline{K}} \backslash \{0, 1, \infty\}$$

be the standard K-rational tangential base point over the $K(t)$-rational point t. We write l_0, l_1, and l_∞ for topological loops with the base point $\overrightarrow{01}$ on $\mathbb{P}^1(\mathbb{C}) \backslash \{0, 1, \infty\}$ as in FIGURE 2.1. Then $\pi_1^{\text{top}}\left(\mathbb{P}^1(\mathbb{C}) \backslash \{0, 1, \infty\}, \overrightarrow{01}\right)$ is the free group of rank 2 with generating system (l_0, l_1):

$$\begin{aligned} \pi_1^{\text{top}}\left(\mathbb{P}^1(\mathbb{C}) \backslash \{0, 1, \infty\}, \overrightarrow{01}\right) &= \langle l_0, l_1, l_\infty \mid l_0 \cdot l_1 \cdot l_\infty = 1 \rangle \\ &= \langle l_0, l_1 \rangle \,. \end{aligned} \tag{2.3.2}$$

Then the pro-ℓ geometric étale fundamental group $\pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{K}} \backslash \{0, 1, \infty\}, \overrightarrow{01}\right)$ is the free pro-ℓ group of rank 2 with topologically generating system (l_0, l_1):

$$\pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{K}} \backslash \{0, 1, \infty\}, \overrightarrow{01}\right) = \overline{\langle l_0, l_1 \rangle}$$

Next we discuss the moduli space $\mathcal{M}_{0,5}$. Let

$$\mathcal{M}_{0,5} = \left\{(a_1, a_2, a_3, a_4, a_5) \in \left(\mathbb{P}^1_{\overline{K}}\right)^5 \mid a_i \neq a_j \ (i \neq j)\right\} / \mathrm{Aut}\left(\mathbb{P}^1_{\overline{K}}\right) \tag{2.3.3}$$

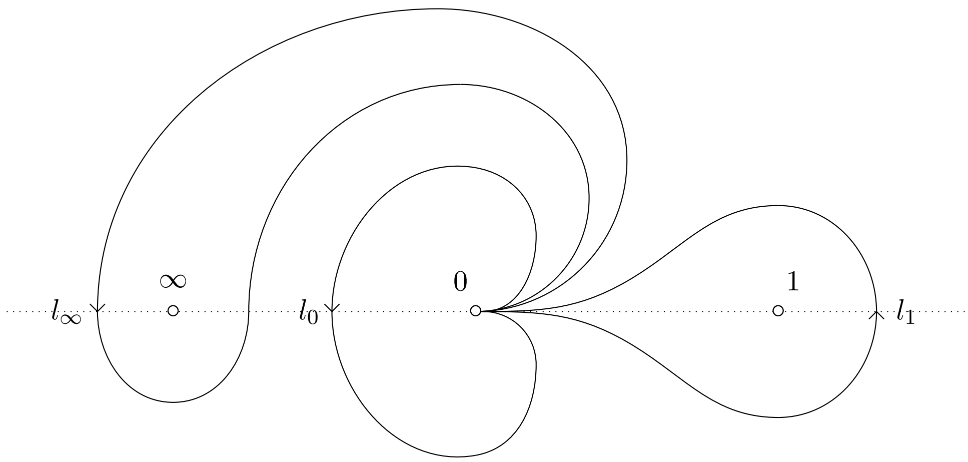
be the moduli space of $\mathbb{P}^1_{\overline{K}}$ with ordered five points. By sending

$$[(a_1, a_2, a_3, a_4, a_5)] = [(1, t_1, t_2, 0, \infty)] \mapsto (t_1, t_2), \tag{2.3.4}$$

we identify $\mathcal{M}_{0,5}$ with the second configuration space $\mathrm{Conf}_2\left(\mathbb{P}^1_{\overline{K}} \backslash \{0, 1, \infty\}\right)$ of $\mathbb{P}^1_{\overline{K}} \backslash \{0, 1, \infty\}$. Note that $\mathrm{Conf}_2\left(\mathbb{P}^1_{\overline{K}} \backslash \{0, 1, \infty\}\right)$ is the complement to the braid arrangement $t_1 t_2 (t_1 - 1)(t_2 - 1)(t_1 - t_2)$.

$$\begin{aligned} \mathcal{M}_{0,5} &\simeq \mathrm{Conf}_2\left(\mathbb{P}^1_{\overline{K}} \backslash \{0, 1, \infty\}\right) \\ &= \mathrm{Spec}\left(\overline{K}\left[t_1, t_2, \frac{1}{t_1 t_2 (t_1 - 1)(t_2 - 1)(t_1 - t_2)}\right]\right). \end{aligned} \tag{2.3.5}$$

Figure 2.1: Topological loops on $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$



We use the $K(t)$-rational point

$$(\tau_1, \tau_2) := \left(\left(\frac{1-t^2}{1+t^2}\right)^2, \left(\frac{1-t}{1+t}\right)^2\right) \in \mathcal{M}_{0,5} \tag{2.3.6}$$

and consider the K-rational tangential base point

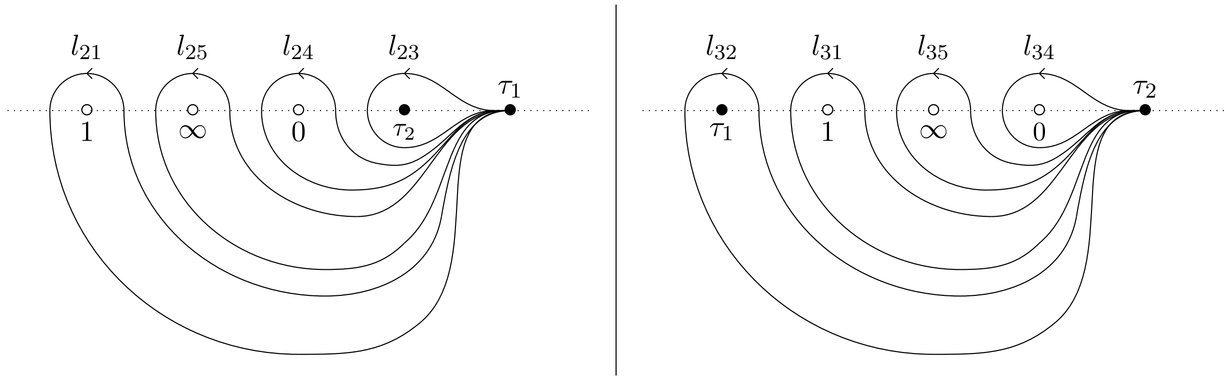
$$\vec{\tau} : \mathrm{Spec}\left(K((t))\right) \to \mathcal{M}_{0,5} \tag{2.3.7}$$

over (τ_1, τ_2). In the present paper, we define homotopy classes of topological loops

$$A_{ij}\,(= A_{ji}) \in \pi_1^{\mathrm{top}}\left(\mathcal{M}_{0,5}^{\mathrm{an}}, \vec{\tau}\right)\ (1 \leq i, j \leq 5)$$

as follows. We set the loops l_{ij} $(i \in \{2,3\},\ j \in \{1,2,3,4,5\},\ i \neq j)$ in FIGURE 2.2.

Figure 2.2: Topological loops on $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$



$$A_{12} := \begin{cases} t_1 = l_{21}, \\ t_2 = \tau_2, \end{cases} \quad A_{13} := \begin{cases} t_1 = \tau_1, \\ t_2 = l_{13}, \end{cases} \quad A_{23} := \begin{cases} t_1 = l_{23}, \\ t_2 = \tau_2 \end{cases} = \begin{cases} t_1 = \tau_1, \\ t_2 = l_{32}, \end{cases}$$

$$A_{24} := \begin{cases} t_1 = l_{24}, \\ t_2 = \tau_2, \end{cases} \quad A_{34} := \begin{cases} t_1 = \tau_1, \\ t_2 = l_{34}, \end{cases} \quad A_{25} := \begin{cases} t_1 = l_{25}, \\ t_2 = \tau_2, \end{cases} \quad A_{35} := \begin{cases} t_1 = \tau_1, \\ t_2 = l_{35}, \end{cases}$$

$$A_{14} := A_{13}^{-1} A_{12}^{-1} A_{34}^{-1} A_{24}^{-1} A_{23}^{-1}, \quad A_{15} := A_{14}^{-1} A_{13}^{-1} A_{12}^{-1}, \quad A_{45} := A_{34}^{-1} A_{24}^{-1} A_{14}^{-1}.$$

$$A_{ii} := 1 \ (1 \leq i \leq 5), \quad A_{ji} := A_{ij} \ (1 \leq i \leq j \leq 5).$$

Table 2.1: Correspondence between each divisor on $\mathcal{M}_{0,5}^{\mathrm{an}}$ and its meridian

divisor	$t_1 = 0$	$t_2 = 0$	$t_1 = t_2$	$t_1 = 1$	$t_2 = 1$	$t_1 = \infty$	$t_2 = \infty$
meridian	A_{24}	A_{34}	A_{23}	A_{12}	A_{13}	A_{25}	A_{35}

The topological fundamental group of $\mathcal{M}_{0,5}^{\mathrm{an}}$ is the mapping class group of type $(0,5)$ (cf. [N94, §3.1]). It coincides with the quotient group of the pure braid group P_4 by its center $\langle \omega_4 \rangle$ where $\omega_4 := A_{12}A_{13}A_{14}A_{23}A_{24}A_{34}$. This group has the following presentation (cf. [Lee10]).

$$\pi_1^{\mathrm{top}}(\mathcal{M}_{0,5}^{\mathrm{an}}, \vec{\tau}) = \left\langle \begin{array}{c} A_{12}, A_{13}, A_{14}, \\ A_{23}, A_{24}, A_{34} \end{array} \middle| \ (R1) \sim (R5) \right\rangle \left(= P_4 / \langle \omega_4 \rangle \right)$$

$$= \left\langle \begin{array}{c} A_{12}, A_{13}, A_{14}, A_{23}, \\ A_{24}, A_{34}, A_{25}, A_{35} \end{array} \middle| \ (R1) \sim (R6) \right\rangle$$

where (Ri) $(i = 1, 2, 3, 4, 5, 6)$ are as follows:

$$\begin{array}{ll} (R1) & A_{ij} = A_{ji}, \ A_{ii} = 1 \ (1 \leq i, j \leq 4), \\ (R2) & A_{12}A_{34} = A_{34}A_{12}, \quad A_{14}A_{23} = A_{23}A_{14}, \\ (R3) & A_{12}A_{13}A_{23} = A_{23}A_{12}A_{13} = A_{13}A_{23}A_{12}, \\ & A_{12}A_{14}A_{24} = A_{14}A_{24}A_{12} = A_{24}A_{12}A_{14}, \\ & A_{23}A_{24}A_{34} = A_{24}A_{34}A_{23} = A_{34}A_{23}A_{24}, \\ & A_{13}A_{14}A_{34} = A_{14}A_{34}A_{13} = A_{34}A_{13}A_{14}, \\ (R4) & A_{34}A_{24}A_{14}A_{13} = A_{13}A_{34}A_{24}A_{14}, \\ (R5) & \omega_4 = 1, \\ (R6) & A_{12}A_{23}A_{24}A_{25} = 1, \ A_{13}A_{23}A_{34}A_{35} = 1. \end{array}$$

Then the pro-ℓ geometric étale fundamental group $\pi_1^{\ell\text{-ét}}(\mathcal{M}_{0,5}, \vec{\tau})$ is the pro-ℓ mapping class group of the five marked pointed Riemann sphere with topologically generating system $\{A_{ij}\}$:

$$\pi_1^{\ell\text{-ét}}(\mathcal{M}_{0,5}, \vec{\tau}) = \overline{\left\langle \begin{array}{c} A_{12}, A_{13}, A_{14}, \\ A_{23}, A_{24}, A_{34} \end{array} \middle| \ (R1) \sim (R5) \right\rangle}$$

Chapter 3

Review of complex associators and polylogarithms

In this section, we recall the definition and some properties of complex associators and polylogarithms.

3.1 Complex multiple polylogarithms

In this subsection, we recall the basic properties of complex iterated integrals and complex polylogarithms [NW12], [F04], [F14], [D90].

Let z be a $\mathbb{C}$-rational base point on $\mathbb{P}^1\backslash\{0,1,\infty\}$. The complex logarithm $\log(z;\gamma)$ in a form depending on $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$ is defined as follows:

$$\log(z;\gamma) := \int_{\delta_{\overrightarrow{10}}^{-1}\cdot\gamma} \frac{dt}{t}, \tag{3.1.1}$$

where $\delta_{\overrightarrow{10}} \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},\overrightarrow{10}\right)$ is the straight path along the unit interval $(0,1) \subset \mathbb{P}^1(\mathbb{R})\backslash\{0,1,\infty\}$. So our complex logarithm may be regarded to be the map

$$\log(z) : \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right) \to \mathbb{C},$$

sending $\gamma \mapsto \log(z;\gamma)$. Note that $\log(z)$ factors through $\pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,\infty\};\overrightarrow{01},z\right)$.

For each $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$, we set

$$\gamma' := \delta_{\overrightarrow{10}} \cdot \phi_{\overrightarrow{10}}(\gamma) \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},1-z\right), \tag{3.1.2}$$

where $\phi_{\overrightarrow{10}} \in \mathrm{Aut}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}\right)$ is given by $\phi_{\overrightarrow{10}}(t) = 1-t$. For a pair $\mathbf{k} = (k_1 \ldots, k_d) \in \mathbb{N}^d$, we shall define the complex multiple polylogarithm $Li_{\mathbf{k}}(z;\gamma)$ associated to $\gamma\,(=\gamma_z) \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$ as the iterated integral

$$Li_{\mathbf{k}}(z;\gamma) := \begin{cases} \int_\gamma \frac{1}{t} Li_{k_1,\cdots,k_d-1}(t;\gamma_t)\,dt & k_d \neq 1, \\ \int_\gamma \frac{1}{1-t} Li_{k_1,\cdots,k_{d-1}}(t;\gamma_t)\,dt & k_d = 1, \end{cases} \tag{3.1.3}$$

$$Li_1(z;\gamma) := -\log(1-z;\gamma') = \int_\gamma \frac{dt}{1-t}, \tag{3.1.4}$$

So our complex multiple polylogarithm may be regarded to be the map

$$Li_{\mathbf{k}}(z) : \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, z\right) \to \mathbb{C},$$

sending $\gamma \mapsto Li_{\mathbf{k}}(z;\gamma)$. In particular, we define the multiple zeta value

$$\zeta(\mathbf{k}) := Li_{\mathbf{k}}\left(\overrightarrow{10};\delta\right) \in \mathbb{R}. \tag{3.1.5}$$

The complex multiple polylogarithm is closely related to the KZ (Knizhnik-Zamolodchikov) equation. The formal KZ equation on $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$ is the differential equation

$$\frac{d}{dz}G(z) = \left(\frac{e_0}{z} + \frac{e_1}{z-1}\right)G(z) \tag{3.1.6}$$

where $G(z)$ is an analytic (i.e. each of whose coefficient is analytic) function with values in the non-commutative formal power series ring $\mathbb{C}\langle\langle e_0, e_1\rangle\rangle$. There exists a solution $\Lambda_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1) \in \mathbb{C}\langle\langle e_0, e_1\rangle\rangle$ attached to $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, z\right)$ defined by the expansion in terms of iterated integrals as

$$\Lambda_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1) := 1 + \sum_{k=1}^{\infty}\int_{\gamma}\underbrace{\omega\ldots\omega}_{k\ \text{times}} \in \mathbb{C}\langle\langle e_0, e_1\rangle\rangle \tag{3.1.7}$$

where $\omega := \frac{dz}{z}e_0 + \frac{dz}{z-1}e_1$ is a 1-form on $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$. It is easily verified that the expansion of (3.1.7) looks like

$$\begin{aligned}\Lambda_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1) =& 1 + \sum_{k=1}^{\infty}\frac{\log^k(z;\gamma)}{k!}e_0^k + \sum_{k=1}^{\infty}\frac{\log^k(1-z;\gamma')}{k!}e_1^k - \sum_{k=2}^{\infty}Li_k(z;\gamma)e_1e_0^{k-1} \\ &+ \sum_{d=2}^{\infty}(-1)^d \sum_{\mathbf{k}=(k_1\ldots,k_d)\in(\mathbb{Z}_{>1})^d} Li_{\mathbf{k}}(z;\gamma)e_1e_0^{k_1-1}\cdots e_1e_0^{k_d-1} + \cdots .\end{aligned} \tag{3.1.8}$$

We remark on the relationship between the fundamental solution of the KZ equation (3.1.6) and $\Lambda_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1)$. Let

$$G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1) \in \mathbb{C}\langle\langle e_0, e_1\rangle\rangle$$

be the fundamental solution of (3.1.6) attached to $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, z\right)$ characterized by the asymptotic behavior

$$G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1) \approx \sum_{i=0}^{\infty}\frac{\log^i(z;\gamma)}{i!}e_0^i \quad (z\to 0).$$

For a word $w = w_1\cdots w_n$ $(w_1,\cdots,w_n \in \{X,Y\})$, we define $w^{\mathrm{op}} := w_n\cdots w_1$. For $\Lambda = \sum_w \mathtt{Coeff}_w(\Lambda)\cdot w \in \mathbb{C}\langle\langle e_0, e_1\rangle\rangle$, we define the dual of Λ by $\Lambda^{\mathrm{op}} := \sum_w \mathtt{Coeff}_w(\Lambda)\cdot \overline{w}$. Then, the following equation holds

$$G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1) = \left(\Lambda_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1)\right)^{\mathrm{op}}, \tag{3.1.9}$$

that is

$$Li_{\mathbf{k}}(z;\gamma) = (-1)^d \cdot \texttt{Coeff}_{e_0^{k_d-1}e_1\cdots e_0^{k_1-1}e_1}\left(G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1)\right). \tag{3.1.10}$$

In particular, we define the Drinfeld associator

$$\Phi(e_0,e_1) := G_{\overrightarrow{01}}^{\overrightarrow{10},\delta_{\overrightarrow{10}}}(e_0,e_1) \in \mathbb{C}\langle\langle e_0,e_1\rangle\rangle.$$

Moreover, the Lie-version of the complex polylogarithm is defined as follows. We denote by

$$\mathrm{Lie}_{\mathbb{C}}\langle\langle e_0,e_1\rangle\rangle$$

the complete free Lie algebra consisting of lie-like elements of $\mathbb{C}\langle\langle e_0,e_1\rangle\rangle$. Since $\Lambda_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1) \in \mathbb{C}\langle\langle e_0,e_1\rangle\rangle$ is group-like, we can take the inverse of it and obtain a Lie formal power series

$$\mathbf{log}\left(\Lambda_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1)^{-1}\right) \in \mathrm{Lie}_{\mathbb{C}}\langle\langle e_0,e_1\rangle\rangle$$

where

$$\mathbf{log} : \Lambda^{-1} \mapsto \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\Lambda^{-1}-1\right)}{n}.$$

We shall write

$$\varphi_k : \mathrm{Lie}_{\mathbb{C}}\langle\langle e_0,e_1\rangle\rangle \to \mathbb{C}$$

for the $\mathbb{C}$-linear form that picks up the coefficient of h_k with respect to the Hall basis

$$h_1 := e_1, \quad h_m := [e_0, h_{m-1}] = \mathrm{ad}(e_0)^{m-1}(e_1).$$

We define

$$\begin{aligned} \mathrm{li}_k(z;\gamma) &:= \frac{1}{\left(2\pi\sqrt{-1}\right)^k} \cdot \varphi_k\left(\mathbf{log}\left(\Lambda_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1)^{-1}\right)\right) \quad (k \geq 1), \qquad (3.1.11)\\ \mathrm{li}_0(z;\gamma) &:= -\frac{1}{2\pi\sqrt{-1}}\log(z;\gamma), \end{aligned}$$

called the Lie-version of the complex polylogarithm. Then [NW12, Proposition 5.2] asserts the following formula

$$\mathrm{li}_n(z;\gamma) = \frac{(-1)^{n+1}}{\left(2\pi\sqrt{-1}\right)^n} \sum_{k=0}^{n-1} \frac{B_k}{k!} \log^k(z;\gamma) Li_{n-k}(z;\gamma) \tag{3.1.12}$$

where $\{B_n\}_n$ is the family of Bernoulli numbers defined by

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} T^n = \frac{T}{e^T - 1}.$$

Note that $B_1 = -\frac{1}{2}$.

3.2 Computations of complex associators

Proposition 3.2.1 (Chain rules)**.** Given a base point z of $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$ and a path $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$, define the paths γ', γ'' associated to γ by

$$\gamma' := \delta_{\overrightarrow{10}} \cdot \phi_{\overrightarrow{10}}(\gamma) \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},1-z\right), \tag{3.2.1}$$

$$\gamma'' := \delta_{\overrightarrow{0\infty}} \cdot \phi_{\overrightarrow{0\infty}}(\gamma) \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},\frac{z}{z-1}\right), \tag{3.2.2}$$

where $\phi_{\overrightarrow{10}}, \phi_{\overrightarrow{0\infty}} \in \mathrm{Aut}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}\right)$ is given by $\phi_{\overrightarrow{10}}(t) = 1-t$, $\phi_{\overrightarrow{0\infty}}(t) = \frac{z}{z-1}$. Let

$$e_\infty := -e_0 - e_1.$$

Then the following holds:

1. $G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1) = G_{\overrightarrow{01}}^{1-z,\gamma'}(e_1,e_0)\cdot\Phi(e_0,e_1)$,
2. $G_{\overrightarrow{01}}^{\frac{z}{z-1},\gamma''}(e_0,e_1) = G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_\infty)\cdot\mathbf{exp}\left(\pi\sqrt{-1}\cdot e_0\right)$.

Proof. We use the following formula

$$\Lambda_b^{b',\alpha}(e_0,e_1)\cdot\Lambda_{b'}^{b'',\beta}(e_0,e_1) = \Lambda_b^{b'',\alpha\cdot\beta}(e_0,e_1),$$

for $\alpha \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};b,b'\right)$ and $\beta \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};b',b''\right)$. First, we show (1). By (3.2.1) and (3.1.9), we compute

$$\begin{aligned}
G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1) &= \left(\Lambda_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1)\right)^{\mathrm{op}} \\
&= \left(\Lambda_{\overrightarrow{01}}^{\overrightarrow{10},\delta_{\overrightarrow{10}}}(e_0,e_1)\cdot\Lambda_{\overrightarrow{10}}^{z,\phi_{\overrightarrow{10}}(\gamma')}(e_0,e_1)\right)^{\mathrm{op}} \\
&= \left(\Lambda_{\overrightarrow{01}}^{\overrightarrow{10},\delta_{\overrightarrow{10}}}(e_0,e_1)\cdot\phi_{\overrightarrow{10}}\left(\Lambda_{\overrightarrow{01}}^{1-z,\gamma'}(e_0,e_1)\right)\right)^{\mathrm{op}} \\
&= \left(\Lambda_{\overrightarrow{01}}^{\overrightarrow{10},\delta_{\overrightarrow{10}}}(e_0,e_1)\cdot\Lambda_{\overrightarrow{01}}^{1-z,\gamma'}(e_1,e_0)\right)^{\mathrm{op}} \\
&= G_{\overrightarrow{01}}^{1-z,\gamma'}(e_1,e_0)\cdot\Phi(e_0,e_1).
\end{aligned}$$

This completes the proof of (1).

Next, we show (2). By (3.2.2) and (3.1.9), we compute

$$\begin{aligned}
G_{\overrightarrow{01}}^{\frac{z}{z-1},\gamma''}(e_0,e_1) &= \left(\Lambda_{\overrightarrow{01}}^{\frac{z}{z-1},\gamma''}(e_0,e_1)\right)^{\mathrm{op}} \\
&= \left(\Lambda_{\overrightarrow{01}}^{\overrightarrow{0\infty},\delta_{\overrightarrow{0\infty}}}(e_0,e_1)\cdot\Lambda_{\overrightarrow{0\infty}}^{\frac{z}{z-1},\phi_{\overrightarrow{0\infty}}(\gamma)}(e_0,e_1)\right)^{\mathrm{op}} \\
&= \left(\Lambda_{\overrightarrow{01}}^{\overrightarrow{0\infty},\delta_{\overrightarrow{0\infty}}}(e_0,e_1)\cdot\phi_{\overrightarrow{0\infty}}\left(\Lambda_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1)\right)\right)^{\mathrm{op}} \\
&= \left(\Lambda_{\overrightarrow{01}}^{\overrightarrow{0\infty},\delta_{\overrightarrow{0\infty}}}(e_0,e_1)\cdot\Lambda_{\overrightarrow{01}}^{z,\gamma}(e_0,e_\infty)\right)^{\mathrm{op}} \\
&= G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_\infty)\cdot G_{\overrightarrow{01}}^{\overrightarrow{0\infty},\delta_{\overrightarrow{0\infty}}}(e_0,e_1) \\
&= G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_\infty)\cdot\mathbf{exp}\left(\pi\sqrt{-1}\cdot e_0\right).
\end{aligned}$$

This completes the proof of (2). ∎

Chapter 4

Review of ℓ-adic Galois associators and polylogarithms

In this section, we recall the definition and some properties of ℓ-adic Galois associators and polylogarithms.

4.1 ℓ-adic Galois multiple polylogarithms

In this subsection, we recall the basic properties of ℓ-adic iterated integrals and ℓ-adic Galois polylogarithms [NW12], [NW20], [NW99], [W0]-[W3].

For an algebraic variety V over $\overline{K}$ and its K-rational base points b and $*$, the absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ acts on the pro-ℓ-finite set $\pi_1^{\ell\text{-ét}}(V; b, *)$ (cf. [N99, 2.8], [NW99, (1.1)]). For $\sigma \in G_K$ and $p \in \pi_1^{\mathrm{top}}(V^{\mathrm{an}}; b, *)$, regarding p as $p \in \pi_1^{\ell\text{-ét}}(V; b, *)$ by the comparison map (2.2.1), we define a pro-ℓ étale loop

$$\mathfrak{f}_\sigma^{*,p} := p \cdot \sigma(p)^{-1} \in \pi_1^{\ell\text{-ét}}(V, b) \tag{4.1.1}$$

called the ℓ-adic Galois associator associated to $p \in \pi_1^{\mathrm{top}}(V^{\mathrm{an}}; b, *)$. When $p = \delta_{\overrightarrow{10}}$, it is called the ℓ-adic Ihara associator in [F07, Definition 2.32].

In the following, we mainly consider the case where $V = \mathbb{P}^1_{\overline{K}} \backslash \{0, 1, \infty\}$ and $b = \overrightarrow{01}$. Recall that $\pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{K}} \backslash \{0, 1, \infty\}, \overrightarrow{01}\right)$ is the free pro-ℓ group F_2^ℓ of rank 2 with topologically generating system (l_0, l_1) as in (2.3.2). The ℓ-adic Galois associator $\mathfrak{f}_\sigma^{\overrightarrow{10}, \delta_{\overrightarrow{10}}}$ forms the main component of the image of G_K in the pro-ℓ Grothendieck-Teichmüller group GT_ℓ which is a closed subgroup of the automorphism group of $\pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{K}} \backslash \{0, 1, \infty\}, \overrightarrow{01}\right) \left(= F_2^\ell = \overline{\langle l_0, l_1 \rangle}\right)$, defined as follows

$$\mathrm{GT}_\ell := \left\{ \sigma \in \mathrm{Aut}\left(F_2^\ell\right) \;\middle|\; \begin{array}{l} \text{There exists unique } (\lambda, \mathfrak{f}) \in \mathbb{Z}_\ell^\times \times [F_2^\ell, F_2^\ell] \text{ such that } \sigma(l_0) = {l_0}^\lambda, \\ \sigma(l_1) = \mathfrak{f}^{-1} {l_1}^\lambda \mathfrak{f}, \text{ the pair } (\lambda, \mathfrak{f}) \text{ holds 2-cycle relation, 3-cycle} \\ \text{relation, 5-cycle relation (cf. [Ihara, 1990] (3.1.1)).} \end{array} \right\}.$$

There exists the natural homomorphism

$$G_K \to \mathrm{GT}_\ell, \quad \sigma \mapsto \left(\chi(\sigma), \mathfrak{f}_\sigma^{\overrightarrow{10}, \delta_{\overrightarrow{10}}}\right)$$

where $\chi : G_K \to \mathbb{Z}_\ell^\times$ is the ℓ-adic cyclotomic character.

Let z be a K-rational base point on $\mathbb{P}^1_{\overline{K}}\backslash\{0,1,\infty\}$. For $\gamma \in \pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z)$ and $\sigma \in G_K$, we get a formal power series

$$\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1) \in \mathbb{Q}_\ell\langle\langle e_0,e_1\rangle\rangle \tag{4.1.2}$$

by taking the image of $\mathfrak{f}_\sigma^{z,\gamma} \in \pi_1^{\ell\text{-ét}}(\mathbb{P}^1_{\overline{K}}\backslash\{0,1,\infty\},\overrightarrow{01})$ under the multiplicative ℓ-adic Magnus embedding into the non-commutative formal power series ring

$$\pi_1^{\ell\text{-ét}}(\mathbb{P}^1_{\overline{K}}\backslash\{0,1,\infty\},\overrightarrow{01}) \hookrightarrow \mathbb{Q}_\ell\langle\langle e_0,e_1\rangle\rangle$$

defined by $l_0 \mapsto \mathbf{exp}(e_0) := \sum_{n=0}^\infty \frac{1}{n!}e_0^n$, $l_1 \mapsto \mathbf{exp}(e_1)$. The power series $\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)$ is a ℓ-adic Galois analog of the KZ solution $G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1) \in \mathbb{C}\langle\langle e_0,e_1\rangle\rangle$ in (3.1.9). For $\mathbf{k} = (k_1,\dots,k_d) \in \mathbb{N}^d$, we define the ℓ-adic Galois polylogarithm and the ℓ-adic Galois zeta value

$$Li_{\mathbf{k}}^\ell(z;\gamma,\sigma) := (-1)^d \cdot \mathtt{Coeff}_{e_0^{k_d-1}e_1\cdots e_0^{k_1-1}e_1}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right), \tag{4.1.3}$$

$$\boldsymbol{\zeta}_{\mathbf{k}}^\ell(\sigma) := Li_k^\ell\left(\overrightarrow{10};\delta_{\overrightarrow{10}},\sigma\right). \tag{4.1.4}$$

Note that $\boldsymbol{\zeta}_{\mathbf{k}}^\ell(\sigma)$ is called the ℓ-adic multiple Soulé element in [F07, Definition 2.32] and $\boldsymbol{\zeta}_k^\ell(\sigma)$ is described by the Soulé character (cf. [F07, Examples 2.33]).

So our ℓ-adic Galois polylogarithm may be regarded to be the map

$$Li_{\mathbf{k}}^\ell(z) : \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right) \times G_K \to \mathbb{Q}_\ell,$$

sending $(\gamma,\sigma) \mapsto Li_{\mathbf{k}}^\ell(z;\gamma,\sigma)$ where z is a K-rational base point of $\mathbb{P}^1\backslash\{0,1,\infty\}$.

Here we shall recall the character version of the ℓ-adic Galois polylogarithm, which is closely related to the Soulé character (cf. [NW99, REMARK 2]). For each $n \in \mathbb{N}$, we denote by $\zeta_n := \exp\left(\frac{2\pi\sqrt{-1}}{n}\right) \in \overline{K}$ a primitive n-th root of unity and choose $z^{1/n}$ ($n \in \mathbb{N}$) as the specific n-th power roots determined by $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$. For each $k \in \mathbb{Z}_{\geq 1}$, the ℓ-adic Galois polylogarithmic character associated to $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$

$$\tilde{\chi}_k^{z,\gamma} : G_K \to \mathbb{Z}_\ell \tag{4.1.5}$$

is defined by the following Kummer property

$$\zeta_{\ell^n}^{\tilde{\chi}_k^{z,\gamma}(\sigma)} = \sigma\left(\prod_{i=0}^{\ell^n-1}(1-\zeta_{\ell^n}^{\chi(\sigma)^{-1}i}z^{1/\ell^n})^{\frac{i^{k-1}}{\ell^n}}\right)\Big/\prod_{i=0}^{\ell^n-1}(1-\zeta_{\ell^n}^{i+\rho_{z,\gamma}(\sigma)}z^{1/\ell^n})^{\frac{i^{k-1}}{\ell^n}}$$

where $\chi : G_K \to \mathbb{Z}_\ell^\times$ is the ℓ-adic cyclotomic character and $\rho_{z,\gamma} : G_K \to \mathbb{Z}_\ell$ is the Kummer 1-cocycle defined by $\sigma(z^{1/\ell^n}) = z^{1/\ell^n}\cdot\zeta_{\ell^n}^{\rho_{z,\gamma}(\sigma)}$. Then, the ℓ-adic Magnus expansion of the Galois associator $\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1) \in \mathbb{Q}_\ell\langle\langle X,Y\rangle\rangle$ looks like

$$\begin{aligned}\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1) = 1 &+ \sum_{k=1}^\infty \frac{(-\rho_{z,\gamma}(\sigma))^k}{k!}e_0^k + \sum_{k=1}^\infty \frac{(-\rho_{1-z,\gamma'}(\sigma))^k}{k!}e_1^k \\ &- \sum_{k=2}^\infty \frac{\tilde{\chi}_k^{z,\gamma}(\sigma)}{(k-1)!}e_1e_0^{k-1} - \sum_{k=2}^\infty Li_k^\ell(z;\gamma,\sigma)\cdot e_0^{k-1}e_1 \\ &+ \sum_{d=2}^\infty(-1)^d \sum_{\mathbf{k}=(k_1\dots,k_d)\in(\mathbb{Z}_{>1})^d} Li_{\mathbf{k}}^\ell(z;\gamma)e_0^{k_d-1}e_1\cdots e_0^{k_1-1}e_1 + \cdots.\end{aligned} \tag{4.1.6}$$

where $\gamma' \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, 1-z\right)$ is as in (6.1.1).

Moreover, the Lie-version of the ℓ-adic Galois polylogarithm is defined as follows. We denote by $\mathrm{Lie}_{\mathbb{Q}_\ell}\langle\langle X, Y\rangle\rangle$ the complete free Lie algebra consisting of lie-like elements of $\mathbb{Q}_\ell\langle\langle X, Y\rangle\rangle$. Since $\mathfrak{f}_\sigma^{z,\gamma}(e_0, e_1) \in \mathbb{Q}_\ell\langle\langle X, Y\rangle\rangle$ is group-like, we can take the inverse of it and obtain

$$\mathbf{log}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0, e_1)^{-1}\right) \in \mathrm{Lie}_{\mathbb{Q}_\ell}\langle\langle e_0, e_1\rangle\rangle.$$

We shall write $\varphi_k : \mathrm{Lie}_{\mathbb{Q}_\ell}\langle\langle e_0, e_1\rangle\rangle \to \mathbb{Q}_\ell$ for the $\mathbb{Q}_\ell$-linear form that picks up the coefficient of h_k with respect to the Hall basis $h_1 := e_1, \quad h_m := [e_0, h_{m-1}] = \mathrm{ad}(e_0)^{m-1}(e_1)$ of $\mathrm{Lie}_{\mathbb{Q}_\ell}\langle\langle e_0, e_1\rangle\rangle$. We define

$$\ell i_k(z;\gamma,\sigma) := \varphi_k\left(\mathbf{log}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0, e_1)^{-1}\right)\right) \quad (k \geq 1), \quad \ell i_0(z;\gamma,\sigma) := \rho_{z,\gamma}(\sigma), \tag{4.1.7}$$

called the Lie-version of the ℓ-adic Galois polylogarithm. By computations similar to [NW12, Proposition 5.2], we have the following formula

$$\ell i_n(z;\gamma,\sigma) = \sum_{k=0}^{n-1} \frac{B_k}{k!} \left(-\rho_{z,\gamma}(\sigma)\right)^k Li_{n-k}^{\ell}(z;\gamma,\sigma). \tag{4.1.8}$$

The ℓ-adic Galois polylogarithm is similar to the complex polylogarithm as in TABLE 4.1.

Table 4.1: Analogy between complex polylogarithms and ℓ-adic Galois polylogarithms

ℓ-adic Galois side	complex side
z : K-ratinal base point on $\mathbb{P}^1\backslash\{0,1,\infty\}$	z : $\mathbb{C}$-rational base point on $\mathbb{P}^1\backslash\{0,1,\infty\}$
$\mathfrak{f}_\sigma^{z,\gamma}(e_0, e_1) \in \mathbb{Q}_\ell\langle\langle e_0, e_1\rangle\rangle$ $(\gamma,\sigma) \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, z\right) \times G_K$	$G_{\overrightarrow{01}}^{z,\gamma}(e_0, e_1) = \left(\Lambda_{\overrightarrow{01}}^{z,\gamma}(e_0, e_1)\right)^{\mathrm{op}} \in \mathbb{C}\langle\langle e_0, e_1\rangle\rangle$ $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, z\right)$
$Li_k^{\ell}(z;\gamma,\sigma) \in \mathbb{Q}_\ell$	$Li_k(z;\gamma) \in \mathbb{C}$
$\boldsymbol{\zeta}_k^{\ell} : G_K \to \mathbb{Q}_\ell$	$\zeta(k) \in \mathbb{R}$
$Li_1^{\ell}(z;\gamma,\sigma) = \rho_{1-z,\gamma'}(\sigma) \in \mathbb{Z}_\ell$	$Li_1(z;\gamma) = -\log(1-z;\gamma') \in \mathbb{C}$
$\tilde{\chi}_k^{z,\gamma}(\sigma) \in \mathbb{Z}_\ell$	$-(k-1)! \cdot \mathtt{Coeff}_{e_0^{k-1}e_1}\left(\Lambda_{\overrightarrow{01}}^{z,\gamma}(e_0, e_1)\right) \in \mathbb{C}$
$\ell i_k(z;\gamma,\sigma) \in \mathbb{Q}_\ell$	$\varphi_k\left(\mathbf{log}\left(G_{\overrightarrow{01}}^{z,\gamma}(e_0, e_1)^{-1}\right)\right) \in \mathbb{C}$
$\varphi_{k,\vec{l}}\left(\mathbf{log}\left(\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0, e_1)^{-1}\right)^{\mathrm{op}}\right)\right) \in \mathbb{Q}_\ell$	$\mathrm{li}_n(z;\gamma) \in \mathbb{C}$

4.2 Computations of ℓ-adic Galois associators

Lemma 4.2.1 (Chain rules)**.** Given a K-rational base point z of $\mathbb{P}^1\backslash\{0,1,\infty\}$ and a path $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$, define the paths γ', γ'' associated to γ by

$$\gamma' := \delta_{\overrightarrow{10}} \cdot \phi_{\overrightarrow{10}}(\gamma) \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},1-z\right), \tag{4.2.1}$$

$$\gamma'' := \delta_{\overrightarrow{0\infty}} \cdot \phi_{\overrightarrow{0\infty}}(\gamma) \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},\frac{z}{z-1}\right), \tag{4.2.2}$$

where $\phi_{\overrightarrow{10}}, \phi_{\overrightarrow{0\infty}} \in \mathrm{Aut}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}\right)$ is given by $\phi_{\overrightarrow{10}}(t) = 1-t$, $\phi_{\overrightarrow{0\infty}}(t) = \frac{z}{z-1}$. Let

$$e_\infty := \log(\exp(-e_1)\exp(-e_0)).$$

Then, the following holds:

1. $\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1) = \mathfrak{f}_\sigma^{1-z,\gamma'}(e_1,e_0) \cdot \mathfrak{f}_\sigma^{\overrightarrow{10},\delta_{\overrightarrow{10}}}(e_0,e_1)$.
2. $\mathfrak{f}_\sigma^{\frac{z}{z-1},\gamma''}(e_0,e_1) = \mathfrak{f}_\sigma^{z,\gamma}(e_0,e_\infty) \cdot \exp\left(\frac{1-\chi(\sigma)}{2}e_0\right)$.

Proof. Let $\sigma \in G_K$. First, we show (1). By (4.2.1) and (4.1.1), we compute

$$\begin{aligned}
\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1) &= \gamma \cdot \sigma(\gamma)^{-1} \\
&= \delta_{\overrightarrow{10}} \cdot \phi_{\overrightarrow{10}}(\gamma') \cdot \sigma\left(\phi_{\overrightarrow{10}}(\gamma')^{-1}\right) \cdot \sigma\left(\delta_{\overrightarrow{10}}\right)^{-1} \\
&= \delta_{\overrightarrow{10}} \cdot \phi_{\overrightarrow{10}}\left(\mathfrak{f}_\sigma^{1-z,\gamma'}(e_0,e_1)\right) \cdot \delta_{\overrightarrow{10}}^{-1} \cdot \mathfrak{f}_\sigma^{\overrightarrow{10},\delta_{\overrightarrow{10}}}(e_0,e_1) \\
&= \mathfrak{f}_\sigma^{1-z,\gamma'}(e_1,e_0) \cdot \mathfrak{f}_\sigma^{\overrightarrow{10},\delta_{\overrightarrow{10}}}(e_0,e_1).
\end{aligned}$$

This completes the proof of (1).

Next, we show (2). By (4.2.2) and (4.1.1), we compute

$$\begin{aligned}
\mathfrak{f}_\sigma^{\frac{z}{z-1},\gamma''}(e_0,e_1) &= \gamma'' \cdot \sigma\left(\gamma''\right)^{-1} \\
&= \delta_{\overrightarrow{0\infty}} \cdot \phi_{\overrightarrow{0\infty}}(\gamma) \cdot \sigma\left(\phi_{\overrightarrow{0\infty}}(\gamma)\right)^{-1} \cdot \delta_{\overrightarrow{0\infty}}^{-1} \\
&= \delta_{\overrightarrow{0\infty}} \cdot \phi_{\overrightarrow{0\infty}}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right) \cdot \delta_{\overrightarrow{0\infty}}^{-1} \cdot \mathfrak{f}_\sigma^{\overrightarrow{0\infty},\delta_{\overrightarrow{0\infty}}}(e_0,e_1) \\
&= \mathfrak{f}_\sigma^{z,\gamma}(e_0,e_\infty) \cdot \mathfrak{f}_\sigma^{\overrightarrow{0\infty},\delta_{\overrightarrow{0\infty}}}(e_0,e_1) \\
&= \mathfrak{f}_\sigma^{z,\gamma}(e_0,e_\infty) \cdot l_0^{\frac{1-\chi(\sigma)}{2}} \\
&= \mathfrak{f}_\sigma^{z,\gamma}(e_0,e_\infty) \cdot \exp(e_0)^{\frac{1-\chi(\sigma)}{2}} \\
&= \mathfrak{f}_\sigma^{z,\gamma}(e_0,e_\infty) \cdot \exp\left(\frac{1-\chi(\sigma)}{2}e_0\right).
\end{aligned}$$

This completes the proof of (2). $\square$

Chapter 5

Duality-reflection foumulas of multiple polylogarithms and their ℓ-adic Galois analogues

In this chapter, we derive formulas of complex and ℓ-adic Galois multiple polylogarithms, which have two aspects: a duality of indexes and a reflection of variables.

5.1 Main results and the proof

In [Oi09] and [OU13], Oi and Ueno showed the the following functional equation:

$$\sum_{j=0}^{m-1} \frac{(-\log(z;\gamma))^j}{j!} Li_{m-j}(z;\gamma) + Li_{\underbrace{1,\ldots,1}_{m-2\text{ times}},2}(1-z;\gamma') = \zeta(m). \tag{5.1.1}$$

Our main result of the complex case is as follows. The following functional equation is a generalization of (5.1.1) and has two aspects: a duality $n \leftrightarrow m$ with respect to indexes and a reflection $z \leftrightarrow 1-z$ with respect to variables.

Theorem 5.1.1 (The duality-reflection formula of complex multiple polylogarithms)**.** Given a base point z of $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$ and a path $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, z\right)$, define

$$\gamma' := \delta_{\overrightarrow{10}} \cdot \phi(\gamma) \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, 1-z\right), \tag{5.1.2}$$

where $\phi \in \mathrm{Aut}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}\right)$ is given by $\phi(t) = 1-t$. For $n, m \in \mathbb{Z}_{\geq 2}$, the following holds:

$$\sum_{j=0}^{m-1} \frac{(-\log(z;\gamma))^j}{j!} Li_{\underbrace{1,\ldots,1}_{n-2\text{ times}},m-j}(z;\gamma) + \sum_{j=0}^{n-2} \frac{(-\log(1-z;\gamma'))^j}{j!} Li_{\underbrace{1,\ldots,1}_{m-2\text{ times}},n-j}(1-z;\gamma') \tag{5.1.3}$$
$$= \zeta(\underbrace{1,\ldots,1}_{n-2\text{ times}}, m).$$

We also deal with the ℓ-adic Galois case for any prime number ℓ. Let K be a subfield of $\mathbb{C}$ and G_K the absolute Galois group of K with respect to its algebraic closure $\overline{K}$ in $\mathbb{C}$. In [NS22] and [N21], Nakamura showed the following functional equation

$$\sum_{j=0}^{m-1} \frac{(\rho_{z,\gamma}(\sigma))^j}{j!} Li^{\ell}_{m-j}(z;\gamma,\sigma) + Li^{\ell}_{\underbrace{1,\dots,1}_{m-2\text{ times}},2}(1-z;\gamma',\sigma) = \boldsymbol{\zeta}^{\ell}_m(\sigma) \quad (\sigma \in G_K). \tag{5.1.4}$$

Our another main result is a generalization of (5.1.4) as follows.

Theorem 5.1.2 (The duality-reflection formula of ℓ-adic Galois multiple polylogarithms)**.** Given a K-rational base point z of $\mathbb{P}^1\backslash\{0,1,\infty\}$ and $\gamma \in \pi_1^{\text{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$, define the path γ' associated to γ as in (5.1.2). For any $\sigma \in G_K$, the following holds:

$$\begin{aligned}\sum_{j=0}^{m-1} \frac{(\rho_{z,\gamma}(\sigma))^j}{j!} Li^{\ell}_{\underbrace{1,\dots,1}_{n-2\text{ times}},m-j}(z;\gamma,\sigma) + \sum_{j=0}^{n-2} \frac{(\rho_{1-z,\gamma'}(\sigma))^j}{j!} Li^{\ell}_{\underbrace{1,\dots,1}_{m-2\text{ times}},n-j}(1-z;\gamma',\sigma) \\ = \boldsymbol{\zeta}^{\ell}_{\underbrace{1,\dots,1}_{n-2\text{ times}},m}(\sigma).\end{aligned} \tag{5.1.5}$$

By the specialization of (5.1.3) and (5.1.5) when $z \to 1$, we obtain the well-known duality formula of multiple zeta values $\zeta(\underbrace{1,\dots,1}_{m-2\text{ times}},n) = \zeta(\underbrace{1,\dots,1}_{n-2\text{ times}},m)$ and its ℓ-adic Galois analog:

Corollary 5.1.3. For any $\sigma \in G_K$, we obtain

$$\boldsymbol{\zeta}^{\ell}_{\underbrace{1,\dots,1}_{m-2\text{ times}},n}(\sigma) = \boldsymbol{\zeta}^{\ell}_{\underbrace{1,\dots,1}_{n-2\text{ times}},m}(\sigma) \tag{5.1.6}$$

Remark 5.1.4. The theory of the p-adic KZ equation is constructed by Furusho in [F04]. He introduced the p-adic crystalline analog of $Li_{\mathbf{k}}(z;\gamma)$. Using the results in [F04], we obtain the p-adic crystalline analog of (5.1.3) in parallel ways as in the proof of (5.1.3).

Proof of Theorem 5.1.1, Theorem 5.1.2. Let $n, m \in \mathbb{Z}_{\geq 2}$ and $\gamma \in \pi_1^{\text{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$. The following proofs are based on a remark given in the Appendix of Furusho's lecture note [F14, A.24] and an insight about the ℓ-adic Oi-Ueno's equation in Nakamura's Oberwolfach Report [N21].

First, we prove Theorem 5.1.1. For $w, w' \in \mathrm{M}$, we write

$$\mathtt{Coeff}_{w+w'} := \mathtt{Coeff}_w + \mathtt{Coeff}_{w'}.$$

Since $G^{z,\gamma}_{\overrightarrow{01}}(e_0,e_1)$ is group-like element in the Hopf algebra $\mathbb{C}\langle\langle X,Y\rangle\rangle$, The shuffle relation

$$\mathtt{Coeff}_{w \sqcup\!\sqcup w'} = \mathtt{Coeff}_w \cdot \mathtt{Coeff}_{w'} \tag{5.1.7}$$

holds for $w, w' \in \mathrm{M}$. By the definition of the shuffle product $\sqcup\!\sqcup$, we get

$$e_0^j \sqcup\!\sqcup e_0^{m-j-1}e_1^{n-1} = e_0(e_1^{j-1} \sqcup\!\sqcup e_0^{m-j-1}e_1^{n-1}) + e_0(e_0^j \sqcup\!\sqcup e_0^{m-j-2}e_1^{n-1}). \tag{5.1.8}$$

Then we obtain

$$\begin{aligned}
&\sum_{j=0}^{m-1}\frac{(-\log(z;\gamma))^j}{j!}Li_{\underbrace{1,\dots,1}_{n-2\ \text{times}},m-j}(z)\\
=&\sum_{j=0}^{m-1}(-1)^{n+j-1}\cdot\texttt{Coeff}_{e_0^j}\left(G_{\overrightarrow{01}}^{z,\gamma}(e_1,e_0)\right)\cdot\texttt{Coeff}_{e_0^{m-j-1}e_1^{n-1}}\left(G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1)\right)\\
=&\sum_{j=0}^{m-1}(-1)^{n+j-1}\cdot\texttt{Coeff}_{e_0^j\sqcup\!\sqcup e_0^{m-j-1}e_1^{n-1}}\left(G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1)\right)\quad(\text{by }(5.1.7))\\
=&(-1)^{n+m-2}\cdot\texttt{Coeff}_{e_1(e_0^{m-1}\sqcup\!\sqcup e_1^{n-2})}\left(G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1)\right)\quad(\text{by }(5.1.8)).
\end{aligned}$$

By the same computation as above, we have

$$\sum_{j=0}^{n-2}\frac{(-\log(1-z;\gamma'))^j}{j!}Li_{\underbrace{1,\dots,1}_{m-2\ \text{times}},n-j}(1-z;\gamma')=(-1)^{n+m-3}\cdot\texttt{Coeff}_{e_0(e_1^{m-1}\sqcup\!\sqcup e_0^{n-2})}\left(G_{\overrightarrow{01}}^{1-z,\gamma'}(e_0,e_1)\right),$$

$$\zeta(\underbrace{1,\dots,1}_{n-2\ \text{times}},m)=(-1)^{n+m-2}\cdot\texttt{Coeff}_{e_1(e_0^{m-1}\sqcup\!\sqcup e_1^{n-2})}\left(G_{\overrightarrow{01}}^{\overrightarrow{10},\delta}(e_1,e_0)\right).$$

By the chain rule (1) of (3.2.1), we get

$$\begin{aligned}
&\texttt{Coeff}_{e_1(e_0^{m-1}\sqcup\!\sqcup e_1^{n-2})}\left(G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1)\right)\\
=&\texttt{Coeff}_{e_1(e_0^{m-1}\sqcup\!\sqcup e_1^{n-2})}\left(G_{\overrightarrow{01}}^{1-z,\gamma'}(e_1,e_0)\right)+\texttt{Coeff}_{e_1(e_0^{m-1}\sqcup\!\sqcup e_1^{n-2})}\left(G_{\overrightarrow{01}}^{\overrightarrow{10},\delta}(e_0,e_1)\right)\\
=&\texttt{Coeff}_{e_0(e_1^{m-1}\sqcup\!\sqcup e_0^{n-2})}\left(G_{\overrightarrow{01}}^{1-z,\gamma'}(e_0,e_1)\right)+\texttt{Coeff}_{e_1(e_0^{m-1}\sqcup\!\sqcup e_1^{n-2})}\left(G_{\overrightarrow{01}}^{\overrightarrow{10},\delta}(e_0,e_1)\right).
\end{aligned}$$

Then we obtain (5.1.3) by above equalities. This completes the proof of Theorem 5.1.1.

Next, we prove Theorem 5.1.2. Let $\sigma\in G_K$. Since $\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)$ is group-like in the Hopf algebra $\mathbb{Q}_\ell\langle\langle e_0,e_1\rangle\rangle$, the shuffle relation $\texttt{Coeff}_{w\sqcup\!\sqcup w'}=\texttt{Coeff}_w\cdot\texttt{Coeff}_{w'}$ holds. By the same computations as above, we have

$$\sum_{j=0}^{m-1}\frac{(\rho_{z,\gamma}(\sigma))^j}{j!}Li^{\ell}_{\underbrace{1,\dots,1}_{n-2\ \text{times}},m-j}(z;\gamma,\sigma)=(-1)^{n+m-2}\cdot\texttt{Coeff}_{e_1(e_0^{m-1}\sqcup\!\sqcup e_1^{n-2})}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right),$$

$$\sum_{j=0}^{n-2}\frac{(\rho_{1-z,\gamma'}(\sigma))^j}{j!}Li^{\ell}_{\underbrace{1,\dots,1}_{m-2\ \text{times}},n-j}(1-z;\gamma',\sigma)=(-1)^{n+m-3}\cdot\texttt{Coeff}_{e_0(e_1^{m-1}\sqcup\!\sqcup e_0^{n-2})}\left(\mathfrak{f}_\sigma^{1-z,\gamma'}(e_0,e_1)\right),$$

$$\boldsymbol{\zeta}^{\ell}_{\underbrace{1,\dots,1}_{n-2\ \text{times}},m}(\sigma)=(-1)^{n+m-2}\cdot\texttt{Coeff}_{e_1(e_0^{m-1}\sqcup\!\sqcup e_1^{n-2})}\left(\mathfrak{f}_\sigma^{\overrightarrow{10},\delta}(e_0,e_1))\right).$$

By the chain rule (1) of (4.2.1), we get

$$\begin{aligned}
&\texttt{Coeff}_{e_1(e_0^{m-1}\sqcup\!\sqcup e_1^{n-2})}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right)\\
=&\texttt{Coeff}_{e_1(e_0^{m-1}\sqcup\!\sqcup e_1^{n-2})}\left(\mathfrak{f}_\sigma^{1-z,\gamma'}(e_1,e_0)\right)+\texttt{Coeff}_{e_1(e_0^{m-1}\sqcup\!\sqcup e_1^{n-2})}\left(\mathfrak{f}_\sigma^{\overrightarrow{10},\delta}(e_0,e_1)\right)\\
=&\texttt{Coeff}_{e_0(e_1^{m-1}\sqcup\!\sqcup e_0^{n-2})}\left(\mathfrak{f}_\sigma^{1-z,\gamma'}(e_0,e_1)\right)+\texttt{Coeff}_{e_1(e_0^{m-1}\sqcup\!\sqcup e_1^{n-2})}\left(\mathfrak{f}_\sigma^{\overrightarrow{10},\delta}(e_0,e_1)\right).
\end{aligned}$$

Then we obtain (5.1.5) by above equalities. This completes the proof of Theorem 5.1.2. □

Chapter 6

Landen's trilogarithm functional equation and its ℓ-adic Galois analog

In this section, we derive Landen's functional equation of trilogarithms and its ℓ-adic Galois analog.

6.1 Main results and the proof

Theorem 6.1.1 (The 3-terms functional equation of the complex trilogarithm)**.** Given a base point z of $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$ and a path $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$, define the paths γ', γ'' associated to γ by

$$\gamma' := \delta_{\overrightarrow{10}} \cdot \phi_{\overrightarrow{10}}(\gamma) \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},1-z\right), \tag{6.1.1}$$

$$\gamma'' := \delta_{\overrightarrow{0\infty}} \cdot \phi_{\overrightarrow{0\infty}}(\gamma) \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},\frac{z}{z-1}\right), \tag{6.1.2}$$

where $\phi_{\overrightarrow{10}}, \phi_{\overrightarrow{0\infty}} \in \mathrm{Aut}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}\right)$ is given by $\phi_{\overrightarrow{10}}(t) = 1-t$, $\phi_{\overrightarrow{0\infty}}(t) = \frac{z}{z-1}$. Then the following holds:

Landen's equation

$$\begin{aligned} &Li_3(z;\gamma) + Li_3(1-z;\gamma') + Li_3\left(\frac{z}{z-1};\gamma''\right) \\ &\quad = \zeta(3) + \zeta(2)\log(1-z;\gamma') - \frac{1}{2}\log(z,\gamma)\log^2(1-z;\gamma') + \frac{1}{6}\log^3(1-z;\gamma'). \end{aligned}$$

We next discuss the ℓ-adic Galois case for any prime number ℓ. Let K be a subfield of $\mathbb{C}$ with the algebraic closure $\overline{K}$ and G_K the absolute Galois group of K.

Theorem 6.1.2 (The 3-terms functional equation of the ℓ-adic Galois trilogarithm)**.** Given a K-rational base point z of $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$ and a path $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$, define the paths γ', γ'' associated to γ as in (6.1.1) and (6.1.2). For any $\sigma \in G_K$, the following holds:

ℓ-adic Landen's equation

$$\begin{aligned}
&Li_3^{\ell}(z;\gamma,\sigma) + Li_3^{\ell}(1-z;\gamma',\sigma) + Li_3^{\ell}\left(\frac{z}{z-1};\gamma'',\sigma\right) \\
&= \boldsymbol{\zeta}_3^{\ell}(\sigma) - \boldsymbol{\zeta}_2^{\ell}(\sigma)\rho_{1-z,\gamma'}(\sigma) + \frac{1}{2}\rho_{z,\gamma}(\sigma)\rho_{1-z,\gamma'}(\sigma)^2 - \frac{1}{6}\rho_{1-z,\gamma'}(\sigma)^3 \\
&\quad - \frac{1}{2}Li_2^{\ell}(z;\gamma,\sigma) - \frac{1}{12}\rho_{1-z,\gamma'}(\sigma) - \frac{1}{4}\rho_{1-z,\gamma'}(\sigma)^2.
\end{aligned}$$

Proof of Theorem 6.1.1. We compare the coefficients of $e_1e_0e_1$ in the chain rule

$$G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1) = G_{\overrightarrow{01}}^{1-z,\gamma'}(e_1,e_0)\cdot\Phi(e_0,e_1)$$

of Proposition 3.2.1 (1). Let $w \in \mathrm{M}$ and $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01},z\right)$. By simple calculation, we then obtain

$$\begin{aligned}
&\mathtt{Coeff}_{e_1e_0e_1}\left(G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1)\right) \\
&= -\zeta(2)\mathtt{Coeff}_{e_0}\left(G_{\overrightarrow{01}}^{1-z,\gamma'}(e_0,e_1)\right) - 2\zeta(3) + \mathtt{Coeff}_{e_0e_1e_0}\left(G_{\overrightarrow{01}}^{1-z,\gamma'}(e_0,e_1)\right) \\
&= -\zeta(2)\mathtt{Coeff}_{e_0}\left(G_{\overrightarrow{01}}^{1-z,\gamma'}(e_0,e_1)\right) - 2\zeta(3) \\
&\quad + \left(\mathtt{Coeff}_{e_0e_1}\left(G_{\overrightarrow{01}}^{1-z,\gamma'}(e_0,e_1)\right)\mathtt{Coeff}_{e_0}\left(G_{\overrightarrow{01}}^{1-z,\gamma'}(e_0,e_1)\right) - 2\mathtt{Coeff}_{e_0^2e_1}\left(G_{\overrightarrow{01}}^{1-z,\gamma'}(e_0,e_1)\right)\right)
\end{aligned}$$

where in the former equality are used known identities

$$\mathtt{Coeff}_{e_0e_1}(\Phi(e_0,e_1)) = -\zeta(2), \quad \mathtt{Coeff}_{e_1e_0e_1}(\Phi(e_0,e_1)) = -2\zeta(3),$$

and in the last equality is used the shuffle relation $e_0e_1 ⧢ e_0 = e_0e_1e_0 + 2e_0^2e_1$. This leads to

$$Li_{2,1}(z;\gamma) = -\frac{\pi^2}{6}\log(1-z;\gamma') - 2\zeta(3) - Li_2(1-z;\gamma')\log(1-z;\gamma') + 2Li_3(1-z;\gamma'). \tag{6.1.3}$$

Now let us compare the coefficients of $e_0^2e_1$ on both sides of the chain rule

$$G_{\overrightarrow{01}}^{\frac{z}{z-1},\gamma''}(e_0,e_1) = G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_\infty)\cdot\mathbf{exp}\left(\pi\sqrt{-1}\cdot e_0\right)$$

from Proposition 3.2.1 (2). It follows easily that

$$\begin{aligned}
&\mathtt{Coeff}_{e_0^2e_1}\left(G_{\overrightarrow{01}}^{\frac{z}{z-1},\gamma''}(e_0,e_1)\right) \\
=&\mathtt{Coeff}_{e_0^2e_1}\left(G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1)\right) - \mathtt{Coeff}_{e_1^3}\left(G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1)\right) \\
&+ \mathtt{Coeff}_{e_0e_1^2}\left(G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1)\right) + \mathtt{Coeff}_{e_1e_0e_1}\left(G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1)\right)
\end{aligned}$$

equivalently,

$$-Li_3\left(\frac{z}{z-1};\gamma''\right) = Li_3(z;\gamma) + Li_{1,1,1}(z;\gamma) + Li_{1,2}(z;\gamma) + Li_{2,1}(z;\gamma). \tag{6.1.4}$$

We know from the case $n=2, m=3$ of (5.1.3) with interchange $z \leftrightarrow 1-z$ that

$$Li_{1,2}(z;\gamma) = \zeta(3) - \left(Li_3(1-z;\gamma') - Li_2(1-z;\gamma')\log(1-z;\gamma') - \frac{1}{2}\log(z;\gamma)\log^2(1-z;\gamma')\right). \tag{6.1.5}$$

Putting (6.1.3) and (6.1.5) into the last two terms of (6.1.4) with noticing $Li_{1,1,1}(z;\gamma) = -\frac{1}{6}\log^3(1-z;\gamma')$, we obtain a proof of Landen's trilogarithm functional equation □

Proof of Theorem 6.1.2. We only have to examine the ℓ-adic Galois versions of the identities (6.1.3), (6.1.4) and (6.1.5) with replacing the role of $G_{\overrightarrow{01}}^{z,\gamma}(e_0,e_1)$ by $\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)$. It turns out that the two identities (6.1.3), (6.1.5) have exactly the parallel counterparts:

$$
\begin{aligned}
Li_{2,1}^{\ell}(z;\gamma,\sigma) =& \boldsymbol{\zeta}_2^{\ell}(\sigma)\rho_{1-z,\gamma'}(\sigma) + \boldsymbol{\zeta}_{2,1}^{\ell}(\sigma) \\
&+ Li_2^{\ell}(1-z;\gamma',\sigma)\rho_{1-z}(\sigma) + 2Li_3^{\ell}(1-z;\gamma',\sigma),
\end{aligned}
\tag{6.1.6}
$$

$$
\begin{aligned}
Li_{1,2}^{\ell}(z;\gamma,\sigma) =& \boldsymbol{\zeta}_3^{\ell}(\sigma) - Li_3^{\ell}(1-z;\gamma',\sigma) \\
&- Li_2^{\ell}(1-z;\gamma,\sigma)\rho_{1-z,\gamma'}(\sigma) - \frac{1}{2}\rho_{z,\gamma}(\sigma)\rho_{1-z,\gamma'}(\sigma)^2
\end{aligned}
\tag{6.1.7}
$$

with $\sigma \in G_K$. There occurs a small difference for (6.1.4) when evaluating the key identity

$$
\mathfrak{f}_\sigma^{\frac{z}{z-1},\gamma''}(e_0,e_1) = \mathfrak{f}_\sigma^{z,\gamma}(e_0,e_\infty)\cdot \exp\left(\frac{1-\chi(\sigma)}{2}e_0\right)
$$

from Proposition 3.2.1 (4) with taking into accounts the Campbell-Hausdorff sum

$$
\begin{aligned}
e_\infty :=& \log(\exp(-e_1)\exp(-e_0)) \\
=& -e_1 - e_0 \quad \underbrace{+\frac{1}{2}(e_1e_0 - e_0e_1) - \frac{1}{12}e_0e_0e_1 + \cdots}_{(*)},
\end{aligned}
\tag{6.1.8}
$$

where do exist nontrivial nonlinear terms $(*)$ in the ℓ-adic Galois case. Then $\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_\infty)$ is calculated as follows:

$$
\begin{aligned}
&\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_\infty) \\
=& 1 + \mathtt{Coeff}_{e_0}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right)e_0 + \mathtt{Coeff}_{e_1}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right)e_\infty \\
&+ \mathtt{Coeff}_{e_0e_1}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right)e_0e_\infty + \mathtt{Coeff}_{e_1^2}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right)e_\infty^2 + \cdots \\
=& 1 + \left(\mathtt{Coeff}_{e_0}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right) - \mathtt{Coeff}_{e_1}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right)\right)e_0 - \mathtt{Coeff}_{e_1}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right)e_1 \\
&\underbrace{+\mathtt{Coeff}_{e_1}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right)\left(\frac{1}{2}e_1e_0 - \frac{1}{2}e_0e_1 - \frac{1}{12}e_0e_0e_1 + \cdots\right)}_{\ell\text{-adic extra terms in } \mathtt{Coeff}_{e_1}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right)e_\infty} \\
&+ \left(\mathtt{Coeff}_{e_1^2}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right) - \mathtt{Coeff}_{e_0e_1}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right)\right)e_0^2 \\
&\underbrace{+\mathtt{Coeff}_{e_0e_1}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right)\left(\frac{1}{2}e_0e_1e_0 - \frac{1}{2}e_0^2e_1 + \cdots\right)}_{\ell\text{-adic extra terms in } \mathtt{Coeff}_{e_0e_1}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right)e_0e_\infty} \\
&+ \left(\mathtt{Coeff}_{e_1^2}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right) - \mathtt{Coeff}_{e_0e_1}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right)\right)e_0e_1 \\
&\underbrace{+\mathtt{Coeff}_{e_1^2}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right)\left(\frac{1}{2}e_0^2e_1 - \frac{1}{2}e_1^2e_0 + \cdots\right)}_{\ell\text{-adic extra terms in } \mathtt{Coeff}_{e_1^2}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right)e_\infty^2} \\
&+ \mathtt{Coeff}_{e_1^2}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right)e_1e_0 + \mathtt{Coeff}_{e_1^2}\left(\mathfrak{f}_\sigma^{z,\gamma}(e_0,e_1)\right)e_1^2 + \cdots \qquad (\sigma \in G_K).
\end{aligned}
\tag{6.1.9}
$$

Summing up, we find that the ℓ-adic Galois analog to identity (6.1.4) turns to get extra additional terms as:

$$Li_3^{\ell}\left(\frac{z}{z-1};\gamma'',\sigma\right) = -Li_3^{\ell}(z;\gamma,\sigma) - Li_{1,1,1}^{\ell}(z;\gamma,\sigma) - Li_{1,2}^{\ell}(z;\gamma,\sigma) - Li_{2,1}^{\ell}(z;\gamma,\sigma) \qquad (6.1.10)$$
$$- \left(\frac{1}{2}Li_2^{\ell}(z;\gamma,\sigma) + \frac{1}{4}\rho_{1-z,\gamma'}(\sigma)^2 + \frac{1}{12}\rho_{1-z,\gamma'}(\sigma)\right)$$

for $\sigma \in G_K$. The asserted formula follows from (6.1.10) after $Li_{1,2}^{\ell}(z;\gamma,\sigma)$, $Li_{2,1}^{\ell}(z;\gamma,\sigma)$ in the RHS are replaced by the equations (6.1.7), (6.1.6) respectively and from knowledge of the lower degree coefficients of $\mathfrak{f}_\sigma^{z,\gamma}(e_0, e_1)$. □

Corollary 6.1.3 (The 3-terms functional equation of the ℓ-adic Galois trilogarithmic charactar)**.** Given a K-rational base point z of $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$ and $\gamma \in \pi_1^{\text{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, z\right)$, define γ', γ'' associated to γ as in (6.1.1) and (6.1.2). For any $\sigma \in G_K$, the following holds:

ℓ-adic Landen's equation

$$\begin{aligned}
&\tilde{\chi}_3^{z,\gamma}(\sigma) + \tilde{\chi}_3^{1-z,\gamma'}(\sigma) + \tilde{\chi}_3^{\frac{z}{z-1},\gamma''}(\sigma) \\
&= \tilde{\chi}_3^{\overrightarrow{10},\delta_{\overrightarrow{10}}}(\sigma) + \chi(\sigma)\tilde{\chi}_2^{z,\gamma}(\sigma) + \rho_{z,\gamma}(\sigma)\rho_{1-z,\gamma'}(\sigma)^2 - \frac{\rho_{1-z,\gamma'}(\sigma)}{12}\left(\chi(\sigma)^2 - 1\right) \\
&\quad - \frac{\rho_{1-z,\gamma'}(\sigma)}{6}\Big(\chi(\sigma) - \rho_{1-z,\gamma'}(\sigma)\Big)\Big(\chi(\sigma) - 2\rho_{1-z,\gamma'}(\sigma)\Big).
\end{aligned}$$

Proof. Putting the conversion formula [NS22, Proposition 4.2 (ii)] into Theorem 6.1.2, we obtain the desired equation. □

Remark 6.1.4. The above polylogarithmic character functional equation enables us to check the $\mathbb{Z}_\ell$-integrality of both sides. First, the term $\dfrac{1-\chi(\sigma)^2}{12}\rho_{1-z,\gamma'}(\sigma)$ is a ℓ-adic integer since $\dfrac{\chi(\sigma)^2-1}{24} = \tilde{\chi}_2^{\overrightarrow{10}}(\sigma) \in \mathbb{Z}_\ell$. Now we need to check the ℓ-adic integralness of the last term

$$\frac{1}{6}\rho_{1-z,\gamma'}(\sigma)(\chi(\sigma) - \rho_{1-z,\gamma'}(\sigma))(\chi(\sigma) - 2\rho_{1-z,\gamma'}(\sigma)). \qquad (6.1.11)$$

Let $v_\ell : \mathbb{Q}_\ell \to \mathbb{Z} \cup \{\infty\}$ be the ℓ-adic valuation. Then $v_\ell\left(\dfrac{1}{6}\right) = \begin{cases} 0 & (\ell \geq 5), \\ -1 & (\ell = 2, 3). \end{cases}$ Therefore (6.1.11) $\in \mathbb{Z}_\ell$ for $\ell \geq 5$. In the case of $\ell = 2$, we obtain

$$v_2(\rho_{1-z,\gamma'}(\sigma)) \geq 1 \quad \text{or} \quad v_2(\chi(\sigma) - \rho_{1-z,\gamma'}(\sigma)) \geq 1,$$

since $\rho_{1-z,\gamma'}(\sigma) \equiv 0, 1 \pmod 2$ and $\chi(\sigma) \equiv 1 \pmod 2$ by $\chi(\sigma) \in \mathbb{Z}_2^\times$. Hence (6.1.11) $\in \mathbb{Z}_2$. In the case of $\ell = 3$, we have

$$v_3(\rho_{1-z,\gamma'}(\sigma)) \geq 1 \quad \text{or} \quad v_3(\chi(\sigma) - \rho_{1-z,\gamma'}(\sigma)) \geq 1 \quad \text{or} \quad v_3(\chi(\sigma) - 2\rho_{1-z,\gamma'}(\sigma)) \geq 1$$

since $\rho_{1-z,\gamma'}(\sigma) \equiv 0, 1, -1 \pmod 3$ and $\chi(\sigma) \equiv 1, -1 \pmod 3$ by $\chi(\sigma) \in \mathbb{Z}_3^\times$. Thus (6.1.11) $\in \mathbb{Z}_3$.

Chapter 7

Spence-Kummer's trilogarithm functional equation and the underlying geometry

7.1 Main results

Let K be a subfield of $\mathbb{C}$ with the algebraic closure $\overline{K}$. The underlying geometry of Spence-Kummer's trilogarithm functional equation is the complement to the non-Fano arrangement

$$V_{\text{non-Fano}} := \operatorname{Spec}\left(\overline{K}\left[s_1, s_2, \frac{1}{s_1 s_2(1-s_1)(1-s_2)(s_1-s_2)(1-s_1 s_2)}\right]\right), \tag{7.1.1}$$

together with nine morphisms $\{f_i\}_{i=1,\ldots,9} : V_{\text{non-Fano}} \to \mathbb{P}^1_{\overline{K}} \backslash \{0,1,\infty\}$ defined as follows:

$$\begin{aligned} f_1(s_1,s_2) &:= \frac{s_1(1-s_2)^2}{s_2(1-s_1)^2}, & f_2(s_1,s_2) &:= s_1 s_2, & f_3(s_1,s_2) &:= \frac{s_1}{s_2}, \\ f_4(s_1,s_2) &:= \frac{s_1(1-s_2)}{s_2(1-s_1)}, & f_5(s_1,s_2) &:= \frac{s_1(1-s_2)}{s_1-1}, & f_6(s_1,s_2) &:= \frac{1-s_2}{1-s_1}, \\ f_7(s_1,s_2) &:= \frac{1-s_2}{s_2(s_1-1)}, & f_8(s_1,s_2) &:= s_1, & f_9(s_1,s_2) &:= s_2. \end{aligned} \tag{7.1.2}$$

We take the K-rational tangential base point

$$\vec{v} : \operatorname{Spec}\Big(K((t))\Big) \to V_{\text{non-Fano}} \tag{7.1.3}$$

over the $K(t)$-rational point (t^2, t). In TABLE 7.1, we identify the images $f_i(\vec{v})$ with standard K-rational tangential base points of $\mathbb{P}^1_{\overline{K}} \backslash \{0,1,\infty\}$ under Galois equivalence $\approx$ (in the sense of [N02, §5.9]). In FIGURE 7.1, the dashed line represents $\mathbb{P}^1(\mathbb{R})\backslash\{0,1,\infty\}$, and the upper half plane is located above. The main result in the complex case is the following theorem.

Theorem 7.1.1 (The 9-term functional equation for the complex trilogarithm)**.** Given a $\mathbb{C}$-rational point $(x,y) \in V_{\text{non-Fano}}(\mathbb{C})$ and a path $\gamma_0 \in \pi_1^{\text{top}}(V^{\text{an}}_{\text{non-Fano}}; \vec{v}, (x,y))$, define the path system $\{\gamma_i\}_{i=1,\ldots,9}$ associated to γ_0 by

$$\gamma_i := \delta_i \cdot f_i^{\text{an}}(\gamma_0) \in \pi_1^{\text{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, f_i^{\text{an}}(x,y)\right), \tag{7.1.4}$$

where $\delta_i \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, f_i^{\mathrm{an}}(\vec{v})\right)$ is as in TABLE 7.1. Then the following functional equation holds:

(d-$\mathbb{C}$) Spence-Kummer's equation

$$
\begin{aligned}
&Li_3\left(\frac{x(1-y)^2}{y(1-x)^2};\gamma_1\right) + Li_3\left(xy;\gamma_2\right) + Li_3\left(\frac{x}{y};\gamma_3\right) - 2Li_3\left(\frac{x(1-y)}{y(1-x)};\gamma_4\right) \\
&- 2Li_3\left(\frac{x(1-y)}{x-1};\gamma_5\right) - 2Li_3\left(\frac{1-y}{1-x};\gamma_6\right) - 2Li_3\left(\frac{1-y}{y(x-1)};\gamma_7\right) - 2Li_3\left(x;\gamma_8\right) \\
&- 2Li_3\left(y;\gamma_9\right) + 2\zeta(3) = \log^2(y;\gamma_9)\log\left(\frac{1-y}{1-x};\gamma_6\right) - \frac{\pi^2}{3}\log(y;\gamma_9) - \frac{1}{3}\log^3(y;\gamma_9),
\end{aligned}
$$

Figure 7.1: Topological paths on $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$

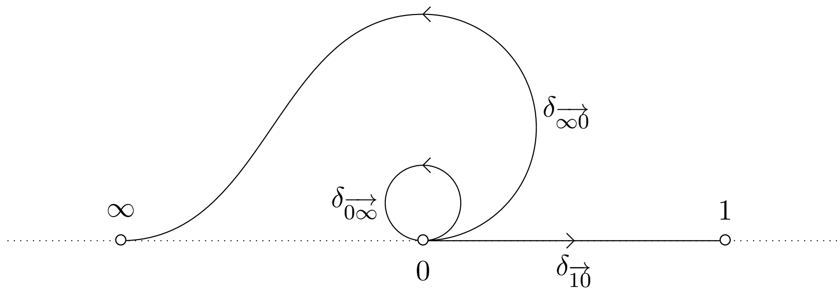


Table 7.1: $\delta_1, \ldots, \delta_9$

i	$f_i(x,y)$	$f_i(t^2,t)$	$f_i(\vec{v})$	$\delta_i \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, f_i^{\mathrm{an}}(\vec{v})\right)$
1	$\frac{x(1-y)^2}{y(1-x)^2}$	$\frac{t}{(1+t)^2}$	$\overrightarrow{01} \approx f_1(\vec{v})$	$\delta_1 := 1$ (= trivial path)
2	xy	t^3	$\overrightarrow{01} \approx f_2(\vec{v})$	$\delta_2 := 1$
3	$\frac{x}{y}$	t	$\overrightarrow{01} = f_3(\vec{v})$	$\delta_3 := 1$
4	$\frac{x(1-y)}{y(1-x)}$	$\frac{t}{1+t}$	$\overrightarrow{01} \approx f_4(\vec{v})$	$\delta_4 := 1$
5	$\frac{x(1-y)}{x-1}$	$\frac{-t^2}{1+t}$	$\overrightarrow{0\infty} \approx f_5(\vec{v})$	$\delta_5 := \delta_{\overrightarrow{0\infty}}$ (= as in FIGURE 7.1)
6	$\frac{1-y}{1-x}$	$\frac{1}{1+t}$	$\overrightarrow{10} \approx f_6(\vec{v})$	$\delta_6 := \delta_{\overrightarrow{10}}$ (= as in FIGURE 7.1)
7	$\frac{1-y}{y(x-1)}$	$\frac{-1}{t(1+t)}$	$\overrightarrow{\infty 0} \approx f_7(\vec{v})$	$\delta_7 := \delta_{\overrightarrow{\infty 0}}$ (= as in FIGURE 7.1)
8	x	t^2	$\overrightarrow{01} \approx f_8(\vec{v})$	$\delta_8 := 1$
9	y	t	$\overrightarrow{01} = f_9(\vec{v})$	$\delta_9 := 1$

The following theorem is the main result in the ℓ-adic Galois case for any fixed prime number ℓ. It is noteworthy that the ℓ-adic Galois version of the Spence-Kummer equation involves two nontrivial Li_2^ℓ terms in contrast to the complex version.

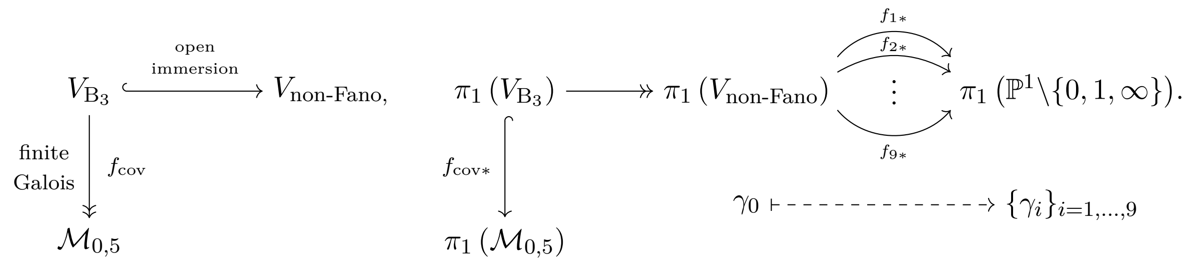
Theorem 7.1.2 (The 9-term functional equation for the ℓ-adic Galois trilogarithm)**.** Given a K-rational point $(x, y) \in V_{\text{non-Fano}}(K)$ and a path $\gamma_0 \in \pi_1^{\text{top}}(V_{\text{non-Fano}}^{\text{an}}; \vec{v}, (x, y))$, define the path system $\{\gamma_i\}_{i=1,\ldots,9}$ associated to γ_0 as in (7.1.4). For any $\sigma \in G_K$, the following holds:

(d-ℓ) ℓ-adic Spence-Kummer's equation

$$
\begin{aligned}
&Li_3^\ell\left(\frac{x(1-y)^2}{y(1-x)^2}; \gamma_1, \sigma\right) + Li_3^\ell(xy; \gamma_2, \sigma) + Li_3^\ell\left(\frac{x}{y}; \gamma_3, \sigma\right) - 2Li_3^\ell\left(\frac{x(1-y)}{y(1-x)}; \gamma_4, \sigma\right) \\
&- 2Li_3^\ell\left(\frac{x(1-y)}{x-1}; \gamma_5, \sigma\right) - 2Li_3^\ell\left(\frac{1-y}{1-x}; \gamma_6, \sigma\right) - 2Li_3^\ell\left(\frac{1-y}{y(x-1)}; \gamma_7, \sigma\right) - 2Li_3^\ell(x; \gamma_8, \sigma) \\
&- 2Li_3^\ell(y; \gamma_9, \sigma) + 2\boldsymbol{\zeta}_3^\ell(\sigma) = -\rho_{y,\gamma_9}(\sigma)^2 \rho_{\frac{1-y}{1-x}, \gamma_6}(\sigma) + 2\boldsymbol{\zeta}_2^\ell(\sigma)\rho_{y,\gamma_9}(\sigma) + \frac{1}{3}\rho_{y,\gamma_9}(\sigma)^3 \\
&- Li_2^\ell\left(\frac{x(1-y)}{x-1}; \gamma_5, \sigma\right) - Li_2^\ell\left(\frac{1-y}{y(x-1)}; \gamma_7, \sigma\right) + \frac{1}{2}\rho_{\frac{1-xy}{1-x}, \gamma_5'}(\sigma) - \frac{1}{3}\rho_{y,\gamma_9}(\sigma).
\end{aligned}
$$

A primary ingredient of the present work is to capture $\pi_1(V_{\text{non-Fano}})$ as an explicit subquotient of the well-known fundamental group $\pi_1(\mathcal{M}_{0,5})$ of the moduli space of the projective line with ordered five points, through an intermediate geometric object V_{B_3}, that is, the complement to the Coxeter arrangement of type B_3:

Figure 7.2: Key diagram to the proofs



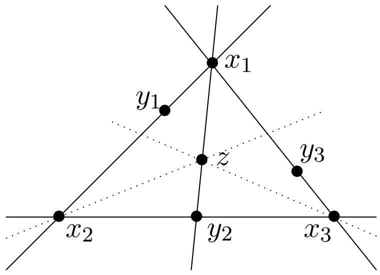
This enables us to compute precisely the lower degree terms of the Spence-Kummer equation in both complex and ℓ-adic Galois cases, where we follow the computational method devised by Nakamura-Wojtkowiak [NW12, Proposition 5.11] plugging Zagier's tensor criterion into the language of fundamental groups [NW12, Theorem 5.7]. In the course of our procedure are also obtained as byproducts the dilogarithm functional equations of Schaeffer's, Kummer's, and Hill's types entried by (a,b,c-$\mathbb{C}$) and (a,b,c-ℓ) of TABLE 7.2. These equations will be used to decrease Li_2-terms to derive Spence-Kummer's equations (d-$\mathbb{C}$), (d-ℓ).

Remark 7.1.3. The reason for the name "non-Fano" of the arrangement $s_1 s_2(1 - s_1)(1 - s_2)(s_1 - s_2)(1 - s_1 s_2)$ is due to the non-Fano matroid after Gino Fano. By the change of variables $s_1' = \frac{1}{1-s_1}$ and $s_2' = \frac{1}{1-s_2}$, the affine variety $V_{\text{non-Fano}}$ is isomorphic to the complement to the arrangement $s_1' s_2'(1 - s_1')(1 - s_2')(s_1' - s_2')(s_1' + s_2' - 1)$. This arrangement is the decone of a realization of the non-Fano matroid F_7^- (cf. [O11, Figure 1.15(a)], [Su01, Example 10.5]).

Table 7.2: Functional equations to be proved

ℓ-adic Galois side	complex side
$Li_3^{\ell}\left(\frac{x(1-y)^2}{y(1-x)^2};\gamma_1,\sigma\right)+Li_3^{\ell}(xy;\gamma_2,\sigma)+Li_3^{\ell}\left(\frac{x}{y};\gamma_3,\sigma\right)$ $-2Li_3^{\ell}\left(\frac{x(1-y)}{y(1-x)};\gamma_4,\sigma\right)-2Li_3^{\ell}\left(\frac{x(1-y)}{x-1};\gamma_5,\sigma\right)$ $-2Li_3^{\ell}\left(\frac{1-y}{1-x};\gamma_6,\sigma\right)-2Li_3^{\ell}\left(\frac{1-y}{y(x-1)};\gamma_7,\sigma\right)$ $-2Li_3^{\ell}(x;\gamma_8,\sigma)-2Li_3^{\ell}(y;\gamma_9,\sigma)+2\boldsymbol{\zeta}_3^{\ell}(\sigma)$ $=-\left(\rho_{y,\gamma_9}(\sigma)\right)^2\rho_{\frac{1-y}{1-x},\gamma_6}(\sigma)$ $+2\boldsymbol{\zeta}_2^{\ell}(\sigma)\rho_{y,\gamma_9}(\sigma)+\frac{1}{3}\left(\rho_{y,\gamma_9}(\sigma)\right)^3$ $-Li_2^{\ell}\left(\frac{x(1-y)}{x-1};\gamma_5,\sigma\right)-Li_2^{\ell}\left(\frac{1-y}{y(x-1)};\gamma_7,\sigma\right)$ $+\frac{1}{2}\rho_{\frac{1-xy}{1-x},\gamma_5'}(\sigma)-\frac{1}{3}\rho_{y,\gamma_9}(\sigma).$ Theorem 7.1.2 (d-ℓ)	$Li_3\left(\frac{x(1-y)^2}{y(1-x)^2};\gamma_1\right)+Li_3(xy;\gamma_2)+Li_3\left(\frac{x}{y};\gamma_3\right)$ $-2Li_3\left(\frac{x(1-y)}{y(1-x)};\gamma_4\right)-2Li_3\left(\frac{x(1-y)}{x-1};\gamma_5\right)$ $-2Li_3\left(\frac{1-y}{1-x};\gamma_6\right)-2Li_3\left(\frac{1-y}{y(x-1)};\gamma_7\right)$ $-2Li_3(x;\gamma_8)-2Li_3(y;\gamma_9)+2\zeta(3)$ $=\log^2(y;\gamma_9)\log\left(\frac{1-y}{1-x};\gamma_6\right)$ $-\frac{\pi^2}{3}\log(y;\gamma_9)-\frac{1}{3}\log^3(y;\gamma_9).$ Theorem 7.1.1 (d-$\mathbb{C}$), Spence [Sp1809], Kummer [K1840]
$Li_2^{\ell}\left(\frac{x(1-y)}{y(1-x)};\gamma_4,\sigma\right)-Li_2^{\ell}(y;\gamma_9,\sigma)+Li_2^{\ell}(x;\gamma_8,\sigma)$ $-Li_2^{\ell}\left(\frac{x}{y};\gamma_3,\sigma\right)-Li_2^{\ell}\left(\frac{1-y}{1-x};\gamma_6,\sigma\right)$ $=\rho_{y,\gamma_9}(\sigma)\rho_{\frac{1-y}{1-x},\gamma_6}(\sigma)-\boldsymbol{\zeta}_2^{\ell}(\sigma).$ Theorem 7.3.2 (a-ℓ)	$Li_2\left(\frac{x(1-y)}{y(1-x)};\gamma_4\right)-Li_2(y;\gamma_9)+Li_2(x;\gamma_8)$ $-Li_2\left(\frac{x}{y};\gamma_3\right)-Li_2\left(\frac{1-y}{1-x};\gamma_6\right).$ $=\log(y;\gamma_9)\log\left(\frac{1-y}{1-x};\gamma_6\right)-\frac{\pi^2}{6}.$ Theorem 7.3.1 (a-$\mathbb{C}$), Schaeffer [Sc1846]
$Li_2^{\ell}\left(\frac{x(1-y)^2}{y(1-x)^2};\gamma_1,\sigma\right)-Li_2^{\ell}\left(\frac{x(1-y)}{x-1};\gamma_5,\sigma\right)$ $-Li_2^{\ell}\left(\frac{1-y}{y(x-1)};\gamma_7,\sigma\right)-Li_2^{\ell}\left(\frac{x(1-y)}{y(1-x)};\gamma_4,\sigma\right)$ $-Li_2^{\ell}\left(\frac{1-y}{1-x};\gamma_6,\sigma\right)=\frac{1}{2}\left(\rho_{y,\gamma_9}(\sigma)\right)^2$ $+\frac{1}{2}\rho_{y,\gamma_9}(\sigma)+\rho_{1-x,\gamma_8'}(\sigma)-\rho_{1-xy,\gamma_2'}(\sigma).$ Theorem 7.3.2 (b-ℓ)	$Li_2\left(\frac{x(1-y)^2}{y(1-x)^2};\gamma_1\right)-Li_2\left(\frac{x(1-y)}{x-1};\gamma_5\right)$ $-Li_2\left(\frac{1-y}{y(x-1)};\gamma_7\right)-Li_2\left(\frac{x(1-y)}{y(1-x)};\gamma_4\right)$ $-Li_2\left(\frac{1-y}{1-x};\gamma_6\right)=\frac{1}{2}\log^2(y;\gamma_9).$ Theorem 7.3.1 (b-$\mathbb{C}$), Kummer [K1840]
$Li_2^{\ell}\left(\frac{1-y}{y(x-1)};\gamma_7,\sigma\right)+Li_2^{\ell}(xy;\gamma_2,\sigma)-Li_2^{\ell}(x;\gamma_8,\sigma)$ $-Li_2^{\ell}(y;\gamma_9,\sigma)-Li_2^{\ell}\left(\frac{x(1-y)}{x-1};\gamma_5,\sigma\right)$ $=-\boldsymbol{\zeta}_2^{\ell}(\sigma)+\rho_{y,\gamma_9}(\sigma)\rho_{\frac{1-y}{1-x},\gamma_6}(\sigma)-\frac{1}{2}\left(\rho_{y,\gamma_9}(\sigma)\right)^2$ $-\frac{1}{2}\rho_{y,\gamma_9}(\sigma).$ Theorem 7.3.2 (c-ℓ)	$Li_2\left(\frac{1-y}{y(x-1)};\gamma_7\right)+Li_2(xy;\gamma_2)-Li_2(x;\gamma_8)$ $-Li_2(y;\gamma_9)-Li_2\left(\frac{x(1-y)}{x-1};\gamma_5\right)$ $=-\frac{\pi^2}{6}+\log(y;\gamma_9)\log\left(\frac{1-y}{1-x};\gamma_6\right)-\frac{1}{2}\log^2(y;\gamma_9).$ Theorem 7.3.1 (c-$\mathbb{C}$), Hill [H1830]

Figure 7.3: Goncharov's 7-points configuration on $\mathbb{P}^2$



Remark 7.1.4. In [Go91],[Go95],[Go00], Goncharov looks at the following space

$$\mathscr{M}_G := \text{the moduli space of certain 7-point configurations on } \mathbb{P}^2_{\overline{K}} \text{ as shown in FIGURE 7.3} \tag{7.1.5}$$

to derive Spence-Kummer's trilogarithm functional equation. In fact, $\mathscr{M}_G$ is isomorphic to $V_{\text{non-Fano}}$ as follows: Since no three of four points x_1, x_2, x_3, z are collinear, there is a unique projective transformation sending the standard frame, namely $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ and $[1:1:1]$, to x_1, x_2, x_3 and z, respectively. By this projective transformation, y_2 is transferred to $[0:1:1]$, and the images of y_1 and y_3 are denoted $[1:u_1:0]$ and $[1:0:u_2]$, respectively. Then, $\mathscr{M}_G$ is identified with the affine variety

$$\mathrm{Spec}\left(\overline{K}\left[u_1, u_2, \frac{1}{u_1u_2(1-u_1)(1-u_2)(u_1+u_2)(u_1u_2-u_1-u_2)}\right]\right) \tag{7.1.6}$$

by sending $[(x_1, x_2, x_3, y_1, y_2, y_3, z)] \mapsto (u_1, u_2)$. This affine variety is isomorphic to $V_{\text{non-Fano}}$ by

$$u_1 = s_1, \quad u_2 = f_5(s_1, s_2) \left(= \frac{s_1(1-s_2)}{s_1 - 1}\right). \tag{7.1.7}$$

Remark 7.1.5. Historically, the study of functional equations of $Li_3(z)$ originated in the late 18th century by Landen [L1780] and others (cf. [Lew81, Chapter 6]). Modern treatments have been presented by Zagier [Z91], Goncharov [Go91], Wojtkowiak [W91], Beilinson-Deligne [BD94], Gangl [Ga03] and so forth, since the last decade of the 20th century, where the Bloch-Wigner-Ramakrishnan polylogarithms are recognized as the main characters for the regulator maps in the motivic cohomology theory. Relations between Spence-Kummer's equation and the non-Fano arrangement are pointed out in more recent works of this century (cf. [Pe12], [Pi05], [Pi21], [Pi22], [R02]) in the context of web geometry and cluster algebras.

7.2 Fundamental groups of V_{B_3} and $V_{\text{non-Fano}}$

We fix a subfield K of $\mathbb{C}$. Let $\overline{K}$ be the algebraic closure of K in $\mathbb{C}$. We shall write

$$V_{\mathrm{B}_3} \;=\; \mathrm{Spec}\left(\overline{K}\left[s_1, s_2, \frac{1}{s_1s_2(1-s_1^2)(1-s_2^2)(s_1-s_2)(1-s_1s_2)}\right]\right) \tag{7.2.1}$$

for the complement to the pseudo-line arrangement $s_1 s_2(1-s_1^2)(1-s_2^2)(s_1-s_2)(1-s_1 s_2)$. By the change of variables $s_1' = \frac{1-s_1}{1+s_1}$ and $s_2' = \frac{1-s_2}{1+s_2}$, the affine variety V_{B_3} is isomorphic to the complement to the Coxeter arrangement $s_1' s_2'(1-{s'}_1^2)(1-{s'}_2^2)(s_1'-s_2')(s_1'+s_2')$ of type B_3. Then

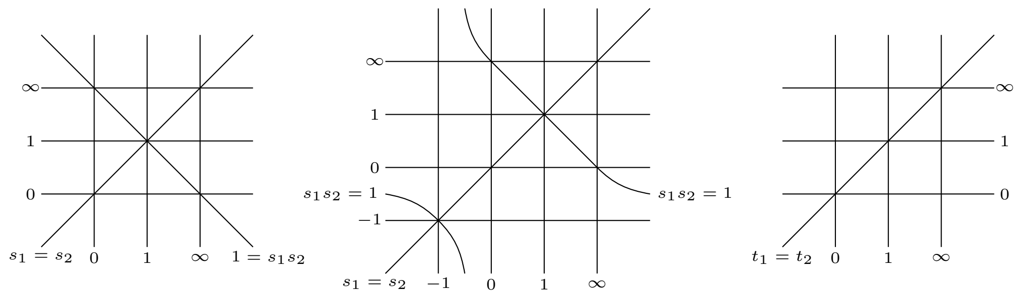
$$f_{\mathrm{cov}} : V_{\mathrm{B}_3} \to \mathcal{M}_{0,5}, \quad (s_1, s_2) \mapsto \left(\left(\frac{1-s_1}{1+s_1}\right)^2, \left(\frac{1-s_2}{1+s_2}\right)^2 \right) = (t_1, t_2) \tag{7.2.2}$$

is a finite étale Galois covering space with Galois group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We remark that a covering space isomorphic to f_{cov} is found in [M97, 3.6.2]. There is a natural open immersion

$$V_{\mathrm{B}_3} \hookrightarrow V_{\text{non-Fano}}. \tag{7.2.3}$$

Through this open immersion, we regard points and (étale) paths on V_{B_3} as those on $V_{\text{non-Fano}}$.

Figure 7.4: Line arrangements appearing in the present paper



non-Fano arragement | B_3 Coxeter arragement | Braid arragement

Remark 7.2.1. The spaces $V^{\mathrm{an}}_{\text{non-Fano}}$, $V^{\mathrm{an}}_{\mathrm{B}_3}$, and $\mathcal{M}_{0,5}(\mathbb{C})$ derive from complex arrangements called Braid arragement, B_3 arragement, and non-Fano arragement, respectively. To be precise, the decones of what are called Braid arrangement, B_3 arrangement, and non-Fano arrangement in [Su01] with certain changes of variables are called Braid arrangement, B_3 arrangement, and non-Fano arrangement in this paper.

We also fix the tangential base points of these spaces. We shall write $K((t))$ for the Laurent power series field over K. Let

$$\vec{v} : \mathrm{Spec}\Big(K((t))\Big) \to V_{\mathrm{B}_3} \tag{7.2.4}$$

be the K-rational tangential base point of V_{B_3} over the $K(t)$-rational point (t^2, t). As the composite of $\vec{v} : \mathrm{Spec}\Big(K((t))\Big) \to V_{\mathrm{B}_3}$ and the open immersion $V_{\mathrm{B}_3} \hookrightarrow V_{\text{non-Fano}}$, we have the K-rational tangential base point of $V_{\text{non-Fano}}$, which is also denoted as $\vec{v}$. The image of $\vec{v} : \mathrm{Spec}\Big(K((t))\Big) \to V_{\mathrm{B}_3} \hookrightarrow V_{\text{non-Fano}}$ under $\mathrm{pr}_2 (= f_9) : V_{\text{non-Fano}} \to \mathbb{P}^1_K \backslash \{0, 1, \infty\}, (s_1, s_2) \mapsto s_2$ is the standard K-rational tangential base point $\overrightarrow{01} : \mathrm{Spec}\Big(K((t))\Big) \to \mathbb{P}^1_K \backslash \{0, 1, \infty\}$. We set

$$(\tau_1, \tau_2) := f_{\mathrm{cov}}(t^2, t) = \left(\left(\frac{1-t^2}{1+t^2}\right)^2, \left(\frac{1-t}{1+t}\right)^2 \right) \in \mathcal{M}_{0,5} \tag{7.2.5}$$

and write

$$\vec{\tau} : \mathrm{Spec}\left(K((t))\right) \to \mathcal{M}_{0,5} \tag{7.2.6}$$

for the K-rational tangential base point of $\mathcal{M}_{0,5}$ over the $K(t)$-rational point (τ_1, τ_2).

$$\begin{array}{ccccccc}
 & & & \mathrm{Spec}\left(K((t))\right) & & & \\
 & \swarrow^{\overrightarrow{01}} & \swarrow^{\vec{v}} & \downarrow^{\vec{v}} & \searrow^{\vec{\tau}} & & \\
\left(\mathbb{P}^1_{\overline{K}} \setminus \{0,1,\infty\}, t\right) & \xleftarrow[\mathrm{pr}_2]{} & (V_{\text{non-Fano}}, (t^2, t)) & \hookrightarrow & (V_{\mathrm{B}_3}, (t^2, t)) & \xrightarrow[f_{\mathrm{cov}}]{} & (\mathcal{M}_{0,5}, (\tau_1, \tau_2)).
\end{array}$$

When discussing the complex case, we regard these tangential base points of the algebraic varieties as those of associated complex analytic spaces by the embedding $\overline{K} \hookrightarrow \mathbb{C}$.

Next we compute the topological fundamental groups of $V_{\mathrm{B}_3}^{\mathrm{an}}$ and $V_{\text{non-Fano}}^{\mathrm{an}}$ by using the Galois theory of $\mathcal{M}_{0,5}(\mathbb{C})$. First, by the Galois theory for covering spaces, the pointed finite Galois covering space $f_{\mathrm{cov}}^{\mathrm{an}} : \left(V_{\mathrm{B}_3}^{\mathrm{an}}, \vec{v}\right) \to (\mathcal{M}_{0,5}(\mathbb{C}), \vec{\tau})$ in (7.2.2) corresponds to the normal subgroup with index 4

$$f_{\mathrm{cov}*}^{\mathrm{an}}\left(\pi_1^{\mathrm{top}}\left(V_{\mathrm{B}_3}^{\mathrm{an}}, \vec{v}\right)\right) \subset \pi_1^{\mathrm{top}}\left(\mathcal{M}_{0,5}(\mathbb{C}), \vec{\tau}\right),$$

where $f_{\mathrm{cov}*}^{\mathrm{an}} : \pi_1^{\mathrm{top}}\left(V_{\mathrm{B}_3}^{\mathrm{an}}, \vec{v}\right) \hookrightarrow \pi_1^{\mathrm{top}}\left(\mathcal{M}_{0,5}(\mathbb{C}), \vec{\tau}\right)$ is the homomorphism induced by $f_{\mathrm{cov}}^{\mathrm{an}}$. We set

$$B_1, ..., B_{10} \in \pi_1^{\mathrm{top}}\left(\mathcal{M}_{0,5}(\mathbb{C}), \vec{\tau}\right)$$

as in TABLE 7.3. For each $i = 1, ..., 10$, one can see that the inverse image of B_i under $f_{\mathrm{cov}}^{\mathrm{an}}$ is also a closed path. Thus, it holds that $B_1, ..., B_{10} \in f_{\mathrm{cov}*}^{\mathrm{an}}\left(\pi_1^{\mathrm{top}}\left(V_{\mathrm{B}_3}^{\mathrm{an}}, \vec{v}\right)\right)$. Hereafter, we identify closed paths of $(\mathcal{M}_{0,5}(\mathbb{C}), \vec{\tau})$ contained in $f_{\mathrm{cov}*}^{\mathrm{an}}\left(\pi_1^{\mathrm{top}}\left(V_{\mathrm{B}_3}^{\mathrm{an}}, \vec{v}\right)\right)$ with those of $\left(V_{\mathrm{B}_3}^{\mathrm{an}}, \vec{v}\right)$ by taking the inverse image under $f_{\mathrm{cov}}^{\mathrm{an}}$. Then $B_1, ..., B_{10} \in \pi_1^{\mathrm{top}}\left(V_{\mathrm{B}_3}^{\mathrm{an}}, \vec{v}\right)$ are meridians of divisors on $V_{\mathrm{B}_3}^{\mathrm{an}}$ as TABLE 8.2.

Table 7.3: $B_1, \ldots, B_{10}$

i	1	2	3	4	5	6	7	8	9	10
B_i	A_{12}	A_{13}	A_{23}	A_{24}^2	A_{34}^2	$A_{34}A_{23}A_{34}^{-1}$	$A_{24}A_{12}A_{24}^{-1}$	$A_{34}A_{13}A_{34}^{-1}$	A_{25}^2	A_{35}^2

Table 7.4: Correspondence between each divisor on $V_{\mathrm{B}_3}^{\mathrm{an}}$ and its meridian B_i

divisor	$s_1 = 0$	$s_2 = 0$	$s_1 = s_2$	$s_1 = 1$	$s_2 = 1$	$s_1 s_2 = 1$	$s_1 = \infty$	$s_2 = \infty$	$s_0 = -1$	$s_1 = -1$
meridian	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}

Moreover, one can check that the subgroup of $\pi_1^{\mathrm{top}}\left(\mathcal{M}_{0,5}(\mathbb{C}), \vec{\tau}\right)$ generated by $B_1, ..., B_{10}$ is a normal subgroup with index 4. Therefore $\pi_1^{\mathrm{top}}\left(V_{\mathrm{B}_3}^{\mathrm{an}}, \vec{v}\right)$ is generated by meridians $B_1, ..., B_{10}$.

We obtain the following presentation of $\pi_1^{\text{top}}\left(V_{\mathrm{B}_3}^{\text{an}}, \vec{v}\right)$ by the Reidemeister-Schreier Rewriting Process:

$$\pi_1^{\text{top}}\left(V_{\mathrm{B}_3}^{\text{an}}, \vec{v}\right) = \left\langle \begin{array}{c} B_1, B_2, B_3, B_4, B_5, \\ B_6, B_7, B_8, B_9, B_{10} \end{array} \middle| \; (R'1) \sim (R'12) \right\rangle \tag{7.2.7}$$

$$= \left\langle \begin{array}{c} B_1, B_2, B_3, B_4, \\ B_5, B_6, B_7, B_8 \end{array} \middle| \; (R'1) \sim (R'10) \right\rangle.$$

The relations $(R'1) \sim (R'12)$ are as follows:

$(R'1)$ $B_1B_5 = B_5B_1, \quad B_4B_8 = B_8B_4,$

$(R'2)$ $B_1B_2B_3 = B_2B_3B_1 = B_3B_1B_2, \quad B_1B_8B_6 = B_8B_6B_1 = B_6B_1B_8,$

$(R'3)$ $B_5B_3B_4B_6 = B_6B_5B_3B_4, \quad B_3B_4B_1B_2 = B_2B_3B_4B_1,$

$(R'4)$ $B_3B_7B_3^{-1} = B_5B_3B_7B_3^{-1}B_5^{-1},$

$B_6B_5B_6^{-1} = B_4^{-1}B_3^{-1}B_5B_3B_4,$

$B_7B_3B_7^{-1} = B_3^{-1}B_8^{-1}B_3B_8B_3,$

$(R'5)$ $B_4B_6B_5B_2B_3 = B_6B_5B_2B_3B_4,$

$(R'6)$ $B_7B_8B_4B_6B_5B_3 = B_8B_3B_7B_4B_6B_5, \quad B_6B_5B_2B_3B_7B_4 = B_5B_2B_3B_7B_4B_6,$

$(R'7)$ $B_3B_7B_3^{-1} = B_6B_5B_2B_3B_7B_3^{-1}B_2^{-1}B_5^{-1}B_6^{-1},$

$B_6B_2B_6^{-1} = B_3B_7^{-1}B_3^{-1}B_8^{-1}B_2B_8B_3B_7B_3^{-1},$

$B_5^{-1}B_8B_5 = B_7^{-1}B_3^{-1}B_2^{-1}B_5^{-1}B_8B_5B_2B_3B_7,$

$(R'8)$ $B_5B_8B_5B_2B_3B_7B_4B_6B_1 = B_3B_7B_4B_6B_1B_5B_8B_5B_2,$

$(R'9)$ $B_1^{-1}B_7B_4B_1B_2^{-1}B_5^{-1}B_6^{-1}B_8^{-1}B_1^{-1}B_4^{-1}B_7^{-1}B_4B_6B_5B_4^{-1}B_1B_2B_7B_5^{-1}B_8B_5B_7^{-1} = 1,$

$B_4B_3^{-1}B_2^{-1}B_5^{-1}B_8^{-1}B_2B_8B_5B_2B_3B_6^{-1}B_4^{-1}B_7^{-1}B_3^{-1}B_2^{-1}B_5^{-1}B_2^{-1}B_6B_5B_2B_3B_7 = 1,$

$(R'10)$ $B_4B_2^{-1}B_5^{-1}B_1^{-1}B_6^{-1}B_4^{-1}B_7^{-1}B_3^{-1}B_8B_5B_2B_3B_7B_4B_6B_1B_4^{-1}B_7^{-1}B_3^{-1}B_5^{-1}B_8^{-1}B_5B_3B_7 = 1,$

$(R'11)$ $B_1B_3B_7B_4B_6B_9 = 1,$

$(R'12)$ $B_2B_3B_8B_6B_5B_{10} = 1.$

The kernel of $\pi_1^{\text{top}}\left(V_{\mathrm{B}_3}^{\text{an}}, \vec{v}\right) \twoheadrightarrow \pi_1^{\text{top}}\left(V_{\text{non-Fano}}^{\text{an}}, \vec{v}\right)$ induced by the inclusion $V_{\mathrm{B}_3}^{\text{an}} \hookrightarrow V_{\text{non-Fano}}^{\text{an}}$ is the free group of rank 2 generated by B_9, B_{10}, which are meridians of $s_0 = -1$, $s_1 = -1$, respectively.

$$1 \to \langle B_9, B_{10} \rangle \to \pi_1^{\text{top}}\left(V_{\mathrm{B}_3}^{\text{an}}, \vec{v}\right) \to \pi_1^{\text{top}}\left(V_{\text{non-Fano}}^{\text{an}}, \vec{v}\right) \to 1.$$

Hence, $\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})$ has a presentation as follows:

$$\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v}) = \left\langle \begin{array}{c} B_1, B_2, B_3, B_4, \\ B_5, B_6, B_7, B_8 \end{array} \middle| \; (R''1) \sim (R''12) \right\rangle \tag{7.2.8}$$
$$= \left\langle \begin{array}{c} B_1, B_2, B_3, \\ B_4, B_5, B_6 \end{array} \middle| \; (R''1) \sim (R''10) \right\rangle .$$

For $i = 1, 2, \cdots, 10$, we set $(R''i) := (R'i)$. The relations $(R''11)$, $(R''12)$ are as follows:

$$(R''11)\ B_1B_3B_7B_4B_6 = 1,$$
$$(R''12)\ B_2B_3B_8B_6B_5 = 1.$$

To obtain a more detailed structure of $\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})$, we consider the following diagram:

$$\begin{array}{ccccc}
V^{\mathrm{an}}_{\text{non-Fano}} & \xrightarrow{f_8^{\mathrm{an}}\,(=\mathrm{pr}_1)} & \mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\} & \ni & t \\
\uparrow & \circlearrowleft & \uparrow & & \updownarrow \\
V^{\mathrm{an}}_{\mathrm{B}_3} & \xrightarrow[\mathrm{pr}_1]{} & \mathbb{P}^1(\mathbb{C})\backslash\{0,1,-1,\infty\} & \ni & t \\
{\scriptstyle f^{\mathrm{an}}_{\mathrm{cov}}}\downarrow & \circlearrowleft & \downarrow & & \downarrow \\
\mathcal{M}_{0,5}(\mathbb{C}) & \xrightarrow[\mathrm{pr}_1]{} & \mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\} & \ni & \left(\dfrac{1-t}{1+t}\right)^2 .
\end{array}$$

Note that pr_1 is the projection to the first component. The image of $\vec{v}$ under the middle pr_1 is Galois equivalent to $\overrightarrow{01}$. Therefore, we obtain the following diagram by taking π_1^{top}:

$$\begin{array}{ccccccccc}
1 & \to & \langle B_2, B_3, B_5, B_6 \rangle & \hookrightarrow & \pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v}) & \underset{s}{\overset{f^{\mathrm{an}}_{8*}}{\rightleftarrows}} & \pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}, \overrightarrow{01}) & \to & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \to & \langle B_2, B_3, B_5, B_6, B_8 \rangle & \hookrightarrow & \pi_1^{\mathrm{top}}\left(V^{\mathrm{an}}_{\mathrm{B}_3}, \vec{v}\right) & \xrightarrow[\mathrm{pr}_{1*}]{} & \pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,-1,\infty\}, \overrightarrow{01}) & \to & 1 \\
 & & \downarrow & & {\scriptstyle f^{\mathrm{an}}_{\mathrm{cov}*}}\downarrow & & \downarrow & & \\
1 & \to & \langle A_{13}, A_{23}, A_{34} \rangle & \hookrightarrow & P_4/\langle \omega_4 \rangle & \xrightarrow[\mathrm{pr}_{1*}]{} & \pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}, \tau_1) & \to & 1
\end{array}$$

We take a group-theoretic section of f_{8*}

$$s : \pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}, \overrightarrow{01}) \to \pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})$$

defined by $l_0 \mapsto B_1,\ l_1 \mapsto B_4$. Then we have

$$\begin{aligned}\pi_1^{\mathrm{top}}\left(V_{\text{non-Fano}}^{\mathrm{an}}, \vec{v}\right) &= \langle B_1, B_4\rangle \ltimes \langle B_2, B_3, B_5, B_6\rangle \\ &= F_2 \ltimes F_4.\end{aligned} \tag{7.2.9}$$

One can see that the conjugation action $F_2^{\mathrm{ab}} \to \mathrm{Aut}\left(F_4^{\mathrm{ab}}\right)$ is trivial by using $(R''1) \sim (R''20)$. Hence we get

$$\begin{aligned}\pi_1^{\mathrm{top}}\left(V_{\text{non-Fano}}^{\mathrm{an}}, \vec{v}\right)^{\mathrm{ab}} &= \left\langle \bar{B}_1, \bar{B}_4\right\rangle \times \left\langle \bar{B}_2, \bar{B}_3, \bar{B}_5, \bar{B}_6\right\rangle \\ &= F_2^{\mathrm{ab}} \times F_4^{\mathrm{ab}}\end{aligned} \tag{7.2.10}$$

where we denote by $\bar{g}$ the image of a element $g \in \pi$ in the abelianization π^{ab} for a group π.

Remark 7.2.2. In [Su01], the topological fundamental groups of $V_{\text{non-Fano}}^{\mathrm{an}}$ and $V_{\mathrm{B}_3}^{\mathrm{an}}$ are computed by the braid monodromy of the arrangement. However, in the present paper, we compute it by using the Galois theory of $\mathcal{M}_{0,5}$. The form (7.2.8) of the topological fundamental group obtained in this paper has the advantage that it is easier to compute the image of B_j under f_{i*}^{an} (See TABLE 7.5).

7.3 Proof of main results

In this section, we prove the main results, Theorem 7.1.1 and Theorem 7.1.2. Throughout this section, we fix a path

$$\gamma_0 \in \pi_1^{\mathrm{top}}\left(V_{\text{non-Fano}}^{\mathrm{an}}; \vec{v}, (x,y)\right).$$

For $i = 1, ..., 9$, we set $\delta_i \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, f_i^{\mathrm{an}}\left(\vec{v}\right)\right)$ and

$$\gamma_i \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, f_i^{\mathrm{an}}(x,y)\right)$$

as in TABLE 7.1 and (7.1.4). For a (topological) group G (resp. Lie algebra L), we denote by $\{\Gamma^k\pi\}_k$ (resp. $\{\Gamma^k L\}_k$) the lower central series of π (resp. L) with $\Gamma^1\pi = \pi$ (resp. $\Gamma^1 L = L$). We set

$$\mathrm{gr}_\Gamma^k \pi := \Gamma^k\pi/\Gamma^{k+1}\pi \text{ (resp. } \mathrm{gr}_\Gamma^k L := \Gamma^k L/\Gamma^{k+1}L).$$

The commutator bracket $(g, g') := gg'g^{-1}g'^{-1}$ $(g, g' \in \pi)$ of π then induces the Lie bracket

$$[\alpha, \beta] := (\alpha, \beta) \bmod \Gamma^{n+m+1}\pi \ \in \mathrm{gr}_\Gamma^{n+m}\pi \quad (\alpha \in \mathrm{gr}_\Gamma^n\pi,\ \beta \in \mathrm{gr}_\Gamma^m\pi)$$

on the graded sum $\oplus_{k=1}^{\infty}\mathrm{gr}_\Gamma^k\pi$.

7.3.1 Complex case

In this subsection, we derive Spence-Kummer's trilogarithm functional equation. For this purpose, we also derive Schaeffer's, Kummer's, and Hill's dilogarithm functional equations. These functional equations are derived by using Nakamura-Wojtkowiak's tensor-homotopy criteria [NW12, Theorem 5.7, Proposition 5.11]. We fix a $\mathbb{C}$-rational point (x, y) of $V_{\text{non-Fano}}$.

First, we make the preparations for checking the tensor-homotopy criteria [NW12, Theorem 5.7, (i)$_{\mathbb{C}}$, (ii)$_{\mathbb{C}}$]. For $i = 1, ..., 9$, we write

$$f^{\mathrm{an}}_{i*} : \pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v}) \to \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}, f^{\mathrm{an}}_i(\vec{v})\right)$$

for the homomorphism induced by f^{an}_i and denote by

$$\iota^{\mathrm{an}}_{\delta_i} : \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}, f^{\mathrm{an}}_i(\vec{v})\right) \xrightarrow{\simeq} \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}, \overrightarrow{01}\right)$$

the change-of-basepoint homomorphism defined by $p \mapsto \delta_i \cdot p \cdot \delta_i{}^{-1}$. The image of B_j $(j = 1, ..., 8)$ in TABLE 7.3 under $\iota^{\mathrm{an}}_{\delta_i} \circ f^{\mathrm{an}}_{i*}$ are calculated as in TABLE 7.5.

Table 7.5: $\iota^{\mathrm{an}}_{\delta_i}(f^{\mathrm{an}}_i(B_j))$

#	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8
$\iota^{\mathrm{an}}_{\delta_1}(f^{\mathrm{an}}_{1*}(\#))$	l_0	$l_0^{-1}\cdot l_1^{-1}$	l_1	l_∞^2	l_0^2	$l_0\cdot l_1\cdot l_0^{-1}$	$l_1^{-1}\cdot l_0\cdot l_1$	l_∞
$\iota^{\mathrm{an}}_{\delta_2}(f^{\mathrm{an}}_{2*}(\#))$	l_0	l_0	1	1	1	l_1	$l_0^{-1}\cdot l_1^{-1}$	$l_0^{-1}\cdot l_1^{-1}$
$\iota^{\mathrm{an}}_{\delta_3}(f^{\mathrm{an}}_{3*}(\#))$	l_0	$l_0^{-1}\cdot l_1^{-1}$	l_1	1	1	1	l_∞	l_0
$\iota^{\mathrm{an}}_{\delta_4}(f^{\mathrm{an}}_{4*}(\#))$	l_0	$l_0^{-1}\cdot l_1^{-1}$	l_1	l_∞	l_0	1	1	1
$\iota^{\mathrm{an}}_{\delta_5}(f^{\mathrm{an}}_{5*}(\#))$	l_0	1	1	l_∞	l_0	$l_0\cdot l_1\cdot l_0^{-1}$	1	l_∞
$\iota^{\mathrm{an}}_{\delta_6}(f^{\mathrm{an}}_{6*}(\#))$	1	1	l_1	l_∞	l_0	1	$l_1^{-1}\cdot l_0\cdot l_1$	l_∞
$\iota^{\mathrm{an}}_{\delta_7}(f^{\mathrm{an}}_{7*}(\#))$	1	l_∞	1	l_∞	l_0	$l_0\cdot l_1\cdot l_0^{-1}$	l_0	1
$\iota^{\mathrm{an}}_{\delta_8}(f^{\mathrm{an}}_{8*}(\#))$	l_0	1	1	l_1	1	1	$l_0^{-1}\cdot l_1^{-1}$	1
$\iota^{\mathrm{an}}_{\delta_9}(f^{\mathrm{an}}_{9*}(\#))$	1	l_0	1	1	l_1	1	1	$l_0^{-1}\cdot l_1^{-1}$

We focus on (7.2.10):

$$\begin{aligned}\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})^{\mathrm{ab}} &= \left\langle \bar{B}_1, \bar{B}_4 \right\rangle \times \left\langle \bar{B}_2, \bar{B}_3, \bar{B}_5, \bar{B}_6 \right\rangle \\ &= F_2^{\mathrm{ab}} \times F_4^{\mathrm{ab}}.\end{aligned}$$

Since $\mathrm{rank}_{\mathbb{Z}}(\mathrm{gr}^2_\Gamma F_2) = 1$ and $\mathrm{rank}_{\mathbb{Z}}(\mathrm{gr}^2_\Gamma F_4) = 6$ by Witt's formula,

$$\mathrm{gr}^2_\Gamma \pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v}) = \mathrm{gr}^2_\Gamma F_2 \times \mathrm{gr}^2_\Gamma F_4$$

has a free generating system

$$[\bar{B}_1, \bar{B}_4], \quad [\bar{B}_2, \bar{B}_3], \quad [\bar{B}_2, \bar{B}_5], \quad [\bar{B}_2, \bar{B}_6], \quad [\bar{B}_3, \bar{B}_5], \quad [\bar{B}_3, \bar{B}_6], \quad [\bar{B}_5, \bar{B}_6] \tag{7.3.1}$$

as a finitely generated free $\mathbb{Z}_\ell$-module. Since $\mathrm{rank}_{\mathbb{Z}}(\mathrm{gr}^3_\Gamma F_2) = 2$ and $\mathrm{rank}_{\mathbb{Z}}(\mathrm{gr}^3_\Gamma F_4) = 20$,

$$\mathrm{gr}^3_\Gamma \pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v}) = \mathrm{gr}^3_\Gamma F_2 \times \mathrm{gr}^3_\Gamma F_4$$

has a free generating system

$$
\begin{gathered}
\left[\bar{B}_1,\left[\bar{B}_1,\bar{B}_4\right]\right],\quad \left[\bar{B}_4,\left[\bar{B}_1,\bar{B}_4\right]\right],\quad \left[\bar{B}_2,\left[\bar{B}_2,\bar{B}_3\right]\right],\quad \left[\bar{B}_3,\left[\bar{B}_2,\bar{B}_3\right]\right],\quad \left[\bar{B}_2,\left[\bar{B}_2,\bar{B}_5\right]\right],\\
\left[\bar{B}_5,\left[\bar{B}_2,\bar{B}_5\right]\right],\quad \left[\bar{B}_2,\left[\bar{B}_2,\bar{B}_6\right]\right],\quad \left[\bar{B}_6,\left[\bar{B}_2,\bar{B}_6\right]\right],\quad \left[\bar{B}_3,\left[\bar{B}_3,\bar{B}_5\right]\right],\quad \left[\bar{B}_5,\left[\bar{B}_3,\bar{B}_5\right]\right],\\
\left[\bar{B}_3,\left[\bar{B}_3,\bar{B}_6\right]\right],\quad \left[\bar{B}_6,\left[\bar{B}_3,\bar{B}_6\right]\right],\quad \left[\bar{B}_5,\left[\bar{B}_5,\bar{B}_6\right]\right],\quad \left[\bar{B}_6,\left[\bar{B}_5,\bar{B}_6\right]\right],\quad \left[\bar{B}_2,\left[\bar{B}_3,\bar{B}_5\right]\right],\\
\left[\bar{B}_3,\left[\bar{B}_5,\bar{B}_2\right]\right],\quad \left[\bar{B}_2,\left[\bar{B}_3,\bar{B}_6\right]\right],\quad \left[\bar{B}_3,\left[\bar{B}_6,\bar{B}_2\right]\right],\quad \left[\bar{B}_2,\left[\bar{B}_5,\bar{B}_6\right]\right],\quad \left[\bar{B}_5,\left[\bar{B}_6,\bar{B}_2\right]\right],\\
\left[\bar{B}_3,\left[\bar{B}_5,\bar{B}_6\right]\right],\quad \left[\bar{B}_5,\left[\bar{B}_6,\bar{B}_3\right]\right].
\end{gathered}
$$

Moreover, for $i = 1, ..., 9$, by considering the composite $V^{\mathrm{an}}_{\text{non-Fano}} \xrightarrow{f_i^{\mathrm{an}}} \mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\} \hookrightarrow \mathbb{G}_m(\mathbb{C})$, we may regard f_i^{an} as an element of the unit group

$$f_i^{\mathrm{an}} \in (\mathcal{O}^{\mathrm{an}})^{\times}$$

where $\mathcal{O}^{\mathrm{an}}$ is the coordinate ring of $V^{\mathrm{an}}_{\text{non-Fano}}$:

$$\mathcal{O}^{\mathrm{an}} = \mathbb{C}\left[s_1, s_2, \frac{1}{s_1s_2(1-s_1)(1-s_2)(s_1-s_2)(1-s_1s_2)}\right].$$

Then $f_i^{\mathrm{an}} - 1 \in (\mathcal{O}^{\mathrm{an}})^{\times}$ holds.

Next, we make the preparations for calculating the functional equations of complex iterated integrals [NW12, Theorem 5.7, $(\mathrm{iii})_{\mathbb{C}}$]. By (3.1.11) and (3.1.12), for $i = 1, ..., 9$, we have

$$
\begin{cases}
\mathrm{li}_0\left(f_i^{\mathrm{an}}(x,y);\gamma_i\right) = & -\dfrac{1}{2\pi i}\log\left(f_i^{\mathrm{an}}(x,y);\gamma_i\right),\\
\mathrm{li}_1\left(f_i^{\mathrm{an}}(x,y);\gamma_i\right) = & -\dfrac{1}{2\pi i}\log\left(1-f_i^{\mathrm{an}}(x,y);\gamma_i'\right),\\
\mathrm{li}_2\left(f_i^{\mathrm{an}}(x,y);\gamma_i\right) = & \dfrac{1}{4\pi^2}\left(Li_2\left(f_i^{\mathrm{an}}(x,y),\gamma_i\right)+\dfrac{1}{2}\log\left(f_i^{\mathrm{an}}(x,y);\gamma_i\right)\log\left(1-f_i^{\mathrm{an}}(x,y);\gamma_i'\right)\right),\\
\mathrm{li}_3\left(f_i^{\mathrm{an}}(x,y);\gamma_i\right) = & \dfrac{-1}{8\pi^3 i}\left(Li_3\left(f_i^{\mathrm{an}}(x,y),\gamma_i\right)-\dfrac{1}{2}\log\left(f_i^{\mathrm{an}}(x,y);\gamma_i\right)Li_2\left(f_i^{\mathrm{an}}(x,y),\gamma_i\right)\right.\\
& \left.-\dfrac{1}{12}\log^2\left(f_i^{\mathrm{an}}(x,y);\gamma_i\right)\log\left(1-f_i^{\mathrm{an}}(x,y);\gamma_i'\right)\right),
\end{cases}
\tag{7.3.2}
$$

where $\gamma_i' \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, f_i^{\mathrm{an}}(x,y)\right)$ is the path associated to γ_i as in (6.1.1). Furthermore, by the calculation of Drinfeld associators (cf. [NW12, §5.4]), we have

$$
\begin{cases}
\left(-\mathrm{li}_j\left(\overrightarrow{01};\delta_i\right)\right)_{0\le j\le 3} = (0,0,0,0)\quad (i=1,2,3,4,8,9),\\
\left(-\mathrm{li}_j\left(\overrightarrow{0\infty};\delta_5\right)\right)_{0\le j\le 3} = \left(\dfrac{1}{2},0,0,0\right),\\
\left(-\mathrm{li}_j\left(\overrightarrow{10};\delta_6\right)\right)_{0\le j\le 3} = \left(0,0,-\mathrm{li}_2\left(\overrightarrow{10};\delta_6\right),-\mathrm{li}_3\left(\overrightarrow{10};\delta_6\right)\right) = \left(0,0,-\dfrac{1}{24},\dfrac{1}{8\pi^3 i}Li_3(1)\right),\\
\left(-\mathrm{li}_j\left(\overrightarrow{\infty 0},\delta_7\right)\right)_{0\le j\le 3} = \left(\dfrac{1}{2},0,\dfrac{1}{24},0\right),
\end{cases}
\tag{7.3.3}
$$

Here, we consider an associator (i.e. generating function of iterated integrals)

$$\Lambda_{f_i^{\mathrm{an}}(\vec{v})}\left(f_i^{\mathrm{an}}(x,y); f_i^{\mathrm{an}}(\gamma_0)\right) := 1 + \sum_{n=1}^{\infty} \int_{f_i^{\mathrm{an}}(\gamma_0)} \underbrace{\omega \ldots \omega}_{n \text{ times}}$$

where $\omega = \frac{dz}{z}X + \frac{dz}{z-1}Y$. Then, by the path composition $\gamma_i = \delta_i \cdot f_i^{\mathrm{an}}(\gamma_0)$, we obtain an algebraic relation

$$\Lambda_{f_i^{\mathrm{an}}(\vec{v})}\left(f_i^{\mathrm{an}}(x,y); f_i^{\mathrm{an}}(\gamma_0)\right) = \Lambda_{\overrightarrow{01}}\left(f_i^{\mathrm{an}}(\vec{v}); \delta_i\right) \cdot \Lambda_{\overrightarrow{01}}\left(f_i^{\mathrm{an}}(x,y); \gamma_i\right) \tag{7.3.4}$$

in $\mathbb{C}\langle\langle X, Y\rangle\rangle$. By (7.3.4), (7.3.3) and the polylog-BCH formula [NW12, Proposition 5.9], we compute Wojtkowiak's complex iterated integrals ([NW12, Definition 4.4])

$$\mathcal{L}_{\mathbb{C}}^{\varphi_k}\left(f_i^{\mathrm{an}}(x,y); f_i^{\mathrm{an}}(\vec{v}), f_i^{\mathrm{an}}(\gamma_0)\right) := \varphi_k\left(\mathbf{log}\left(\Lambda_{f_i^{\mathrm{an}}(\vec{v})}\left(f_i^{\mathrm{an}}(x,y); f_i^{\mathrm{an}}(\gamma_0)\right)^{-1}\right)\right)$$

associated to $f_i^{\mathrm{an}}(\gamma_0)$ as follows:

$$\begin{cases} \mathcal{L}_{\mathbb{C}}^{\varphi_2}\left(f_i^{\mathrm{an}}(x,y); f_i^{\mathrm{an}}(\vec{v}), f_i^{\mathrm{an}}(\gamma_0)\right) = & \mathrm{li}_2\left(f_i^{\mathrm{an}}(x,y); \gamma_i\right) \qquad (i = 1,2,3,4,8,9), \\ \mathcal{L}_{\mathbb{C}}^{\varphi_2}\left(f_5^{\mathrm{an}}(x,y); f_5^{\mathrm{an}}(\vec{v}), f_5^{\mathrm{an}}(\gamma_0)\right) = & \mathrm{li}_2(f_5^{\mathrm{an}}(x,y); \gamma_5) + \frac{1}{2}\mathrm{li}_0\left(\overrightarrow{0\infty}; \delta_5\right)\mathrm{li}_1(z_5; \gamma_5), \\ \mathcal{L}_{\mathbb{C}}^{\varphi_2}\left(f_6^{\mathrm{an}}(x,y); f_i^{\mathrm{an}}(\vec{v}), f_6^{\mathrm{an}}(\gamma_0)\right) = & \mathrm{li}_2(f_6^{\mathrm{an}}(x,y); \gamma_6) - \mathrm{li}_2\left(\overrightarrow{10}; \delta_6\right), \\ \mathcal{L}_{\mathbb{C}}^{\varphi_2}\left(f_7^{\mathrm{an}}(x,y); f_7^{\mathrm{an}}(\vec{v}), f_7^{\mathrm{an}}(\gamma_0)\right) = & \mathrm{li}_2(f_7^{\mathrm{an}}(x,y); \gamma_7) - \mathrm{li}_2\left(\overrightarrow{10}; \delta_7\right) \\ & -\frac{1}{2}\mathrm{li}_0(f_7^{\mathrm{an}}(x,y); \gamma_7)\mathrm{li}_1\left(\overrightarrow{10}; \delta_7\right) \\ & +\frac{1}{2}\mathrm{li}_1(f_7^{\mathrm{an}}(x,y); \gamma_7)\mathrm{li}_0\left(\overrightarrow{10}; \delta_7\right), \end{cases} \tag{7.3.5}$$

$$\begin{cases} \mathcal{L}_{\mathbb{C}}^{\varphi_3}\left(f_i^{\mathrm{an}}(x,y); f_i^{\mathrm{an}}(\vec{v}), f_i^{\mathrm{an}}(\gamma_0)\right) = & \mathrm{li}_3\left(f_i^{\mathrm{an}}(x,y); \gamma_i\right) \qquad (i = 1,2,3,4,8,9), \\ \mathcal{L}_{\mathbb{C}}^{\varphi_3}\left(f_5^{\mathrm{an}}(x,y); f_5^{\mathrm{an}}(\vec{v}), f_5^{\mathrm{an}}(\gamma_0)\right) = & \mathrm{li}_3\left(f_5^{\mathrm{an}}(x,y); \gamma_5\right) + \frac{1}{2}\mathrm{li}_0\left(\overrightarrow{0\infty}; \delta_5\right)\mathrm{li}_2\left(f_5^{\mathrm{an}}(x,y); \gamma_5\right) \\ & +\frac{1}{12}\mathrm{li}_0\left(\overrightarrow{0\infty}; \delta_5\right)\mathrm{li}_0\left(f_5^{\mathrm{an}}(x,y); \gamma_5\right)\mathrm{li}_1\left(f_5^{\mathrm{an}}(x,y); \gamma_5\right) \\ & +\frac{1}{12}\left(\mathrm{li}_0\left(\overrightarrow{0\infty}; \delta_5\right)\right)^2\mathrm{li}_1\left(f_5^{\mathrm{an}}(x,y); \gamma_5\right), \\ \mathcal{L}_{\mathbb{C}}^{\varphi_3}\left(f_6^{\mathrm{an}}(x,y); f_i^{\mathrm{an}}(\vec{v}), f_6^{\mathrm{an}}(\gamma_0)\right) = & \mathrm{li}_3\left(f_6^{\mathrm{an}}(x,y); \gamma_6\right) - \mathrm{li}_3\left(\overrightarrow{10}; \delta_6\right) \\ & -\frac{1}{2}\mathrm{li}_0\left(f_6^{\mathrm{an}}(x,y); \gamma_6\right)\mathrm{li}_2\left(\overrightarrow{10}; \delta_6\right) \\ \mathcal{L}_{\mathbb{C}}^{\varphi_3}\left(f_7^{\mathrm{an}}(x,y); f_7^{\mathrm{an}}(\vec{v}), f_7^{\mathrm{an}}(\gamma_0)\right) = & \mathrm{li}_3\left(f_7^{\mathrm{an}}(x,y); \gamma_7\right) - \mathrm{li}_3\left(\overrightarrow{10}; \delta_7\right) \\ & -\frac{1}{2}\mathrm{li}_0\left(f_7^{\mathrm{an}}(x,y); \gamma_7\right)\mathrm{li}_2\left(\overrightarrow{10}; \delta_7\right) \\ & +\frac{1}{2}\mathrm{li}_2\left(f_7^{\mathrm{an}}(x,y); \gamma_7\right)\mathrm{li}_0\left(\overrightarrow{10}; \delta_7\right) \\ & +\frac{1}{12}\left(\mathrm{li}_0\left(\overrightarrow{\infty 0}; \delta_7\right)\right)^2\mathrm{li}_1\left(f_7^{\mathrm{an}}(x,y); \gamma_7\right) \\ & +\frac{1}{12}\mathrm{li}_0\left(\overrightarrow{\infty 0}; \delta_7\right)\mathrm{li}_0\left(f_7^{\mathrm{an}}(x,y); \gamma_7\right)\mathrm{li}_1\left(f_7^{\mathrm{an}}(x,y); \gamma_7\right). \end{cases} \tag{7.3.6}$$

We also examine the relationships among $\{\log\left(f_i^{\mathrm{an}}(x,y); \gamma_i\right), \log\left(1 - f_i^{\mathrm{an}}(x,y); \gamma_i'\right)\}_{i=1,\ldots,9}$. Let

$$L\left(\pi_1^{\mathrm{top}}\left(V_{\text{non-Fano}}^{\mathrm{an}}, \vec{v}\right)\right) \quad \left(\text{resp. } U\left(\pi_1^{\mathrm{top}}\left(V_{\text{non-Fano}}^{\mathrm{an}}, \vec{v}\right)\right)\right)$$

be the complete Lie algebra of $\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})$ over $\mathbb{C}$ (resp. the complete Hopf algebra given as the universal enveloping algebra of $L\left(\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})\right)$). Then, there is a natural inclusion

$$\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v}) \hookrightarrow L\left(\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})\right), \ B_i \mapsto X_i := \mathbf{log}(B_i).$$

Each element of $L\left(\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})\right)$ has an expansion as a formal Lie series in $X_1, ..., X_6$. Let

$$\Lambda_{\vec{v}}((x,y);\gamma_0) \in U\left(\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})\right)$$

be a horizontal section along $\gamma_0 \in \pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}; \vec{v}, (x,y))$ of the trivial principal bundle $V^{\mathrm{an}}_{\text{non-Fano}} \times U\left(\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})\right) \to V^{\mathrm{an}}_{\text{non-Fano}}$ (cf. [W97, §1], [NW12, §4.1]). The associated horizontal section starting from $(\vec{v}, 1)$ over γ_0 terminates at the point $((x,y), \Lambda_{\vec{v}}((x,y);\gamma_0))$. We denote by $L_n\left(\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})\right)$ the homogeneous degree n part and $L_{<n}\left(\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})\right)$ the part whose homogeneous degree is less than n. Then, we have a decomposition

$$L\left(\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})\right) = L_{<n}\left(\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})\right) \oplus \Gamma^n L\left(\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})\right)$$

and

$$\mathbf{log}\left(\Lambda_{\vec{v}}((x,y);\gamma_0)^{-1}\right) = \left[\mathbf{log}\left(\Lambda_{\vec{v}}((x,y);\gamma_0)^{-1}\right)\right]_{<n} \oplus \left[\mathbf{log}\left(\Lambda_{\vec{v}}((x,y);\gamma_0)^{-1}\right)\right]_{\geq n}.$$

There is an isomorphism

$$L_n\left(\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})\right) \simeq \mathrm{gr}^n_{\Gamma}\left(\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})\right) \otimes \mathbb{C}$$

induced by $L_1\left(\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})\right) \ni X_i \mapsto \bar{B}_i \in \mathrm{gr}^1_{\Gamma}\left(\pi_1^{\mathrm{top}}(V^{\mathrm{an}}_{\text{non-Fano}}, \vec{v})\right)$. So we write

$$\left[\mathbf{log}\left(\Lambda_{\vec{v}}((x,y);\gamma_0)^{-1}\right)\right]_{<2} = C_1X_1 + C_2X_2 + C_3X_3 + C_4X_4 + C_5X_5 + C_6X_6.$$

Using TABLE 7.5, we compute

$$\begin{aligned}
&\left[\mathbf{log}\left(\Lambda_{f_i^{\mathrm{an}}(\vec{v})}(f_i^{\mathrm{an}}(x,y); f_i^{\mathrm{an}}(\gamma_0))^{-1}\right)\right]_{<2} \qquad (7.3.7)\\
&= f_{i*}^{\mathrm{an}}\left(\left[\mathbf{log}\left(\Lambda_{\vec{v}}((x,y);\gamma_0)^{-1}\right)\right]_{<2}\right)\\
&= \begin{cases}
(C_1 - C_2 - 2C_4 + 2C_5)X + (-C_2 + C_3 - 2C_4 + C_6)Y & (i=1),\\
(C_1 + C_2)X + C_6 Y & (i=2),\\
(C_1 - C_2)X + (-C_2 + C_3)Y & (i=3),\\
(C_1 - C_2 - C_4 + C_5)X + (-C_2 + C_3 - C_4)Y & (i=4),\\
(C_1 - C_4 + C_5)X + (-C_4 + C_6)Y & (i=5),\\
(-C_4 + C_5)X + (C_3 - C_4)Y & (i=6),\\
(-C_2 - C_4 + C_5)X + (-C_2 - C_4 + C_6)Y & (i=7),\\
C_1X + C_4Y & (i=8),\\
C_2X + C_5Y & (i=9).
\end{cases}
\end{aligned}$$

By (7.3.7), (7.3.4), (7.3.2) and (7.3.3), we obtain

$$C_1 = -\log(x;\gamma_8), \qquad C_2 = -\log(y;\gamma_9), \qquad C_3 = -\log\left(1-\frac{x}{y};\gamma_3'\right) - \log(y;\gamma_9),$$
$$C_4 = -\log(1-x;\gamma_8'), \qquad C_5 = -\log(1-y;\gamma_9'), \qquad C_6 = -\log(1-xy;\gamma_2'),$$

$$\begin{cases}
\log\left(\frac{x(1-y)^2}{y(1-x)^2};\gamma_1\right) = \log(x;\gamma_8) + 2\log(1-y;\gamma_9') - \log(y;\gamma_9) - 2\log(1-x;\gamma_8'), \\
\log\left(1-\frac{x(1-y)^2}{y(1-x)^2};\gamma_1'\right) = \log\left(1-\frac{x}{y};\gamma_3'\right) + \log(1-xy;\gamma_2') - 2\log(1-x;\gamma_8'), \\
\log(xy;\gamma_2) = \log(x;\gamma_8) + \log(y;\gamma_9), \\
\log\left(\frac{x}{y};\gamma_3\right) = \log(x;\gamma_8) - \log(y;\gamma_9), \\
\log\left(\frac{x(1-y)}{y(1-x)};\gamma_4\right) = \log(x;\gamma_8) + \log(1-y;\gamma_9') - \log(y;\gamma_9) - \log(1-x;\gamma_8'), \\
\log\left(1-\frac{x(1-y)}{y(1-x)};\gamma_4'\right) = \log\left(1-\frac{x}{y};\gamma_3'\right) - \log(1-x;\gamma_8'), \\
\log\left(\frac{x(1-y)}{x-1};\gamma_5\right) = \log(x;\gamma_8) + \log(1-y;\gamma_9') - \log(1-x;\gamma_8') + \pi i, \\
\log\left(1-\frac{x(1-y)}{x-1};\gamma_5'\right) = \log(1-xy;\gamma_2') - \log(1-x;\gamma_8'), \\
\log\left(\frac{1-y}{1-x};\gamma_6\right) = \log(1-y;\gamma_9') - \log(1-x;\gamma_8'), \\
\log\left(\frac{1-y}{y(x-1)};\gamma_7\right) = \log(1-y;\gamma_9') - \log(y;\gamma_9) - \log(1-x;\gamma_8') + \pi i, \\
\log\left(1-\frac{1-y}{y(x-1)};\gamma_7'\right) = \log(1-xy;\gamma_2') - \log(y;\gamma_9) - \log(1-x;\gamma_8').
\end{cases} \tag{7.3.8}$$

By combining these formulas, we prove functional equations. As in TABLE 7.6, we set

$$a_i, b_i, c_i, d_i \in \mathbb{Z},$$

which will be used for the coefficients in the functional equations to be proved.

Table 7.6: a_i, b_i, c_i, d_i

i	1	2	3	4	5	6	7	8	9
$f_i(x,y)$	$\frac{x(1-y)^2}{y(1-x)^2}$	xy	$\frac{x}{y}$	$\frac{x(1-y)}{y(1-x)}$	$\frac{x(1-y)}{x-1}$	$\frac{1-y}{1-x}$	$\frac{1-y}{y(1-x)}$	x	y
a_i	0	0	-1	1	0	-1	0	1	-1
b_i	1	0	0	-1	-1	-1	-1	0	0
c_i	0	1	0	0	-1	0	1	-1	-1
d_i	1	1	1	-2	-2	-2	-2	-2	-2

Theorem 7.3.1 (Functional equations for complex dilogarithms)**.** Given a $\mathbb{C}$-rational point $(x,y) \in V_{\text{non-Fano}}(\mathbb{C})$ and a path $\gamma_0 \in \pi_1^{\text{top}}(V_{\text{non-Fano}}^{\text{an}}; \vec{v}, (x,y))$, define the path system $\{\gamma_i\}_{i=1,\dots,9}$ associated to γ_0 as in (7.1.4). Then the following holds.

(a-$\mathbb{C}$) Schaeffer's equation

$$Li_2\left(\frac{x(1-y)}{y(1-x)}; \gamma_4\right) - Li_2(y;\gamma_9) + Li_2(x;\gamma_8) - Li_2\left(\frac{x}{y};\gamma_3\right)$$

$$-Li_2\left(\frac{1-y}{1-x};\gamma_6\right) = \log(y;\gamma_9)\log\left(\frac{1-y}{1-x};\gamma_6\right) - \frac{\pi^2}{6}.$$

(b-$\mathbb{C}$) Kummer's equation

$$Li_2\left(\frac{x(1-y)^2}{y(1-x)^2};\gamma_1\right) - Li_2\left(\frac{x(1-y)}{x-1};\gamma_5\right) - Li_2\left(\frac{1-y}{y(x-1)};\gamma_7\right)$$

$$-Li_2\left(\frac{x(1-y)}{y(1-x)};\gamma_4\right) - Li_2\left(\frac{1-y}{1-x};\gamma_6\right) = \frac{1}{2}\log^2(y;\gamma_9).$$

(c-$\mathbb{C}$) Hill's equation

$$Li_2\left(\frac{1-y}{y(x-1)};\gamma_7\right) + Li_2(xy;\gamma_2) - Li_2(x;\gamma_8) - Li_2(y;\gamma_9) - Li_2\left(\frac{x(1-y)}{x-1};\gamma_5\right)$$

$$= -\frac{\pi^2}{6} + \log(y;\gamma_9)\log\left(\frac{1-y}{1-x};\gamma_6\right) - \frac{1}{2}\log^2(y;\gamma_9).$$

Proof. Let $k_i \in \{a_i, b_i, c_i\}$ where a_i, b_i, c_i $(i = 1,\dots,9)$ are shown in TABLE 7.6. First, by TABLE 7.7, TABLE 7.8, TABLE 7.9, we can check the homotopy criteria [NW12, Theorem 5.7, (i)$_{\mathbb{C}}$]:

$$\sum_{i=1}^{9} k_i \cdot \varphi_2\left(\mathrm{gr}_\Gamma^2\left(\iota_{\delta_i}^{\text{an}} \circ f_i^{\text{an}}\right)\right) = 0 \quad \text{in } \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{gr}_\Gamma^2\left(\pi_1^{\text{top}}(V_{\text{non-Fano}}^{\text{an}}, \vec{v})\right), \mathbb{Z}\right).$$

Next, by simple calculations, the tensor criteria [NW12, Theorem 5.7, (ii)$_{\mathbb{C}}$]:

$$\sum_{i=1}^{9} k_i \cdot (f_i^{\text{an}} \wedge (f_i^{\text{an}} - 1)) = 0 \quad \text{in } \left((\mathcal{O}^{\text{an}})^\times / \mathbb{C}^\times\right) \wedge \left((\mathcal{O}^{\text{an}})^\times / \mathbb{C}^\times\right)$$

holds. Therefore, we have the functional equation [NW12, Theorem 5.7, (iii)$_{\mathbb{C}}$]:

$$\sum_{i=1}^{9} k_i \cdot \mathcal{L}_{\mathbb{C}}^{\varphi_2}\left(f_i^{\text{an}}(x,y), f_i^{\text{an}}(\vec{v}); f_i^{\text{an}}(\gamma_0)\right) = 0. \tag{7.3.9}$$

For each $k_i \in \{a_i, b_i, c_i\}$, putting (7.3.5) into (7.3.9) and applying (7.3.2), (7.3.3) and (7.3.8), we obtain the desired equations (a-$\mathbb{C}$), (b-$\mathbb{C}$), (c-$\mathbb{C}$). This completes the proof of Theorem 7.3.1. □

Table 7.7: Homotopy criteria for Schaeffer's equation

#	$[\bar{B}_1, \bar{B}_4]$	$[\bar{B}_2, \bar{B}_3]$	$[\bar{B}_2, \bar{B}_5]$	$[\bar{B}_2, \bar{B}_6]$	$[\bar{B}_3, \bar{B}_5]$	$[\bar{B}_3, \bar{B}_6]$	$[\bar{B}_5, \bar{B}_6]$
$\mathrm{gr}^2\left(\iota^{\mathrm{an}}_{\delta_4} \circ f^{\mathrm{an}}_{4*}\right)(\#)$	$-[\bar{l}_0, \bar{l}_1]$	$-[\bar{l}_0, \bar{l}_1]$	$[\bar{l}_0, \bar{l}_1]$	0	$-[\bar{l}_0, \bar{l}_1]$	0	0
$\mathrm{gr}^2\left(\iota^{\mathrm{an}}_{\delta_9} \circ f^{\mathrm{an}}_{9*}\right)(\#)$	0	0	$[\bar{l}_0, \bar{l}_1]$	0	0	0	0
$\mathrm{gr}^2\left(\iota^{\mathrm{an}}_{\delta_8} \circ f^{\mathrm{an}}_{8*}\right)(\#)$	$[\bar{l}_0, \bar{l}_1]$	0	0	0	0	0	0
$\mathrm{gr}^2\left(\iota^{\mathrm{an}}_{\delta_3} \circ f^{\mathrm{an}}_{3*}\right)(\#)$	0	$-[\bar{l}_0, \bar{l}_1]$	0	0	0	0	0
$\mathrm{gr}^2\left(\iota^{\mathrm{an}}_{\delta_6} \circ f^{\mathrm{an}}_{6*}\right)(\#)$	0	0	0	0	$-[\bar{l}_0, \bar{l}_1]$	0	0

Table 7.8: Homotopy criteria for Kummer's equation

#	$[\bar{B}_1, \bar{B}_4]$	$[\bar{B}_2, \bar{B}_3]$	$[\bar{B}_2, \bar{B}_5]$	$[\bar{B}_2, \bar{B}_6]$	$[\bar{B}_3, \bar{B}_5]$	$[\bar{B}_3, \bar{B}_6]$	$[\bar{B}_5, \bar{B}_6]$
$\mathrm{gr}^2\left(\iota^{\mathrm{an}}_{\delta_1} \circ f^{\mathrm{an}}_{1*}\right)(\#)$	$-2[\bar{l}_0, \bar{l}_1]$	$-[\bar{l}_0, \bar{l}_1]$	$2[\bar{l}_0, \bar{l}_1]$	$-[\bar{l}_0, \bar{l}_1]$	$-2[\bar{l}_0, \bar{l}_1]$	0	$2[\bar{l}_0, \bar{l}_1]$
$\mathrm{gr}^2\left(\iota^{\mathrm{an}}_{\delta_5} \circ f^{\mathrm{an}}_{5*}\right)(\#)$	$-[\bar{l}_0, \bar{l}_1]$	0	0	0	0	0	$[\bar{l}_0, \bar{l}_1]$
$\mathrm{gr}^2\left(\iota^{\mathrm{an}}_{\delta_7} \circ f^{\mathrm{an}}_{7*}\right)(\#)$	0	0	$[\bar{l}_0, \bar{l}_1]$	$-[\bar{l}_0, \bar{l}_1]$	0	0	$[\bar{l}_0, \bar{l}_1]$
$\mathrm{gr}^2\left(\iota^{\mathrm{an}}_{\delta_4} \circ f^{\mathrm{an}}_{4*}\right)(\#)$	$-[\bar{l}_0, \bar{l}_1]$	$-[\bar{l}_0, \bar{l}_1]$	$[\bar{l}_0, \bar{l}_1]$	0	$-[\bar{l}_0, \bar{l}_1]$	0	0
$\mathrm{gr}^2\left(\iota^{\mathrm{an}}_{\delta_6} \circ f^{\mathrm{an}}_{6*}\right)(\#)$	0	0	0	0	$-[\bar{l}_0, \bar{l}_1]$	0	0

Table 7.9: Homotopy criteria for Hill's equation

#	$[\bar{B}_1, \bar{B}_4]$	$[\bar{B}_2, \bar{B}_3]$	$[\bar{B}_2, \bar{B}_5]$	$[\bar{B}_2, \bar{B}_6]$	$[\bar{B}_3, \bar{B}_5]$	$[\bar{B}_3, \bar{B}_6]$	$[\bar{B}_5, \bar{B}_6]$
$\mathrm{gr}^2\left(\iota^{\mathrm{an}}_{\delta_7} \circ f^{\mathrm{an}}_{7*}\right)(\#)$	0	0	$[\bar{l}_0, \bar{l}_1]$	$-[\bar{l}_0, \bar{l}_1]$	0	0	$[\bar{l}_0, \bar{l}_1]$
$\mathrm{gr}^2\left(\iota^{\mathrm{an}}_{\delta_2} \circ f^{\mathrm{an}}_{2*}\right)(\#)$	0	0	0	$[\bar{l}_0, \bar{l}_1]$	0	0	0
$\mathrm{gr}^2\left(\iota^{\mathrm{an}}_{\delta_8} \circ f^{\mathrm{an}}_{8*}\right)(\#)$	$[\bar{l}_0, \bar{l}_1]$	0	0	0	0	0	0
$\mathrm{gr}^2\left(\iota^{\mathrm{an}}_{\delta_9} \circ f^{\mathrm{an}}_{9*}\right)(\#)$	0	0	$[\bar{l}_0, \bar{l}_1]$	0	0	0	0
$\mathrm{gr}^2\left(\iota^{\mathrm{an}}_{\delta_5} \circ f^{\mathrm{an}}_{5*}\right)(\#)$	$-[\bar{l}_0, \bar{l}_1]$	0	0	0	0	0	$[\bar{l}_0, \bar{l}_1]$

Proof of Theorem 7.1.1. First, by TABLE 7.5, we can check the homotopy criteria [NW12, Theorem 5.7, $(\mathrm{i})_{\mathbb{C}}$]:

$$\sum_{i=1}^{9} d_i \cdot \varphi_3 \left(\mathrm{gr}_{\Gamma}^{3} \left(\iota_{\delta_i}^{\mathrm{an}} \circ f_i^{\mathrm{an}} \right) \right) = 0 \quad \text{in} \quad \mathrm{Hom}_{\mathbb{Z}} \left(\mathrm{gr}_{\Gamma}^{3} \left(\pi_1^{\mathrm{top}} \left(V_{\text{non-Fano}}^{\mathrm{an}}, \vec{v} \right) \right), \mathbb{Z} \right). \tag{7.3.10}$$

where $d_1, d_2, \ldots, d_9$ are shown in TABLE 7.6. Next, by simple calculations, the tensor criteria [NW12, Theorem 5.7, $(\mathrm{ii})_{\mathbb{C}}$]:

$$\sum_{i=1}^{9} d_i \cdot \left(f_i^{\mathrm{an}} \otimes \left(f_i^{\mathrm{an}} \wedge \left(f_i^{\mathrm{an}} - 1 \right) \right) \right) = 0 \quad \text{in} \quad \left(\left(\mathcal{O}^{\mathrm{an}} \right)^{\times} / \mathbb{C}^{\times} \right) \otimes \left(\left(\left(\mathcal{O}^{\mathrm{an}} \right)^{\times} / \mathbb{C}^{\times} \right) \wedge \left(\left(\mathcal{O}^{\mathrm{an}} \right)^{\times} / \mathbb{C}^{\times} \right) \right) \tag{7.3.11}$$

holds. Therefore, we have the functional equation [NW12, Theorem 5.7, $(\mathrm{iii})_{\mathbb{C}}$]:

$$\sum_{i=1}^{9} d_i \cdot \mathcal{L}_{\mathbb{C}}^{\varphi_3} \left(f_i^{\mathrm{an}}(x, y), f_i^{\mathrm{an}}(\vec{v}); f_i^{\mathrm{an}}(\gamma_0) \right) = 0. \tag{7.3.12}$$

Putting (7.3.6) into (7.3.12) and applying (7.3.2), (7.3.3) and (7.3.8), we obtain the desired equation (d-$\mathbb{C}$). In this process, the nine Li_2-terms appearing in the left-hand side of (7.3.12) are canceled by using (a-$\mathbb{C}$), (b-$\mathbb{C}$) and (c-$\mathbb{C}$). This completes the proof of Theorem 7.1.1.

□

7.3.2 ℓ-adic Galois case

In this subsection, we derive Spence-Kummer's ℓ-adic Galois trilogarithm functional equation. For this purpose, we also derive Schaeffer's, Kummer's, and Hill's ℓ-adic Galois dilogarithm functional equations. These functional equations are derived by using Nakamura-Wojtkowiak's tensor-homotopy criteria [NW12, Theorem 5.7, Proposition 5.11]. We fix a K-rational point (x, y) of $V_{\text{non-Fano}}$.

First, we make the preparations for checking the tensor-homotopy criteria [NW12, Theorem 5.7, $(\mathrm{i})_{\ell}$, $(\mathrm{ii})_{\ell}$]. For $i = 1, \ldots, 9$, we write

$$f_{i*} : \pi_1^{\ell\text{-ét}} \left(V_{\text{non-Fano}}, \vec{v} \right) \to \pi_1^{\ell\text{-ét}} \left(\mathbb{P}_{\overline{K}}^1 \backslash \{0, 1, \infty\}, f_i(\vec{v}) \right)$$

for the homomorphism induced by f_i and denote by

$$\iota_{\delta_i} : \pi_1^{\ell\text{-ét}} \left(\mathbb{P}_{\overline{K}}^1 \backslash \{0, 1, \infty\}, f_i(\vec{v}) \right) \xrightarrow{\simeq} \pi_1^{\ell\text{-ét}} \left(\mathbb{P}_{\overline{K}}^1 \backslash \{0, 1, \infty\}, \overrightarrow{01} \right)$$

the change-of-basepoint homomorphism defined by $p \mapsto \delta_i \cdot p \cdot \delta_i^{-1}$. Then $\left(\iota_{\delta_i} \circ f_{i*} \right)(B_j)$ are calculated as well as the complex case (TABLE 7.5). Moreover, for $i = 1, \ldots, 9$, by considering the composite $V_{\text{non-Fano}} \xrightarrow{f_i} \mathbb{P}_{\overline{K}}^1 \backslash \{0, 1, \infty\} \hookrightarrow \mathbb{G}_{m, \overline{K}}$, we may regard f_i as an element of the unit group

$$f_i \in \mathcal{O}^{\times}$$

where $\mathcal{O}$ is the coordinate ring $\overline{K}\left[s_1, s_2, \frac{1}{s_1 s_2 (1 - s_1)(1 - s_2)(s_1 - s_2)(1 - s_1 s_2)} \right]$ of $V_{\text{non-Fano}}$. Then

$$f_i - 1 \in \mathcal{O}^{\times}$$

holds.

Next, we make the preparations for calculating the functional equations of ℓ-adic iterated integrals [NW12, Theorem 5.7, (iii)$_\ell$]. Let $\sigma \in G_K$. By (4.1.8), for $i = 1, ..., 9$, we have

$$\begin{cases} \ell i_0\left(f_i(x,y);\gamma_i,\sigma\right) = & \rho_{f_i(x,y),\gamma_i}(\sigma), \\ \ell i_1\left(f_i(x,y);\gamma_i,\sigma\right) = & \rho_{1-f_i(x,y),\gamma_i'}(\sigma), \\ \ell i_2\left(f_i(x,y);\gamma_i,\sigma\right) = & Li_2^{\ell}\left(f_i(x,y);\gamma_i,\sigma\right) + \dfrac{1}{2}\rho_{f_i(x,y),\gamma_i}(\sigma)\rho_{1-f_i(x,y),\gamma_i'}(\sigma), \\ \ell i_3\left(f_i(x,y);\gamma_i,\sigma\right) = & Li_3^{\ell}\left(f_i(x,y);\gamma_i,\sigma\right) + \dfrac{1}{2}\rho_{f_i(x,y),\gamma_i}(\sigma)Li_2^{\ell}\left(f_i(x,y);\gamma_i,\sigma\right) \\ & +\dfrac{1}{12}\left(\rho_{f_i(x,y),\gamma_i}(\sigma)\right)^2\rho_{1-f_i(x,y),\gamma_i'}(\sigma), \end{cases} \tag{7.3.13}$$

where $\gamma_i' \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\};\overrightarrow{01}, f_i^{\mathrm{an}}(x,y)\right)$ is the path associated to γ_i as in (6.1.1). By (4.1.7), we obtain the following relations of ℓ-adic Galois associators (cf. [I90, p.106])

$$\begin{cases} \mathfrak{f}_{\overrightarrow{01},\sigma}^{f_5(\vec{v}),\delta_5}(l_0,l_1) = l_0^{\frac{1-\chi(\sigma)}{2}}, \\ \mathfrak{f}_{\overrightarrow{01},\sigma}^{f_7(\vec{v}),\delta_7}(l_0,l_1) = \mathfrak{f}_{\overrightarrow{01},\sigma}^{f_6(\vec{v}),\delta_6}(l_0,l_\infty)\cdot\mathfrak{f}_{\overrightarrow{01},\sigma}^{f_5(\vec{v}),\delta_5}(l_0,l_1), \end{cases} \tag{7.3.14}$$

where

$$\mathfrak{f}(l_*,l_{*'}) \in \pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_K\backslash\{0,1,\infty\},\overrightarrow{01}\right)$$

is the image of $\mathfrak{f}$ under the map $\pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_K\backslash\{0,1,\infty\},\overrightarrow{01}\right) \to \pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_K\backslash\{0,1,\infty\},\overrightarrow{01}\right)$ given by $l_0, l_1 \mapsto l_*, l_{*'}$. Then we get

$$\begin{cases} \mathbf{log}\left(\left(\mathfrak{f}_{\overrightarrow{01},\sigma}^{f_i(\vec{v}),\delta_i}(l_0,l_1)\right)^{-1}\right) = & 0 \qquad (i=1,2,3,4,8,9), \\ \mathbf{log}\left(\left(\mathfrak{f}_{\overrightarrow{01},\sigma}^{\overrightarrow{10},\delta_5}(l_0,l_1)\right)^{-1}\right) = & \left(\frac{\chi(\sigma)-1}{2}\right)X, \\ \mathbf{log}\left(\left(\mathfrak{f}_{\overrightarrow{01},\sigma}^{\overrightarrow{0\infty},\delta_6}(l_0,l_1)\right)^{-1}\right) = & \ell i_2\left(\overrightarrow{10};\delta_6,\sigma\right)[X,Y] + \ell i_3\left(\overrightarrow{10};\delta_6,\sigma\right)[X,[X,Y]] + \cdots, \\ \mathbf{log}\left(\left(\mathfrak{f}_{\overrightarrow{01},\sigma}^{\overrightarrow{\infty0},\delta_7}(l_0,l_1)\right)^{-1}\right) = & \left(\frac{\chi(\sigma)-1}{2}\right)X - \ell i_2\left(\overrightarrow{10};\delta_6,\sigma\right)[X,Y] \\ & -\frac{1}{2}\left(\frac{\chi(\sigma)+1}{2}\right)\ell i_2\left(\overrightarrow{10};\delta_6,\sigma\right)[X,[X,Y]] + \cdots, \end{cases} \tag{7.3.15}$$

that is,

$$\begin{cases} \left(-\ell i_j(\overrightarrow{01};\delta_i,\sigma)\right)_{0\le j\le 3} = (0,0,0,0) \qquad (i=1,2,3,4,8,9), \\ \left(-\ell i_j(\overrightarrow{10};\delta_5,\sigma)\right)_{0\le j\le 3} = \left(\frac{1-\chi(\sigma)}{2},0,0,0\right), \\ \left(-\ell i_j(\overrightarrow{0\infty};\delta_6,\sigma)\right)_{0\le j\le 3} = \left(0,0,-\ell i_2\left(\overrightarrow{10};\delta_6,\sigma\right),-\ell i_3\left(\overrightarrow{10};\delta_6,\sigma\right)\right), \\ \left(-\ell i_j(\overrightarrow{\infty0};\delta_7,\sigma)\right)_{0\le j\le 3} = \left(\frac{1-\chi(\sigma)}{2},0,\ell i_2\left(\overrightarrow{10};\delta_6,\sigma\right),\frac{1}{2}\left(\frac{1+\chi(\sigma)}{2}\right)\ell i_2\left(\overrightarrow{10};\delta_6,\sigma\right)\right), \end{cases} \tag{7.3.16}$$

where $\chi : G_K \to \mathbb{Z}_\ell^\times$ is the ℓ-adic cyclotomic character.

We consider the path $\gamma_i \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, f_i^{\mathrm{an}}(x,y)\right)$ in (7.1.4) as a pro-ℓ etale path

$$\gamma_i \in \pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{K}}\backslash\{0,1,\infty\}; \overrightarrow{01}, f_i(x,y)\right)$$

by the comparison map (2.2.1). Then, we have $\gamma_i = \delta_i \cdot f_i(\gamma_0) \in \pi_1^{\ell\text{-ét}}\left(\mathbb{P}^1_{\overline{K}}\backslash\{0,1,\infty\}; \overrightarrow{01}, f_i(x,y)\right)$. Through this path composition, we obtain an algebraic relation

$$\mathfrak{f}_{\overrightarrow{01},\sigma}^{f_i(x,y),\gamma_i} = \left(\left(\iota_{\delta_i} \circ f_{i*}\right)\left(\mathfrak{f}_{\vec{v},\sigma}^{(x,y),\gamma_0}\right)\right) \cdot \mathfrak{f}_{\overrightarrow{01},\sigma}^{f_i(\vec{v}),\delta_i}. \tag{7.3.17}$$

By (7.3.17), (7.3.16) and the polylog-BCH formula [NW12, Proposition 5.9], we compute Wojtkowiak's native ℓ-adic iterated integral (cf. [NW12, Definition 4.7])

$$\mathcal{L}_{\mathrm{nv}}^{\varphi_k(f_i)_{\vec{l}}}\left(f_i(x,y), f_i(\vec{v}); f_i(\gamma_0), \sigma\right) := \varphi_{k,\vec{l}}\left(\mathbf{log}\left(\left(\iota_{\delta_i} \circ f_{i*}\right)\left(\mathfrak{f}_{\vec{v},\sigma}^{(x,y),\gamma_0}\right)^{-1}\right)\right)$$

associated to $f_i(\gamma_0)$ as follows:

$$\begin{cases} \mathcal{L}_{\mathrm{nv}}^{\varphi_2(f_i)_{\vec{l}}}\left(f_i(x,y), f_i(\vec{v}); f_i(\gamma_0), \sigma\right) & = \ell i_2(f_i(x,y); \gamma_i, \sigma) \qquad (i = 1,2,3,4,8,9), \\ \mathcal{L}_{\mathrm{nv}}^{\varphi_2(f_5)_{\vec{l}}}\left(z_5, f_5(\vec{v}); f_5(\gamma_0), \sigma\right) & = \ell i_2(z_5; \gamma_5, \sigma) - \frac{1}{2}\ell i_0\left(\overrightarrow{0\infty}; \delta_5, \sigma\right)\ell i_1(z_5; \gamma_5, \sigma), \\ \mathcal{L}_{\mathrm{nv}}^{\varphi_2(f_6)_{\vec{l}}}\left(z_6, f_6(\vec{v}); f_6(\gamma_0), \sigma\right) & = \ell i_2(z_6; \gamma_6, \sigma) - \ell i_2\left(\overrightarrow{10}; \delta_6, \sigma\right) + \frac{1}{2}\ell i_0(z_6; \gamma_6, \sigma)\ell i_1\left(\overrightarrow{10}; \delta_6, \sigma\right), \\ \mathcal{L}_{\mathrm{nv}}^{\varphi_2(f_7)_{\vec{l}}}\left(z_7, f_7(\vec{v}); f_7(\gamma_0), \sigma\right) & = \ell i_2(z_7; \gamma_7, \sigma) - \ell i_2\left(\overrightarrow{\infty 0}; \delta_7, \sigma\right) - \frac{1}{2}\ell i_1(z_7; \gamma_7, \sigma)\ell i_0\left(\overrightarrow{\infty 0}; \delta_7, \sigma\right) \\ & \quad -\frac{1}{2}\ell i_0(z_7; \gamma_7, \sigma)\ell i_1\left(\overrightarrow{\infty 0}; \delta_7, \sigma\right), \end{cases} \tag{7.3.18}$$

$$\begin{cases} \mathcal{L}_{\mathrm{nv}}^{\varphi_2(f_i)_{\vec{l}}}\left(f_i(x,y), f_i(\vec{v}); f_i(\gamma_0), \sigma\right) & = \ell i_2(f_i(x,y); \gamma_i, \sigma) \qquad (i = 1,2,3,4,8,9), \\ \mathcal{L}_{\mathrm{nv}}^{\varphi_2(f_5)_{\vec{l}}}\left(z_5, f_5(\vec{v}); f_5(\gamma_0), \sigma\right) & = \ell i_3(z_5; \gamma_5, \sigma) - \frac{1}{2}\ell i_0\left(\overrightarrow{0\infty}; \delta_5, \sigma\right)\ell i_2(z_5; \gamma_5, \sigma) \\ & \quad +\frac{1}{12}\left(\ell i_0\left(\overrightarrow{0\infty}; \delta_5, \sigma\right)\right)^2 \ell i_1(z_5; \gamma_5, \sigma) \\ & \quad +\frac{1}{12}\ell i_0\left(\overrightarrow{0\infty}; \delta_5, \sigma\right)\ell i_0(z_5; \gamma_5, \sigma)\ell i_1(z_5; \gamma_5, \sigma), \\ \mathcal{L}_{\mathrm{nv}}^{\varphi_2(f_6)_{\vec{l}}}\left(z_6, f_6(\vec{v}); f_6(\gamma_0), \sigma\right) & = \ell i_3(z_6; \gamma_6, \sigma) - \ell i_3\left(\overrightarrow{10}; \delta_6, \sigma\right) + \frac{1}{2}\ell i_0(z_6; \gamma_6, \sigma)\ell i_2\left(\overrightarrow{10}; \delta_6, \sigma\right), \\ \mathcal{L}_{\mathrm{nv}}^{\varphi_2(f_7)_{\vec{l}}}\left(z_7, f_7(\vec{v}); f_7(\gamma_0), \sigma\right) & = \ell i_3(z_7; \gamma_7, \sigma) - \ell i_3\left(\overrightarrow{\infty 0}; \delta_7, \sigma\right) - \frac{1}{2}\ell i_2(z_7; \gamma_7, \sigma)\ell i_0\left(\overrightarrow{\infty 0}; \delta_7, \sigma\right) \\ & \quad -\frac{1}{2}\ell i_0(z_7; \gamma_7, \sigma)\ell i_2\left(\overrightarrow{\infty 0}; \delta_7, \sigma\right) + \frac{1}{12}\left(\ell i_0\left(\overrightarrow{\infty 0}; \delta_7, \sigma\right)\right)^2 \ell i_1(z_7; \gamma_7, \sigma) \\ & \quad +\frac{1}{12}\ell i_0\left(\overrightarrow{\infty 0}; \delta_7, \sigma\right)\ell i_0(z_7; \gamma_7, \sigma)\ell i_1(z_7; \gamma_7, \sigma). \end{cases} \tag{7.3.19}$$

We shall write

$$L\left(\pi_1^{\ell\text{-ét}}\left(V_{\text{non-Fano}}, \vec{v}\right)\right)$$

for the complete ℓ-adic Lie algebra of $\pi_1^{\ell\text{-ét}}(V_{\text{non-Fano}}, \vec{v})$ over $\mathbb{Q}_\ell$. Then, there is a natural inclusion

$$\pi_1^{\ell\text{-ét}}\left(V_{\text{non-Fano}}, \vec{v}\right) \hookrightarrow L\left(\pi_1^{\ell\text{-ét}}\left(V_{\text{non-Fano}}, \vec{v}\right)\right), \ B_i \mapsto X_i := \mathbf{log}(B_i).$$

Each element of $L\left(\pi_1^{\ell\text{-ét}}\left(V_{\text{non-Fano}}, \vec{v}\right)\right)$ has an expansion as a formal Lie series in $X_1, ..., X_6$. We denote by

$$L_n\left(\pi_1^{\ell\text{-ét}}\left(V_{\text{non-Fano}}, \vec{v}\right)\right)$$

the homogeneous degree n part and

$$L_{<n}\left(\pi_1^{\ell\text{-ét}}\left(V_{\text{non-Fano}},\vec{v}\right)\right)$$

the part whose homogeneous degree is less than n. Then, we have a decomposition

$$L\left(\pi_1^{\ell\text{-ét}}\left(V_{\text{non-Fano}},\vec{v}\right)\right)=L_{<n}\left(\pi_1^{\ell\text{-ét}}\left(V_{\text{non-Fano}},\vec{v}\right)\right)\oplus\Gamma^n L\left(\pi_1^{\ell\text{-ét}}\left(V_{\text{non-Fano}},\vec{v}\right)\right)$$

and

$$\mathbf{log}\left(\left(\mathfrak{f}_{\vec{v},\sigma}^{(x,y),\gamma_0}\right)^{-1}\right)=\left[\mathbf{log}\left(\left(\mathfrak{f}_{\vec{v},\sigma}^{(x,y),\gamma_0}\right)^{-1}\right)\right]_{<n}+\left[\mathbf{log}\left(\left(\mathfrak{f}_{\vec{v},\sigma}^{(x,y),\gamma_0}\right)^{-1}\right)\right]_{\geq n}.$$

Here, there is an isomorphism

$$L_n\left(\pi_1^{\ell\text{-ét}}\left(V_{\text{non-Fano}},\vec{v}\right)\right)\simeq\mathrm{gr}_\Gamma^n\left(\pi_1^{\ell\text{-ét}}\left(V_{\text{non-Fano}},\vec{v}\right)\right)\otimes\mathbb{Q}_\ell$$

induced by $L_1\left(\pi_1^{\ell\text{-ét}}\left(V_{\text{non-Fano}},\vec{v}\right)\right)\ni X_i\mapsto\bar{B}_i\in\mathrm{gr}_\Gamma^1\left(\pi_1^{\ell\text{-ét}}\left(V_{\text{non-Fano}},\vec{v}\right)\right)$. Therefore, we write

$$\begin{aligned}\left[\mathbf{log}\left(\left(\mathfrak{f}_{\vec{v},\sigma}^{(x,y),\gamma_0}\right)^{-1}\right)\right]_{<2}&=C_1X_1+C_2X_2+C_3X_3+C_4X_4+C_5X_5+C_6X_6,\\ \left[\mathbf{log}\left(\left(\mathfrak{f}_{\vec{v},\sigma}^{(x,y),\gamma_0}\right)^{-1}\right)\right]_{<3}&=C_1X_1+C_2X_2+C_3X_3+C_4X_4+C_5X_5+C_6X_6\\ &\quad+C_7[X_1,X_4]+C_8[X_2,X_3]+C_9[X_2,X_5]+C_{10}[X_2,X_6]\\ &\quad+C_{11}[X_3,X_5]+C_{12}[X_3,X_6]+C_{13}[X_5,X_6].\end{aligned}$$

Using the Baker–Campbell–Hausdorff formula, TABLE 7.5, (7.3.17) and (7.3.15), we compute

$$(\iota_{\delta_i}\circ f_{i*})\left(\left[\mathbf{log}\left(\left(\mathfrak{f}_{\vec{v},\sigma}^{(x,y),\gamma_0}\right)^{-1}\right)\right]_{<n}\right)\in\mathrm{Lie}_{\mathbb{Q}_\ell}\langle\langle X,Y\rangle\rangle$$

for $n=2,3$. Consequently, we get TABLE 7.10. By (7.3.17), (7.3.15) and TABLE 7.10, we obtain

$$\begin{cases}C_1=\rho_{x,\gamma_8}(\sigma), \qquad C_2=\rho_{y,\gamma_9}(\sigma),\\ C_3=\rho_{1-\frac{x}{y},\gamma_3'}(\sigma)+\rho_{y,\gamma_9}(\sigma),\\ C_4=\rho_{1-x,\gamma_8'}(\sigma), \qquad C_5=\rho_{1-y,\gamma_9'}(\sigma),\\ C_6=\rho_{1-xy,\gamma_2'}(\sigma), \qquad C_7=\ell i_2(x;\gamma_8,\sigma),\\ C_8=\dfrac{1}{2}\rho_{y,\gamma_9}(\sigma)-\ell i_2\left(\dfrac{x}{y};\gamma_3,\sigma\right),\\ C_9=\ell i_2(y;\gamma_9,\sigma), \qquad C_{10}=\ell i_2(xy;\gamma_2,\sigma),\\ C_{11}=-\dfrac{1}{2}\rho_{1-x,\gamma_8'}(\sigma)-\ell i_2\left(\dfrac{1-y}{1-x};\gamma_6,\sigma\right),\\ C_{13}=\dfrac{1}{2}\rho_{1-x,\gamma_8'}(\sigma)-\rho_{1-xy,\gamma_2'}(\sigma)+\ell i_2(x;\gamma_8,\sigma)+\ell i_2\left(\dfrac{x(1-y)}{x-1};\gamma_5,\sigma\right),\end{cases} \tag{7.3.20}$$

Table 7.10: Computation of $\mathbf{log}(\mathfrak{f}_\sigma^{-1})$

i	$\left(\iota_{\delta_i} \circ f_{i*}\right)\left(\left[\mathbf{log}\left(\left(\mathfrak{f}_{\vec{v},\sigma}^{(x,y),\gamma_0}\right)^{-1}\right)\right]_{<2}\right)$	$\left(\iota_{\delta_i} \circ f_{i*}\right)\left(\left[\mathbf{log}\left(\left(\mathfrak{f}_{\vec{v},\sigma}^{(x,y),\gamma_0}\right)^{-1}\right)\right]_{<3}\right)$
1	$(C_1 - C_2 - 2C_4 + 2C_5)\,X$ $+ (-C_2 + C_3 - 2C_4 + C_6)\,Y$ $+ \left(\frac{1}{2}C_2 - C_4 + C_6 - 2C_7 - C_8\right)[X,Y]$ $+\cdots$	$(C_1 - C_2 - 2C_4 + 2C_5)\,X + (-C_2 + C_3 - 2C_4 + C_6)\,Y$ $+ \left(\frac{1}{2}C_2 - C_4 + C_6 - 2C_7 - C_8 + 2C_9 - C_{10} - 2C_{11} + 2C_{13}\right)[X,Y]$ $+ \left(\frac{1}{12}C_2 - \frac{1}{6}C_4 + \frac{1}{2}C_6 - C_7 - C_9 - C_{10} + 2C_{13}\right)[X,[X,Y]] + \cdots$
2	$(C_1 + C_2)\,X + C_6Y$	$(C_1 + C_2)\,X + C_6Y + C_{10}[X,Y]$
3	$(C_1 - C_2)\,X + (-C_2 + C_3)\,Y$ $+\frac{1}{2}C_2[X,Y] + \cdots$	$(C_1 - C_2)\,X + (-C_2 + C_3)\,Y + \left(\frac{1}{2}C_2 - C_8\right)[X,Y]$ $-\frac{1}{12}C_2[X,[X,Y]] + \cdots$
4	$(C_1 - C_2 - C_4 + C_5)\,X$ $+ (-C_2 + C_3 - C_4)\,Y$ $+ \left(\frac{1}{2}C_2 - \frac{1}{2}C_4 - C_7\right)[X,Y] + \cdots$	$(C_1 - C_2 - C_4 + C_5)\,X + (-C_2 + C_3 - C_4)\,Y$ $+ \left(\frac{1}{2}C_2 - \frac{1}{2}C_4 - C_7 - C_8 + C_9 - C_{11}\right)[X,Y]$ $+ \left(\frac{1}{12}C_2 - \frac{1}{12}C_4 - \frac{1}{2}C_7 - \frac{1}{2}C_9\right)[X,[X,Y]] + \cdots$
5	$(C_1 - C_4 + C_5)\,X + (-C_4 + C_6)\,Y$ $+ \left(-\frac{1}{2}C_4 + C_6\right)[X,Y] + \cdots$	$(C_1 - C_4 + C_5)\,X + (-C_4 + C_6)\,Y + \left(-\frac{1}{2}C_4 + C_6 - C_7 + C_{13}\right)[X,Y]$ $+ \left(-\frac{1}{12}C_4 + \frac{1}{2}C_6 - \frac{1}{2}C_7 + C_{13}\right)[X,[X,Y]] + \cdots$
6	$(-C_4 + C_5)\,X + (C_3 - C_4)\,Y$ $-\frac{1}{2}C_4[X,Y] + \cdots$	$(-C_4 + C_5)\,X + (C_3 - C_4)\,Y + \left(-\frac{1}{2}C_4 - C_{11}\right)[X,Y]$ $-\frac{1}{12}C_4[X,[X,Y]] + \cdots$
7	$(-C_2 - C_4 + C_5)\,X$ $+ (-C_2 - C_4 + C_6)\,Y$ $+ \left(-\frac{1}{2}C_2 - \frac{1}{2}C_4 + C_6\right)[X,Y] + \cdots$	$(-C_2 - C_4 + C_5)\,X + (-C_2 - C_4 + C_6)\,Y$ $+ \left(-\frac{1}{2}C_2 - \frac{1}{2}C_4 + C_6 + C_9 - C_{10} + C_{13}\right)[X,Y]$ $+ \left(-\frac{1}{12}C_2 - \frac{1}{12}C_4 + \frac{1}{2}C_6 + \frac{1}{2}C_9 - C_{10} + C_{13}\right)[X,[X,Y]] + \cdots$
8	$C_1X + C_4Y$	$C_1X + C_4Y + C_7[X,Y]$
9	$C_2X + C_5Y$	$C_2X + C_5Y + C_9[X,Y]$

and

$$\begin{cases}
\rho_{\frac{x(1-y)^2}{y(1-x)^2};\gamma_1}(\sigma) & = \rho_{x,\gamma_8}(\sigma) + 2\rho_{1-y,\gamma_9'}(\sigma) - \rho_{y,\gamma_9}(\sigma) - 2\rho_{1-x,\gamma_8'}(\sigma), \\
\rho_{1-\frac{x(1-y)^2}{y(1-x)^2};\gamma_1'}(\sigma) & = \rho_{1-\frac{x}{y},\gamma_3'}(\sigma) + \rho_{1-xy,\gamma_2'}(\sigma) - 2\rho_{1-x,\gamma_8'}(\sigma), \\
\rho_{xy;\gamma_2}(\sigma) & = \rho_{x,\gamma_8}(\sigma) + \rho_{y,\gamma_9}(\sigma), \\
\rho_{\frac{x}{y};\gamma_3}(\sigma) & = \rho_{x,\gamma_8}(\sigma) - \rho_{y,\gamma_9}(\sigma), \\
\rho_{\frac{x(1-y)}{y(1-x)};\gamma_4}(\sigma) & = \rho_{x,\gamma_8}(\sigma) + \rho_{1-y,\gamma_9'}(\sigma) - \rho_{y,\gamma_9}(\sigma) - \rho_{1-x,\gamma_8'}(\sigma), \\
\rho_{1-\frac{x(1-y)}{y(1-x)};\gamma_4'} & = \rho_{1-\frac{x}{y},\gamma_3'}(\sigma) - \rho_{1-x,\gamma_8'}(\sigma), \\
\rho_{\frac{x(1-y)}{x-1};\gamma_5}(\sigma) & = \rho_{x,\gamma_8}(\sigma) + \rho_{1-y,\gamma_9'}(\sigma) - \rho_{1-x,\gamma_8'}(\sigma) + \left(\frac{\chi(\sigma)-1}{2}\right), \\
\rho_{1-\frac{x(1-y)}{x-1};\gamma_5'}(\sigma) & = \rho_{1-xy,\gamma_2'}(\sigma) - \rho_{1-x,\gamma_8'}(\sigma), \\
\rho_{\frac{1-y}{1-x};\gamma_6}(\sigma) & = \rho_{1-y,\gamma_9'}(\sigma) - \rho_{1-x,\gamma_8'}(\sigma), \\
\rho_{\frac{1-y}{y(x-1)};\gamma_7}(\sigma) & = \rho_{1-y,\gamma_9'}(\sigma) - \rho_{y,\gamma_9}(\sigma) - \rho_{1-x,\gamma_8'}(\sigma) + \left(\frac{\chi(\sigma)-1}{2}\right), \\
\rho_{1-\frac{1-y}{y(x-1)};\gamma_7'}(\sigma) & = \rho_{1-xy,\gamma_2'}(\sigma) - \rho_{y,\gamma_9}(\sigma) - \rho_{1-x,\gamma_8'}(\sigma).
\end{cases} \tag{7.3.21}$$

By combining these formulas, we prove functional equations of ℓ-adic Galois polylogarithms.

Theorem 7.3.2 (Functional equations for ℓ-adic Galois dilogarithms)**.** Given a K-rational point $(x,y) \in V_{\text{non-Fano}}(K)$ and a path $\gamma_0 \in \pi_1^{\text{top}}(V_{\text{non-Fano}}^{\text{an}}; \vec{v}, (x,y))$, define the path system $\{\gamma_i\}_{i=1,\dots,9}$ associated to γ_0 as in (7.1.4). For any $\sigma \in G_K$, the following holds.

(a-ℓ) ℓ-adic Schaeffer's equation

$$Li_2^{\ell}\left(\frac{x(1-y)}{y(1-x)};\gamma_4,\sigma\right) - Li_2^{\ell}(y;\gamma_9,\sigma) + Li_2^{\ell}(x;\gamma_8,\sigma) - Li_2^{\ell}\left(\frac{x}{y};\gamma_3,\sigma\right)$$

$$-Li_2^{\ell}\left(\frac{1-y}{1-x};\gamma_6,\sigma\right) = \rho_{y,\gamma_9}(\sigma)\rho_{\frac{1-y}{1-x},\gamma_6}(\sigma) - \boldsymbol{\zeta}_2^{\ell}(\sigma).$$

(b-ℓ) ℓ-adic Kummer's equation

$$Li_2^{\ell}\left(\frac{x(1-y)^2}{y(1-x)^2};\gamma_1,\sigma\right) - Li_2^{\ell}\left(\frac{x(1-y)}{x-1};\gamma_5,\sigma\right) - Li_2^{\ell}\left(\frac{1-y}{y(x-1)};\gamma_7,\sigma\right)$$

$$-Li_2^{\ell}\left(\frac{x(1-y)}{y(1-x)};\gamma_4,\sigma\right) - Li_2^{\ell}\left(\frac{1-y}{1-x};\gamma_6,\sigma\right)$$

$$= \frac{1}{2}\left(\rho_{y,\gamma_9}(\sigma)\right)^2 + \frac{1}{2}\rho_{y,\gamma_9}(\sigma) + \rho_{1-x,\gamma_8'}(\sigma) - \rho_{1-xy,\gamma_2'}(\sigma).$$

(c-ℓ) ℓ-adic Hill's equation

$$Li_2^{\ell}\left(\frac{1-y}{y(x-1)};\gamma_7,\sigma\right) + Li_2^{\ell}(xy;\gamma_2,\sigma) - Li_2^{\ell}(x;\gamma_8,\sigma)$$

$$-Li_2^{\ell}(y;\gamma_9,\sigma) - Li_2^{\ell}\left(\frac{x(1-y)}{x-1};\gamma_5,\sigma\right)$$
$$= -\boldsymbol{\zeta}_2^{\ell}(\sigma) + \rho_{y,\gamma_9}(\sigma)\rho_{\frac{1-y}{1-x},\gamma_6}(\sigma) - \frac{1}{2}\left(\rho_{y,\gamma_9}(\sigma)\right)^2 - \frac{1}{2}\rho_{y,\gamma_9}(\sigma).$$

Proof. Let $\sigma \in G_K$ and $k_i \in \{a_i, b_i, c_i\}$ where a_i, b_i, c_i $(i = 1, \ldots, 9)$ are shown in TABLE 7.6. As well as the complex case, the homotopy criteria [NW12, Theorem 5.7 (i)$_\ell$]:

$$\sum_{i=1}^{9} k_i \cdot \varphi_2\left(\mathrm{gr}_\Gamma^2\left(\iota_{\delta_i} \circ f_i\right)\right) = 0 \quad \text{in} \quad \mathrm{Hom}_{\mathbb{Z}_\ell}\left(\mathrm{gr}_\Gamma^2\left(\pi_1^{\ell\text{-ét}}\left(V_{\text{non-Fano}}\right), \vec{v}\right), \mathbb{Z}_\ell\right),$$

and the tensor criteria [NW12, Theorem 5.7 (ii)$_\ell$]:

$$\sum_{i=1}^{9} k_i \cdot \left(f_i \wedge (f_i - 1)\right) = 0 \quad \text{in} \quad \left(\mathcal{O}^\times/\overline{K}^\times\right) \wedge \left(\mathcal{O}^\times/\overline{K}^\times\right)$$

hold. Therefore, we have the functional equation [NW12, Theorem 5.7 (iii)$_\ell$, Corollary 5.8]:

$$\sum_{i=1}^{9} k_i \cdot \mathcal{L}_{\mathrm{nv}}^{\varphi_2(f_i)_{\vec{l}}}\left(f_i(x,y), f_i(\vec{v}); f_i(\gamma_0), \sigma\right) = \sum_{i=1}^{9} k_i \cdot \varphi_{2,\vec{l}}\left(\left(\iota_{\delta_i} \circ f_{i*}\right)\left(\left[\mathbf{log}\left(\left(\mathfrak{f}_{\vec{v},\sigma}^{(x,y),\gamma_0}\right)^{-1}\right)\right]_{<2}\right)\right). \tag{7.3.22}$$

By TABLE 7.10 and (7.3.20), the right-hand side of (7.3.22) is equal to

$$\begin{cases} 0 \qquad (\text{if } k_i = a_i), \\ \frac{1}{2}\rho_{y,\gamma_9}(\sigma) + \rho_{1-x,\gamma_8'}(\sigma) - \rho_{1-xy,\gamma_2'}(\sigma) \qquad (\text{if } k_i = b_i), \\ \frac{1}{2}\rho_{y,\gamma_9}(\sigma) \qquad (\text{if } k_i = c_i). \end{cases} \tag{7.3.23}$$

By (7.3.13), (7.3.18), (7.3.16) and (7.3.21), the left-hand side of (7.3.22) is equal to

$$\begin{cases} Li_2^{\ell}\left(\frac{x(1-y)}{y(1-x)};\gamma_4,\sigma\right) - Li_2^{\ell}(y;\gamma_9,\sigma) + Li_2^{\ell}(x;\gamma_8,\sigma) - Li_2^{\ell}\left(\frac{x}{y};\gamma_3,\sigma\right) \\ \quad -Li_2^{\ell}\left(\frac{1-y}{1-x};\gamma_6,\sigma\right) - \rho_{y,\gamma_9}(\sigma)\rho_{\frac{1-y}{1-x},\gamma_6}(\sigma) + \boldsymbol{\zeta}_2^{\ell}(\sigma) \qquad (\text{if } k_i = a_i), \\ \\ Li_2^{\ell}\left(\frac{x(1-y)^2}{y(1-x)^2};\gamma_1,\sigma\right) - Li_2^{\ell}\left(\frac{x(1-y)}{x-1};\gamma_5,\sigma\right) - Li_2^{\ell}\left(\frac{1-y}{y(x-1)};\gamma_7,\sigma\right) \\ -Li_2^{\ell}\left(\frac{x(1-y)}{y(1-x)};\gamma_4,\sigma\right) - Li_2^{\ell}\left(\frac{1-y}{1-x};\gamma_6,\sigma\right) - \frac{1}{2}\left(\rho_{y,\gamma_9}(\sigma)\right)^2 \qquad (\text{if } k_i = b_i), \\ \\ Li_2^{\ell}\left(\frac{1-y}{y(x-1)};\gamma_7,\sigma\right) + Li_2^{\ell}(xy;\gamma_2,\sigma) - Li_2^{\ell}(x;\gamma_8,\sigma) - Li_2^{\ell}(y;\gamma_9,\sigma) \\ -Li_2^{\ell}\left(\frac{x(1-y)}{x-1};\gamma_5,\sigma\right) + \boldsymbol{\zeta}_2^{\ell}(\sigma) - \rho_{y,\gamma_9}(\sigma)\rho_{\frac{1-y}{1-x},\gamma_6}(\sigma) + \frac{1}{2}\left(\rho_{y,\gamma_9}(\sigma)\right)^2 \qquad (\text{if } k_i = c_i). \end{cases} \tag{7.3.24}$$

Putting (7.3.23) and (7.3.24) into (7.3.22), we obtain the desired equations (a-ℓ), (b-ℓ) and (c-ℓ). This completes the proof of Theorem 7.3.2. □

Proof of Theorem 7.1.2. Let $\sigma \in G_K$. As well as the complex case, the tensor-homotopy criteria [NW12, Theorem 5.7, (i)$_\ell$, (ii)$_\ell$] hold:

$$\sum_{i=1}^{9} d_i \cdot \varphi_3\left(\mathrm{gr}^3_\Gamma\left(\iota_{\delta_i} \circ f_i\right)\right) = 0 \quad \text{in} \quad \mathrm{Hom}_{\mathbb{Z}_\ell}\left(\mathrm{gr}^3_\Gamma\left(\pi_1^{\ell\text{-ét}}\left(V_{\text{non-Fano}}\right), \vec{v}\right), \mathbb{Z}_\ell\right), \tag{7.3.25}$$

$$\sum_{i=1}^{9} d_i\left(f_i \otimes \left(f_i \wedge \left(f_i - 1\right)\right)\right) = 0 \quad \text{in} \quad \left(\mathcal{O}^\times/\overline{K}^\times\right) \otimes \left(\left(\mathcal{O}^\times/\overline{K}^\times\right) \wedge \left(\mathcal{O}^\times/\overline{K}^\times\right)\right), \tag{7.3.26}$$

where d_i ($i = 1, \ldots, 9$) are shown in TABLE 7.6. Therefore, we have the functional equation [NW12, Theorem 5.7 (iii)$_\ell$, Corollary 5.8]:

$$\sum_{i=1}^{9} d_i \cdot \mathcal{L}_{\text{nv}}^{\varphi_3(f_i)_{\vec{l}}}\left(f_i(x,y), f_i\left(\vec{v}\right); f_i\left(\gamma_0\right), \sigma\right) = \sum_{i=1}^{9} d_i \cdot \varphi_{3,\vec{l}}\left(\left(\iota_{\delta_i} \circ f_{i*}\right)\left(\left[\mathbf{log}\left(\left(\mathfrak{f}_{\vec{v},\sigma}^{(x,y),\gamma_0}\right)^{-1}\right)\right]_{<3}\right)\right). \tag{7.3.27}$$

By TABLE 7.10 and (7.3.20), the right-hand side of (7.3.27) is equal to

$$\begin{aligned} &- Li_2^\ell\left(\frac{x(1-y)}{x-1}; \gamma_5, \sigma\right) - Li_2^\ell\left(\frac{1-y}{y(x-1)}; \gamma_7, \sigma\right) - \frac{1}{2}\rho_{\frac{1-xy}{1-x}, \gamma_5'}(\sigma)\left(\rho_{\frac{x(1-y)^2}{y(1-x)^2}, \gamma_1}(\sigma) - 1\right) \\ &- \frac{1}{6}\rho_{y,\gamma_9}(\sigma)\left(2 - 3\rho_{\frac{1-y}{1-x}, \gamma_6}(\sigma) + 3\rho_{y,\gamma_9}(\sigma)\right) - \boldsymbol{\zeta}_2^\ell(\sigma). \end{aligned} \tag{7.3.28}$$

Putting (7.3.19) and (7.3.28) into (7.3.27) and applying (7.3.13), (7.3.16) and (7.3.21), we obtain the desired equation (d-$\mathbb{C}$). In the process, the nine Li_2^ℓ-terms appearing in the left-hand side of (7.3.27) are canceled by using (a-ℓ), (b-ℓ) and (c-ℓ). This completes the proof of Theorem 7.1.2. □

Remark 7.3.3. Let us consider a specialization of ℓ-adic Spence-Kummer's equation (*d*-ℓ) when $x \to 0$ (i.e. taking x as a tangential base point at 0) : We write

$$\begin{aligned} \hat{\gamma}_i &:= \gamma_i|_{x\to 0} \in \pi_1^{\text{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, f_i^{\text{an}}(x,y)|_{x\to 0}\right), \\ \hat{\gamma}_8'' &:= \delta_{\overrightarrow{0\infty}} \cdot \phi_{\overrightarrow{0\infty}}\left(\hat{\gamma}_8\right) \in \pi_1^{\text{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, \frac{y}{y-1}\right) \end{aligned}$$

where $\phi_{\overrightarrow{0\infty}} \in \mathrm{Aut}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}\right)$ is given by $\phi_{\overrightarrow{0\infty}}(t) = \frac{t}{t-1}$. Then we get Landen's ℓ-adic trilogarithm functional equation equivalent to [NS22, Theorem 1.1]

$$\begin{aligned} &Li_3^\ell\left(y; \hat{\gamma}_8, \sigma\right) + Li_3^\ell\left(1-y; \hat{\gamma}_8', \sigma\right) + Li_3^\ell\left(\frac{y}{y-1}; \hat{\gamma}_8'', \sigma\right) \\ &\quad = \boldsymbol{\zeta}_3^\ell(\sigma) - \boldsymbol{\zeta}_2^\ell(\sigma)\rho_{1-y,\hat{\gamma}_8'}(\sigma) + \frac{1}{2}\rho_{y,\hat{\gamma}_8}(\sigma)\left(\rho_{1-y,\hat{\gamma}_8'}(\sigma)\right)^2 - \frac{1}{6}\left(\rho_{1-y,\hat{\gamma}_8'}(\sigma)\right)^3 \\ &\quad\quad - \frac{1}{2}Li_2^\ell\left(y; \hat{\gamma}_8, \sigma\right) - \frac{1}{12}\rho_{1-y,\hat{\gamma}_8'}(\sigma) - \frac{1}{4}\left(\rho_{1-y,\hat{\gamma}_8'}(\sigma)\right)^2 \end{aligned} \tag{7.3.29}$$

by combining ℓ-adic Spence-Kummer's equation (*d*-ℓ) setting $x \to 0$, the inversion formula [NW12, (6.31)] and the conversion formula [NS22, Proposition 4.2]. By a similar computation, we can obtain Landen's complex trilogarithm functional equation [L1780], [NS22, (1.3)] from a specialization of Spence-Kummer's equation (*d*-$\mathbb{C}$) setting $x \to 0$ and the inversion formula [NW12, (6.26)].

Corollary 7.3.4 (Functional equations for ℓ-adic Galois polylogarithmic characters)**.** Given a K-rational point $(x, y) \in V_{\text{non-Fano}}(K)$ and a path $\gamma_0 \in \pi_1^{\text{top}}(V^{\text{an}}_{\text{non-Fano}}; \vec{v}, (x, y))$, define the path system $\{\gamma_i\}_{i=1,\ldots,9}$ associated to γ_0 as in (7.1.4). For any $\sigma \in G_K$, the following holds.

(a'-ℓ) ℓ-adic Schaeffer's equation

$$\begin{aligned}
&\tilde{\chi}_2^{\frac{x(1-y)}{y(1-x)},\gamma_4}(\sigma) - \tilde{\chi}_2^{y,\gamma_9}(\sigma) + \tilde{\chi}_2^{x,\gamma_8}(\sigma) - \tilde{\chi}_2^{\frac{x}{y},\gamma_3}(\sigma) - \tilde{\chi}_2^{\frac{1-y}{1-x},\gamma_6}(\sigma) \\
&= \rho_{\frac{1-y}{1-x},\gamma_6}(\sigma)\rho_{y,\gamma_9}(\sigma) - \tilde{\chi}_2^{\overrightarrow{10},\delta_{\overrightarrow{10}}}(\sigma).
\end{aligned}$$

(b'-ℓ) ℓ-adic Kummer's equation

$$\begin{aligned}
&\tilde{\chi}_2^{\frac{x(1-y)^2}{y(1-x)^2},\gamma_1}(\sigma) - \tilde{\chi}_2^{\frac{x(1-y)}{x-1},\gamma_5}(\sigma) - \tilde{\chi}_2^{\frac{1-y}{y(x-1)},\gamma_7}(\sigma) - \tilde{\chi}_2^{\frac{x(1-y}{)}y(1-x),\gamma_4}(\sigma) - \tilde{\chi}_2^{\frac{1-y}{1-x},\gamma_6}(\sigma) \\
&= \frac{1}{2}\left(\rho_{y,\gamma_9}(\sigma)\right)^2 - \frac{1}{2}\rho_{y,\gamma_9}(\sigma) - \rho_{1-x,\gamma_8'}(\sigma) + \rho_{1-xy,\gamma_2'}(\sigma) \\
&\quad + \left(\frac{\chi(\sigma)-1}{2}\right)\left(2\rho_{1-xy,\gamma_2'}(\sigma) - 2\rho_{1-x,\gamma_8'}(\sigma) - \rho_{y,\gamma_9}(\sigma)\right).
\end{aligned}$$

(c'-ℓ) ℓ-adic Hill's equation

$$\begin{aligned}
&\tilde{\chi}_2^{\frac{1-y}{y(x-1)},\gamma_7}(\sigma) + \tilde{\chi}_2^{xy,\gamma_2}(\sigma) - \tilde{\chi}_2^{x,\gamma_8}(\sigma) - \tilde{\chi}_2^{y,\gamma_9}(\sigma) - \tilde{\chi}_2^{\frac{1-y}{y(x-1)},\gamma_7}(\sigma) \\
&= -\tilde{\chi}_2^{\overrightarrow{10},\delta_{\overrightarrow{10}}}(\sigma) + \rho_{\frac{1-y}{1-x},\gamma_6}(\sigma)\rho_{y,\gamma_9}(\sigma) - \frac{1}{2}\left(\rho_{y,\gamma_9}(\sigma)\right)^2 + \frac{1}{2}\rho_{y,\gamma_9}(\sigma) \\
&\quad + \left(\frac{\chi(\sigma)-1}{2}\right)\rho_{y,\gamma_9}(\sigma).
\end{aligned}$$

(d'-ℓ) ℓ-adic Spence-Kummer's equation

$$\begin{aligned}
&\tilde{\chi}_3^{\frac{x(1-y)^2}{y(1-x)^2},\gamma_1}(\sigma) + \tilde{\chi}_3^{xy,\gamma_2}(\sigma) + \tilde{\chi}_3^{\frac{x}{y},\gamma_3}(\sigma) - 2\tilde{\chi}_3^{\frac{x(1-y)}{y(1-x)},\gamma_4}(\sigma) - 2\tilde{\chi}_3^{\frac{x(1-y)}{x-1},\gamma_5}(\sigma) \\
&- 2\tilde{\chi}_3^{\frac{1-y}{1-x},\gamma_6}(\sigma) - 2\tilde{\chi}_3^{\frac{1-y}{y(x-1)},\gamma_7}(\sigma) - 2\tilde{\chi}_3^{x,\gamma_8}(\sigma) - 2\tilde{\chi}_3^{y,\gamma_9}(\sigma) + 2\tilde{\chi}_3^{\overrightarrow{10},\delta_{\overrightarrow{10}}}(\sigma) \\
&= -2\rho_{y,\gamma_9}(\sigma)^2\rho_{\frac{1-y}{1-x},\gamma_6}(\sigma) + \left(\frac{1-\chi(\sigma)^2}{2}\right)\rho_{y,\gamma_9}(\sigma) + \frac{2}{3}\rho_{y,\gamma_9}(\sigma)^3 \\
&\quad + 2\chi(\sigma)\left(\tilde{\chi}_2^{\frac{x(1-y)}{x-1},\gamma_5}(\sigma) + \tilde{\chi}_2^{\frac{1-y}{y(x-1)},\gamma_7}(\sigma)\right) + \chi(\sigma)^2\rho_{\frac{1-xy}{1-x},\gamma_5'}(\sigma) - \frac{2}{3}\rho_{y,\gamma_9}(\sigma).
\end{aligned}$$

Proof. Putting the conversion formula [NS22, Proposition 4.2 (ii)] into (a-ℓ), (b-ℓ), (c-ℓ) and (d-ℓ), we obtain the desired equations (a'-ℓ), (b'-ℓ), (c'-ℓ) and (d'-ℓ). □

The above functional equations (a',b',c',d'-ℓ) enable us to check the $\mathbb{Z}_\ell$-integrality of both sides of each equation. By definition and $\chi(\sigma) \equiv 1 \bmod 2$, we can easily see that the right-hand sides of $(a', b', c'\text{-}\ell)$ have no denominator. The right-hand side of $(d'\text{-}\ell)$ is equal to

$$\begin{aligned}
&-2\rho_{y,\gamma_9}(\sigma)^2\rho_{\frac{1-y}{1-x},\gamma_6}(\sigma) - 12\tilde{\chi}_2^{\overrightarrow{10},\delta_{\overrightarrow{10}}}(\sigma)\rho_{y,\gamma_9}(\sigma) + 2\chi(\sigma)\left(\tilde{\chi}_2^{\frac{x(1-y)}{x-1},\gamma_5}(\sigma) + \tilde{\chi}_2^{\frac{1-y}{y(x-1)},\gamma_7}(\sigma)\right) \\
&+\chi(\sigma)^2\rho_{\frac{1-xy}{1-x},\gamma_5'}(\sigma) - \frac{2}{3}\rho_{y,\gamma_9}(\sigma)\left(1-\rho_{y,\gamma_9}(\sigma)\right)\left(1+\rho_{y,\gamma_9}(\sigma)\right),
\end{aligned}$$

so it has also no denominator.

Chapter 8

Special values of mod ℓ Galois dilogarithms and triple ℓ-th power residue symbols

8.1 Main results

Let K be a number field with its algebraic closure $\overline{K}$ in the complex number field $\mathbb{C}$. for a prime number ℓ. let $\zeta_\ell := \exp\left(\frac{2\pi\sqrt{-1}}{\ell}\right)$ the primitive ℓ-th root of unity in $\overline{K}$.

Following the analogy between knots and primes, M. Morishita introduced the mod ℓ Milnor invariant

$$\mu_\ell(123) \in \mathbb{Z}/\ell\mathbb{Z}$$

for prime ideals $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ of $\mathbb{Q}(\zeta_\ell)$ satisfying certain conditions for $\ell = 2, 3$, as the arithmetic analog of the Milnor invariant of links ([Mo], [AMM]). The *triple ℓ-th power residue symbol* is defined by

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_\ell := \zeta_\ell^{\mu_\ell(123)},$$

which controls the decomposition law of $\mathfrak{p}_3$ in a certain Heisenberg extension $R^{(\ell)}_{\mathfrak{p}_1,\mathfrak{p}_2}/\mathbb{Q}(\zeta_\ell)$.

On the other hand, we consider the mod ℓ Galois polylogarithm

$$\tilde{\chi}^{z,\gamma}_{\ell,n} : G_K \to \mathbb{Z}/\ell\mathbb{Z},$$

that is the composite of the ℓ-adic Galois polylogarithmic character $\tilde{\chi}^{z,\gamma}_n : G_K \to \mathbb{Z}_\ell$ introduced in (4.1.5) and the natural projection $\mathbb{Z}_\ell \twoheadrightarrow \mathbb{Z}/\ell\mathbb{Z}$. In this chapter, we relate $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_\ell$ to the mod ℓ Galois dilogarithm $\tilde{\chi}^{z,\gamma}_{\ell,2} : G_K \to \mathbb{Z}/\ell\mathbb{Z}$ for $\ell = 2, 3$ as follows:

Theorem 8.1.1 (Main formula)**.** For $\ell \in \{2, 3\}$, let $\mathfrak{p}_i = (p_i)$ $(i = 1, 2, 3)$ be three prime ideals of $\mathbb{Q}(\zeta_\ell)$ satisfying certain conditions. Let $R^{(\ell)}_{p_1,p_2}$ be the unique finite Galois extensiton of $\mathbb{Q}(\zeta_\ell)$ in $\mathbb{C}$ in which only primes $\mathfrak{p}_1, \mathfrak{p}_2$ are ramified with ramification index ℓ and whose Galois group is the Hisenberg group $H_3(\mathbb{Z}/\ell\mathbb{Z})$. Take a $\mathbb{Q}(\zeta_\ell)$-rational point (x, y, w) of

$$\mathrm{Spec}\left(\frac{\mathbb{Q}(\zeta_\ell)[x, y, w]}{(x^\ell + (-1)^{\ell+1}p_1 y^\ell = p_2 w^\ell)}\right).$$

We put

$$K := \mathbb{Q}\left(\zeta_\ell, \sqrt[\ell]{p_1}, \sqrt[\ell]{p_2}\right)$$

and choose a K-rational point of $\mathbb{P}^1\backslash\{0,1,\infty\}$

$$z := p_1\left(-\frac{y}{x}\right)^\ell \in K\backslash\{0,1\}.$$

Let

$$\tilde{\sigma}_{\mathfrak{p}_3} \in \mathrm{Gal}(\overline{K}/K) \tag{8.1.1}$$

be an extension of the Frobenius substitution $\sigma_{\mathfrak{p}_3} := \mathrm{Frob}_{\tilde{\mathfrak{p}}_3} \in \mathrm{Gal}\left(R^{(\ell)}_{p_1,p_2}/K\right)$ where $\tilde{\mathfrak{p}}_3$ is a prime ideal of K above $\mathfrak{p}_3$. Then the following holds:
(1) The special value $\tilde{\chi}^{z,\gamma}_{\ell,2}(\tilde{\sigma}_{\mathfrak{p}_3}) \in \mathbb{Z}/\ell\mathbb{Z}$ does not depend on $\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, z\right)$;
(2) We have

$$[\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3]_\ell = \frac{\tilde{\sigma}_{\mathfrak{p}_3}\left(x^{\frac{1}{2}(\ell-1)}\right)}{x^{\frac{1}{2}(\ell-1)}} \cdot \zeta_\ell^{\tilde{\chi}^{z,\gamma}_{\ell,2}(\tilde{\sigma}_{\mathfrak{p}_3})}. \tag{8.1.2}$$

As an arithmetic application of this result (8.1.2), we derive a reciprocity law of $[\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3]_\ell$ due to Rédei [Ré] and Amano-Mizusawa-Morishita [AMM] in the form

$$[\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3]_\ell \cdot [\mathfrak{p}_2,\mathfrak{p}_1,\mathfrak{p}_3]_\ell = 1 \qquad (\ell = 2,3) \tag{8.1.3}$$

from a ℓ-adic Galois dilogarithm functional equation due to Nakamura-Wojtkowiak [NW2]:

$$\tilde{\chi}^{z,\gamma}_2(\sigma) + \tilde{\chi}^{1-z,\gamma'}_2(\sigma) + \rho_{z,\gamma}(\sigma)\rho_{1-z,\gamma'}(\sigma) = \frac{1}{24}\left(\chi_\ell(\sigma)^2 - 1\right). \tag{8.1.4}$$

(See Corollary 8.3.9 for details.) Thus, by using a functional equation of ℓ-adic Galois polylogarithms, we have another proof of a reciprocity law of triple ℓ-th power residue symbols. This fact is an indication that the Galois action mentioned at the beginning of this introduction has abundant arithmetic information.

8.2 Triple ℓ-th power residue symbols for $\ell = 2,3$

For $\ell = 2,3$, the triple ℓ-th power residue symbol is well-defined at present in [Mo], [AMM]. Following [HM; Section 4], [Mo], [AMM], we recall the definition and some properties of triple ℓ-th power residue symbols for $\ell = 2,3$.

8.2.1 Case of $\ell = 2$

Take distinct prime numbers p_1, p_2 satisfying

$$\left(\frac{p_i}{p_j}\right) = 1, \quad p_i \equiv 1 \bmod 4 \qquad (1 \le i \ne j \le 2). \tag{8.2.1}$$

Then there exist $x, y, w \in \mathbb{Z}$ satisfying the conditions [Am; Lemma 1.1]:

$$x^2 - p_1 y^2 = p_2 w^2, \quad \gcd(x,y,w) = 1, \tag{8.2.2}$$

$$x \equiv y+1 \bmod 4, \quad y \equiv 0 \bmod 2.$$

The choice of $(x, y, w) \in \mathbb{Z}^3$ is not unique. For such a pair (x, y), we set

$$\theta^{(2)}_{p_1,p_2} := x + \sqrt{p_1}y. \tag{8.2.3}$$

Let

$$R^{(2)}_{p_1,p_2} := \mathbb{Q}\left(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\theta^{(2)}_{p_1,p_2}}\right), \tag{8.2.4}$$

$$K^{(2)}_{p_1,p_2} := \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}). \tag{8.2.5}$$

Theorem 8.2.1 ([Am; Theorem 1.2, Corollary 1.5])**.** The field extension $R^{(2)}_{p_1,p_2}/\mathbb{Q}$ is finite Galois and satisfies the following properties:
(i) $\mathrm{Gal}\left(R^{(2)}_{p_1,p_2}/\mathbb{Q}\right)$ is the mod 2 Heisenberg group

$$H_3(\mathbb{Z}/2\mathbb{Z}) := \left\{ \left(\begin{array}{ccc} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{array} \right) \middle| \, a, b, c \in \mathbb{Z}/2\mathbb{Z} \right\};$$

(ii) p_1, p_2 are only prime numbers ramified in $R^{(2)}_{p_1,p_2}/\mathbb{Q}$ with ramification index 2;
(iii) $R^{(2)}_{p_1,p_2}$ does not depend on the choice of (x, y, w). Thus, $R^{(2)}_{p_1,p_2}/\mathbb{Q}$ depends only on the set $\{p_1, p_2\}$.

Theorem 8.2.2 (An arithmetic characterization of $R^{(2)}$; [Am; Theorem 2.1])**.** Take prime numbers p_1, p_2 ($p_1 \neq p_2$) satisfying (8.2.1). For a field extension $L/\mathbb{Q}$, the following conditions are equivalent:
(1) $L = R^{(2)}_{p_1,p_2}$;
(2) $L/\mathbb{Q}$ is a finite Galois extension with Galois group $H_3(\mathbb{Z}/2\mathbb{Z})$ and p_1, p_2 are only prime numbers that ramified with a ramification index 2.

Take another prime number p_3 such that $p_3 \equiv 1 \bmod 4$ and

$$\left(\frac{p_i}{p_j}\right) = 1 \qquad (1 \leq i \neq j \leq 3). \tag{8.2.6}$$

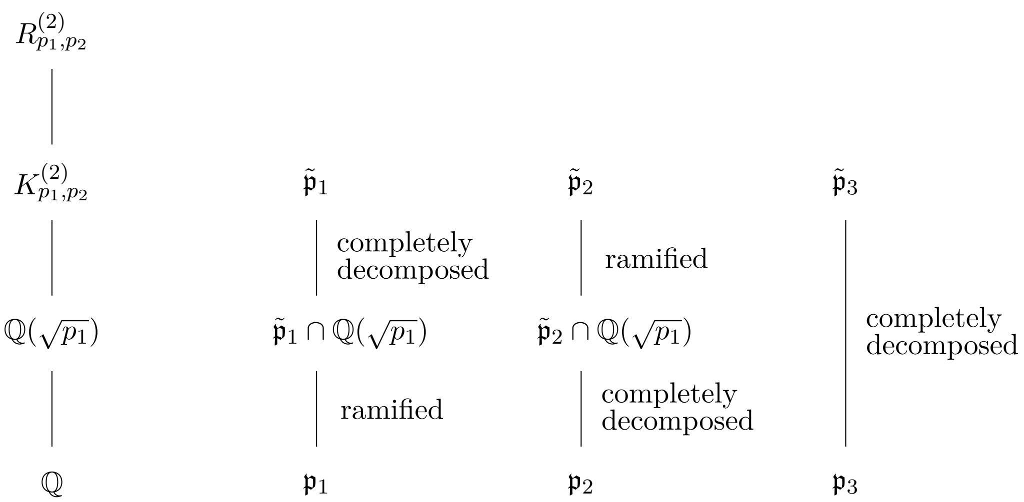
We define an arithmetic symbol which controls the decomposition of p_3 in the Heisenberg extension $R^{(2)}/\mathbb{Q}$. Set $\mathfrak{p}_i := (p_i)$ for $i \in \{1, 2, 3\}$.

Definition 8.2.3 (Triple quadratic residue symbol; [Mo; Section 8.4])**.** For a triple $(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$ of prime ideals of $\mathbb{Z}$ satisfying (8.2.1) and (8.2.6), the triple quadratic residue symbol is defined by

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_2 := (-1)^{\mu_2(123)} \in \{1, -1\},$$

where $\mu_2(123) \in \mathbb{Z}/2\mathbb{Z}$ is the mod 2 Milnor invariant of $(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$ (cf. [Mo; Section 8.4]).

Let $\tilde{\mathfrak{p}}_i$ be a prime ideal of $K^{(2)}_{p_1,p_2}$ lying over $\mathfrak{p}_i$. By (8.2.1) and (8.2.6), the prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are ramified or completely decomposed in $K^{(2)}_{p_1,p_2}/\mathbb{Q}$ as follows.



Theorem 8.2.4 ([Mo; Section 8.4, Theorem 8.25])**.** Let $\sigma_{\mathfrak{p}_3} := \mathrm{Frob}_{\tilde{\mathfrak{p}}_3} \in \mathrm{Gal}\left(R^{(2)}_{p_1,p_2}/K^{(2)}_{p_1,p_2}\right)$ be the Frobenius substitution of $\tilde{\mathfrak{p}}_3$ in $R^{(2)}_{p_1,p_2}/K^{(2)}_{p_1,p_2}$. Then we have

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_2 = \frac{\sigma_{\mathfrak{p}_3}(\sqrt{\theta^{(2)}_{p_1,p_2}})}{\sqrt{\theta^{(2)}_{p_1,p_2}}}.$$

In particular, $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_2 = 1$ if and only if $\mathfrak{p}_3$ is completely decomposed in $R^{(2)}_{p_1,p_2}/\mathbb{Q}$.

Remark 8.2.5. The triple symbol $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_2$ is nothing but the Rédei symbol [Ré]:

$$[p_1, p_2, p_3]_{\text{Rédei}} := \frac{\sigma_{\mathfrak{p}_3}(\sqrt{\theta^{(2)}_{p_1,p_2}})}{\sqrt{\theta^{(2)}_{p_1,p_2}}}.$$

L. Rédei proved the reciprocity law of the triple symbol ([Ré]). F. Aamano gave another simple proof of Rédei's reciprocity law ([Am]).

Theorem 8.2.6 (Reciprocity law of triple quadratic residue symbols; [Ré], [Am])**.** For any permutation $\rho \in S_3$ of $\{1, 2, 3\}$, we have

$$[(p_1), (p_2), (p_3)]_2 = [(p_{\rho(1)}), (p_{\rho(2)}), (p_{\rho(3)})]_2.$$

Thus $[(p_1), (p_2), (p_3)]_2 \cdot [(p_{\rho(1)}), (p_{\rho(2)}), (p_{\rho(3)})]_2 = 1$.

8.2.2 Case of $\ell = 3$

In this section, we follow [HM, Section 4.2] for various assumptions. Let the Eisenstein field $k := \mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$ where $\zeta_3 := \exp(\frac{2\pi\sqrt{-1}}{3}) = \frac{-1+\sqrt{-3}}{2}$. Take distinct prime ideals $\mathfrak{p}_i = (p_i)$ $(i = 1, 2)$ of k satisfying

$$\mathrm{N}\mathfrak{p}_i \equiv 1 \bmod 9, \quad \left(\frac{p_i}{p_j}\right)_3 = 1 \qquad (1 \le i \ne j \le 2). \tag{8.2.7}$$

Following [AMM, Corollary 5.9], [HM, Section 4.2], we suppose that

$$\text{each } \mathfrak{p}_i \text{ is generated by a rational prime number.} \tag{8.2.8}$$

Let p_i be the unique prime element such that

$$p_i \equiv 1 \bmod (3\sqrt{-3}). \tag{8.2.9}$$

Let $K_i := k(\sqrt[3]{p_i})$. Let ϕ be the generator of $\mathrm{Gal}(K_1/k)$ defined by $\phi(\sqrt[3]{p_1}) = \zeta_3\sqrt[3]{p_1}$. By (8.2.7) and (8.2.9), there is

$$\alpha_{p_1,p_2} \in \mathcal{O}_{K_1} \tag{8.2.10}$$

together with $w \in \mathbb{Z}[\zeta_3]$ and prime ideals $\mathfrak{P}, \mathfrak{B}$ of K_1 satisfying

$$N_{K_1/k}(\alpha_{p_1,p_2}) = p_2 w^3, \;\; (\alpha_{p_1,p_2}) = \mathfrak{P}^e\mathfrak{B}^f, \tag{8.2.11}$$
$$f \equiv 0 \bmod 3, \; (e,3) = 1, \; (\mathfrak{B}, 3) = 1.$$

The choice of α_{p_1,p_2} is not unique. We set

$$\theta^{(3)}_{p_1,p_2} := \phi(\alpha_{p_1,p_2})(\phi^2(\alpha_{p_1,p_2}))^2. \tag{8.2.12}$$

Let

$$R^{(3)}_{p_1,p_2} := k(\sqrt[3]{p_1}, \sqrt[3]{p_2}, \sqrt[3]{\theta^{(3)}_{p_1,p_2}}), \tag{8.2.13}$$

$$K^{(3)}_{p_1,p_2} := k(\sqrt[3]{p_1}, \sqrt[3]{p_2}). \tag{8.2.14}$$

By (8.2.8), we have the following theorem.

Theorem 8.2.7 ([AMM; Theorem 5.11, Corollary 5.12])**.** The field extension $R^{(3)}_{p_1,p_2}/k$ is finite Galois and satisfies the following properties:
(i) $\mathrm{Gal}\left(R^{(3)}_{p_1,p_2}/k\right)$ is the mod 3 Heisenberg group

$$H_3(\mathbb{Z}/3\mathbb{Z}) := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \; a, b, c \in \mathbb{Z}/3\mathbb{Z} \right\};$$

(ii) $\mathfrak{p}_1, \mathfrak{p}_2$ are only prime ideals that ramified in $R^{(3)}_{p_1,p_2}/k$ with ramification index 3;
(iii) $R^{(3)}_{p_1,p_2}$ does not depend on the choice of α_{p_1,p_2}. Thus $R^{(3)}_{p_1,p_2}/k$ depends only on $\{\mathfrak{p}_1, \mathfrak{p}_2\}$.

Theorem 8.2.8 (An arithmetic characterization of $R^{(3)}$; cf. [AMM; Corollary 5.12])**.** Take prime ideals $\mathfrak{p}_1 = (p_1), \mathfrak{p}_2 = (p_2)$ ($\mathfrak{p}_1 \neq \mathfrak{p}_2$) satisfying (8.2.7), (8.2.8), (8.2.9) and (8.2.15). For a finite extension L/k, the following conditions are equivalent:
(1) $L = R^{(3)}_{p_1,p_2}$;
(2) L/k is a finite Galois extension with Galois group $H_3(\mathbb{Z}/3\mathbb{Z})$ and $\mathfrak{p}_1, \mathfrak{p}_2$ are only prime ideals that ramified in L/k with ramification index 3.

Take another prime ideal $\mathfrak{p}_3 = (p_3)$ of k such that $\mathrm{N}\mathfrak{p}_3 \equiv 1 \bmod 9$ and

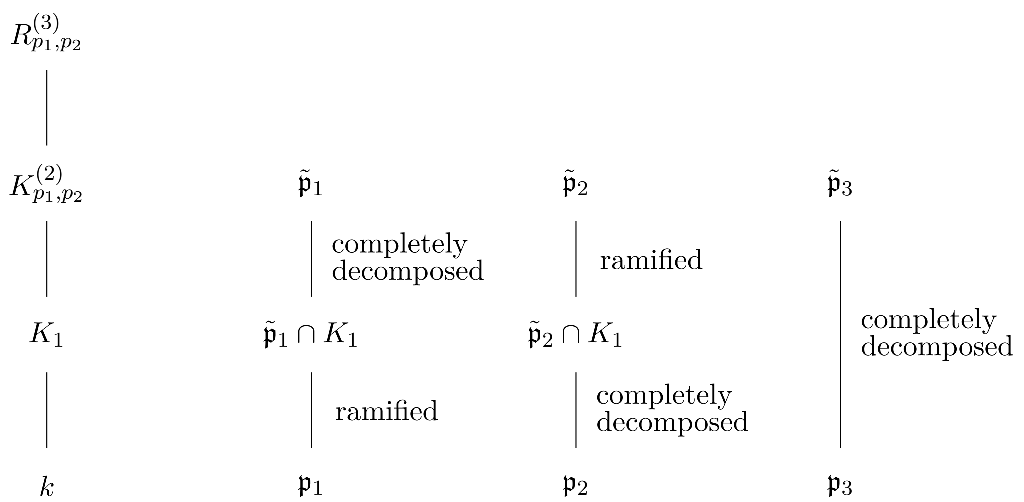
$$\left(\frac{p_i}{p_j}\right)_3 = 1 \quad (1 \leq i \neq j \leq 3). \tag{8.2.15}$$

Definition 8.2.9 (Triple cubic residue symbol ; [AMM; Definition 6.2])**.** For a triple $(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$ of prime ideals of k satisfying (8.2.7) and (8.2.15), the triple cubic residue symbol is defined by

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 := \zeta_3^{\mu_3(123)} \in \{1, \zeta_3, \zeta_3^{-1}\},$$

where $\mu_3(123) \in \mathbb{Z}/3\mathbb{Z}$ is the mod 3 Milnor invariant of $(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$ (cf. [AMM; (2.3) of Chapter 2, Theorem 4.4]).

Take a prime ideal $\tilde{\mathfrak{p}}_i$ of $K^{(3)}_{p_1,p_2}$ lying over $\mathfrak{p}_i$. By (8.2.7) and (8.2.15), prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are ramified or completely decomposed in $K^{(3)}_{p_1,p_2}/k$ as follows.



Theorem 8.2.10 ([AMM; Theorem.6.3])**.** Let $\sigma_{\mathfrak{p}_3} := \mathrm{Frob}_{\tilde{\mathfrak{p}}_3} \in \mathrm{Gal}\left(R^{(3)}_{p_1,p_2}/K^{(3)}_{p_1,p_2}\right)$ be the Frobenius substitution of $\tilde{\mathfrak{p}}_3$ in $R^{(3)}_{p_1,p_2}/K^{(3)}_{p_1,p_2}$. Then

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 = \frac{\sigma_{\mathfrak{p}_3}(\sqrt[3]{\theta^{(3)}_{p_1,p_2}})}{\sqrt[3]{\theta^{(3)}_{p_1,p_2}}}.$$

Hence, $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 = 1$ if and only if $\mathfrak{p}_3$ is completely decomposed in $R^{(3)}_{p_1,p_2}/k$.

Theorem 8.2.11 (a reciprocity law of triple cubic residue symbols; [AMM; Proposition 6.5])**.** We have

$$[\mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_3]_3 = [\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3^{-1},$$

that is $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 \cdot [\mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_3]_3 = 1$.

Following Hirano-Morishita [HM, Section 4.2], we restrict to the case with

<u>Assumption (A)</u> : The element $\alpha_{p_1,p_2} \in \mathcal{O}_{K_1}$ in (8.2.10) and (8.2.11) has the form $\alpha_{p_1,p_2} = x + y\sqrt[3]{p_1} \quad (x, y \in k)$.

By this assumption (A), the equalities (8.2.11), (8.2.12) are equivalent to

$$x^3 + p_1 y^3 = p_2 w^3, \tag{8.2.16}$$

$$\theta^{(3)}_{p_1,p_2} = (x + \zeta_3 y\sqrt[3]{p_1})(x + \zeta_3^2 y\sqrt[3]{p_1})^2. \tag{8.2.17}$$

These will play an important role in the next section.

8.3 Triple ℓ-th power residue symbols and mod ℓ Galois dilogarithms

In this section, we describe triple ℓ-th power residue symbols in terms of mod ℓ Galois dilogarithms for $\ell = 2, 3$. As a result, we derive a reciprocity law of triple ℓ-th power residue symbols from the Euler-type functional equation of the ℓ-adic Galois dilogarithm.

8.3.1 Main formula

Let $\ell \in \{2, 3\}$. Consider $k := \begin{cases} \mathbb{Q} & (\text{if } \ell = 2), \\ \mathbb{Q}(\zeta_3) & (\text{if } \ell = 3) \end{cases}$ and

$$K := \mathbb{Q}(\zeta_\ell)\left(\sqrt[\ell]{p_1}, \sqrt[\ell]{p_2}\right). \tag{8.3.1}$$

We set

$$p_i \in \mathbb{Z}[\zeta_\ell] \quad (i = 1, 2, 3), \quad x, y, w \in k, \quad \theta^{(\ell)}_{p_1,p_2}, \ R^{(\ell)}_{p_1,p_2}, \ K^{(\ell)}_{p_1,p_2}$$

as in Section 8.2.1 and Section 8.2.2 with the assumption (A). By (8.2.5) and (8.2.14), we have $K = K^{(\ell)}_{p_1,p_2}$. By (8.2.3), (8.2.17) and (8.2.2), (8.2.16), we obtain

$$x^\ell - (-y)^\ell p_1 = w^\ell p_2, \tag{8.3.2}$$

$$\theta^{(\ell)}_{p_1,p_2} = \prod_{i=0}^{\ell-1} (x + \zeta_\ell^i y\sqrt[\ell]{p_1})^i. \tag{8.3.3}$$

Let $\mathfrak{p}_i = (p_i)$ $(i = 1, 2, 3)$. Then as discussed in Section 8.2, the triple ℓ-th power residue symbol $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_\ell$ is defined. We choose

$$z := p_1 \left(-\frac{y}{x}\right)^\ell \in K\backslash\{0, 1\}. \tag{8.3.4}$$

We regard z as a K-rational point of $\mathbb{P}^1\backslash\{0,1,\infty\}$. For a prime ideal $\tilde{\mathfrak{p}}_3$ of K lying over $\mathfrak{p}_3$, let

$$\tilde{\sigma}_{\mathfrak{p}_3} \in \mathrm{Gal}(\overline{K}/K) \tag{8.3.5}$$

be an extension of $\sigma_{\mathfrak{p}_3} := \mathrm{Frob}_{\tilde{\mathfrak{p}}_3} \in \mathrm{Gal}\left(R^{(\ell)}_{p_1,p_2}/K\right)$. Take a path

$$\gamma \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, z\right).$$

Since z is K-rational, the mod ℓ Galois dilogarithm $\tilde{\chi}^{z,\gamma}_{\ell,2} : G_K \to \mathbb{Z}_\ell$ is defined.

Proposition 8.3.1. Let the assumptions and notations be as above. For any $\tau \in G_K$, the special value

$$\tilde{\chi}^{z,\gamma}_{\ell,2}(\tau) \in \mathbb{Z}/\ell\mathbb{Z}$$

does not depend on the choice of γ.

Proof. For $\tau \in G_K$, we have $\chi(\tau) \equiv 1, \rho_{z,\gamma}(\tau) \equiv 0 \bmod \ell$ by (8.3.1). It follows from Definition 4.1.5 that

$$\zeta_\ell^{\tilde{\chi}^{z,\gamma}_{\ell,2}(\tau)} = \tau\left(\prod_{i=0}^{\ell-1}(1-\zeta_\ell^i z^{1/\ell})^{\frac{i}{\ell}}\right) \Big/ \prod_{i=0}^{\ell-1}(1-\zeta_\ell^i z^{1/\ell})^{\frac{i}{\ell}}. \tag{8.3.6}$$

Let $\gamma_0, \gamma_1 \in \pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, z)$. For $\epsilon \in \{0,1\}$, we take $z_\epsilon^{1/\ell}$, $(1-\zeta_\ell^i z_\epsilon^{1/\ell})$ along γ_ϵ (cf. [NW1]). In order to show that $\tilde{\chi}^{z,\gamma}_{\ell,2}(\tau)$ is independent of the choice of γ, it suffices to show

$$\zeta_\ell^{\tilde{\chi}^{z,\gamma_0}_{\ell,2}(\tau)} = \zeta_\ell^{\tilde{\chi}^{z,\gamma_1}_{\ell,2}(\tau)} \tag{8.3.7}$$

by comparing the right hand side of (8.3.6) for $\gamma = \gamma_0, \gamma_1$. Now we show (8.3.7). Let

$$A_\epsilon := \prod_{i=0}^{\ell-1}(1-\zeta_\ell^i z_\epsilon^{1/\ell})^{\frac{i}{\ell}} \ (\epsilon \in \{0,1\}).$$

Let $s \in \mathbb{Z}/\ell\mathbb{Z}$ such that $z_1^{1/\ell} = \zeta_\ell^{-s} \cdot z_0^{1/\ell}$. For each $i \in \mathbb{Z}/\ell\mathbb{Z}$, take $t_i \in \mathbb{Z}/\ell\mathbb{Z}$ such that

$$(1-\zeta_\ell^i z_1^{1/\ell})^{\frac{1}{\ell}} = \zeta_\ell^{t_i}(1-\zeta_\ell^{i-s} z_0^{1/\ell})^{\frac{1}{\ell}}.$$

Then

$$\begin{aligned}
\frac{A_1}{A_0} &= \frac{\prod_{i=0}^{\ell-1}(1-\zeta_\ell^i z_0^{1/\ell})^{\frac{i}{\ell}} \cdot \zeta_\ell^{\sum_{i=0}^{\ell-1} it_i}}{\prod_{i=0}^{\ell-1}(1-\zeta_\ell^i z_0^{1/\ell})^{\frac{i}{\ell}}} \\
&= \frac{\prod_{j=0}^{\ell-1}(1-\zeta_\ell^j z_0^{1/\ell})^{\frac{j+s}{\ell}}}{\prod_{i=0}^{\ell-1}(1-\zeta_\ell^i z_0^{1/\ell})^{\frac{i}{\ell}}} \cdot \zeta_\ell^{\sum_{i=0}^{\ell-1} it_i} \\
&= \left(\prod_{j=0}^{\ell-1}(1-\zeta_\ell^j z_0^{1/\ell})^{\frac{1}{\ell}}\right)^s \cdot \zeta_\ell^{\sum_{i=0}^{\ell-1} it_i}.
\end{aligned}$$

By (8.3.1), (8.3.2), and (8.3.4), we have $\prod_{j=0}^{\ell-1}(1-\zeta_\ell^j z_0^{1/\ell}) = 1-z = \left(\frac{w}{x}\right)^\ell p_2$ and $K = \mathbb{Q}(\zeta_\ell)(\sqrt[\ell]{p_1}, \sqrt[\ell]{p_2})$. So it follows that $\frac{A_1}{A_0} \in K$. hence

$$\zeta_\ell^{\tilde{\chi}_{\ell,2}^{z,\gamma_0}(\tau)} = \frac{\tau(A_0)}{A_0} = \frac{\tau(A_1)}{A_1} = \zeta_\ell^{\tilde{\chi}_{\ell,2}^{z,\gamma_1}(\tau)}.$$

This completes the proof of (8.3.7).

□

Definition 8.3.2. Let the assumptions and notations be as above. Based on Proposition 8.3.1, we let

$$\tilde{\chi}_{\ell,2}^{z}(\tau) := \tilde{\chi}_{\ell,2}^{z,\gamma}(\tau) \quad (\tau \in G_K).$$

Now we shall describe $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_\ell$ by a special value of the mod ℓ Galois dilogarithm.

Theorem 8.3.3. Let the notations and assumptions be as above. For $\ell \in \{2, 3\}$, we have

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_\ell = \frac{\tilde{\sigma}_{\mathfrak{p}_3}(x^{\frac{1}{2}(\ell-1)})}{x^{\frac{1}{2}(\ell-1)}} \cdot \zeta_\ell^{\tilde{\chi}_{\ell,2}^{z}(\tilde{\sigma}_{\mathfrak{p}_3})}.$$

Proof. Let $\ell \in \{2, 3\}$. Then

$$\begin{aligned}
[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_\ell &= \sigma_{\mathfrak{p}_3}\left(\sqrt[\ell]{\theta_{p_1,p_2}^{(\ell)}}\right) / \sqrt[\ell]{\theta_{p_1,p_2}^{(\ell)}} \quad \text{(by Theorem 8.2.4, Theorem 8.2.10)} \\
&= \tilde{\sigma}_{\mathfrak{p}_3}\left(\sqrt[\ell]{\theta_{p_1,p_2}^{(\ell)}}\right) / \sqrt[\ell]{\theta_{p_1,p_2}^{(\ell)}} \\
&= \tilde{\sigma}_{\mathfrak{p}_3}\left(\prod_{i=0}^{\ell-1}(x+\zeta_\ell^i y\sqrt[\ell]{p_1})^{\frac{i}{\ell}}\right) \Big/ \prod_{i=0}^{\ell-1}\left(x+\zeta_\ell^i y\sqrt[\ell]{p_1}\right)^{\frac{i}{\ell}} \quad \text{(by (8.3.3))} \\
&= \frac{\tilde{\sigma}_{\mathfrak{p}_3}\left(\prod_{i=0}^{\ell-1} x^{\frac{i}{\ell}}\right)}{\prod_{i=0}^{\ell-1} x^{\frac{i}{\ell}}} \cdot \frac{\tilde{\sigma}_{\mathfrak{p}_3}\left(\prod_{i=0}^{\ell-1}\left(1+\zeta_\ell^i \frac{y}{x} p_1^{1/\ell}\right)^{\frac{i}{\ell}}\right)}{\prod_{i=0}^{\ell-1}\left(1+\zeta_\ell^i \frac{y}{x} p_1^{1/\ell}\right)^{\frac{i}{\ell}}} \\
&= \frac{\tilde{\sigma}_{\mathfrak{p}_3}\left(x^{\frac{1}{2}(\ell-1)}\right)}{x^{\frac{1}{2}(\ell-1)}} \cdot \frac{\tilde{\sigma}_{\mathfrak{p}_3}\left(\prod_{i=0}^{\ell-1}\left(1+\zeta_\ell^i \frac{y}{x} p_1^{1/\ell}\right)^{\frac{i}{\ell}}\right)}{\prod_{i=0}^{\ell-1}\left(1+\zeta_\ell^i \frac{y}{x} p_1^{1/\ell}\right)^{\frac{i}{\ell}}}.
\end{aligned}$$

By (8.3.4), we have $z = p_1\left(-\frac{y}{x}\right)^\ell$. The second factor of the above last side coincides with

$$\frac{\tilde{\sigma}_{\mathfrak{p}_3}\left(\prod_{i=0}^{\ell-1}(1-\zeta_\ell^i z^{1/\ell})^{\frac{i}{\ell}}\right)}{\prod_{i=0}^{\ell-1}(1-\zeta_\ell^i z^{1/\ell})^{\frac{i}{\ell}}} = \zeta_\ell^{\tilde{\chi}_{\ell,2}^{z}(\tilde{\sigma}_{\mathfrak{p}_3})} \quad \text{(by (8.3.6))}.$$

□

Corollary 8.3.4 (Case of $\ell = 2$)**.** Let the assumptions and notations be as above. Let $\ell = 2$. Then we have

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_2 = (-1)^{\rho_x(\tilde{\sigma}_{\mathfrak{p}_3}) + \tilde{\chi}^z_{\ell,2}(\tilde{\sigma}_{\mathfrak{p}_3})},$$

where $\rho_x(\tilde{\sigma}_{\mathfrak{p}_3}) \in \mathbb{Z}/2\mathbb{Z}$ is characterized by $\tilde{\sigma}_{\mathfrak{p}_3}(\sqrt{x})/\sqrt{x} = (-1)^{\rho_x(\tilde{\sigma}_{\mathfrak{p}_3})}$. Hence, we obtain

$$\mu_2(123) = \rho_x(\tilde{\sigma}_{\mathfrak{p}_3}) + \tilde{\chi}^z_{\ell,2}(\tilde{\sigma}_{\mathfrak{p}_3}).$$

Proof. The assertion follows immediately from Theorem 8.3.3 and Definition 8.2.3. □

Corollary 8.3.5 (Case of $\ell = 3$)**.** Let the assumptions and notations be as above. Let $\ell = 3$. Then we have

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 = {\zeta_3}^{\tilde{\chi}^z_{\ell,2}(\tilde{\sigma}_{\mathfrak{p}_3})}.$$

Hence, we obtain

$$\mu_3(123) = \tilde{\chi}^z_{\ell,2}(\tilde{\sigma}_{\mathfrak{p}_3}).$$

Proof. The assertion follows immediately from Theorem 8.3.3 and Definition 8.2.9. □

8.3.2 Deriving a reciprocity law

Let the assumptions and notations be as in the previous section. We set $\gamma' = \delta_{\overrightarrow{10}} \cdot \phi_{\overrightarrow{10}}(\gamma) \in \pi_1^{\mathrm{top}}\left(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}; \overrightarrow{01}, 1-z\right)$ as in (6.1.1).

Proposition 8.3.6. For any $\tau \in G_K$, the special value

$$\tilde{\chi}^{1-z,\gamma'}_{\ell,2}(\tau) \in \mathbb{Z}/\ell\mathbb{Z}$$

does not depend on the choice of γ.

Proof. The proof can be done in the same way as the proof of Proposition 8.3.1. □

Definition 8.3.7. Based on Proposition 8.3.6, we let

$$\tilde{\chi}^{1-z}_{\ell,2}(\tau) := \tilde{\chi}^{1-z,\gamma'}_{2}(\tau) \quad (\tau \in G_K).$$

Firstly we describe the triple symbol $[\mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_3]_\ell$ by the mod ℓ Galois dilogarithm to derive a reciprocity law of triple ℓ-th power residue symbols.

Theorem 8.3.8. For $\ell \in \{2, 3\}$, we have

$$[\mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_3]_\ell = \frac{\tilde{\sigma}_{\mathfrak{p}_3}(x^{\frac{1}{2}(\ell-1)})}{x^{\frac{1}{2}(\ell-1)}} \cdot \zeta_\ell^{\tilde{\chi}^{1-z}_{\ell,2}(\tilde{\sigma}_{\mathfrak{p}_3})}.$$

Proof. Let $\ell \in \{2, 3\}$. By (8.3.2), we obtain

$$x^\ell - (-y)^\ell p_1 = w^\ell p_2 \iff x^\ell - w^\ell p_2 = (-y)^\ell p_1.$$

We take

$$\theta^{(\ell)}_{p_2,p_1} = \prod_{i=0}^{\ell-1}(x - \zeta_\ell^i w \sqrt[\ell]{p_2})^i \tag{8.3.8}$$

by replacing p_1, p_2, and y in (8.3.3) with p_2, p_1, and $-w$. Then, as with Theorem 8.3.3, we compute the triple symbol $[\mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_3]_\ell$ as follows:

$$[\mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_3]_\ell = \frac{\tilde{\sigma}_{\mathfrak{p}_3}(x^{\frac{1}{2}(\ell-1)})}{x^{\frac{1}{2}(\ell-1)}} \cdot \frac{\tilde{\sigma}_{\mathfrak{p}_3}\left(\prod_{i=0}^{\ell-1}\left(1-\zeta_\ell^i \frac{w}{x} p_2^{1/\ell}\right)^{\frac{i}{\ell}}\right)}{\prod_{i=0}^{\ell-1}\left(1-\zeta_\ell^i \frac{w}{x} p_2^{1/\ell}\right)^{\frac{i}{\ell}}}.$$

By (8.3.2), we have

$$1 - z = \frac{x^\ell - (-y)^\ell p_1}{x^\ell} = \frac{w^\ell}{x^\ell} p_2.$$

The second factor of the above last side coincides with

$$\frac{\tilde{\sigma}_{\mathfrak{p}_3}\left(\prod_{i=0}^{\ell-1}\left(1-\zeta_\ell^i (1-z)^{1/\ell}\right)^{\frac{i}{\ell}}\right)}{\prod_{i=0}^{\ell-1}\left(1-\zeta_\ell^i (1-z)^{1/\ell}\right)^{\frac{i}{\ell}}} = \zeta_\ell^{\tilde{\chi}_{\ell,2}^{1-z}(\tilde{\sigma}_{\mathfrak{p}_3})} \quad \text{(by Definition 4.1.5, (8.3.1), (8.3.5))}.$$

Therefore we obtain the assertion of the theorem. □

We derive a reciprocity law of triple ℓ-th power residue symbols from the functional equation (8.1.4) of the ℓ-adic Galois dilogarithm.

Corollary 8.3.9 (a reciprocity law)**.** Let the assumptions and notations be as above. For $\ell \in \{2, 3\}$, we have

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_\ell \cdot [\mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_3]_\ell = 1.$$

Proof. By Theorem 8.3.3 and Theorem 8.3.8,

$$\begin{aligned}
[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_\ell \cdot [\mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_3]_\ell &= \begin{cases} \frac{\tilde{\sigma}_{\mathfrak{p}_3}(\sqrt{x})}{\sqrt{x}}(-1)^{\tilde{\chi}_{\ell,2}^{z}(\tilde{\sigma}_{\mathfrak{p}_3})} \cdot \frac{\tilde{\sigma}_{\mathfrak{p}_3}(\sqrt{x})}{\sqrt{x}}(-1)^{\tilde{\chi}_{\ell,2}^{1-z}(\tilde{\sigma}_{\mathfrak{p}_3})} & (\text{if } \ell = 2), \\ \zeta_3^{\tilde{\chi}_{\ell,2}^{z}(\tilde{\sigma}_{\mathfrak{p}_3})} \cdot \zeta_3^{\tilde{\chi}_{\ell,2}^{1-z}(\tilde{\sigma}_{\mathfrak{p}_3})} & (\text{if } \ell = 3) \end{cases} \\
&= \begin{cases} (-1)^{\tilde{\chi}_{\ell,2}^{z}(\tilde{\sigma}_{\mathfrak{p}_3}) + \tilde{\chi}_{\ell,2}^{1-z}(\tilde{\sigma}_{\mathfrak{p}_3})} & (\text{if } \ell = 2), \\ \zeta_3^{\tilde{\chi}_{\ell,2}^{z}(\tilde{\sigma}_{\mathfrak{p}_3}) + \tilde{\chi}_{\ell,2}^{1-z}(\tilde{\sigma}_{\mathfrak{p}_3})} & (\text{if } \ell = 3) \end{cases} \\
&= \zeta_\ell^{\tilde{\chi}_{\ell,2}^{z}(\tilde{\sigma}_{\mathfrak{p}_3}) + \tilde{\chi}_{\ell,2}^{1-z}(\tilde{\sigma}_{\mathfrak{p}_3})} \\
&= \zeta_\ell^{\tilde{\chi}_{\ell,2}^{z,\gamma}(\tilde{\sigma}_{\mathfrak{p}_3}) + \tilde{\chi}_{\ell,2}^{1-z,\gamma'}(\tilde{\sigma}_{\mathfrak{p}_3})}.
\end{aligned}$$

By the functional equation (8.1.4), the above last side is equal to

$$\zeta_\ell^{-\rho_{z,\gamma}(\tilde{\sigma}_{\mathfrak{p}_3})\rho_{1-z,\gamma'}(\tilde{\sigma}_{\mathfrak{p}_3}) + \frac{1}{24}(\chi(\tilde{\sigma}_{\mathfrak{p}_3})^2 - 1)} = 1 \quad (\text{by } (8.3.1), (8.3.5)).$$

This completes the proof. □

8.4 Examples for $\ell = 3$ under the assumption (A)

In this appendix, let us introduce examples of $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3$ and $\tilde{\chi}^{z}_{\ell,2}(\tilde{\sigma}_{\mathfrak{p}_3})$ for some (p_1, p_2) with α_{p_1,p_2} satisfying the assumption (A) in Section 8.2.2. The set of rational prime numbers p satisfying $1 \leq -p \leq 1000$ and $p \equiv 1 \bmod 9$ is

$$\mathbf{P} := \begin{Bmatrix} -17, -53, -71, -89, -107, -179, -197, -233, -251, -269, \\ -359, -431, -449, -467, -503, -521, -557, -593, -647, -683, \\ -701, -719, -773, -809, -827, -863, -881, -953, -971 \end{Bmatrix}. \tag{8.4.1}$$

For any $\{p_1, p_2\} \subset \mathbf{P}$ $(p_1 \neq p_2)$, prime ideals $\mathfrak{p}_i = (p_i)$ $(i = 1, 2)$ of $\mathbb{Q}(\zeta_3)$ satisfy the conditions (8.2.7), (8.2.8) and (8.2.9).

Consider the case $(p_1, p_2) = (-17, -593)$. We take $(x, y, w) = (9, 2, -1)$ as a solution of (8.2.16) and $\alpha_{p_1,p_2} = 9 + 2\sqrt[3]{-17}$ satisfying (A). Hence $\theta^{(3)}_{p_1,p_2} = (9 + 2\zeta_3\sqrt[3]{-17})(9 + 2\zeta_3^2\sqrt[3]{-17})^2$. For a prime ideal $\mathfrak{p}_3 = (p_3)$ of $\mathbb{Q}(\zeta_3)$ satisfying (8.2.15), we obtain

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 = \frac{\sigma_{\mathfrak{p}_3}\left(\sqrt[3]{\theta^{(3)}_{p_1,p_2}}\right)}{\sqrt[3]{\theta^{(3)}_{p_1,p_2}}} \equiv \theta^{(3)\,\frac{p_3^2-1}{3}}_{p_1,p_2} \bmod \tilde{\mathfrak{p}}_3,$$

where $\tilde{\mathfrak{p}}_3$ is a prime ideal of $K^{(3)}_{p_1,p_2}$ lying over $\mathfrak{p}_3$. Since $\theta^{(3)\,\frac{p_3^2-1}{3}}_{p_1,p_2} \in K_1 = \mathbb{Q}(\zeta_3, \sqrt[3]{p_1})$, we obtain

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 = \zeta_3^c \iff N_{K_1/\mathbb{Q}}\left(\theta^{(3)\,\frac{p_3^2-1}{3}}_{p_1,p_2} - \zeta_3^c\right) \equiv 0 \bmod p_3 \quad (c \in \{0, 1, -1\}). \tag{8.4.2}$$

On the other hand, we take $\alpha_{p_2,p_1} = 9 + \sqrt[3]{-593}$ satisfying (A). Thus $\theta^{(3)}_{p_2,p_1} = (9 + \zeta_3\sqrt[3]{-593})(9 + \zeta_3^2\sqrt[3]{-593})^2$. By replacing $\theta^{(3)}_{p_1,p_2}$ (resp. K_1) with $\theta^{(3)}_{p_2,p_1}$ (resp. $K_2 = \mathbb{Q}(\zeta_3, \sqrt[3]{p_2})$) in (8.4.2), we obtain

$$[\mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_3]_3 = \zeta_3^c \iff N_{K_2/\mathbb{Q}}\left(\theta^{(3)\,\frac{p_3^2-1}{3}}_{p_2,p_1} - \zeta_3^c\right) \equiv 0 \bmod p_3 \quad (c \in \{0, 1, -1\}). \tag{8.4.3}$$

Checking the right hand condition of (8.4.2) and (8.4.3) by using the computer algebra system PARI/GP, we obtain TABLE 8.1 for $p_3 \in \mathbf{L}\backslash\{-17, -593\}$ and

$$z\left(= -p_1\frac{y^3}{x^3}\right) = \frac{136}{729}.$$

Taking a solution of (8.2.16) and checking (8.4.2) for each

$$\{p_1, p_2, p_3\} = \{-17, -53, -431\},\ \{-17, -557, -773\},\ \{-17, -593, -773\},$$

we get TABLE 8.2. Based on TABLE 8.2, for $(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$ satisfying the conditions (8.2.7), (8.2.8), (8.2.9) and (8.2.15), it may be plausible to expect that

$$[\mathfrak{p}_{\rho(1)}, \mathfrak{p}_{\rho(2)}, \mathfrak{p}_{\rho(3)}]_3 = [\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3^{\mathrm{sgn}(\rho)}, \tag{8.4.4}$$

where $\mathrm{sgn} : S_3 \to \{1, -1\}$ is the signature.

Table 8.1: Table of $[\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3]_3$, $[\mathfrak{p}_2,\mathfrak{p}_1,\mathfrak{p}_3]_3$, $\tilde{\chi}^{z}_{\ell,2}(\tilde{\sigma}_{\mathfrak{p}_3})$ and $\tilde{\chi}^{1-z}_{\ell,2}(\tilde{\sigma}_{\mathfrak{p}_3})$ for $(p_1,p_2) = (-17,-593), p_3 \in \mathbf{P}\backslash\{p_1,p_2\}$

p_3	$[\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3]_3$	$[\mathfrak{p}_2,\mathfrak{p}_1,\mathfrak{p}_3]_3$	$\tilde{\chi}^{z}_{\ell,2}(\tilde{\sigma}_{\mathfrak{p}_3})$	$\tilde{\chi}^{1-z}_{\ell,2}(\tilde{\sigma}_{\mathfrak{p}_3})$
-53	ζ_3	ζ_3^{-1}	1	-1
-71	ζ_3^{-1}	ζ_3	-1	1
-89	ζ_3	ζ_3^{-1}	1	-1
-107	ζ_3^{-1}	ζ_3	-1	1
-179	ζ_3	ζ_3^{-1}	1	-1
-197	ζ_3	ζ_3^{-1}	1	-1
-233	1	1	0	0
-251	ζ_3	ζ_3^{-1}	1	-1
-269	ζ_3	ζ_3^{-1}	1	-1
-359	ζ_3^{-1}	ζ_3	-1	1
-431	ζ_3^{-1}	ζ_3	-1	1
-449	1	1	0	0
-467	ζ_3	ζ_3^{-1}	1	-1
-503	ζ_3^{-1}	ζ_3	-1	1
-521	ζ_3	ζ_3^{-1}	1	-1
-557	ζ_3	ζ_3^{-1}	1	-1
-647	ζ_3^{-1}	ζ_3	-1	1
-683	ζ_3^{-1}	ζ_3	-1	1
-701	ζ_3^{-1}	ζ_3	-1	1
-719	ζ_3^{-1}	ζ_3	-1	1
-773	ζ_3^{-1}	ζ_3	-1	1
-809	ζ_3^{-1}	ζ_3	-1	1
-827	1	1	0	0
-863	ζ_3	ζ_3^{-1}	1	-1
-881	ζ_3^{-1}	ζ_3	-1	1
-953	ζ_3	ζ_3^{-1}	1	-1
-971	1	1	0	0

Table 8.2: Table of $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3$ and $\tilde{\chi}^z_{\ell,2}(\tilde{\sigma}_{\mathfrak{p}_3})$ for the cases of $\{p_1, p_2, p_3\} = \{-17, -53, -431\}$, $\{-17, -557, -773\}$, $\{-17, -593, -773\}$

(p_1, p_2)	$\alpha_{p_1,p_2} = x + y\sqrt[3]{p_1}$	$z = -p_1\frac{y^3}{x^3}$	p_3	$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3$	$\tilde{\chi}^z_{\ell,2}(\tilde{\sigma}_{\mathfrak{p}_3})$
$(-17, -53)$	$8 + 3\sqrt[3]{-17}$	$\frac{459}{512}$	-431	1	0
$(-53, -17)$	$8 + \sqrt[3]{-53}$	$\frac{53}{512}$	-431	1	0
$(-17, -431)$	$31 + 15\sqrt[3]{-17}$	$\frac{57375}{29791}$	-51	1	0
$(-431, -53)$	$10 + 3\sqrt[3]{-431}$	$\frac{11637}{1000}$	-17	1	0
$(-53, -431)$	$10 - \sqrt[3]{-53}$	$-\frac{53}{1000}$	-17	1	0
$(-431, -17)$	$31 - 4\sqrt[3]{-431}$	$-\frac{27584}{29791}$	-53	1	0
$(-17, -557)$	$-42 - 16\sqrt[3]{-17}$	$\frac{8704}{9261}$	-773	ζ_3	1
$(-557, -17)$	$-42 - 2\sqrt[3]{-557}$	$\frac{557}{9261}$	-773	ζ_3^{-1}	-1
$(-17, -773)$	$-23 + 8\sqrt[3]{-17}$	$-\frac{8704}{12167}$	-557	ζ_3^{-1}	-1
$(-773, -557)$	$-6 - \sqrt[3]{-773}$	$\frac{773}{216}$	-17	ζ_3^{-1}	-1
$(-557, -773)$	$-6 + \sqrt[3]{-557}$	$-\frac{557}{216}$	-17	ζ_3	1
$(-773, -17)$	$-23 - 3\sqrt[3]{-773}$	$\frac{20871}{12167}$	-557	ζ_3	1
$(-17, -593)$	$9 + 2\sqrt[3]{-17}$	$\frac{136}{729}$	-773	ζ_3^{-1}	-1
$(-593, -17)$	$9 + \sqrt[3]{-593}$	$\frac{593}{729}$	-773	ζ_3	1
$(-17, -773)$	$-23 + 8\sqrt[3]{-17}$	$-\frac{8704}{12167}$	-593	ζ_3	1
$(-773, -593)$	$-55 - 6\sqrt[3]{-773}$	$\frac{166968}{166375}$	-17	ζ_3	1
$(-593, -773)$	$-55 + \sqrt[3]{-593}$	$-\frac{593}{166375}$	-17	ζ_3^{-1}	-1
$(-773, -17)$	$-23 - 3\sqrt[3]{-773}$	$\frac{20871}{12167}$	-593	ζ_3^{-1}	-1

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