Algebraic Aspects of Multiplier Maps on Algebraic Dynamical Systems on the Projective Line

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1 Introduction

Throughout this paper, we fix a field k of characteristic zero. Without any mention, any objects are defined over k.

The study of algebraic dynamical system on the projective line $\mathbb{P}^1$, endomorphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ and behaviour of its iterations ϕ^n, is one of central objects of the theory of complex dynamical system ([Mil11]). Many complex analytic and algebraic tools are developed to study dynamical system. For a given endomorphism ϕ, one of the most important property of ϕ as a dynamical system is the periodic points and the multiplier for each periodic point. A point $z \in \mathbb{P}^1$ is called *a periodic point of period n*, or *an n-th periodic point*, of ϕ if $\phi^n(z) = z$. Periodic point of period 1 are also called *fixed point*. For a periodic point z of period n, *the multiplier* of ϕ at z (as a n-th periodic point) is defined by

$$\lambda_z(\phi) := (\phi^n)'(z).$$

The multiplier determines the behaviour of ϕ around the periodic point z. For example, if k is a topological field, the periodic point is called attracting if $|\lambda_z(f)| < 1$, and this is because any point y of a neighbourhood of z satisfies $\phi^N(y) \to z$ $(N \to \infty)$. Another property of multiplier is that the multiplier is invariant under the coordinate transformation, that is,

$$\lambda_\phi(z) = \lambda_{\gamma\circ\phi\circ\gamma^{-1}}(\gamma(z))$$

for any $\gamma \in \mathrm{Aut}(\mathbb{P}^1)$. On the moduli space Rat_d of parameters of endomorphism of degree d ([Sil98]), *the n-th multiplier map* is defined as

$$\lambda_n : \mathrm{Rat}_d \ni [\phi] \mapsto \{\lambda_\phi(z) \mid \phi^n(z) = z\} \in (\mathbb{A}^1)^N/\mathfrak{S}_N.$$

Under the conjugation action $\gamma \cdot [\phi] := [\gamma \circ \phi \circ \gamma^{-1}]$ of PGL_2 on Rat_d, the n-th multiplier map factors as a rational map

$$\lambda_n : \mathrm{Rat}_d \dashrightarrow \mathrm{rat}_d = \text{“}\mathrm{Rat}_d\,/\,\mathrm{PGL}_2\text{”} \dashrightarrow (\mathbb{A}^1)^N/\mathfrak{S}_N.$$

Here the quotient space rat_d is defined by the geometric invariant theory.

The purpose of this thesis is to observe algebraic aspects of the multiplier map by using invariant theory with respect to this conjugation action. The contents of this thesis are based on the preprints [Got22] and [Got23b] and stated on the fundamental property of moduli spaces of dynamical systems of correspondences defined in the article [Got23a].

We can consider effective divisorial self-correspondences [Smi05] instead of endomorphism. In [Got23a], the author considered the moduli space $\mathrm{Corr}_{d,e}$ of self-correspondences on $\mathbb{P}^1$ and its quotient $\mathrm{Dyn}_{d,e}$ defined using geometric invariant theory. Then $\mathrm{Dyn}_{d,e}$ parameterizes effective divisors of degree (d,e) with only mild (i.e. of multiplicity $\leq \frac{d+e}{2}$) singularities on the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$ ([Got23a, Theorem 1.1]), up to conjugation by PGL_2. As essential structures of moduli spaces of dynamical systems, iteration maps $\Psi_n : \mathrm{Dyn}_{d,e} \dashrightarrow \mathrm{Dyn}_{d^n,e^n}$ and fixed point multiplier map $\lambda_{1,(d,e)} : \mathrm{Dyn}_{d,e} \dashrightarrow \mathbb{P}^{d+e}$ were also introduced. These respectively indicate n-th iteration $C \mapsto C \circ \cdots \circ C$ and the fixed point multipliers

$$C : (f=0) \mapsto \left\{ \lambda_z(f) := -\frac{\partial_y f(z,z)}{\partial_x f(z,z)} \,\middle|\, z \in \mathbb{P}^1 : f(z,z) = 0 \right\}$$

under the isomorphism $\{(d+e)$ (possibly multiple) points in $\mathbb{P}^1\} = \mathrm{Sym}_{d+e}\,\mathbb{P}^1 \simeq \mathbb{P}^{d+e}$. Fixed point multiplier map has an outstanding property with respect to the invariant theory. Let V_n be the SL_2-representation on the binary n-forms. For a pair of binary forms $(f_n, f_{n-2}) \in V_n \oplus V_{n-2}$, we define a system of invariants.

Definition 1.1. The r-th discriminant-resultants $DR_{n,r} \in k[V_n \oplus V_{n-2}]_{(2n-2-r,r)}$ for $0 \leq r \leq n$ is the polynomials which satisfy

$$\sum_{r=0}^{n} DR_{n,r}(f_n, f_{n-2}) t^r = \mathrm{res}(f_n(x,y), x\partial_x f_n(x,y) + txy f_{n-2}(x,y))/a_0 a_n.$$

This is an example of resultant system ([vdW30]) including the discriminant $DR_{n,0} = \Delta(f_n)$ and resultant $DR_{n,n} = \mathrm{res}(f_n, f_{n-2})$. In [Got23a, Section 7], it is shown that the fixed point multiplier map is given by discriminant-resultants up to coordinate transformations on $\mathrm{Corr}_{d,e}$ and the codomain $\mathbb{P}^{d+e}$. In [Got22], we applied a universal tool called *bracket polynomial* we will introduce in Section 4 to express SL_2-invariant. Then we can give the following bracket polynomial expression of discriminant-resultants.

Theorem 1.2. (Theorem 4.11) We have $DR_{2,2} = f_0^2$ and

$$DR_{n,r} = \sum_{\substack{I \sqcup J = [n], \\ |I| = r}} \left(\prod_{\substack{j \in J \\ i \in [n] \setminus \{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{\substack{i \in I \\ k \in [n-2]}} [\beta_k, \alpha_i] \right) \tag{1}$$

except $(n,r) = (2,2)$, where $[m] := \{1, 2, \ldots, m\}$.

As an application of Theorem 1.2, we can give purely algebraic proofs for the properties of fixed point multiplier maps obtained from the theory of dynamical systems in [Got23a]. One is the vanishing of the first discriminant-resultant.

Theorem 1.3. (Corollary 4.12) We have $DR_{n,1} = 0$.

The other is the algebraic independence of discriminant-resultants, which was shown as the non-degeneracy of fixed point multiplier map of dynamical systems over the projective line. The case of $n = 3$ is shown by Milnor ([Mil93]) and $n \geq 4$ is shown by Fujimura ([Fuj06]). In [Got23a], the author translated their result into the terms of discriminant-resultant.

Theorem 1.4. (Theorem 4.19) The discriminant-resultants

$$\{DR_{n,r} \mid r = 0, 2, 3, \ldots, n\}$$

are algebraically independent.

These results about bracket polynomial expressions are the results from [Got22]. The results of [Got23b] is about multiplier maps of other periodic points. In [Got23a], n-th multiplier maps $\lambda_{n,(d,e)} := \lambda_{1,(d^n,e^n)} \circ \Psi_n$ were also introduced. Well-definedness of $\lambda_{n,(d,e)}$ for general (n, d, e) remains as an open problem ([Got23a, Problem 1.9, Remark 7.2]). If the n-th multiplier map $\lambda_{n,(d,e)}$ is well-defined, then it indicates

$$C : (f = 0) \mapsto \left\{ \lambda_{(z_i)}(f) := \prod_{i=0}^{n-1} \left(-\frac{\partial_y f(z_i, z_{i+1})}{\partial_x f(z_i, z_{i+1})} \right) \middle| \begin{array}{c} (z_i)_{i=0}^{n} \in (\mathbb{P}^1)^{n+1} : \\ f(z_i, z_{i+1}) = 0, z_0 = z_n \end{array} \right\}.$$

A purpose to define morphisms $\lambda_n := \lambda_{n,(d,e)}$ is to consider the inverse problem of multipliers, that is, to what extant information of multipliers determines morphisms. The results about the inverse problem of multipliers are rephrased as the properties of multiplier maps to their images. Let

$$\Lambda_n := \prod_{m:m|n} \lambda_m : \mathrm{Dyn}_{d,e} \dashrightarrow \prod_{m:m|n} \mathbb{P}^{d^m+e^m} \text{ and}$$
$$\Lambda_\infty := \prod_{m\geq 1} \lambda_m : \mathrm{Dyn}_{d,e} \dashrightarrow \prod_{m\geq 1} \mathbb{P}^{d^m+e^m},$$

where $m|n$ means that m divides n. Let $\mathrm{PDyn}_d \subset \mathrm{Dyn}_{1,d}$ be the locus of polynomial maps, the conjugation classes of rational maps with any totally ramified fixed point. Table 1 is a brief review of known results about the inverse problem of multipliers, in the form of degrees of multiplier maps to their images. All known results are about cases of usual morphisms, that is, the degree as self-correspondence is $(1, d)$.

Remark 1.5. There are some more precise results about the degrees for the loci where the degrees of multiplier maps changes from generic behaviour ([McM87], [Sil07], [Fuj06], [Fuj07], [Sug17], [Sug20], [GOV20]).

Reference	locus X	degree of morphisms	map F	degree of $F : X \dashrightarrow F(X)$
[McM87]	$\mathrm{Dyn}_{1,d}$	$d \geq 2$	Λ_∞	$< \infty$
[Gor15]			Λ_n $(n \geq 3)$	$< \infty$
[Sch16]				$<$ Recursive formula
[Gor15, Conjecture]			Λ_2	?
[Mil93],[Sil98]		$d = 2$	$\Lambda_1 (= \lambda_1)$	1
[JX23]		$d \geq 4$	Λ_∞	1
(From dimension)		$d \geq 3$	Λ_1	∞
[HT13]		$d = 3$	Λ_2	$a_{3,2}$
[Fuj06]	PDyn_d	$d \geq 2$	Λ_1	$(d-2)!$
[HT13]		$d = 4, 5$	Λ_2	1

Table 1: Results about degree of multiplier maps onto their images

In [HT13], $a_{3,2} = 12$ was stated, but the author corrected it to $a_{3,2} = 1$ ([Hut21]). In this thesis, we also give precise proofs of the correction.

Theorem 1.6 (Theorem 7.1)**.** The rational map

$$\Lambda_{2,(1,3)} = \lambda_{1,(1,3)} \times \lambda_{2,(1,3)} : \mathrm{Dyn}_{1,3} \dashrightarrow \mathbb{P}^3 \times \mathbb{P}^9$$

is birational to its image.

In Section 7, we give two proofs of this theorem. In Subsection 7.1, we prove Theorem 7.1 by continuing the computation done in [HT13]. This proof is the proof mentioned in [Hut21] and independent from other parts (except the programs in Subsection A.1 used for the proof) of this paper.

The other proof in Subsection 7.2 is by a direct computation on the invariant ring given in [Wes15], which is the coordinate ring of $\mathrm{Dyn}_{1,3}$. The computation is done by an interpolation method in Subsection A.2, and some unexpectedly simple relations among the coordinate functions (Remark 7.2). A merit of this method is that the part without the direct computation of structure of the coordinate rings can be used for more general cases of $\Lambda_{n,(d,e)}$. The other aim of this paper is to give a primitive estimation of the degree of multiplier maps on moduli spaces of correspondences along this method. This gives a very rough, but explicit upper bound.

Theorem 1.7. Let p be a prime number. If the p-th multiplier map is well-defined and

$$\Lambda_p := \lambda_{1,(d,e)} \times \lambda_{p,(d,e)} : \mathrm{Dyn}_{d,e} \dashrightarrow \Lambda_p(\mathrm{Dyn}_{d,e}) \subset \mathbb{P}^{d+e} \times \mathbb{P}^{d^p+e^p}$$

is generically finite to its image, then its degree is at most

$$\frac{\gcd(d+e,2) N^{de+d+e-3} \cdot (d+e-3)!(de-3)!}{2(d+e) \cdot (de+d+e-3)!},$$

where

$$N := 2(d+e-1) + \frac{2((d^p-1)(d^p-d)-(e^p-1)(e^p-e))}{p(d-e)}.$$

Remark 1.8. We can give upper bounds in similar method for generic Λ_n with a similar assumption, but N becomes a slightly more complicated polynomial. We also note that Λ_2 can be generically finite to its image only if $(d-e)^2 \geq d+e-2$ (Remark 6.9 and Remark 5.12).

Combining with the finiteness result ([Gor15] in Table 1), we can see the following:

Corollary 1.9. For $d \geq 2$, the degree of $\Lambda_{3,(1,d)}$ is at most

$$\frac{\gcd(d+1,2)2^{2d-2}(d^5+d^4-d^2+2d)^{2d-2}\cdot(d-2)!(d-3)!}{2\cdot 3^{2d-2}\cdot(d+1)\cdot(2d-1)!}.$$

For $d=3$, this only gives the evaluation $\deg \Lambda_{3,(1,3)} \leq 4369320$.

Remark 1.10. In [Sch16], an algorithm to count the degree of multiplier maps using equivariant Gromov-Witten invariant is given. The author has not completed the evaluation of the order of the recursion formula.

This paper is organized as follows. In Section 2, we set up notation and terminology. In Section 3, we define Hilbert series for generic case in a categorical way. In Section 4, we introduce symbolic method and show the results about symbolic polynomial expressions of discriminant-resultants. In Section 5, we introduce moduli spaces of dynamical systems of correspondence and multiplier maps. In Section 6, we evaluate the degree of multiplier maps using the evaluation in Subsection 3.5. In Section 7, we give two proofs of Theorem 1.6.

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2 Notation and Terminology

Throughout this paper, we refer to [Liu06] for the terminology of algebraic geometry.

We fix a field k of characteristic zero. Unless otherwise stated, we consider any scheme as a scheme over k.

For a ring R and a free R-module M of finite rank, we denote by $R[M]$ the polynomial ring generated by a basis of M (with suitable identifications between different choices of basis). If a group G and a representation $\rho : G \to \mathrm{Aut}_R(M)$ are also given, we write $I(M) = R[M]^G$ for the invariant ring.

3 Some Structures on Categories for Hilbert Series

3.1 Monoidal structure and Green ring

In this subsection, we introduce monoidal category and Green ring, which is a well-known tool to consider Hilbert series functorially.

Definition 3.1. *Monoidal category* is a tuple $(\mathcal{C}, \otimes, I, \alpha, \rho, \lambda)$ such that

(i) $\mathcal{C}$ is a category,

(ii) $\otimes$ is a (bi-covariant) bifunctor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called *product functor* or *tensor functor*,

(iii) I is an object of $\mathcal{C}$ called *unit object*,

(iv) α is a natural isomorphism $\otimes \circ (\otimes \times \mathrm{id}) \to \otimes \circ (\mathrm{id} \times \otimes)$ (object-wisely, it can be written as $\alpha(A, B, C) : (A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$ for each tuple of objects (A, B, C) of $\mathcal{C}$) called *associator*,

(v) λ is a natural isomorphism $I \otimes - \to \mathrm{id}_{\mathcal{C}}$ and ρ is a natural isomorphism $- \otimes I \to \mathrm{id}_{\mathcal{C}}$, called *left unitor* and *right unitor* respectively,

(vi) α satisfies the pentagon relation, the commutativity of the diagram

$$\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha(A,B,C\otimes D)"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha(A\otimes B,C,D)"] \arrow[d, "\alpha(A,B,C)\otimes \mathrm{id}"] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha(A,B\otimes C,D)"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "\mathrm{id} \otimes \alpha(B,C,D)"']
\end{tikzcd}$$

and

(vii) (α, ρ, λ) satisfies the following trivialization commutativity:

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha(A,I,B)"] \arrow[dr, "\rho\otimes \mathrm{id}"] & & A \otimes (I \otimes B) \arrow[dl, "\mathrm{id} \otimes \lambda"] \\
 & A \otimes B &
\end{tikzcd}$$

As in other algebraic structures, we sometimes abbreviate members in the tuple of monoidal category. For example, we write $(\mathcal{C}, \otimes)$ for a monoidal category $(\mathcal{C}, \otimes, I, \alpha, \rho, \lambda)$ if the unit object, the associator and the unitor are appropriately recovered from $(\mathcal{C}, \otimes)$.

Remark 3.2. (Coherence) The pentagon relation ensures that for any pair of orderings of applications of tensor functors on $A_1 \otimes \cdots \otimes A_n$ are cannonically identified by associators.

Definition 3.3. For monoidal categories $(\mathcal{C}, \otimes, I, \alpha, \rho, \lambda)$ and $(\mathcal{C}', \otimes', I', \alpha', \rho', \lambda')$, a *strict monoidal functor* $(\mathcal{C}, \otimes) \to (\mathcal{C}', \otimes')$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ such that compatible with all other structures of monoidal categories, that is,

$$\otimes' \circ (F \times F) = F \circ \otimes,\ F(I) = I',\ \alpha' \circ (F \times F \times F) = F \circ \alpha$$
$$\rho' \circ (F \times F) = F \circ \rho,\ \lambda' \circ (F \times F) = F \circ \lambda'.$$

A strict monoidal functor is called self-equivalence if self-equivalence as a functor. A *monoidal functor* is a functor $F : \mathcal{C} \to \mathcal{C}'$ which is strict monoidal up to strict monoidal self-equivalence functors.

Definition 3.4. *Symmetric monoidal category* is a tuple $(\mathcal{C}, \otimes, I, \alpha, \rho, \lambda, \sigma)$ such that

(i) $(\mathcal{C}, \otimes, I, \alpha, \rho, \lambda)$ is a monoidal category,

(ii) σ called *commutator* is a natural isomorphism $\otimes \circ \Sigma \to \otimes$, where $\Sigma : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is the swapping isomorphism of the two components (objectwisely, it can be written as $\sigma(A, B) : (A \otimes B) \simeq (B \otimes A)$ for each tuple of objects (A, B) of $\mathcal{C}$), such that

(iii) σ is involutive, that is, $(\sigma \circ \Sigma) \circ \sigma = \mathrm{id}_{\otimes}$,

(iv) σ and α satisfies the hexagon relation, the commutativity of the diagram

$$\begin{tikzcd}
& A \otimes (B \otimes C) \arrow[dr, "\mathrm{id} \otimes \sigma(B,C)"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha(A,B,C)"] \arrow[d, "\sigma(A,B) \otimes \mathrm{id}"] & & A \otimes (C \otimes B) \\
(B \otimes A) \otimes C \arrow[dr, "\alpha(B,A,C)"] & & (A \otimes C) \otimes B \arrow[u, "\alpha(A,C,B)"'] \\
& B \otimes (A \otimes C) \arrow[ur, "\sigma(A \otimes C, B)"] &
\end{tikzcd}$$

and

(v) ρ and λ are identified by the commutator σ, that is, the diagram

$$\begin{tikzcd}
A \otimes I \arrow[rr, "\sigma(A,I)"] \arrow[dr, "\lambda"'] & & I \otimes A \arrow[dl, "\rho"] \\
& A &
\end{tikzcd}$$

is commutative for any object A of $\mathcal{C}$.

Example 3.5. The tuple $(\mathcal{C}, \times, *)$ of a category $\mathcal{C}$, the product $\times$ as the product functor and the final object $*$ as the unit object is a symmetric monoidal category, with the natural transformations made from universal morphisms as an appropriate associator, unitors and a commutator. Dually, the tuple $(\mathcal{C}, \sqcup, \emptyset)$ of a category $\mathcal{C}$ and the coproduct $\sqcup$ and the initial object $\emptyset$ is also a symmetric monoidal category.

Example 3.6. For a commutative ring R, the tuple $(\mathrm{Mod}_R, \otimes_R, R)$ of a category of Mod_R and the tensor product $\otimes_R$ as the product functor is a symmetric monoidal category, with other structures induced from universal property of tensor product.

Definition 3.7. A monoidal structure $(\mathcal{C}, \otimes)$ on an abelian category $\mathcal{C}$ is called *exact* if the functors $- \otimes A$ and $A \otimes -$ are exact for any object A of $\mathcal{C}$.

Example 3.8. For a k-group scheme G, we write $\mathrm{rep}_k(G)$ for the category of finite dimensional representations of G. The tuple $(\mathrm{rep}_k(G), \otimes_k, k)$ is an exact symmetric monoidal category.

Remark 3.9. (Coherence) The hexagon relation ensures that any permutation of factor orderings are realized canonically by σ and α. We sometimes write the identification by α, σ, or abbreviate the ordering of taking product as $A \otimes B \otimes C$ and identify the products.

Definition 3.10. For an abelian category $\mathcal{A}$, the K_0-group is $K_0(\mathcal{A})$ is the abelian group defined by

$$K_0(\mathcal{A}) := \bigoplus_{V \in \mathrm{objclass}(\mathcal{C})} \mathbb{Z}[V] \Big/ \left(\begin{array}{c} [V] - [V'] - [V''] \\ \text{for exact } 0 \to V' \to V \to V'' \to 0 \end{array} \right),$$

where $\mathrm{objclass}(\mathcal{A})$ is the set of isomorphism classes of objects of $\mathcal{A}$.

If an abelian category $\mathcal{A}$ has exact monoidal structure $\otimes$, the K_0-group has a natural ring structure with the multiplication $[V] \cdot [W] := [V \otimes W]$. The ring $K_0(\mathcal{A})$ is called *Green ring* of $\mathcal{A}$.

The image of the map

$$\mathrm{objclass}(\mathcal{A}) \ni V \mapsto [V] \in K_0(\mathcal{A})$$

is closed under the addition, and the multiplication if defined. The image is denoted by $K_0^+(\mathcal{A})$ and called *effective semigroup* or *Green semiring* of $\mathcal{A}$.

Example 3.11. For any monoid S, any S-graded algebra on $\mathcal{A}$ is called *lax monoidal functor* $S \to \mathcal{A}$, the monoidal functor with the compatibility up to the natural transformations.

Example 3.12. The Green ring of the category of finite dimensional vector space $\mathrm{vect}(k)$ is isomorphic to $\mathbb{Z}$. With the monoidal structure defined by the tensor operation, the Green semiring is isomorphic to the semiring of nonnegative integers $\mathbb{N}$, indeed the isomorphism is given by

$$K_0^+(\mathrm{vect}(k)) \ni [V] \mapsto \dim V \in \mathbb{N}.$$

Example 3.13. The Green ring of $\mathrm{rep}_k(\mathbb{G}_m)$ is isomorphic to the Laurent polynomial ring $\mathbb{Z}[q, q^{-1}]$. For any integer n, let k_n be the one-dimensional representation of $\mathbb{G}_m$ which are defined as $t \cdot v := t^n v$. Then the isomorphism is given by

$$K_0^+(\mathrm{rep}_k(\mathbb{G}_m)) \ni [k_n] \mapsto q^n \in \mathbb{Z}[q].$$

Remark 3.14. Any exact functor $F : \mathcal{A} \to \mathcal{B}$ induces the morphism $F_* : K_0(\mathcal{A}) \to K_0(\mathcal{B})$ and $F_*(K_0^+(\mathcal{A})) \subset K_0^+(\mathcal{B})$. If F is monoidal for exact monoidal structures of $\mathcal{A}$ and $\mathcal{B}$, then F_* (resp. $F_*|_{K_0^+(\mathcal{A})}$) is a homomorphism of ring (resp. semiring) with respect to the products induced from the monoidal structures.

3.2 Green Ring of SL_2

We briefly review some fundamental results of the representation theory of SL_2 used to consider the moduli space of dynamical system. The contents in this subsection is on [Got23a, Section 4].

On the vector space of binary n-forms $V_n := H^0(\mathbb{P}^1, \mathcal{O}(n)) = \left\{\sum_{i=0}^n a_n x_0^{d-i} x_1^i\right\}$, the natural GL_2-action is defined by the transformation

$$\gamma \cdot f(x_0, x_1) := f(\gamma \cdot (x_0, x_1)).$$

.

We fix a homomorphism of group scheme $c : \mathbb{G}_m \to \mathrm{SL}_2$ as

$$c : \mathbb{G}_m(k) = k^\times \ni t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in \mathrm{SL}_2(k).$$

Proposition 3.15. The followings are true:

(i) The functor of the morphism of induced representation $c_* : \mathrm{rep}_k(\mathrm{SL}_2) \to \mathrm{rep}_k(\mathbb{G}_m)$ is exact.

(ii) Any object of $\mathrm{rep}_k(\mathrm{SL}_2)$ is semisimple (i.e. direct sum of irreducible objects) and the irreducible objects are $\{V_n \mid n \in \mathbb{N}\}$.

(iii) The morphism c_* between Green semirings induced by c is injective. In fact, under the isomorphism $K_0(\mathrm{rep}_k(\mathbb{G}_m)) \simeq \mathbb{Z}[q]$, we have

$$c_*([V_n]) = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

3.3 Hilbert Series

We recall that a monoid is a set S with associative binary operator $\mu_S : S \times S \to S$ and a unit $e \in S$.

Definition 3.16. A monoid (S, μ_S) is *convolutive* if $\mu_S^{-1}(s)$ is finite for any s in S.

Example 3.17. (i) Any finitely generated commutative monoid with finite invertible elements is a convolutive monoid.

(ii) The positive integers with the multiplication is a convolutive monoid. This is generated by the prime numbers and not finitely generated.

In commutative case, here we note some examples of non-convolutive monoid:

(i) Any monoid with infinite invertible elements, for instance $(\mathbb{Z}, +)$, is not convolutive.

(ii) The positive rational numbers with the addition is not convolutive.

Definition 3.18. For any convolutive monoid S and a ring R, the formal series ring $R[[S]]$ is the ring defined on $\mathrm{Map}(S, R)$ by the pointwise addition and the convolution product.

In the formal series ring $R[[S]]$, we write t^s for the characteristic function of $s \in S$.

Definition 3.19. Let S be a convolutive semigroup and $\mathcal{A}$ be an abelian category.

An *S-graded object* of $\mathcal{A}$ is an S-family of objects of $\mathcal{A}$. For an S-graded object $A := (A_s)_{s\in S}$ of $\mathcal{A}$, *the Hilbert series of* A is the series

$$h_A(t) := \sum_{s\in S} [A_s]t^s \in K_0(\mathcal{A})[[S]].$$

We use Hilbert series for graded algebras and graded modules on monoidal abelian category.

Definition 3.20. Let $\mathcal{A}$ be a monoidal category and S be a monoid. An *S-graded algebra* on $\mathcal{A}$ is an S-graded object $A = (A_s)_{s\in S}$ with the multiplication morphisms

$$\mu_{s,t} : A_s \otimes A_t \to A_{s\cdot t} \text{ for any } s, t \in S,$$

such that the diagram

$$\begin{tikzcd}
 & A_s \otimes (A_t \otimes A_u) \arrow[r, "\mathrm{id} \otimes \mu_{t,u}"] & A_s \otimes A_{t\cdot u} \arrow[dd, "\mu_{s,t\cdot u}"] \\
(A_s \otimes A_t) \otimes A_u \arrow[ur, "\alpha(A_s, A_t, A_u)"] \arrow[d, "\mu_{s,t}\otimes \mathrm{id}"] & & \\
A_{s\cdot t} \otimes A_u \arrow[rr, "\mu_{s\cdot t,u}"] & & A_{s\cdot t\cdot u}
\end{tikzcd}$$

is commutative for any $s, t, u \in S$. If $\mathcal{A}$ is symmetric monoidal and $\mu_{s,t} = \mu_{t,s} \circ \sigma(A_s, A_t)$ for any $s, t \in S$, A is said to be *commutative*.

For an S-set P and an S-graded algebra $A = (A_s)_{s\in S}$ on $\mathcal{A}$, P-graded module of A is a P-graded object $(M_p)_{p\in P}$ with the actions

$$\mu_{s,p} : A_s \otimes M_p \to M_{s\cdot p} \text{ for any } s \in S \text{ and } p \in P$$

such that

$$\begin{tikzcd}
 & A_s \otimes (A_t \otimes M_p) \arrow[r, "\mathrm{id} \otimes \mu_{t,p}"] & A_s \otimes M_{t\cdot p} \arrow[dd, "\mu_{s,t\cdot p}"] \\
(A_s \otimes A_t) \otimes M_p \arrow[ur, "\alpha(A_s, A_t, M_p)"] \arrow[d, "\mu_{s,t} \otimes \mathrm{id}"] & & \\
A_{s\cdot t} \otimes M_p \arrow[rr, "\mu_{s\cdot t,p}"] & & M_{s\cdot t\cdot p}
\end{tikzcd}$$

is commutative for any $s, t \in S$ and $p \in P$.

Example 3.21. (Tensor Product) Let $\mathcal{A}$ be symmetric monoidal. For an S-graded algebra A and a T-graded algebra B both on $\mathcal{A}$, $A \otimes B := (A_s \otimes B_t)_{(s,t)\in S\times T}$ is an $S \times T$-graded algebra under the multiplication

$$(A_s \otimes B_t) \otimes (A_{s'} \otimes B_{t'}) \xrightarrow{\alpha,\sigma} (A_s \otimes A_{s'}) \otimes (B_t \otimes B_{t'}) \xrightarrow{\mu_{s,s'} \otimes \mu_{t,t'}} A_{s\cdot s'} \otimes B_{t\cdot t'}.$$

Example 3.22. Let X be a proper algebraic variety on a field k. For a convolutive submonoid S of the monoid of effective Cartier divisors on X, the partial Cox ring

$$H^0(\mathcal{O}_X(S)) := \left(H^0(X, \mathcal{O}_X(D))\right)_{D\in S}$$

is an S-graded algebra on $\mathrm{vec}(k)$.

For any coherent sheaf $\mathcal{E}$ on X, the Hilbert series of $\mathcal{E}$ with respect to S is

$$h_{\mathcal{E},S}(t) := \sum_{D\in S} \dim H^0(X, \mathcal{E}(D)) \cdot t^D.$$

Definition 3.23. Let $\phi : S \to T$ be a morphism of monoid. For an S-graded algebra A and T-graded algebra B both on $\mathcal{A}$, *a morphism* $f : A \to B$ *of degree* ϕ is a family of morphisms $(f_s : A_s \to B_{\phi(s)})_{s\in S}$ such that

$$\begin{tikzcd}
A_s \otimes A_{s'} \arrow[r, "f_s \otimes f_{s'}"] \arrow[d, "\mu_{s,s'}"] & B_{\phi(s)} \otimes B_{\phi(s')} \arrow[d, "\mu_{\phi(s),\phi(s')}"] \\
A_{s\cdot s'} \arrow[r, "f_{s\cdot s'}"] & B_{\phi(s),\phi(s')}
\end{tikzcd}$$

is commutative for any $s, s' \in S$.

Example 3.24. (Restriction) Let A be an S-graded algebra and $\phi : T \to S$ be a morphism of monoid. Then, $A|_\phi := (A_{\phi(t)})_{t\in T}$ is a T-graded algebra. $A|_\phi$ has a natural morphism $A|_\phi \to A$ of degree ϕ.

Example 3.25. (Contraction) Let A be an S-graded algebra and $\phi : S \to T$ be a morphism of monoid such that $\phi^{-1}(t)$ is finite for any T. Then, $A^\phi := \left(\bigoplus_{s\in\phi^{-1}(t)} A_s\right)_{t\in T}$ is a T-graded algebra. A^ϕ has a natural morphism $A \to A^\phi$ of degree ϕ.

Example 3.26. Let W_1 and W_2 be any representation in $\mathrm{rep}_k(G)$. Then the symmetric tensor algebras $S(W_i) = (\mathrm{Sym}^n W_i)_{n\in\mathbb{N}}(i = 1, 2)$ are an $\mathbb{N}$-graded algebra on $\mathrm{rep}_k(G)$. By the tensor product, we obtain $\mathbb{N}^2$-graded algebra $(\mathrm{Sym}^n W_1 \otimes \mathrm{Sym}^m W_2)_{(n,m)\in\mathbb{N}^2}$. For the morphism $\pi : \mathbb{N}^2 \to \mathbb{N}, (n, m) \mapsto n+m$, the contraction

$$(S(W_1) \otimes S(W_2))^\pi = \left(\bigoplus_{m_1+m_2=n} \mathrm{Sym}^{m_1} W_1 \otimes \mathrm{Sym}^{m_2} W_2 \right)_n$$

is naturally identified with $S(W_1 \oplus W_2)$. In this case, the Hilbert series of $S(W_1) \otimes S(W_2)$ is given by

$$h(t_1, t_2) = h_{S(W_1)}(t_1) \cdot h_{S(W_2)}(t_2) \text{ where } t_1 := t^{(1,0)} \text{ and } t_2 := t^{(0,1)},$$

and the Hilbert series of $(S(W_1 \oplus W_2))^\pi$ is given by $h(t, t)$.

3.4 Covariant Ring

Proposition 3.27. Let A be a T-graded algebra and C an S-graded coalgebra both on a monoidal abelian category $\mathcal{A}$. Then,

$$\langle C, A\rangle := (\mathrm{Hom}_{\mathcal{A}}(C_s, A_t))_{s\in S,t\in T}$$

is an $S \times T$-graded algebra on the category Ab under the multiplication defined by

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}}(C_s, A_t) \otimes \mathrm{Hom}_{\mathcal{A}}(C_{s'}, A_{t'}) &\to \mathrm{Hom}_{\mathcal{A}}(C_s \otimes C_{s'}, A_t \otimes A_{t'}) \\ &\xrightarrow{h\mapsto \mu_{t,t'}\circ h\circ \mu_{s,s'}} \mathrm{Hom}_{\mathcal{A}}(C_{s\cdot s'}, A_{t\cdot t'}).\end{aligned}$$

If $\mathcal{A}$ is $\mathcal{B}$-enriched, then $\mathrm{Hom}_{\mathcal{A}}(C_s, A_t)$ is $S\times T$-graded algebra on $\mathcal{B}$. Moreover, if M and N are an T'-graded A-module and an S'-graded C-comodule respectively,

$$\langle N, M\rangle := (\mathrm{Hom}_{\mathcal{A}}(N_s, M_t))_{s\in S',t\in T'}$$

is an $S' \times T'$-graded $\langle C, A\rangle$-module.

Proof. We show the latter assertion, $\langle N, M\rangle$ is an $S' \times T'$-graded $\langle C, A\rangle$-module. For any

$$(f, f', f'') \in \mathrm{Hom}_{\mathcal{A}}(C_s, A_t) \times \mathrm{Hom}_{\mathcal{A}}(C_{s'}, A_{t'}) \times \mathrm{Hom}_{\mathcal{A}}(N_{s''}, M_{t''}),$$

we have

$$(\mu_{t,t'} \circ (f \otimes f') \circ \mu_{s,s'}) \otimes f'' = (\mu_{t,t'} \otimes \mathrm{id}) \circ (f \otimes f' \otimes f'') \circ (\mu_{s,s'} \otimes \mathrm{id}) \text{ and}$$
$$f \otimes (\mu_{t',t''} \circ (f' \otimes f'') \circ \mu_{s',s''}) = (\mathrm{id} \otimes \mu_{t',t''}) \circ (f \otimes f' \otimes f'') \circ (\mathrm{id} \otimes \mu_{s',s''}).$$

Therefore, the compatibility of the action is reduced to the compatibility of the A-module M and the C-comodule N. The associativity of $\langle C, A\rangle$ is similar. $\square$

Example 3.28. The unit object I of any symmetric monoidal category $\mathcal{A}$ is a coalgebra by the inverse of the unitor $I \to I \otimes I$. The algebra $\langle I, A \rangle$ is *the unit object part* of the algebra A. Since the trivial representation $k \in \mathrm{rep}_k(G)$ is the monoidal unit on $\mathrm{rep}_k(G)$, therefore k is an algebra on $\mathrm{rep}_k(G)$.

Example 3.29. In $\mathrm{rep}_k(G)$, the dual $A^\vee$ of an S-graded algebra $A = (A_s)_{s \in S}$ is an S-graded coalgebra. If A_s is irreducible representation for each s in S, then we obtain algebraic structure of $A_s^\vee$-parts of any graded algebra B. If moreover any part of the algebra B is semisimple, then we have $A \otimes B \simeq \langle A^\vee, B \rangle$.

3.5 Volume of Algebra and Rational Field of Projective Variety

To evaluate degrees of multiplier maps, we need to evaluate the degree of a rational map to its image. In [Got23a], the moduli space $\mathrm{Dyn}_{d,e}$ of dynamical systems is given by the projective scheme $\mathrm{Proj}\, I(V_d \otimes V_e)$ of the naturally graded invariant ring $I(V_d \otimes V_e) := k[V_d \otimes V_e]^{\mathrm{SL}_2}$. Here we have a problem that the graded ring is not fully generated by linear terms. Moreover, full generator (secondary invariants) and relations (syzygies) are not known for generic cases ([Oli17]). Furthermore, we only have little information about multiplier maps. So we use an evaluation only using Hilbert series. We use a trivial evaluation (Proposition 3.34), maybe well-known for experts.

Definition 3.30. The *Gelfand-Kirillov dimension* of an $\mathbb{N}$-graded algebra $A = (A_n)_{n \in \mathbb{N}}$ on $\mathrm{vect}(k)$ is defined by

$$\limsup_{n \to \infty} \log_n \dim A_n.$$

For an $\mathbb{N}$-graded algebra of the Gelfand-Kirillov dimension d, *the volume of* A is defined by

$$\mathrm{Vol}(A) := \lim_{t \to 1} (1-t)^d h_A(t).$$

In this subsection, by a graded k-algebra we call a commutative $\mathbb{N}$-graded algebra over $\mathrm{vect}(k)$ of the Gelfand-Kirillov dimension d. In this case, the Gelfand-Kirillov dimension is equal to the Krull dimension of the commutative algebra $A = \bigoplus_{n \in \mathbb{N}} A_n$.

Remark 3.31. In [DK15], the degree of A is used instead of the volume of A. We choose the word "the volume of A" to avoid confusing with the extension degree of algebras.

We fix a graded k-algebra $A := \bigoplus_{i=0}^\infty A_i$ which is an integral domain. We also fix a sub-graded k-algebra B of A. For any graded k-algebra C, we write $KP(C)$ for the rational function field $K(\mathrm{Proj}\, C)$. We have

$$KP(A) = \bigcup_{i=0}^\infty \left\{ \frac{a_i}{a_i'} \,\middle|\, a_i \in A_i,\ a_i' \in A_i \setminus \{0\} \right\}.$$

We assume that $KP(A)$ is a finite extension of $KP(B)$ and write D for the degree of extension.

Proposition 3.32. There exists a $KP(B)$-basis of $KP(A)$ of the following form:

$$\left\{ \frac{a_i}{b_i} \,\middle|\, i = 1, 2, \ldots, D,\ b_i \in B_{n_i}, a_i \in A_{n_i} \right\}.$$

Proof. We write K for the field $KP(B)$. Let $\{\frac{a_i}{a'_i} \mid i = 1, 2, \ldots, D,\ a_i, a'_i \in A_{n_i}\}$ be a K-basis of $KP(A)$. If $B_{n_i} = 0$ we replace (a_i, a'_i) by (aa_i, aa'_i) for $a \in A$ of sufficiently large degree, and then we can assume $B_{n_i} \neq 0$ and take $b_i \in A_{n_i} \backslash \{0\}$. We have

$$K\left(\frac{a_1}{b_1}, \frac{a'_1}{b_1}, \ldots, \frac{a_D}{b_D}, \frac{a'_D}{b_D}\right) = KP(A)$$

and each $\frac{a_i}{b_i}$ or $\frac{a'_i}{b_i}$ is integral over K. Therefore, for sufficiently large N,

$$\left\{ \frac{a_1^{e_1} \cdots a_D^{e_D} a_1'^{e'_1} \cdots a_D'^{e'_D}}{b_1^{e_1+e'_1} \cdots b_D^{e_D+e'_D}} \,\middle|\, \sum_i e_i + \sum_i e'_i \leq N \right\}$$

is a generator of $KP(A)$ as a K-vector space. $\square$

By reducing to a common denominator, we obtain the following.

Corollary 3.33. There exists a $KP(B)$-basis of $KP(A)$ which is the form

$$\left\{ \frac{a_i}{b_0} \,\middle|\, i = 1, 2, \ldots, D,\ b_0 \in B_n, a_i \in A_n \right\}.$$

We use the following evaluation for extension degrees of rational function fields.

Proposition 3.34. We have

$$\frac{\mathrm{Vol}(A)}{\mathrm{Vol}(B)} \geq [KP(A) : KP(B)].$$

Proof. Let $\{\frac{a_i}{b_0} \mid i = 1, 2, \ldots, D,\ b_0 \in B_m, a_i \in A_m\}$ be a $KP(B)$-basis of $KP(A)$ given by Corollary 3.33. Then, the morphism

$$B^{\oplus D} \ni (b_i) \mapsto \sum_{i=1}^{D} a_i b_i \in A$$

is injective. Thus, we have $\dim A_{m+n} \geq D \dim B_m$ for an arbitrary m. From the definition of volume, we have an asymptotic formula

$$\dim B_m = \frac{\mathrm{Vol}(B) m^{d-1}}{(d-1)!} + o(m^{d-1}) (m \to \infty)$$

and the similar formula for A. By substituting this asymptotic forms, we obtain

$$\frac{\mathrm{Vol}(A)(m+n)^{d-1}}{(d-1)!} + o((m+n)^{d-1}) \geq D\frac{\mathrm{Vol}(B)m^{d-1}}{(d-1)!} + o(m^{d-1})(m \to \infty).$$

This implies that

$$\mathrm{Vol}(A) \geq D\,\mathrm{Vol}(B).$$

□

4 Discriminant-Resultant

4.1 Grassmannian Algebra

Let U_n be the vector space with the basis $\{e_1, \dots, e_n\}$.

Definition 4.1. *The Grassmannian algebra* of a vector space V is the invariant ring

$$G_n(V) := k[U_n \otimes V]^{\mathrm{SL}(U_n)}.$$

For a finite set S, we denote $G(S)$ for the Grassmannian algebra $G(kS)$ of kS.

We denote the element $e_i \otimes v \in U_n \otimes V$ by v_i. For any indexed vector $v_i \in V$, we write $v_{i,j}$ for $(v_i)_j = e_j \otimes v_i \in U_n \otimes V$.

Proposition 4.2. ("Fundamental theorem of invariant theory" [Ful97]) Let V be a vector space. Then the Grassmannian algebra $G_n(V)$ is generated by the invariants

$$\det(v_{i,j})_{i,j=1,\dots n}(v_1, \dots, v_n \in V).$$

In particular, the Grassmannian algebra $G(S)$ is generated by

$$[s_1, \dots, s_n] := \det(s_{i,j})_{i,j=1,\dots n}$$

for n-tuples $(s_1, \dots, s_n)$ of distinct elements of S.

The relations among the generators are given by the sign relations

$$[s_{\sigma(1)}, \dots, s_{\sigma(n)}] = \mathrm{sgn}\,\sigma \cdot [s_1, \dots, s_n]\ (\sigma \in \mathfrak{S}_n)$$

the Plücker relations

$$\sum_{i=1}^{n+1} (-1)^i [s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_{n+1}][s_i, t_1, \dots, t_{n-1}].$$

In this case, elements of S are sometimes called symbols.

Remark 4.3. Grassmannian algebra is the coordinate ring of the affine cone of the Grassmannian space, the moduli space of the linear subspaces of a fixed vector space.

Example 4.4. For $n = 2$, the Grassmannian algebra $G(S)$ is generated by the brackets $\{[\alpha, \beta] \mid \alpha, \beta \in S\}$ and the relations among the bracket symbols are generated by the anti-commutation relations $[\alpha, \beta] = -[\beta, \alpha]$ and the Plücker relations

$$[\alpha, \beta][\gamma, \delta] + [\alpha, \gamma][\delta, \beta] + [\alpha, \delta][\beta, \gamma] = 0 \tag{2}$$

for any symbols α, β, γ, and δ.

4.2 Cayley's symbol

Let $V = \bigoplus_{i=1}^m V_{n_i}$ be an SL_2-representation. We introduce a method to generate the invariant ring $k[V]^{\mathrm{SL}_2}$ from the Grassmannian algebra $G_2(S)$ of $\sum_{i=1}^m n_i$ symbols. In this subsection, we fix the set of symbols as $S = S_V := \{\alpha_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n_i\}$. On the set of symbols, we define the function

$$\mathrm{i} : S \ni \alpha_{i,j} \mapsto i \in \{1, \dots, m\}$$

and the symmetric group $\mathfrak{S}_V$ preserving i, which is

$$\mathfrak{S}_V := \{\sigma : S \to S \mid \sigma \text{ is bijective and } \mathrm{i} \circ \sigma = \mathrm{i}\}.$$

Proposition 4.5. Let V and W be SL_2-representations, $W \to k[V]$ an SL_2-morphism. Then it induces an SL_2-equivariant morphism $k[W] \to k[V]$ of algebra.

Proof. The morphism of commutative algebra is defined from the universality of symmetric polynomial algebra. SL_2-equivariance is obvious. $\square$

Let $\pi_n^\vee : V_n^\vee \to (U_2^{\otimes n})^\vee$ be the dual of the defining morphism of the symmetric quotient $V_n = \mathrm{Sym}^n U_2$. For each nonnegative integer n, there is non-cannonical isomorphism $V_n^\vee \simeq V_n$ ([Got23a, Proposition 4.5, Remark 4.6]). We fix an injective morphism

$$\iota_n : V_n \simeq V_n^\vee \xrightarrow{\pi_n^\vee} (U_2^{\otimes n})^\vee \simeq U_2^{\otimes n}$$

defined from the non-cannonical isomorphisms of dual representations.

Corollary 4.6. The morphism

$$I_n : k[V_n] \to k[U_2^{\oplus n}], V_n \xrightarrow{\iota_n} U_2^{\otimes n} \simeq k[U_2^{\oplus n}]_{(1,1,\dots,1)}$$

of algebra is SL_2-equivariant and injective.

Proof. It is enough to show that the images of a basis of $V_n^\vee$ are algebraically independent in $k[(U_2^{\oplus n})^\vee]$. It is obvious from the definition of $\pi_n^\vee$. $\square$

Corollary 4.7. The morphism $I_V := \bigoplus_{i=1}^m \iota_{n_i}$ induces an SL_2-equivariant injective morphism of algebra

$$I_V : k[V] \to k[U_2 \otimes kS], k[V] \simeq \bigotimes_{i=1}^m k[V_{n_i}] \xrightarrow{I_{n_i}} \bigotimes_{i=1}^m k[U_2^{\oplus n}] \simeq k[U_2 \otimes kS].$$

Moreover, σ is a morphism of an $\mathbb{N}^m$-graded algebra to an $\mathbb{N}^{S_V}$-graded algebra on $\mathrm{rep}(\mathrm{SL}_2)$ of degree the homomorphism $-\circ \mathrm{i} : \mathrm{Map}(\{1,\dots,m\},\mathbb{N}) \to \mathrm{Map}(S,\mathbb{N})$.

Proposition 4.8. Let $\overline{d} \in \mathbb{N}^m$ be a sequence of nonnegative integers. Then we have

$$I_V(k[V]_{\overline{d}}^{\mathrm{SL}_2}) = k[U_2 \otimes kS]_{\mathrm{i}(\overline{d})}^{\mathfrak{S}_V \times \mathrm{SL}_2} = G_2(kS)_{\mathrm{i}(\overline{d})}^{\mathfrak{S}_V}.$$

Proof. The second equality is obvious from the definition of the Grassmannian algebra. For a G-representation V and H-representation W, we have $(V \otimes W)^{G\times H} \simeq V^G \otimes W^H$. Therefore it is enough to show the case of $V = V_n$. This case is the fundamental theorem of the symmetric polynomial. □

Combining this Proposition with the structure of Grassmannian algebra (Proposition 4.2), we obtain the following.

Corollary 4.9. Any SL_2-invariant can be written by bracket polynomials of symbols and any equation among SL_2-invariants are induced from the Plücker relation.

Lemma 4.10. (Laurent Phenomena of Cluster Algebra) Let S be a finite set of symbols and assume that $\#S \geq 3$. Let P be a convex n-gon on a plane with vertices indexed by the elements of S. Let T be a triangulation of P, D_T the $n-3$ diagonals of P defining the triangulation T and E_T the $2n-3$ line segments of the edges of the triangles consists T, that is, $E_T = \{\text{edges of } P\} \cup D_T$. Then, we have the following:

(i) The Plücker relations leads the natural inclusion

$$k[E_T] \subset G(S) \subset k[E_T, D_T^{-1}]. \tag{3}$$

(ii) Let $\alpha = \delta_0, \delta_1, \dots, \beta = \delta_k$ and γ be distinct symbols in S. Then we have

$$[\alpha,\beta] = \sum_{i=0}^{k-1} \frac{[\gamma,\alpha][\gamma,\beta][\delta_i,\delta_{i+1}]}{[\gamma,\delta_i][\gamma,\delta_{i+1}]}.$$

Proof. (i) See [FZ03]. (ii) This is obvious from (2) and $[\alpha,\beta] = -[\beta,\alpha]$. □

4.3 Poisson Formula

Theorem 4.11. We have $DR_{2,2} = f_0^2$ and

$$DR_{n,r} = \sum_{\substack{I \sqcup J = [n],\\ |I| = r}} \left(\prod_{\substack{j\in J\\ i \in [n]\setminus\{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{\substack{i \in I\\ k\in[n-2]}} [\beta_k, \alpha_i] \right) \tag{4}$$

except $(n,r) = (2,2)$, where $[m] := \{1,2,\dots,m\}$.

Proof. We show the assertion by an induction on n. The base cases are the cases $r=0$ of discriminants, the cases $r=n$ of resultants and the cases $(n,r)=(2,1),(3,2)$. The cases $r=0$ and $r=n$ are already introduced. The cases $(n,r)=(2,1),(3,2)$ are shown by the direct computations.

We assume that (4) holds true for $DR_{n-1,r}$ for any r. Firstly, we consider the case that f_n and f_{n-2} have a common linear factor $h=(px+qy)$. In this case, the assertion is obtained by direct computations on the both hand of the equation (4), as the following. We put f_{n-1} and f_{n-3} as $f_n = f_{n-1}\cdot h$ and $f_{n-2}=f_{n-3}\cdot h$ respectively. Then, properties of resultant admits us to compute the resultant with f_n and any polynomial F as

$$\begin{aligned}\operatorname{res}(f_n,F) &= \operatorname{res}(f_{n-1}h,F)\\ &= \operatorname{res}(f_{n-1},F)\cdot\operatorname{res}(h,F)\\ &= \operatorname{res}(f_{n-1},F \mod f_{n-1})\cdot\operatorname{res}(h,F \mod h).\end{aligned}$$

We apply this for $F = x\partial_x f_n + txyf_{n-2}$. Here, we have

$$\begin{aligned}x\partial_x f_n + txyf_{n-2} &= x\partial_x(hf_{n-1}) + txyhf_{n-3}\\ &= x((\partial_x h)\cdot f_{n-1} + h(\partial_x f_{n-1})) + txyhf_{n-3}\\ &= \begin{cases} h(x\partial_x f_{n-1} + txyf_{n-3}) & \mod f_{n-1}\\ x(\partial_x h)\cdot f_{n-1} & \mod h.\end{cases}\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\operatorname{res}(f_n, x\partial_x f_n + txyf_{n-2}) =&\operatorname{res}(f_{n-1}, h(x\partial_x f_{n-1}+txyf_{n-3}))\cdot\operatorname{res}(h, x(\partial_x h)\cdot f_{n-1})\\ =&\operatorname{res}(h,x(\partial_x h))\cdot\operatorname{res}(h,f_{n-1})\cdot\operatorname{res}(f_{n-1},h)\\ &\cdot\operatorname{res}(f_{n-1}, x\partial_x f_{n-1}+txyf_{n-3})\\ =&(-1)^{n-1}pq\cdot\operatorname{res}(h,f_{n-1})^2\cdot\operatorname{res}(f_{n-1},x\partial_x f_{n-1}+txyf_{n-3}).\end{aligned}$$

We write $f_n=\sum_{i=0}^n a_i x^i y^{n-i}$ and $f_{n-1}=\sum_{i=0}^{n-1} a'_i x^i y^{n-1-i}$. Then $f_n = f_{n-1}\cdot h$ leads to $a_0 = pa'_0$ and $a_n = qa'_{n-1}$. Therefore, we have

$$\begin{aligned}\sum_{r=0}^n DR_{n,r}t^r &= \operatorname{res}(f_n, x\partial_x f_n + txyf_{n-2})/(a_0a_n)\\ &= (-1)^{n-1}pq\cdot\operatorname{res}(h,f_{n-1})^2\cdot\operatorname{res}(f_{n-1},x\partial_x f_{n-1}+txyf_{n-3})/(a_0a_n)\\ &= (-1)^{n-1}\operatorname{res}(h,f_{n-1})^2\cdot\operatorname{res}(f_{n-1},x\partial_x f_{n-1}+txyf_{n-3})/(a'_0a'_{n-1})\\ &= (-1)^{n-1}\operatorname{res}(h,f_{n-1})^2\cdot\sum_{r=0}^{n-1}DR_{n-1,r}t^r.\end{aligned}$$

Therefore, we obtain

$$DR_{n,r} = (-1)^{n-1}\operatorname{res}(h,f_{n-1})^2\cdot DR_{n-1,r}.$$

On the RHS of (4), we have $\alpha_n = \beta_{n-2} = h$ and $[\alpha_n, \beta_{n-2}] = 0$ from the assumption. Thus, we have

$$\text{(the RHS of (4))} = \sum_{\substack{I \sqcup J = [n], \\ \# I = r \\ n \notin I}} \left(\prod_{\substack{j \in J \\ i \in [n] \setminus \{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{\substack{i \in I \\ k \in [n-2]}} [\beta_k, \alpha_i] \right). \tag{5}$$

For each term in (5), we have $n \in J$. We put $J' := J \setminus \{n\}$, then we have

$$\prod_{\substack{j \in J \\ i \in [n] \setminus \{j\}}} [\alpha_i, \alpha_j] = \prod_{\substack{j \in J' \\ i \in [n-1] \setminus \{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{j \in J'} [\alpha_n, \alpha_j] \cdot \prod_{i \in [n-1]} [\alpha_i, \alpha_n] \text{ and}$$
$$\prod_{\substack{i \in I \\ k \in [n-2]}} [\beta_k, \alpha_i] = \prod_{\substack{i \in I \\ k \in [n-3]}} [\beta_k, \alpha_i] \cdot \prod_{i \in I} [\beta_{n-2}, \alpha_i].$$

Here we have

$$\prod_{j \in J'} [\alpha_n, \alpha_j] \cdot \prod_{i \in [n-1]} [\alpha_i, \alpha_n] \cdot \prod_{i \in I} [\beta_{n-2}, \alpha_i] = (-1)^{n-1} \left(\prod_{i \in [n-1]} [\alpha_i, h]^2 \right)$$

in any term, thus

$$\begin{aligned}(5) &= (-1)^{n-1} \left(\prod_{i \in [n-1]} [\alpha_i, h]^2 \right) \cdot \sum_{\substack{I \sqcup J' = [n-1], \\ \# I = r}} \left(\prod_{\substack{j \in J' \\ i \in [n] \setminus \{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{\substack{i \in I \\ k \in [n-3]}} [\beta_k, \alpha_i] \right) \\ &= (-1)^{n-1} \operatorname{res}(f_{n-1}, h)^2 \cdot DR_{n-1,r}.\end{aligned}$$

Therefore, the assertion is true if f_n and f_{n-2} have a common linear factor. Since the pair f_n and f_{n-2} have a common linear factor if and only if $\operatorname{res}(f_n, f_{n-2}) = 0$, and the fact that resultant is an irreducible polynomial (see [GKZ94]), we have

$$DR_{n,r} - \sum_{\substack{I \sqcup J = [n], \\ |I| = r}} \left(\prod_{\substack{j \in J \\ i \in [n] \setminus \{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{\substack{i \in I \\ k \in [n-2]}} [\beta_k, \alpha_i] \right) = g \cdot \operatorname{res}(f_n, f_{n-2}) \tag{6}$$

for some polynomial g. As a polynomial of coefficients of f_{n-2}, all terms on the LHS of (6) is homogeneous of degree r and $\operatorname{res}(f_n, f_{n-2})$ is of degree n. Therefore, we have $g = 0$ for $r \leq n-1$. This completes the induction. □

4.4 Holomorphic Lefschetz Vanishing

We show Theorem 1.3 by a direct computation using Theorem 1.2.

Corollary 4.12. We have $DR_{n,1} = 0$.

Proof. We show the assertion by the induction on n. By a direct computation, we have $DR_{2,1} = 0$. We assume $DR_{n-1,r} = 0$. By Theorem 1.2, we have

$$\sum_{l=1}^{n-1} \prod_{\substack{j\in[n-1]\setminus\{l\}\\ i\in[n-1]\setminus\{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{k\in[n-3]} [\beta_k, \alpha_l] = 0.$$

We transpose this as

$$\prod_{\substack{j\in[n-1]\setminus\{n-1\}\\ i\in[n-1]\setminus\{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{k\in[n-3]} [\beta_k, \alpha_{n-1}] = -\sum_{l=1}^{n-2} \prod_{\substack{j\in[n-1]\setminus\{l\}\\ i\in[n-1]\setminus\{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{k\in[n-3]} [\beta_k, \alpha_l]. \tag{7}$$

Again by Theorem 1.2, we have

$$DR_{n,1} = \sum_{l=1}^{n} \prod_{\substack{j\in[n]\setminus\{l\}\\ i\in[n]\setminus\{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{k\in[n-2]} [\beta_k, \alpha_l]. \tag{8}$$

We note that, for any $1 \leq l \leq n-1$,

$$\begin{aligned} \prod_{\substack{j\in[n]\setminus\{l\}\\ i\in[n]\setminus\{j\}}} [\alpha_i, \alpha_j] &= \prod_{\substack{j\in[n-1]\setminus\{l\}\\ i\in[n-1]\setminus\{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{i\in[n-1]} [\alpha_i, \alpha_n] \cdot \prod_{j\in[n-1]\setminus\{l\}} [\alpha_n, \alpha_j] \\ &= \frac{\prod_{i\in[n-1]} \left(-[\alpha_n, \alpha_i]^2\right)}{[\alpha_n, \alpha_l]} \cdot \prod_{\substack{j\in[n-1]\setminus\{l\}\\ i\in[n-1]\setminus\{j\}}} [\alpha_i, \alpha_j]. \end{aligned}$$

In particular, the term of $l = n-1$ in the RHS of (8) have the LHS of (7) as a factor. By substituting it, we obtain

$$\begin{aligned} &\prod_{\substack{j\in[n]\setminus\{n-1\}\\ i\in[n]\setminus\{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{k\in[n-2]} [\beta_k, \alpha_{n-1}] \\ &= \frac{\prod_{i\in[n-1]} \left(-[\alpha_n, \alpha_i]^2\right)}{[\alpha_n, \alpha_{n-1}]} \cdot \prod_{\substack{j\in[n-1]\setminus\{n-1\}\\ i\in[n-1]\setminus\{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{k\in[n-2]} [\beta_k, \alpha_{n-1}] \\ &= \frac{\prod_{i\in[n-1]} \left(-[\alpha_n, \alpha_i]^2\right)}{[\alpha_n, \alpha_{n-1}]} \cdot \left(-\sum_{l=1}^{n-2} \prod_{\substack{j\in[n-1]\setminus\{l\}\\ i\in[n-1]\setminus\{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{k\in[n-3]} [\beta_k, \alpha_l] \right) \cdot [\beta_{n-2}, \alpha_{n-1}] \\ &= \left(-\sum_{l=1}^{n-2} \frac{[\alpha_n, \alpha_l]}{[\alpha_n, \alpha_{n-1}]} \prod_{\substack{j\in[n]\setminus\{l\}\\ i\in[n]\setminus\{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{k\in[n-3]} [\beta_k, \alpha_l] \right) \cdot [\beta_{n-2}, \alpha_{n-1}]. \end{aligned} \tag{9}$$

By the similar substitution for the term of $l = n$ in the RHS of (8), we obtain

$$\begin{aligned}&\prod_{\substack{j\in[n]\setminus\{n\}\\ i\in[n]\setminus\{j\}}}[\alpha_i,\alpha_j]\cdot\prod_{k\in[n-2]}[\beta_k,\alpha_{n-1}]\\ &=\left(-\sum_{l=1}^{n-2}\frac{[\alpha_{n-1},\alpha_l]}{[\alpha_{n-1},\alpha_n]}\prod_{\substack{j\in[n]\setminus\{l\}\\ i\in[n]\setminus\{j\}}}[\alpha_i,\alpha_j]\cdot\prod_{k\in[n-3]}[\beta_k,\alpha_l]\right)\cdot[\beta_{n-2},\alpha_n]. \qquad (10)\end{aligned}$$

By the Plücker relation (2), we have

$$\frac{[\alpha_n,\alpha_l][\beta_{n-2},\alpha_{n-1}]}{[\alpha_n,\alpha_{n-1}]}+\frac{[\alpha_{n-1},\alpha_l][\beta_{n-2},\alpha_n]}{[\alpha_{n-1},\alpha_n]}=[\beta_{n-2},\alpha_l]$$

for each $1 \le l \le n-2$. Therefore, we have

$$(9)+(10)=-\sum_{l=1}^{n-2}\prod_{\substack{j\in[n]\setminus\{l\}\\ i\in[n]\setminus\{j\}}}[\alpha_i,\alpha_j]\cdot\prod_{k\in[n-3]}[\beta_k,\alpha_l]\cdot[\beta_{n-2},\alpha_l],$$

this shows that $DR_{n,1} = 0$. □

4.5 Algebraic independence

To show Theorem 4.19, we use the polygon whose vertices are marked by the symbols $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_{n-2}$ counter-clockwise, and the triangulation T given by drawing all diagonals through β_{n-2}. We put

$$\begin{aligned}
A_i &:= [\beta_{n-2},\alpha_i] && (i=1,\dots,n)\\
B_i &:= [\beta_{n-2},\beta_i] && (i=1,\dots,n-3)\\
C_i &:= \begin{cases}[\alpha_i,\alpha_{i+1}] & (i=1,\dots,n-1)\\ [\alpha_n,\beta_1] & (i=n)\end{cases}\\
D_i &:= [\beta_i,\beta_{i+1}] && (i=1,\dots,n-4).
\end{aligned}$$

The Laurent polynomial ring corresponding to the triangulation T is

$$L := k[A_i, B_i, C_i, D_i, A_2^{-1}, \dots, A_n^{-1}, B_1^{-1}, \dots, B_{n-4}^{-1}].$$

By Lemma 4.10 (i), we regard $G(\{\alpha_i, \beta_i\})$ as a subring of L.

Our strategy is to fix a monomial ordering on L and see the leading monomials of $DR_{n,r}$. For a Laurent polynomial $f \in L$ and a monomial ordering $\preceq$ on L, we write the leading monomial of f with respect to the ordering $\preceq$ by $\mathrm{lm}_{\preceq}(f)$.

Proposition 4.13. Let $\preceq$ be a monomial ordering and $\{f_s\}_{s\in S}$ be a finite family of Laurent polynomials in L. If there is a polynomial f_s such that $\mathrm{lm}_{\preceq}(f_s) \succ \mathrm{lm}_{\preceq}(f_{s'})$ for any other $s' \in S$, then we have $\mathrm{lm}_{\preceq}(\sum_{s\in S} f_s) = \mathrm{lm}_{\preceq}(f_s)$.

Definition 4.14. A subset S of L is said to be *multiplicatively independent* if the map

$$\bigoplus_{s\in S}\mathbb{Z} \ni (a_s)_{s\in S} \mapsto \prod_{\substack{s\in S\\ a_s\neq 0}} s^{a_s} \in L$$

is injective.

Corollary 4.15. A family of Laurent polynomials $\{f_s\}_{s\in S}$ are algebraically independent if there exists a monomial ordering $\preceq$ on L such that $\mathrm{lm}_{\preceq}(f_s)$ are multiplicatively independent.

We fix the monomial ordering $\preceq$ on L as the lexicographic order, where the variables are sorted as

$$(A_1, \dots, A_n, B_1, \dots, B_{n-3}, C_1, \dots, C_n, D_1, \dots, D_{n-4})$$

from maximal to minimal. To show Theorem 4.19, we see the degree of A_i's and C_i's on $\mathrm{lm}_{\preceq}(DR_{n,r})$.

Lemma 4.16. We have

$$\mathrm{lm}_{\preceq}([\alpha_i, \alpha_j]) = A_i A_{j-1}^{-1} C_{j-1} \ (i < j) \text{ and}$$
$$\mathrm{lm}_{\preceq}([\alpha_i, \beta_k]) = \begin{cases} A_i A_n^{-1} C_n & (k = 1) \\ A_i B_{k-1}^{-1} D_{k-1} & (2 \leq k \leq n-3) \\ A_i & (k = n-2). \end{cases}$$

Proof. The assertion is obvious from Lemma 4.10 (ii). $\square$

Lemma 4.17. Let $l \in [n]$ and $I \subset [n]$. We put $J := [n] \setminus I$ and $r := \#I$. Then, we have

$$\deg_{A_l} \mathrm{lm}_{\preceq}\left(\prod_{\substack{j\in J\\ i\in[n]\setminus\{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{\substack{i\in I\\ k\in[n-2]}} [\beta_k, \alpha_i] \right)$$
$$= \begin{cases} n-2 & (l \in I) \\ n-l & (l \in J) \end{cases} - \begin{cases} 0 & (l+1 \in I) \\ l & (l+1 \in J) \end{cases} + \begin{cases} n-r-2\cdot\#(J\cap[l]) & (l \leq n-1) \\ -r & (l = n). \end{cases}$$

Proof. For convenience, we put $B_0 := A_n$ and $D_0 := C_n$. Then we have

$$\begin{aligned}
&\mathrm{lm}_{\preceq}\left(\prod_{\substack{j\in J\\ i\in[n]\setminus\{j\}}}[\alpha_i,\alpha_j]\cdot\prod_{\substack{i\in I\\ k\in[n-2]}}[\beta_k,\alpha_i]\right)\\
&=\prod_{\substack{j\in J\\ i\in[n]\setminus\{j\}}}\mathrm{lm}_{\preceq}([\alpha_i,\alpha_j])\cdot\prod_{\substack{i\in I\\ k\in[n-2]}}\mathrm{lm}_{\preceq}([\beta_k,\alpha_i])\\
&=\prod_{j\in J}\left(\prod_{i=1}^{j-1}\mathrm{lm}_{\preceq}([\alpha_i,\alpha_j])\cdot\prod_{i=j+1}^{n}\mathrm{lm}_{\preceq}([\alpha_i,\alpha_j])\right)\cdot\prod_{\substack{i\in I\\ k\in[n-3]}}A_iB_{k-1}^{-1}D_{k-1}\cdot\prod_{i\in I}A_i\\
&=\prod_{j\in J}\left(\prod_{i=1}^{j-1}A_iA_{j-1}^{-1}C_{j-1}\cdot\prod_{i=j+1}^{n}A_jA_{i-1}^{-1}C_{i-1}\right)\cdot\left(\prod_{k\in[n-3]}B_{k-1}^{-1}D_{k-1}\right)^r\cdot\prod_{i\in I}A_i^{n-2}.
\end{aligned}\tag{11}$$

Here, the first factor is

$$\begin{aligned}
&\prod_{j\in J}\left(\prod_{i=1}^{j-1}A_iA_{j-1}^{-1}C_{j-1}\cdot\prod_{i=j+1}^{n}A_jA_{i-1}^{-1}C_{i-1}\right)\\
&=\prod_{j\in J}(A_{j-1}^{-1}C_{j-1})^{j-1}A_j^{n-j}\left(\prod_{i=1}^{j-1}A_i\cdot\prod_{i=j+1}^{n}A_{i-1}^{-1}C_{i-1}\right).
\end{aligned}$$

The degree of A_l in this factor is given by

$$\begin{aligned}
&\deg_{A_l}\prod_{j\in J}(A_{j-1}^{-1}C_{j-1})^{j-1}A_j^{n-j}\left(\prod_{i=1}^{j-1}A_i\cdot\prod_{i=j+1}^{n}A_{i-1}^{-1}C_{i-1}\right)\\
&=\begin{cases}n-l & (l\in J)\\ 0 & (l\in I)\end{cases}-\begin{cases}l & (l+1\in J)\\ 0 & (l+1\in I)\end{cases}+\deg_{A_l}\prod_{j\in J}\prod_{i=1}^{j-1}A_i\cdot\prod_{i=j}^{n-1}A_i^{-1}\\
&=\begin{cases}n-l & (l\in J)\\ 0 & (l\in I)\end{cases}-\begin{cases}l & (l+1\in J)\\ 0 & (l+1\in I)\end{cases}+\begin{cases}\#(J\cap[n]\setminus[l])-\#(J\cap[l]) & (l\le n-1)\\ 0 & (l=n)\end{cases}
\end{aligned}$$

Since we put $B_0 = A_n$, the degree of A_l on the remaining factor is

$$\deg_{A_l}\left(\prod_{k\in[n-3]}B_{k-1}^{-1}D_{k-1}\right)^r\cdot\prod_{i\in I}A_i^{n-2}=\begin{cases}-r & (l=n)\\ 0 & (l\le n-1)\end{cases}+\begin{cases}n-2 & (l\in I)\\ 0 & (l\in J).\end{cases}$$

By summing up these degrees, we obtain the assertion. $\square$

To apply Corollary 4.15 for discriminant-resultants, we use the expression of Theorem 1.2. We firstly show that only the term of $I = [r]$ in the RHS of (1) gives the maximal leading monomial in the terms of $DR_{n,r}$.

Lemma 4.18. For any integer $r = 0$ or $2 \le r \le n$, we have

$$\mathrm{lm}_{\preceq}(DR_{n,r}) = \mathrm{lm}_{\preceq}\left(\prod_{\substack{j\in[n]\setminus[r]\\ i\in[n]\setminus\{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{\substack{i\in[r]\\ k\in[n-2]}} [\beta_k, \alpha_i]\right).$$

Proof. Since the monomial ordering $\preceq$ is the lexicographic order, it is enough to show the following claim for $1 \le l \le r-1$.

($C_{r,l}$) In the terms of the RHS of (1) corresponding to the sets I such that $I \supset [l-1]$, the degree of A_l on their leading monomials is maximal if $l, l+1 \in I$.

By applying Lemma 4.17 for the case $I \supset [l-1]$ and $r \le l$, we obtain

$$\begin{aligned}&\deg_{A_l} \mathrm{lm}_{\preceq}\left(\prod_{\substack{j\in J\\ i\in[n]\setminus\{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{\substack{i\in I\\ k\in[n-2]}} [\beta_k, \alpha_i]\right)\\ &= 2n - r - 2 - \begin{cases} 0 & (l \in I)\\ l & (l \in J)\end{cases} - \begin{cases} 0 & (l+1 \in I)\\ l & (l+1 \in J).\end{cases}\end{aligned}$$

This shows the claim ($C_{r,l}$). □

Theorem 4.19. (Theorem 1.4) The discriminant-resultants

$$\{DR_{n,r} \mid r = 0, 2, 3, \ldots, n\}$$

are algebraically independent.

Proof. The assertion is obvious for $n = 2$, so we assume $n \ge 3$. By Corollary 4.15, it is sufficient to show that the matrix

$$P := \left(\deg_X \mathrm{lm}_{\preceq}(DR_{n,r})\right)_{\substack{r=0,2,\ldots,n,\\ X:\text{variable of } L}}$$

is of full rank. By Lemma 4.18 and (11),

$$\begin{aligned}\mathrm{lm}_{\preceq}(DR_{n,r}) &= \mathrm{lm}_{\preceq}\left(\prod_{\substack{j\in J\\ i\in[n]\setminus\{j\}}} [\alpha_i, \alpha_j] \cdot \prod_{\substack{i\in I\\ k\in[n-2]}} [\beta_k, \alpha_i]\right)\\ &= \prod_{j=r+1}^{n}\left(\prod_{i=1}^{j-1} A_i A_{j-1}^{-1} C_{j-1} \cdot \prod_{i=j+1}^{n} A_j A_{i-1}^{-1} C_{i-1}\right) \cdot \left(\prod_{k=1}^{n-3} B_{k-1}^{-1} D_{k-1}\right)^r \cdot \prod_{i=1}^{r} A_i^{n-2}.\end{aligned} \tag{12}$$

Using (12), we compute

$$c_{r,l} := \deg_{C_l} \operatorname{lm}_{\preceq}(DR_{n,r}) = \begin{cases} 0 & (l \leq r-1) \\ 2l - r & (r \leq l \leq n-1) \\ r & (l = n) \end{cases}$$

and

$$\begin{aligned} a'_{r,l} &:= \deg_{A_l} \operatorname{lm}_{\preceq}(DR_{n,r}) - \deg_{C_l} \operatorname{lm}_{\preceq}(DR_{n,r}) \\ &= \deg_{A_l} \prod_{j=r+1}^{n} \left(\prod_{i=1}^{j-1} A_i \cdot \prod_{i=j+1}^{n} A_j \right) \cdot \prod_{i=1}^{r} A_i^{n-2} \\ &= \begin{cases} 2n - r - 2 & (l \leq r) \\ 2n - 2l & (r+1 \leq l \leq n). \end{cases} \end{aligned}$$

Thus, we have

$$a'_{r,2} - a'_{r,3} = \begin{cases} 2 & (r \leq 2) \\ 0 & (r \geq 3). \end{cases}$$

Therefore, for the vectors

$$v_r := (a'_{r,2} - a'_{r,3} - c_{r,2}, c_{r,2}, \ldots, c_{r,n-1})$$

for $r = 0, 2, 3, \ldots, n$, the matrix

$$(v_0, v_2, v_3, \ldots, v_n)$$

is triangular with the diagonal entries $(-2, 2, 3, \ldots, n)$. Thus, the matrix P is of full rank. This is what we wanted to show. $\square$

5 Moduli Spaces of Dynamical Systems of Correspondence

In this section, we review results about moduli spaces of dynamical systems of correspondence in [Got23a].

5.1 Geometric invariant theory

In this subsection, we briefly introduce results in geometric invariant theory used in [Got23a].

Definition 5.1. ([MFK94, Definition 1.6]) Let G be a reductive group scheme and X be a scheme with G-action $\sigma : G \times X \to X$. For an invertible sheaf $\mathcal{L}$ over X, an isomorphism $\phi : \sigma^*\mathcal{L} \simeq p_2^*\mathcal{L}$ is said to be *G-linearization* if ϕ satisfies

$$p_{23}^*\phi \circ (\mathrm{id}_G \times \sigma)^*\phi = (\mu \times \mathrm{id}_X)^*\phi \text{ (on } G \times G \times X).$$

Remark 5.2. If $\mathcal{L}$ is very ample and G is affine, then G-linearization is described as the $G(\mathcal{O}(X))$-action on $\mathcal{L}(X)$ compatible with σ. Moreover, $H^0(X, \mathcal{L})$ is a rational $G(k)$-representation.

Remark 5.3. For a G-linearization ϕ of an invertible sheaf $\mathcal{L}$ over a normal scheme X, $\phi^n : \sigma^*\mathcal{L}^n \simeq p_2^*\mathcal{L}^n$ is a G-linearization of $\mathcal{L}^n$.

Remark 5.4. ([MFK94, Proposition 1.4]) If there exists no surjective homomorphism $G \to \mathbb{G}_m$ of group schemes and $X \times_k \bar{k}$ is normal, G-linearization ϕ of an invertible sheaf $\mathcal{L}$ is unique if exists.

Definition 5.5. ([MFK94, Definition 1.7]) Let G be a reductive group, X an algebraic variety with G-action and P a geometric point of X.

(i) P is said to be *pre-stable* if the stabilizer group of P is finite and there exists a G-stable affine open neighborhood of P.

Moreover, we suppose that $\mathcal{L}$ be an ample invertible sheaf over X with G-linearization.

(ii) P is said to be *$\mathcal{L}$-semistable* if for some positive integer $n > 0$, there exists $f \in H^0(X, \mathcal{L}^n)^G$ such that $f(P) \neq 0$ and X_f is affine.

(iii) P is said to be *(proper) $\mathcal{L}$-stable* if P is $\mathcal{L}$-semistable and pre-stable.

The set of pre-stable (resp. $\mathcal{L}$-semistable, $\mathcal{L}$-stable) geometric points is the set of geometric points of an open subscheme of X called *pre-stable (resp. $\mathcal{L}$-semistable, $\mathcal{L}$-stable) locus*. We denote the loci by $X^s(\mathrm{Pre})$ (resp. $X^{ss}(\mathcal{L})$, $X^s(\mathcal{L})$).

Theorem 5.6. [MFK94, p.40] Let G be a reductive group scheme, X be a projective G-scheme with G-action and $\mathcal{L}$ be an ample line bundle over X with G-linearization. Then the rational map induced from the inclusion of invariant ring

$$X = \operatorname{Proj} \bigoplus_{i \in \mathbb{N}} H^0(\mathcal{L}^i) \dashrightarrow Y := \operatorname{Proj} \bigoplus_{i \in \mathbb{N}} H^0(\mathcal{L}^i)^G$$

is a categorical quotient morphism on the semistable locus $X^{ss}(\mathcal{L}) \to X^{ss}(\mathcal{L}) /\!\!/ G = Y$.

5.2 Definition of Moduli Spaces

In this subsection, we define moduli spaces of correspondences over the projective line along [Got23a].

For any symbol x, we define the projective space $\mathbb{P}^n_x$ to $[x_0, x_1, \dots, x_n]$ be the ordered basis of $H^0(\mathbb{P}^n_x, \mathcal{O}(1))$ and we abbreviate this ordered basis by $[x]$ or x. Throughout this section, we fix a pair (d, e) of positive integers such that $(d, e) \neq (1, 1)$. Let

$$\mathrm{Corr}_{d,e} := \mathbb{P}(H^0(\mathbb{P}^1_x \times \mathbb{P}^1_y, \mathcal{O}(d, e)))$$

be the projective space which parameterize the correspondences of degree (d, e) on $\mathbb{P}^1_x \times \mathbb{P}^1_y$. Then we can write

$$H^0(\mathcal{O}(d,e)) = \left\{ f_{\overline{a}}([x],[y]) := \sum_{\substack{0 \le i \le d \\ 0 \le j \le e}} a_{ij} x_0^{d-i} x_1^i y_0^{e-j} y_1^j \,\middle|\, a_{ij} \in k \right\}$$

for any field k. Any k-point of $\mathrm{Corr}_{d,e}$ is given by the linear span of the tuple of coefficients

$$[\overline{a}] := [a_{ij} : 0 \le i \le d,\ 0 \le j \le e, a_{ij} \in k].$$

Remark 5.7. We note that the degree of correspondence is sometimes defined as the degree of projection, as in [Sch17]. For instance, the first degree is given by the degree of the first projection

$$\Gamma_f = \Gamma_{x,y} = V_+(f_{\overline{a}}([x],[y])) \to \mathbb{P}_x$$

is e, because f has e roots for a fixed $[a]$ and $[x]$. In this manner we have degree (d, e) correspondence as a morphism.

We define the $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$-action on $\mathrm{Corr}_{d,e}$ to keep the polynomial $f_{\overline{a}}([x],[y])$ to be invariant. Since the diagonal action of SL_2 on $\mathbb{P}^1 \times \mathbb{P}^1$ yields

$$H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(d,e)) \simeq H^0(\mathbb{P}^1, \mathcal{O}(d)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(e)) = V_d \otimes V_e,$$

we have

$$H^0(\mathrm{Corr}_{d,e}, \mathcal{O}(1)) = (V_d \otimes V_e)^\vee \text{ and thus } \mathrm{Corr}_{d,e} = \mathrm{Proj}\, k[(V_d \otimes V_e)^\vee].$$

Under the (non-canonical) isomorphism $(V_d \otimes V_e)^\vee \simeq V_d \otimes V_e$, we regard $\mathrm{Corr}_{d,e} = \mathrm{Proj}\, k[V_d \otimes V_e]$ with respect to the SL_2-action.

Definition 5.8. The moduli space of the dynamical system of correspondences of degree (d, e) is $\mathrm{Dyn}_{d,e} := \mathrm{Proj}\, k[V_d \otimes V_e]^{\mathrm{SL}_2}$.

By Theorem 5.6, the rational map induced from the inclusion of the invariant ring

$$\mathrm{Corr}_{d,e} = \mathrm{Proj}\, k[V_d \otimes V_e] \dashrightarrow \mathrm{Dyn}_{d,e} = \mathrm{Proj} \bigoplus_{i \in \mathbb{N}} k[V_d \otimes V_e]^{\mathrm{SL}_2}$$

is a categorical quotient morphism $\mathrm{Corr}^{ss}_{d,e} \to \mathrm{Dyn}_{d,e} = \mathrm{Corr}^{ss}_{d,e} /\!\!/ \mathrm{SL}_2$ on the semistable locus $\mathrm{Corr}^{ss}_{d,e} = \mathrm{Corr}^{ss}_{d,e}(\mathcal{O}(1))$. A geometric characterisation of the semistable locus and the stable locus is one of the main result of [Got23a].

Theorem 5.9. ([Got23a, Theorem 1.1]) The point of $\mathrm{Corr}_{d,e}$ which represents a correspondence C is a stable point (resp. a semistable point) if and only if C has no point of multiplicity $\ge \frac{d+e}{2}$ (resp. of multiplicity $> \frac{d+e}{2}$) on the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$.

5.3 Some Covariant Cycles

In this subsection, we introduce some closed subschemes on $\mathrm{Corr}_{d,e} \times (\mathbb{P}^1)^m$, which are covariant with respect to the diagonal action of PGL_2. We first define the diagonal graph, which is given as the divisor of degree $(0,1,1)$,

$$\Delta_{x,y} := \mathrm{Corr}_{d,e} \times \Delta_{\mathbb{P}^1} \xrightarrow{\mathrm{id} \times \iota} \mathrm{Corr}_{d,e} \times \mathbb{P}^1_x \times \mathbb{P}^1_y.$$

The universal graph, the scheme Γ which gives the correspondences indicated by the $\mathrm{Corr}_{d,e}$, is the hypersurface

$$\Gamma_{x,y} := V_+(f_{\overline{a}}(x,y)) \subset \mathrm{Corr}_{d,e} \times \mathbb{P}^1_x \times \mathbb{P}^1_y$$

of degree $(1,d,e)$. From these schemes, we can define some objects which appears in the theory of dynamical system. We firstly note that the composition of self-correspondences $C_{\overline{a}} = V_+(f_{\overline{a}}(x,y))$ and $D_{\overline{a}} = V_+(g_{\overline{a}}(x,y))$ over $\mathbb{P}^n_a$ is given by

$$C_{\overline{a}} \circ D_{\overline{a}} = V_+(\mathrm{res}_z(f(x,z), g(z,y))).$$

From the Sylvester formula, if $C_{\overline{a}}$ and $D_{\overline{a}}$ are of degree (n,d,e) and (n',d',e') respectively, then the composition has the degree $(nd'+n'e, dd', ee')$.

The n-th iteration $\Psi_n\Gamma$ of the graph is defined by

$$\Psi_0\Gamma_{x,y} = \Delta_{x,y}, \Psi_n\Gamma := \Psi_{n-1}\Gamma \circ \Gamma.$$

The n-th iteration graph $\Psi_n\Gamma$ is an hypersurface of degree $(\frac{d^n-e^n}{d-e}, d^n, e^n)$ in $\mathrm{Corr}_{d,e} \times \mathbb{P}^1_x \times \mathbb{P}^1_y$.

Critical loci are also defined. We remark that for correspondences, critical points and critical values are exist in both direction. From the implicit function theorem, an x-critical point (y-critical value) (ξ, η) of a correspondence defined by $f_{\overline{a}} = f_{\overline{a}}([x],[y]) = 0$ is a solution of the equations $f_{\overline{a}}([x],[y]) = 0$ and $\partial_x f_{\overline{a}}([x],[y]) = 0$ for any derivative operator $\partial_x := g_0([x])\partial_{x_0} + g_1([x])\partial_{x_1}$. This system of equations are reduced to the equations

$$\partial_{x_0} f_{\overline{a}} = \partial_{x_1} f_{\overline{a}} = 0$$

since $(x_0\partial_{x_0} + x_1\partial_{x_1})f_{\overline{a}} = d \cdot f_{\overline{a}}$. We can obtain the y-coordinates of the x-critical points (y-critical values) by a computation of the discriminant,

$$x\,\mathrm{Crit}_y := V_+(\Delta_{[x]}(f_{\overline{a}}([x],[y]))).$$

From the definition of discriminant, this gives a hypersurface

$$x\,\mathrm{Crit}_y \subset \mathrm{Corr}_{d,e} \times \mathbb{P}^1_y$$

of degree $(2d-2, 2de-2e)$. The hypersurface of x-coordinates of x-critical points is given by

$$x\,\mathrm{Crit}_x := V_+(\mathrm{res}_{[y]}(\partial_{x_0} f, \partial_{x_1} f)) \subset \mathrm{Corr}_{d,e} \times \mathbb{P}^1_x$$

and the degree is $(2e, 2de-2e)$. The hypersurfaces $y\,\mathrm{Crit}_x \subset \mathrm{Corr}_{d,e} \times \mathbb{P}^1_y$ of x-critical values and y-critical values $y\,\mathrm{Crit}_y \subset \mathrm{Corr}_{d,e} \times \mathbb{P}^1_x$ of degree respectively $(2e-2, 2de-2d)$ and $(2d, 2de-2d)$ are also defined by swapping the coordinates x and y in the definitions.

The hypersurface of periodic points Per_n of period n is given by

$$\mathrm{Per}_n := V_+(\Psi_n(f)(z,z)) \subset \mathrm{Corr}_{d,e} \times \mathbb{P}^1_z.$$

We can regard Per_n as $\Psi_n \Gamma_{x,y}|_{\Delta_{x,y}}$ under the isomorphism $\Delta_{x,y} \simeq \mathrm{Corr}_{d,e} \times \mathbb{P}^1_z$. From the degree of $\Psi_n \Gamma$, the degree of Per_n is $(\frac{d^n-e^n}{d-e}, d^n+e^n)$.

We remark that the cycle of periodic points Per_n includes the cycles of fixed points, and moreover the cycles Per_m for $m|n$. We define the scheme Per^*_n of periodic points of formal period n by extracting the periodic points of shorter periods. More explicitly, we define effective divisors Per^*_n inductively as

$$\mathrm{Per}^*_1 := \mathrm{Per}_1, \mathrm{Per}^*_n := \mathrm{Per}_n - \sum_{m<n, m|n} \mathrm{Per}^*_m.$$

Let $\nu_n(x)$ be the family of polynomials, asymptotically defined by

$$\nu_1(x) = x, \nu_n(x) = x^n - \sum_{m<n, m|n} \nu_m(x).$$

In a closed form, ν_n is written by using the Möbius function μ,

$$\nu_n(x) = \sum_{m|n} \mu\left(n/m\right) x^n.$$

Then the degree of Per^*_n is given by

$$\left(\frac{\nu_n(d) - \nu_n(e)}{d-e}, \nu_n(d) + \nu_n(e) \right).$$

We write $\Pi^*_n f(z)$ for a defining form of the divisor Per^*_n.

Remark 5.10. We have $\nu_n(1) = 0$ for $n > 1$. For the cases only considering rational maps, $\nu_n(d) + \nu_n(1)$ is sometimes used instead of $\nu_n(d)$ (for instance, $\nu_n(d)$ in [Sil07, Remark 4.3] and $N_n(d)$ in [DM06, Chapter 4]).

5.4 Degrees of Linear Systems of Multiplier Maps

Proposition 5.11. If the multiplier map

$$\lambda_{n,(d,e)} : \mathrm{Corr}_{d,e} \dashrightarrow \mathrm{Corr}_{d^n,e^n} \dashrightarrow \mathbb{P}^{d^n+e^n}$$

is well-defined, then it is given by a linear system of degree

$$2(d^n + e^n - 1)\frac{d^n - e^n}{d-e}.$$

Moreover, in this case, we can define the multiplier map of the periodic orbits of period n,

$$\lambda^{\circ}_{n,(d,e)} : \mathrm{Corr}_{d,e} \dashrightarrow \mathbb{P}^{(\nu_n(d)+\nu_n(e))/n}$$

and it is given by a linear system of degree at most

$$\frac{2((d^n-1)\nu_n(d)-(e^n-1)\nu_n(e))}{n(d-e)}.$$

Proof. By [Got23a, Section 7] and the Sylvester formula, the degree of fixed point multiplier map is $2(d+e-1)$. Since the morphism $\Psi_n : \mathrm{Corr}_{d,e} \dashrightarrow \mathrm{Corr}_{d^n,e^n}$ is given by a linear system of degree $\frac{d^n-e^n}{d-e}$, we obtain the first assertion. Moreover, the morphism $\lambda_n := \lambda_{n,(d,e)}$ is given by

$$\begin{aligned} \lambda_n([f]) &= \lambda_{1,(d,e)}([\Psi_n f]) \\ &= \left[\prod_{z:\Psi_n f(z,z)=0} (\partial_x \Psi_n f(z,z)dx + \partial_y \Psi_n f(z,z)dy) \right] \in \mathbb{P}(D_N), \end{aligned} \tag{13}$$

where $N = \deg_z \Psi_n f(z,z) = d^n + e^n$. The well-definedness of λ_n implies that the factors of (13) are not zero. Therefore, we can define $\lambda^{\bullet}_n([f])$ as

$$\lambda^{\bullet}_n([f]) := \left[\prod_{z:\Pi^*_n f(z)=0} (\partial_x \Psi_n f(z,z)dx + \partial_y \Psi_n f(z,z)dy) \right] \in \mathbb{P}(D_M), \tag{14}$$

where $M = \deg_z \Pi^*_n f(z) = \nu_n(d) + \nu_n(e)$. By [Got23a, Remark 7.6], we can write this multiplier map as

$$\begin{aligned} \lambda^{\bullet}_n([f]) &= [\mathrm{res}_z\,(\Pi^*_n f(z), \partial_x \Psi_n f(z,z)dx + \partial_y \Psi_n f(z,z)dy)] \\ &= \left[\mathrm{res}_z \left(\Pi^*_n f(z), d_z \Psi_n f(z,z)dz_0 + z_0 z_1 \Omega^1 \Psi_n f(z)dz_1\right)/A_{n,0}A_{n,1}\right], \end{aligned} \tag{15}$$

where $A_{n,0}$ and $A_{n,1}$ are the coefficients of respectively z_0^M and z_1^M of $\Pi^*_n f(z)$. Therefore, by the Sylvester formula, the rational map $\lambda^{\circ}_n([f])$ is given by the linear system given by the coefficients of $dz_0^i dz_1^{M-i}$ of

$$\mathrm{res}_z \left(\Pi^*_n f(z), d_z \Psi_n f(z,z)dz_0 + z_0 z_1 \Omega^1 \Psi_n f(z)dz_1\right)/A_{n,0}A_{n,1}, \tag{16}$$

and their degree is

$$(d^n+e^n)\frac{\nu_n(d)-\nu_n(e)}{d-e} + (\nu_n(d)+\nu_n(e))\frac{d^n-e^n}{d-e} - 2\frac{\nu_n(d)-\nu_n(e)}{d-e}. \tag{17}$$

Any periodic orbit of formal period n, of a correspondence defined by $f(x,y)$ is given by a tuple of points $(z_0, z_1, \ldots, z_n = z_0)$ such that $f(z_i, z_{i+1}) = 0$.

From the differential of composite functions, for the periodic points of the same periodic orbits, the factor in (14) takes the same value, that is,

$$\lambda_n^\bullet([f]) = \left[\prod_{\substack{(z_0,\ldots,z_{n-1}):\\ \text{Periodic orbits of } f(x,y)}} \left(dx + \frac{\partial_y \Psi_n f(z_0,z_0)}{\partial_x \Psi_n f(z_0,z_0)} dy \right)^n \right]. \tag{18}$$

This leads that the map $\lambda^\bullet_{n,(d,e)}([f])$ is given by an n-th power of some rational function. Therefore, we can define λ_n° as an n-th root of some quotient of (16). □

Remark 5.12. By definition, for the Veronese embedding

$$\epsilon_n : \mathbb{P}^{M/n} \simeq \mathbb{P}(k[x,y]_{M/n}) \to \mathbb{P}(k[x,y]_M) \simeq \mathbb{P}^M : f \mapsto f^n,$$

we have $\lambda_n^\bullet = \epsilon_n \circ \lambda_n^\circ$. Moreover, since for any periodic orbit $(z_0,\ldots,z_{n-1},z_n = z_0)$ we have

$$\frac{\partial_y \Psi_{mn} f(z_0,z_0)}{\partial_x \Psi_{mn} f(z_0,z_0)} = \left(\frac{\partial_y \Psi_n f(z_0,z_0)}{\partial_x \Psi_n f(z_0,z_0)} \right)^m,$$

we can see that

$$\operatorname{Im}\Lambda_n \simeq \operatorname{Im}\Lambda_n^\bullet \simeq \operatorname{Im}\Lambda_n^\circ,$$

where

$$\Lambda_n^\alpha := \prod_{m:m|n} \lambda_m^\alpha \text{ for } \alpha \in \{\circ, \bullet\}.$$

Remark 5.13. Despite the form in (15) is given by an n-th power of some polynomial, it is difficult to obtain more explicit form of the n-th root λ_n°. This phenomenon happens in computing resultant by Cayley's formula ([ESW03]). If $n = 2$, Pfaffian is sometimes used to compute Cayley's formula. Whether analogous method exist for the second multiplier map is a problem. As we see in (26), if we have a method to choose a specified branch of the roots, we can compute the root directly by interpolation.

6 Degree Bound of Multiplier Map

The volumes of the invariant algebras of irreducible representations of SL_2 are classically calculated by Hilbert and the reducible cases are done in [dCCPHHS20].

6.1 Schur Polynomial

In [dCCPHHS20], Schur Polynomials are used to express the volumes of invariant rings. We briefly introduce the polynomials in a form that we can instantly give an explicit evaluation of the volumes.

Definition 6.1. (Schur Polynomial) For a sequence of nonnegative integers (d_i) of length l, the *Schur polynomial* $s_{(d_i)}(x_i)$ of (d_i) is the symmetric polynomial of l variables such that

$$s_{(d_i)}(x_i) = \frac{\det(x_i^{j+d_j-1})_{i,j=1}^l}{\det(x_i^{j-1})_{i,j=1}^l}.$$

Definition 6.2. For a nonincreasing sequence of nonnegative integers (d_i), the Young tableau of (d_i) is the set of lattice points

$$T(d_i) := \{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq d_j\}.$$

The set of n-semistandard tableau is the set

$$\mathrm{SST}_n(d_i) := \{f : T(d_i) \to \{1, \ldots, n\} \mid f(i,j) \leq f(i+1,j), f(i,j) < f(i,j+1)\}.$$

Theorem 6.3. (Kostka's Definition [Pra19, Corollary 12.5]) For a decreasing sequence of nonnegative integers $(d_1, \ldots, d_l)$, we have

$$s_{(d_1,\ldots,d_l,0,0,\ldots,0)}(x_1, \ldots, x_{l+k}) = \sum_{f \in \mathrm{SST}_{l+k}(d_i)} \prod_{(i,j) \in T(d_i)} x_{f(i,j)}.$$

6.2 Evaluation

We use the following result to calculate the volume.

Theorem 6.4. ([dCCPHHS20]) Let $V \in \mathrm{Rep}_k(\mathrm{SL}_2)$ be a representation of SL_2 and $\dim_k V = n \geq 5$. Then the Laurent expansion of the Hilbert function of the invariant ring $I(V) := k[V]^{\mathrm{SL}_2(k)}$ around $t = 1$ has the form

$$h_{I(V)}(t) = (1-t)^{-n+3} \cdot \sum_{i=0}^{\infty} \gamma_i (1-t)^i.$$

Let (a_i) be the positive weights of V and l be the length of the sequence. Then we have

$$\gamma_0 = \gcd(2, a_1, \ldots, a_l) \frac{s_{(l-3,l-3,l-3,l-4,l-5,\ldots,2,1,0)}(a_1, a_2, \ldots, a_l)}{s_{(l-1,l-2,l-3,l-4,l-5,\ldots,2,1,0)}(a_1, a_2, \ldots, a_l)}.$$

Remark 6.5. In [dCCPHHS20], higher terms $(\gamma_1, \gamma_2, \gamma_3)$ are also computed.

Throughout this subsection, we put $n := d + e$. We give a rough estimate of γ_0 for the case $V = V_d \otimes V_e$ of the moduli space of dynamical systems.

Lemma 6.6. We have

$$\mathrm{Vol}\, I(V_d \otimes V_e) \leq \frac{\gcd(n,2)}{2(n-2)(n-1)n}.$$

Proof. We put $\alpha := (l-3, l-3, l-3, l-4, l-5, \dots, 1)$ and $\delta := (l-1, l-2, \dots, 2, 1)$. Let $\Phi_k : \mathrm{SST}_l(\alpha) \to \mathrm{SST}_l(\delta)$ for $k = 1, 2$ be the map such that for any $f \in \mathrm{SST}_l(\alpha)$

$$\Phi_k(f)(i,j) := \begin{cases} f(i,j) & ((i,j) \in T(\alpha)) \\ l-1 & ((i,j) = (1, l-2)) \\ l & ((i,j) = (2, l-2)) \\ l-1 & ((i,j) = (1, l-1),\ k = 1) \\ l & ((i,j) = (1, l-1),\ k = 2) \end{cases}.$$

The maps Φ_1 and Φ_2 are both injective and the images are disjoint. Therefore by Theorem 6.3 we have

$$\begin{aligned} s_\delta(x_1, \dots, x_l) =& (x_{l-1}x_l^2 + x_{l-1}^2 x_l) s_\alpha(x_1, \dots, x_l) \\ &+ (\text{polynomial with nonnegative coefficients}). \end{aligned} \tag{19}$$

The three largest among the weights of the representation $V_d \otimes V_e$ are $(n-2, n-2, n)$ and other weights are smaller than $n-2$. By substituting the weights into (19), we obtain

$$s_\delta(\nu, n-2, n-2, n) \geq 2(n-2)(n-1)n \cdot s_\alpha(\nu, n-2, n-2, n),$$

where we put the sequence of positive weights smaller than $n-2$ by ν. By Theorem 6.4, we have

$$\begin{aligned} \mathrm{Vol}(I(V_d \otimes V_e)) &= \gcd(2, n) \frac{s_\alpha(\nu, n-2, n-2, n)}{s_\delta(\nu, n-2, n-2, n)} \\ &\leq \gcd(2, n) \frac{1}{2(n-2)(n-1)n}. \end{aligned}$$

□

Theorem 6.7. Let p be a prime number. If the first and the p-th multiplier map to the image

$$\Lambda_p^\circ := \lambda_{1,(d,e)}^\circ \times \lambda_{p,(d,e)}^\circ : \mathrm{Dyn}_{d,e} \dashrightarrow \Lambda(\mathrm{Dyn}_{d,e}) \subset \mathbb{P}(D_{d+e}) \times \mathbb{P}(D_M)$$

is finite, then its degree is at most

$$\frac{\gcd(n,2) N^{de+n-3} (n-3)!(de-3)!}{2n \cdot (de+n-3)!},$$

where

$$N := 2(d+e-1) + \frac{2((d^p-1)(d^p-d) - (e^p-1)(e^p-e))}{p(d-e)}.$$

Proof. We put $A := I(V_d \otimes V_e)$. Let L_1 and L_p be the linear systems which gives λ_1° and λ_p°. By Proposition 5.11, we can take the linear systems in A such that whose degrees are respectively at most

$$2(d+e-1) \text{ and } \frac{2((d^p-1)(d^p-d) - (e^p-1)(e^p-e))}{p(d-e)}.$$

By assumption and algebraic independence of discriminant-resultant [Gor15], we can take n elements $\sigma_1, \dots, \sigma_n$ in L_1 and $\dim \mathrm{Dyn}_{d,e} - (n-1) = de - 2$ elements $\delta_1, \dots, \delta_{de-2}$ in L_p to be algebraically independent.

Let $k[L_1 \otimes L_p] \to A$ be the morphism of graded k-algebras defined by $L_1 \otimes L_p \ni f \otimes g \mapsto fg \in A$ and $B_{(1,p)}$ be its image. Then we have the degree of the morphism Λ is the extension degree $[KP(A) : KP(B_{(1,p)})]$ of rational function fields.

The graded subalgebra $B_{(1,p)} \supset B := k[f_i g_j \mid 1 \le i \le n,\ 1 \le j \le de-2]$ of A has the Hilbert series

$$h_B(t) = \sum_{i=0}^{\infty} \binom{i+n-1}{n-1} \binom{i+de-3}{de-3} t^{iN}.$$

This leads to

$$\mathrm{Vol}(B) = \frac{1}{N^{de+n-3}} \cdot \frac{(de+n-3)!}{(de-3)!(n-1)!}.$$

Therefore we have

$$\begin{aligned}
[KP(A) : KP(B_{(1,p)})] &\le [KP(A) : KP(B)] \\
&\le \frac{\mathrm{Vol}(A)}{\mathrm{Vol}(B)} \\
&\le \frac{\gcd(n,2)}{2n(n-1)(n-2)} \cdot \frac{N^{de+n-3}(n-1)!(de-3)!}{(de+n-3)!} \qquad (20) \\
&= \frac{\gcd(n,2) N^{de+n-3}(n-3)!(de-3)!}{2n \cdot (de+n-3)!}
\end{aligned}$$

from Proposition 3.34. □

Remark 6.8. By skipping Lemma 6.6, we can use

$$\frac{s_\delta(\text{positive weights of } V_d \otimes V_e)}{s_\alpha(\text{positive weights of } V_d \otimes V_e)} \qquad (21)$$

instead of $1/2n(n-1)(n-2)$ in (20). Experimentally (21) looks like of order $O(n^{-(4+O(1))})$, but this difference of orders may be very small comparing to $N^{de+d+e-3}$.

Remark 6.9. From number of periodic orbits, and Holomorphic Lefschetz formula ([Ill77], [Got23a]) for C and $\Psi_2 C$, the dimension of fiber of Λ_2° is at least

$$\begin{aligned}
&(d+e-1) + \frac{(d^2-d)+(e^2-e)}{2} - 1 - (de+d+e-3) \\
&= \frac{(d-e)^2 - (d+e) + 2}{2}. \qquad (22)
\end{aligned}$$

In particular, the map Λ_2° can be generically finite to its image only if $(d-e)^2 \ge d+e-2$.

7 Birationality of the Second Multiplier Map of Cubic Maps

In this section, we give two proofs of the following theorem.

Theorem 7.1. The multiplier map $\Lambda_{2,(1,3)}$ is birational to its image.

7.1 Finite field reduction

The counting of the degree of

$$\Lambda_2 := \lambda_{1,(1,3)} \times \lambda_{2,(1,3)} : \mathrm{Dyn}_{1,3} \to \mathbb{P}^3 \times \mathbb{P}^9$$

to its image is done in [HT13] by the following method. First, we fix a point $P \in \lambda_1(\mathrm{Dyn}_{1,3})$ and consider the inverse image $l := \lambda_1^{-1}(P)$. An explicit morphism $\phi_P : \mathbb{P}^1 \to \mathrm{Corr}_{1,3}$ such that l is birational to the image of $\mathbb{P}^1 \xrightarrow{\phi_P} \mathrm{Corr}_{1,3} \xrightarrow{\pi} \mathrm{Dyn}_{1,3}$ is given in [HT13]. We denote the endomorphism on $\mathbb{P}^1$ indicated by the point $\phi_P(a)$ by $\phi_{P,a}$. Then we will solve the equations in the two variables a and b,

$$\phi_{P,a}^2(b) = b \tag{23}$$

$$(\phi_{P,a}^2)'(b) = \lambda \tag{24}$$

$$\phi_{P,a}(b) \neq b \tag{25}$$

for a given P and λ. The equations (23) and (24) are of degree 9 and 16 respectively, in variables a and b. By a MAGMA computation over a finite field, we obtain the solutions as a 0-dimensional closed subscheme Z of degree 144 on $\mathbb{P}^2$. Under a base-change to the algebraically closed field, the support of Z consists of 18 points. Six of them are non-reduced and does not satisfy the inequality (25). Remaining 12 points satisfies (25), moreover the MAGMA computation shows that they are reduced.

That was the computation done in [HT13]. We continue computation from here. At first, we note that for a solution (a, b) of (23),(24) and (25), the points $(a, \phi_{P,a}(b))$ is also a solution of equation. Therefore, we obtain 6 rational maps $\phi_{P,a}$ with periodic points of period two. For a value λ, the solutions are given by explicit values of (a, b). For the 6 rational maps, we compute multipliers of other periodic points of period 2. Then we obtain that the values of other multipliers are mutually different, so we obtain that $\Lambda_{3,2}$ is injective.

7.2 Direct computation with Graded-decomposition and interpolation

In this section, we show Theorem 7.1 by computing the full formula of the second multiplier map $\Lambda^{\circ}_{2,(1,3)}$.

In the computational process of the explicit expression, we use the information that the resulting polynomials are SL_2-invariant. Our algorithm (Subsection A.2) of graded-piece-wise computation is applied for limited cases, but this method makes the computation much faster.

We use the expression of generators of the invariant ring $A := I(V_1 \otimes V_3) = I(V_4 \oplus V_2)$ given in [Wes15]. The invariant ring is given by

$$A \simeq k[d, i, j, a, b, c]/r,$$

where d, i, j, a, b, c are generators of degree respectively $2, 2, 3, 3, 4, 6$ and r is the relation

$$2c^2 = \frac{1}{54}d^3i^3 - \frac{1}{9}d^3j^2 - \frac{1}{12}di^2a^2 - \frac{1}{3}ja^3 + djab + \frac{1}{2}ia^2b - \frac{1}{2}dib^2 - b^3.$$

Let $f_4(z)$ and $f_2(z)$ be the fundamental covariants of V_4 and V_2 respectively (denoted by $\mathbf{f}$ and $\mathbf{g}$ in [Wes15] respectively). By [Got23a, Remark 7.6], the first multiplier map λ_1 is given by the linear system consists of discriminant-resultants (named in [Got22])

$$\sigma_r := DR_{4,r}(f_4, f_2) \ (r = 0, 2, 3, 4) \text{ of degree } (6 - r, r),$$

which are defined by

$$\sum_{r=0}^{4} DR_{4,r}(f_4, f_2)t^r = \mathrm{res}_z(f_4, \partial_z f_4 + z f_2 t).$$

We put

$$\Sigma_\pm := \sigma_0 + \sigma_2 \pm \sigma_3 + \sigma_4.$$

By Proposition 5.11, we can have a linear system of λ_2° of degree at most 24. Let

$$L_2(t) := \mathrm{res}_z(\Pi_2^* f(z), d_z\Omega^0(\Psi_2 f)(z) + t \cdot \Omega^1(\Psi_2 f)(z)).$$

In this case, $L_2(t)$ has a divisor Σ_-^4 and we can take a square root of $L_2(t)/\Sigma_-^4$. We put the square root as

$$\sqrt{L_2(t)/\Sigma_-^4} =: \delta(t) =: \delta_0 + \delta_1 t + \delta_2 t^2 + \delta_3 t^3.$$

The forms δ_i's are invariants in A of degree $(48 - 4 \cdot 6)/2 = 12$. We have

$$\begin{aligned} \delta_0^2 \cdot \Sigma_-^4 &= \mathrm{res}_z(\Pi_2^* f(z), d_z\Omega^0(\Psi_2 f)(z)) \\ &= \Delta_z(\Pi_2^* f(z)) \cdot \mathrm{res}_z(\Pi_2^* f(z), \Pi_1^* f(z)) \end{aligned}$$

and by a direct computation we obtain

$$\begin{aligned} \Delta_z(\Pi_2^* f(z)) &= \Sigma_+ \cdot \Sigma_-^2 \cdot \phi^2 \text{ and} \\ \mathrm{res}_z(\Pi_2^* f(z), \Pi_1^* f(z)) &= \Sigma_+ \cdot \Sigma_-^2, \text{ where} \end{aligned}$$

$$\phi = 2^{-27} \cdot (d^3 - 12d^2i + 48di^2 - 64i^3 + 384j^2 + 288ja + 54a^2),$$

so we have $\delta_0 = \pm\Sigma_+ \cdot \phi$. By fixing the sign to be $+$, we can compute $\delta(t)$ and obtain

$$\delta_1 = -\frac{1}{2}(-9\sigma_0 + \sigma_2 + 6\sigma_3 + 11\sigma_4) \cdot \phi, \delta_2, \delta_3 \in A_{12}. \tag{26}$$

Here we remark that

$$K_2 := K(\Lambda_2(\mathrm{Dyn}_{1,3})) = k\left(\frac{\sigma_i}{\sigma_j}, \frac{\delta_i}{\delta_j}\right) \subset KP(A).$$

So we start from the algebra

$$B_1 := k[\sigma_i \cdot \phi (i = 0, 2, 3, 4), \delta_2, \delta_3] (\subset A)$$

to larger sub-graded-algebras of A with keeping the condition $KP(B_i) = K_2$. By seeking factorizable linear combinations of the generators of B_1, we find

$$\begin{aligned} &\delta_2 + \phi(-10\sigma_0 + \sigma_2 + 10\sigma_3) = \Sigma_- \cdot \psi, \\ &\text{where } \psi = \frac{1}{2^{31}}(-26048i^3 + 9936i^2d - 884id^2 + 7d^3 \\ &\qquad\qquad - 102912j^2 - 38784ja - 72a^2 - 9600ib + 2400db) \text{ and} \\ &(\delta_2 + \delta_3) - \frac{1}{2}(11\sigma_0 + \sigma_2 - 4\sigma_3 - 9\sigma_4)\phi = \frac{5^3}{2^{19}}(\Sigma_- + \sqrt{2}\sigma_3)(\Sigma_- - \sqrt{2}\sigma_3). \end{aligned}$$

Therefore, we can replace B_1 by

$$B_2 := k[\sigma_i, \phi, \psi].$$

By computing the elimination ideal of generators of B_2, we obtain that the only relation among the generators is a relation of degree 60 (degree 10 polynomial of σ_i, ϕ, ψ), so we can see that the Hilbert series of B_2 is given by

$$H_{B_2}(t) = \frac{1 - t^{60}}{(1 - t^6)^6}.$$

Here we have

$$H_A(t) = \frac{1 + t^6}{(1 - t^2)^2(1 - t^3)^2(1 - t^4)}.$$

We recall that for any graded algebra C, $C^{[n]} := \bigoplus_{i \geq 0} C_{in}$ with $\deg C_{in} = i$. By a direct computation, we have

$$H_{A^{[6]}}(t) = \frac{(1 + t)(1 + 5t + 9t^2 + 4t^3)}{(1 - t)^4(1 - t^2)} \text{ and } H_{B_2^{[6]}}(t) = \frac{1 - t^{10}}{(1 - t)^6}.$$

By Proposition 3.34, we have

$$\deg(\Lambda_2) = [KP(A^{[6]}) : KP(B_2^{[6]})] \leq \frac{\mathrm{Vol}(A^{[6]})}{\mathrm{Vol}(B_2^{[6]})} = \frac{9}{5},$$

this shows that $\deg(\Lambda_2) = 1$.

Remark 7.2. Throughout this ad hoc proof, there are three steps completed unexpectedly easily. The first is that there are factorizable linear combinations including δ_2 and δ_3. The second is that a linear combination moreover belongs to $k[\sigma_i]_{12}$. The third is that the relation among σ_i, ϕ, ψ was of degree 60. Because of this small degree (the expected degree from the Hilbert series is 108), we can obtain the result in a few minutes by simply computing the elimination ideal. Moreover, this degree is also the lower bound to determine the extension degree to be 1.

A Appendix: Programs

A.1 Programs used in Subsection 7.1

The MAGMA program run in [HT13] were the following.

```
l0:=3;
l1:=2;
l8:=4;
lB:=-5;
R<a,B,z>:=ProjectiveSpace(GF(101),2);
function h(P,d) Q:=0;
for i:=0 to d do for j:=0 to d-i do Q:=Q+ Term(Term(P,a,i),B,j)*z^(d-i-j);
end for;
end for;
return(Q);
end function;
function f(x,y) return((((l0 - 1)*l1 + (-l0 + 1))*x^3 + ((a*l0*l1
+ (-l0 + (-a + 1)))*l8 + (((-a - 1)*l0 +1)*l1 + (2*l0 + (a - 2))))*x^2*y
+ ((-a*l0*l1 + a*l0)*l8 + (a*l0*l1 - a*l0))*x*y^2));
end function;
function g(x,y) return((((l0 - 1)*l1 + (-l0 + 1))*l8*x^2*y + (((-l0
+ (a + 1))*l1 + (a*l0 - 2*a))*l8 + (-a*l1 + ((-a + 1)*l0 +(2*a
- 1))))*x*y^2 + ((-a*l1 + a)*l8 + (a*l1 - a))*y^3));
end function;
f1:=f(f(B,1),g(B,1));
g1:=g(f(B,1),g(B,1));
F1:=f1-B*g1;
F2:=g1*Derivative(f1,B) - f1*Derivative(g1,B) - lB*g1*g1;
G1:=h(F1,9);
G2:=h(F2,16);
C:=Scheme(R,[G1,G2]);
D:=ReducedSubscheme(C);
Degree(D);
```

After running this computation, we firstly compute the coordinates of the points of D.

```
PointsOverSplittingField(D);
Output:
{@ (0 :  0 :  1), (0 :  4 :  1), (1 :  1 :  1), (1 :  49 :  1),
(4 :  93*$.1^7 + 44*$.1^6 + 24*$.1^5 + 23*$.1^4 + 48*$.1^3 + 26*$.1^2
+ 65*$.1 + 90 :  1), (4 :  8*$.1^7 + 57*$.1^6 + 77*$.1^5 + 78*$.1^4
+ 53*$.1^3 + 75*$.1^2 + 36*$.1 + 79 :  1), (96 :  27 :  1), (96
:  6 :  1), (18*$.1^7 + 50*$.1^6 + 68*$.1^5 + 24*$.1^4 + 59*$.1^3
+ 22*$.1^2 + 93*$.1 + 93 :  55*$.1^7 + 55*$.1^6 + 80*$.1^5 + 72*$.1^4
+ 5*$.1^3 + 89*$.1^2 + 52*$.1 + 10 :  1), (18*$.1^7 + 50*$.1^6
+ 68*$.1^5 + 24*$.1^4 + 59*$.1^3 + 22*$.1^2 + 93*$.1 + 93 :  15*$.1^7
+ 56*$.1^6 + 83*$.1^5 + 3*$.1^4 + 62*$.1^3 + 95*$.1^2 + 21*$.1
+ 80 :  1), (14*$.1^7 + 52*$.1^6 + 48*$.1^5 + 18*$.1^4 + 53*$.1^3
+ 62*$.1^2 + 42*$.1 + 50 :  70*$.1^7 + 7*$.1^6 + 3*$.1^5 + 27*$.1^4
+ 47*$.1^3 + 32*$.1^2 + 64*$.1 + 28 :  1), (14*$.1^7 + 52*$.1^6
+ 48*$.1^5 + 18*$.1^4 + 53*$.1^3 + 62*$.1^2 + 42*$.1 + 50 :  88*$.1^7
+ 85*$.1^6 + 73*$.1^5 + 50*$.1^4 + 65*$.1^3 + 59*$.1^2 + 96*$.1
+ 70 :  1), (75*$.1^7 + 95*$.1^6 + 57*$.1^5 + 100*$.1^4 + 90*$.1^3
+ 4*$.1^2 + 73*$.1 + 55 :  5*$.1^7 + 72*$.1^6 + 70*$.1^5 + 39*$.1^4
+ 32*$.1^3 + 31*$.1^2 + 74*$.1 + 26 :  1), (75*$.1^7 + 95*$.1^6
+ 57*$.1^5 + 100*$.1^4 + 90*$.1^3 + 4*$.1^2 + 73*$.1 + 55 :  86*$.1^7
+ 93*$.1^6 + 92*$.1^5 + 67*$.1^4 + 46*$.1^3 + 95*$.1^2 + 22*$.1
+ 78 :  1), (95*$.1^7 + 5*$.1^6 + 29*$.1^5 + 60*$.1^4 + 13*$.1^2
+ 95*$.1 + 87 :  66*$.1^7 + 33*$.1^6 + 62*$.1^5 + 50*$.1^4 + 83*$.1^3
+ 87*$.1^2 + 29*$.1 + 25 :  1), (95*$.1^7 + 5*$.1^6 + 29*$.1^5
+ 60*$.1^4 + 13*$.1^2 + 95*$.1 + 87 :  19*$.1^7 + 3*$.1^6 + 42*$.1^5
+ 96*$.1^4 + 64*$.1^3 + 17*$.1^2 + 46*$.1 + 2 :  1), (47 :  1
:  0), (1 :  0 :  0) @}
Finite field of size 101^8
```

These are the coordinates $(a : b : z)$ of the solutions of (24) and (23) on $\mathbb{P}^2$, with the homogenizing variable z. The parameters a of 12 reduced points, consisted of 6 values as expected are the following.

```
{4,96,18*$.1^7 + 50*$.1^6 + 68*$.1^5 + 24*$.1^4 + 59*$.1^3 + 22*$.1^2
+ 93*$.1 + 93, 14*$.1^7 + 52*$.1^6 + 48*$.1^5 + 18*$.1^4 + 53*$.1^3
+ 62*$.1^2 + 42*$.1 + 50, 75*$.1^7 + 95*$.1^6 + 57*$.1^5 + 100*$.1^4
+ 90*$.1^3 + 4*$.1^2 + 73*$.1 + 55, 95*$.1^7 + 5*$.1^6 + 29*$.1^5
+ 60*$.1^4 + 13*$.1^2 + 95*$.1 + 87},
```

The multipliers are given by:

```
l0:=3;
l1:=2;
l8:=4;
lB:=-5;

F<w>:= GF(101,8);
```

```
R<a,B,z>:=PolynomialRing(F,3);

function f(x,y)
return((((l0 - 1)*l1 + (-l0 + 1))*x^3 + ((a*l0*l1 + (-l0 + (-a
+ 1)))*l8 + (((-a - 1)*l0 +1)*l1 + (2*l0 + (a - 2))))*x^2*y +
((-a*l0*l1 + a*l0)*l8 + (a*l0*l1 - a*l0))*x*y^2));
end function;
function g(x,y)
return((((l0 - 1)*l1 + (-l0 + 1))*l8*x^2*y + (((-l0 + (a + 1))*l1
+ (a*l0 - 2*a))*l8 + (-a*l1 + ((-a + 1)*l0 +(2*a - 1))))*x*y^2
+ ((-a*l1 + a)*l8 + (a*l1 - a))*y^3));
end function;
f2:=f(f(B,1),g(B,1));
g2:=g(f(B,1),g(B,1));
F2:=f2-B*g2;
redF2:=R!(F2/(f(B,1) - B * g(B,1)));
dF2:=g2*Derivative(f2,B) - f2*Derivative(g2,B) - z*g2*g2;

function mult(c);
return(Resultant(Evaluate(redF2,a,c),Evaluate(dF2,a,c),B));
end function;

result:
> Factorization(mult(4));
[ <z + 5, 2>, <z + 50, 2>, <z + 90, 2> ]
> Factorization(mult(96));
[ <z + 5, 2>, <z + 26, 2>, <z + 66, 2> ]
> Factorization(mult(18*w^7 + 50*w^6 + 68*w^5 + 24*w^4 + 59*w^3
+ 22*w^2 + 93*w + 93));
[ <z + 5, 2>, <z + 78*w^7 + 93*w^6 + 47*w^5 + 57*w^4 + 23*w^3
+ 80*w^2 + 73*w + 52, 2>, <z + 70*w^7 + 79*w^6 + 53*w^5 + 6*w^4
+ 42*w^3 + 86*w^2 + 96*w + 53, 2> ]
> Factorization(mult(14*w^7 + 52*w^6 + 48*w^5 + 18*w^4 + 53*w^3
+ 62*w^2 + 42*w + 50));
[ <z + 5, 2>, <z + 60*w^7 + 98*w^6 + 74*w^4 + 4*w^3 + 24*w^2 +
24*w + 21, 2>, <z + 27*w^7 + 14*w^6 + 59*w^5 + 73*w^4 + 55*w^3
+ 12*w^2 + 37*w + 11, 2> ]
> Factorization(mult(75*w^7 + 95*w^6 + 57*w^5 + 100*w^4 + 90*w^3
+ 4*w^2 + 73*w + 55));
[ <z + 5, 2>, <z + 100*w^7 + 39*w^6 + 25*w^5 + 12*w^4 + 20*w^3
+ 99*w^2 + 21*w + 73, 2>, <z + 84*w^7 + 84*w^6 + 91*w^5 + 31*w^4
+ 44*w^3 + 70*w^2 + 92*w + 13, 2> ]
> Factorization(mult(95*w^7 + 5*w^6 + 29*w^5 + 60*w^4 + 13*w^2
+ 95*w + 87));
[ <z + 5, 2>, <z + 89*w^7 + 42*w^6 + 58*w^5 + 91*w^4 + 11*w^3
+ 22*w^2 + 91*w + 80, 2>, <z + 98*w^7 + 56*w^6 + 71*w^5 + 60*w^4
```

```
    + 3*w^3 + 11*w^2 + 71*w + 81, 2> ]
```

This computation shows the other multipliers of period two orbits are mutually different for the six solutions of (23).

A.2 An algorithm for Subsection 7.2

In order to make up block-decomposed interpolation matrix, we used the following algorithm. The program file written by SAGE[The22] is attached, or at [Got].

Algorithm 1 Degree-wise random-sampling interpolation method (probabilistic)

Require:
  Algorithms to compute $g_1, g_2, \ldots, g_\beta$ and $H = h(g_1, \ldots, g_\beta)$,
  the set of monomials $M = \left\{ \mathbf{y}^{\mathbf{d}} \middle| \mathbf{d} \in \{\mathbf{d}_1, \ldots, \mathbf{d}_l\} \right\}$
  such that $h = \sum c_{\mathbf{d}} \mathbf{y}^{\mathbf{d}}$ ($c_{\mathbf{d}} \in \mathbb{Q}^\beta$)
  A map $\sigma : [n] \to [m]$ such that $g_i(a_1 x_{\sigma(1)}, \ldots, a_n x_{\sigma(n)})$ is a monomial with coefficient for each g_i and $\mathbf{a} \in \mathbb{Q}^n$.
Ensure: [Probably] The polynomial $h(y_1, \ldots, y_\beta)$
  Separate M by the degree of $(\mathbf{g}(\mathbf{x}_\sigma))^{\mathbf{d}}$ into $M_1, \ldots, M_p$
  $l_p := \max \# M_i$
  for j from 1 to $l_p + l'$ **do**
    Take a random vector $\mathbf{a}_j \in \mathbb{Q}^n$
    Compute $H(\mathbf{a}_j \mathbf{x}_\sigma)$
    Compute $g_i(\mathbf{a}_j)$'s and $\mathbf{g}(\mathbf{a}_j)^{\mathbf{d}}$
  end for
  for k from 1 to p **do**
    Let $H_{j,k}$ be the coefficient of the term of degree $\mathbf{g}(\mathbf{x}_\sigma)^{\mathbf{d}}$ of $H(\mathbf{a}_j \mathbf{x}_\sigma)$ for $\mathbf{d}$ in M_k
    Solve the system of linear equations $H_{j,k} = \sum_{\mathbf{d} \in M_k} c_{\mathbf{d}} \mathbf{g}(\mathbf{a}_j)^{\mathbf{d}}$ ($j = 1, \ldots, l_p + l'$) for $c_{\mathbf{d}}$'s.
  end for
  $h(\mathbf{y}) = \sum_{\mathbf{d} \in M} c_{\mathbf{d}} \mathbf{y}^{\mathbf{d}}$.

In our case, we only use addition and multiplication of the polynomials degree less than H to compute H, so the numbers of terms appears in the computation are $O(p)$, thus it costs $O(p^2 t_H)$ to compute $H(\mathbf{a}_j \mathbf{x}_\sigma)$ par once. Therefore, the computational complexity of this algorithm is $O(l_p p^2 t_H + l_p^c p) = O(N(p t_H + l_p^{c-1}))$. In our case $l_p = 70$ and the constants are sufficiently small. Moreover, we set $l' = 5$ in the computation.

A.3 Data for Subsection 7.2

Explicit formula of δ_2 and δ_3 are

$$\begin{aligned}
\delta_2 =& \frac{-1}{2^{31}\cdot 3}(306d^5i - 2072d^4i^2 - 20544d^3i^3 + 300800d^2i^4 - 1691136di^5 \\
&+ 3483648i^6 + 96192d^3j^2 - 1286400d^2ij^2 + 10146816di^2j^2 - 16920576i^3j^2 \\
&+ 1008d^3ja + 42432d^2ija + 1430784di^2ja - 4451328i^3ja + 459d^3a^2 \\
&+ 14316d^2ia^2 - 6768di^2a^2 - 775104i^3a^2 - 1836d^4b + 6624d^3ib \\
&+ 173568d^2i^2b - 1850880di^3b + 4672512i^4b - 23887872j^4 - 10616832j^3a \\
&- 156672j^2a^2 + 244224ja^3 + 26136a^4 - 589824dj^2b + 9289728ij^2b \\
&- 672768djab + 4202496ijab - 111744da^2b + 39168ia^2b + 28800d^2b^2 \\
&- 460800dib^2 + 1382400i^2b^2 - 2208d^3c + 103296d^2ic - 1027584di^2c \\
&+ 2598912i^3c + 9289728j^2c + 3280896jac - 76032a^2c - 230400dbc + 921600ibc),
\end{aligned}$$

$$\begin{aligned}
\delta_3 =& \frac{1}{2^{31}\cdot 3^3}(1458d^5i + 2904d^4i^2 - 43072d^3i^3 + 2453760d^2i^4 - 11570688di^5 \\
&+ 40310784i^6 - 358464d^3j^2 - 10056960d^2ij^2 + 69424128di^2j^2 - 259780608i^3j^2 \\
&- 1296d^3ja - 730944d^2ija + 12379392di^2ja - 58973184i^3ja + 2187d^3a^2 \\
&+ 100188d^2ia^2 - 730224di^2a^2 - 1881792i^3a^2 - 8748d^4b - 95328d^3ib \\
&+ 2674944d^2i^2b - 20113920di^3b + 82861056i^4b + 107495424j^4 + 17915904j^3a \\
&- 3856896j^2a^2 - 2521728ja^3 + 143748a^4 + 18413568dj^2b - 161243136ij^2b \\
&+ 8280576djab - 21399552ijab - 693792da^2b + 5664384ia^2b + 475200d^2b^2 \\
&- 12787200dib^2 + 43545600i^2b^2 + 7776d^3c + 1173888d^2ic - 7921152di^2c \\
&+ 49268736i^3c - 6912000b^3 - 71663616j^2c - 3981312jac + 1672704a^2c \\
&- 5529600dbc + 49766400ibc).
\end{aligned}$$

The relation among σ_i, ϕ, ψ has 1261 terms. The data is at [Got].

References

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The magma algebra system i: The user language*, Journal of Symbolic Computation **24** (1997), no. 3-4, 235–265.

[dCCPHHS20] Pedro de Carvalho Cayres Pinto, Hans-Christian Herbig, Daniel Herden, and Christopher Seaton, *The hilbert series of sl 2-invariants*, Communications in Contemporary Mathematics **22** (2020), no. 07, 1950017.

[DK15] Harm Derksen and Gregor Kemper, *Computational invariant theory*, Encyclopaedia of Mathematical Sciences 130. Invariant

Theory and Algebraic Transformation Groups VIII, Springer, 2015, Second enlarged edition. With two appendices by Vladimir L. Popov, and an addendum by Norbert A'Campo and Popov.

[DM06] Valery Dolotin and Alexei Yurievich Morozov, *The universal mandelbrot set: Beginning of the story*, World Scientific, 2006.

[ESW03] David Eisenbud, Frank-Olaf Schreyer, and Jerzy Weyman, *Resultants and chow forms via exterior syzygies*, Journal of the American Mathematical Society **16** (2003), no. 3, 537–579.

[Fuj06] Masayo Fujimura, *Projective moduli space for the polynomials*, Dynamics of Continuous, Discrete and Impulsive Systems Series A **13** (2006), no. 6, 787.

[Fuj07] ———, *The moduli space of rational maps and surjectivity of multiplier representation*, Computational Methods and Function Theory **7** (2007), 345–360.

[Ful97] William Fulton, *Young tableaux: with applications to representation theory and geometry*, no. 35, Cambridge University Press, 1997.

[FZ03] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras ii: Finite type classification*, Inventiones mathematicae **154** (2003), no. 1, 63–121.

[GKZ94] I. M. Gel'fand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1994.

[Gor15] Igors Gorbovickis, *Algebraic independence of multipliers of periodic orbits in the space of rational maps of the riemann sphere*, Moscow Mathematical Journal **15** (2015), no. 1, 73–87.

[Got] Rin Gotou, *Program files of "dynamical systems of correspondences on the projective line ii: Number of isospectral maps"*, `https://github.com/Rin-Gotou/dyn_corr_p1_II_multiplier_map`.

[Got22] ———, *Bracket polynomial expression of discriminant-resultants as* SL_2*-invariant*, 2022.

[Got23a] ———, *Dynamical systems of correspondences on the projective line i: Moduli spaces and multiplier maps*, Conformal Geometry and Dynamics of the American Mathematical Society **27** (2023), no. 08, 294–321.

[Got23b] ———, *Dynamical systems of correspondences on the projective line II: Degrees of multiplier maps*, 2023.

[GOV20] Thomas Gauthier, Yûsuke Okuyama, and Gabriel Vigny, *Approximation of non-archimedean lyapunov exponents and applications over global fields*, Transactions of the American Mathematical Society **373** (2020), no. 12, 8963–9011.

[HT13] Benjamin Hutz and Michael Tepper, *Multiplier spectra and the moduli space of degree 3 morphisms on* $\mathbb{P}^1$, JP Journal of Algebra, Number Theory and Applications **29** (2013), no. 2, 189–206.

[Hut21] Benjamin Hutz, *Errata for multiplier spectra and the moduli space of degree 3 morphisms on p1*, 2021.

[Ill77] Luc Illusie, *Séminaire de géométrie algébrique du bois-marie 1965–66 - cohomologie l-adique et fonctions l - (sga 5)*, Lecture Notes in Mathematics (1977), 252.

[JX23] Zhuchao Ji and Junyi Xie, *The multiplier morphism is generically injective*, 2023.

[Liu06] Qing Liu, *Algebraic geometry and arithmetic curves*, Translated from the French by Reinie Erné. Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, 2006.

[McM87] Curt McMullen, *Families of rational maps and iterative root-finding algorithms*, Annals of Mathematics **125** (1987), no. 1, 467–493.

[MFK94] David Munford, John Fogarty, and Frances Kirwan, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994.

[Mil93] John Milnor, *Geometry and dynamics of quadratic rational maps*, Experimental mathematics **2** (1993), no. 1, 37–83, With an appendix by the author and Lei Tan.

[Mil11] ———, *Dynamics in one complex variable*, vol. 160, Princeton University Press, 2011.

[Oli17] Marc Olive, *About gordan's algorithm for binary forms*, Foundations of Computational Mathematics **17** (2017), 1407–1466.

[Pra19] Amritanshu Prasad, *An introduction to schur polynomials*, Graduate Journal of Mathematics **4** (2019), 62–84.

[Sch16] Johannes Schmitt, *A compactification of the moduli space of self-maps of* $\mathbb{CP}^1$ *via stable maps*, 2016.

[Sch17] ———, *A compactification of the moduli space of self-maps of* $\mathbb{CP}^1$ *via stable maps*, Conformal Geometry and Dynamics of the American Mathematical Society **21** (2017), no. 11, 273–318.

[Sil98] Joseph H Silverman, *The space of rational maps on $\mathbb{P}^1$*, Duke mathematical journal **94** (1998), 41–77.

[Sil07] ———, *The arithmetic of dynamical systems*, Graduate Texts in Mathematics **241** (2007).

[Smi05] Benjamin Andrew Smith, *Explicit endomorphisms and correspondences*, Ph.D. thesis, University of Sydney, 2005.

[Sug17] Toshi Sugiyama, *The moduli space of polynomial maps and their fixed-point multipliers*, Advances in Mathematics **322** (2017), 132–185.

[Sug20] ———, *The moduli space of polynomial maps and their holomorphic indices: I. generic properties in the case of having multiple fixed points*, arXiv preprint arXiv:2009.11815 (2020).

[The22] The Sage Developers, *SageMath, the Sage Mathematics Software System*, 2022, DOI 10.5281/zenodo.6259615.

[vdW30] B. L. van der Waerden, *Moderne algebra. teil II*, Die Grundlehren der mathematischen Wissenschaften (in German). Vol. 34, Springer-Verlag, 1930.

[Wes15] Lloyd William West, *The moduli space of rational maps*, Ph.D. thesis, City University of New York, 2015, p. 88.