Huang, G. and Chen, W.
Osaka J. Math.
45 (2008), 609–614

UNIQUENESS FOR THE BREZIS-NIRENBERG PROBLEM ON COMPACT EINSTEIN MANIFOLDS

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(Received March 27, 2007, revised May 18, 2007)

Abstract

We consider the positive solution of the following semi-linear elliptic equation on the compact Einstein manifolds M^n with positive scalar curvature R_0

$$\Delta_0 u - \lambda u + f(u)u^{(n+2)/(n-2)} = 0,$$

where Δ_0 is the Laplace-Beltrami operator on M^n. We prove that for $0 < \lambda \leq (n-2)R_0/(4(n-1))$ and $f'(u) \leq 0$, and at least one of two inequalities is strict, the only positive solution to the above equation is constant. The method here is intrinsic.

1. Introduction

Let (M^n, g_0) be the compact Einstein manifold with positive scalar curvature R_0 and $n \geq 3$. In this paper we consider the following nonlinear elliptic equation

$$\begin{cases} \Delta_0 u - \lambda u + f(u)u^{(n+2)/(n-2)} = 0, & \text{on } M^n; \\ u > 0, & \text{on } M^n, \end{cases} \tag{1.1}$$

where Δ_0 is the Laplace-Beltrami operator on M^n related to g_0. In the case of f a constant and $\lambda = (n-2)R_0/(4(n-1))$ with R_0 the scalar curvature of Riemannian manifold M^n, the problem (1.1) is just the Yamabe problem in the conformal geometry. If $M^n = \mathbf{S}^n$, there are infinitely many solutions for the Yamabe problem because the conformal group of the sphere is also infinite. For the Einstein manifold which is conformally distinct from sphere, Obata [11] shown that the Yamabe problem has unique solution, and Schoen pointed out there are more than three solutions for Yamabe problem on $\mathbf{S}^1 \times \mathbf{S}^{n-1}$, $n \geq 3$. Recently, Brezis and Li [4] consider problem (1.1) and specially the following problem by using moving planes and blow-up analysis.

$$\begin{cases} \Delta_0 u - \lambda u + u^p = 0, & \text{on } \mathbf{S}^n; \\ u > 0, & \text{on } \mathbf{S}^n, \end{cases} \tag{1.2}$$

and obtain that

2000 Mathematics Subject Classification. Primary 35J60; Secondary 35C21.

(A): for $M = \mathbf{S}^n$, $0 < \lambda < n(n-2)/4$ and f decreasing on $(0, +\infty)$, the only positive solution to (1.1) is constant,
(B): for $0 < \lambda \leq n(n-2)/4$ and $1 < p \leq (n+2)/(n-2)$, and at least one of two inequalities is strict, the only positive solution to (1.2) is constant,
(C): for M compact, $n = 3$ and $f = 1$, there exists a constant $\lambda_0 = \lambda_0(M, g) > 0$ such for $0 < \lambda < \lambda_0$, the only positive solution to (1.1) is constant.

In this paper our conclusions rely on the remarkable identity established by intrinsic properties. For related problems, see e.g. [2], [3], [5], [6], [8], [9], [12], [13].

Our main results are as follows.

Theorem 1.1. *Suppose M be the compact Einstein manifold, $0 < \lambda \leq (n-2)R_0/(4(n-1))$ and $f'(u) \leq 0$, and at least one of two inequalities is strict. Then the only positive solution to* (1.1) *is constant.*

As a consequence, we prove the following theorem.

Theorem 1.2. *Suppose M be the compact Einstein manifold, $0 < \lambda \leq (n-2)R_0/(4(n-1))$ and $1 < p \leq (n+2)/(n-2)$, and at least one of two inequalities is strict. Then the only solution of the equation*

$$\begin{cases} \Delta_0 u - \lambda u + u^p = 0, & \textit{on} \quad M^n; \\ u > 0, & \textit{on} \quad M^n, \end{cases}$$

is the constant solution $u = \lambda^{1/(p-1)}$.

REMARK. Clearly, Theorem 1.1 and Theorem 1.2 can be seen a generalization of Brezis and Li's results (see Theorem 1 in [4]). On the other hand, Theorem 1.2 also answers the Brezis and Li's problem 2 in [4] for compact Einstein manifolds with positive scalar curvature.

2. Proof of Theorems

Let (M^n, g_0) be the compact Einstein manifold with positive scalar curvature R_0. Define the conformal transformation $g = u^{4/(n-2)} g_0$ on M^n, then Δ_0 is related with the scalar curvatur R of g by

$$\Delta_0 u - \frac{(n-2)R_0}{4(n-1)} u + \frac{(n-2)R}{4(n-1)} u^{(n+2)/(n-2)} = 0,$$

which combing with (1.1) gives

$$R = \frac{4(n-1)}{n-2} \left(f(u) + \left(\frac{(n-2)R_0}{4(n-1)} - \lambda \right) u^{-4/(n-2)} \right).$$

Setting $\overline{\lambda} = \lambda - (n-2)R_0/(4(n-1))$, then

$$R = \frac{4(n-1)}{n-2}(f(u) - \overline{\lambda} u^{-4/(n-2)}).$$

In what follows, the Einstein summation convention will be used. Let

$$\varphi = \varphi_{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$$

be a symmetric tensor defined on M^n, and

$$\varphi_{ij} = \frac{R}{2} g^{ij} - R_{kl} g^{ik} g^{jl}.$$

It follows from [7] that the operator $\Box$ associated to φ acting on any C^2-function f defined by

$$\Box f = \varphi_{ij} f_{,ij} = \left(\frac{R}{2} g^{ij} - R_{kl} g^{ik} g^{jl} \right) f_{,ij} \tag{2.1}$$

is self-adjoint relative to the L^2 inner product of M^n, that is

$$\int_{M^n} (\Box f) g \, dV_g = \int_{M^n} f (\Box g) \, dV_g.$$

Lemma 2.1. *Let $g_0 = \varphi^{-2} g$, B_0 and B the trace free Ricci tensor of the metric g_0 and g on M^n, respectively. Then we have*

$$B_0 = B + \frac{n-2}{\varphi} \left(D \, d\varphi - \frac{\Delta \varphi}{n} g \right).$$

This formula was studied in particular by Obata [10]. One can find a proof also in Besse's book [1], Theorem 1.159.

For an Einstein metric g_0, its trace free Ricci tensor is nothing but zero. Let $\varphi = u^{2/(n-2)}$, then the above lemma shows that, in the local coordinate system,

$$B_{ij} = (n-2) \left(\frac{1}{n} \Delta (u^{2/(n-2)}) g_{ij} - (u^{2/(n-2)})_{,ij} \right) u^{-2/(n-2)}, \tag{2.2}$$

where the covariant derivatives are with respect to g, and

$$R_{ij} = B_{ij} + \frac{R}{n} g_{ij}. \tag{2.3}$$

(2.2) can be written as

$$(u^{2/(n-2)})_{,ij} = \frac{1}{n}\Delta(u^{2/(n-2)})g_{ij} - \frac{1}{n-2}u^{2/(n-2)}B_{ij}. \tag{2.4}$$

Substituting f in (2.1) with $u^{2/(n-2)}$ and using (2.4), we have

$$\begin{aligned}\Box(u^{2/(n-2)}) &= \left(\frac{R}{2}g^{ij} - R_{kl}g^{ik}g^{jl}\right)(u^{2/(n-2)})_{,ij}\\ &= \frac{R}{2}\Delta(u^{2/(n-2)}) - R_{kl}(u^{2/(n-2)})_{,ij}g^{ik}g^{jl}\\ &= \frac{R}{2}\Delta(u^{2/(n-2)}) - R_{kl}\left(\frac{1}{n}\Delta(u^{2/(n-2)})g_{ij} - \frac{1}{n-2}u^{2/(n-2)}B_{ij}\right)g^{ik}g^{jl}\\ &= \frac{(n-2)R}{2n}\Delta(u^{2/(n-2)}) + \frac{1}{n-2}u^{2/(n-2)}R_{kl}B_{ij}g^{ik}g^{jl}.\end{aligned} \tag{2.5}$$

Therefore, (2.5) together with (2.3) gives

$$\begin{aligned}\Box(u^{2/(n-2)}) &= \frac{(n-2)R}{2n}\Delta(u^{2/(n-2)}) + \frac{1}{n-2}u^{2/(n-2)}\left(B_{kl} + \frac{R}{n}g_{kl}\right)B_{ij}g^{ik}g^{jl}\\ &= \frac{(n-2)R}{2n}\Delta(u^{2/(n-2)}) + \frac{1}{n-2}u^{2/(n-2)}|B|^2 + \frac{R}{n(n-2)}u^{2/(n-2)}B_{ij}g^{ij}\\ &= \frac{(n-2)R}{2n}\Delta(u^{2/(n-2)}) + \frac{1}{n-2}u^{2/(n-2)}|B|^2.\end{aligned}$$

Note that

$$\int_{M^n}\Box(u^{2/(n-2)})\,dV_g = 0, \quad dV_g = u^{2n/(n-2)}\,dV_{g_0}.$$

Integrating the above equality and using the divergence theorem, we obtain

$$\begin{aligned}\int_{M^n} u^{2/(n-2)}|B|^2\,dV_g &= \frac{(n-2)^2}{2n}\int_{M^n}\langle\nabla(u^{2/(n-2)}), \nabla R\rangle\,dV_g\\ &= \frac{(n-2)^2}{2n}\int_{M^n}u^2\langle\nabla_0(u^{2/(n-2)}), \nabla_0 R\rangle\,dV_{g_0}\\ &= \frac{2(n-1)(n-2)}{n}\int_{M^n}u^2\langle\nabla_0(u^{2/(n-2)}), \nabla_0(f(u) - \bar{\lambda}u^{-4/(n-2)})\rangle\,dV_{g_0}\\ &= \frac{4(n-1)}{n}\left(\int_{M^n}f'(u)u^{n/(n-2)}|\nabla_0 u|^2\,dV_{g_0}\right.\\ &\qquad\left. + \frac{4\bar{\lambda}}{n-2}\int_{M^n}u^{-2/(n-2)}|\nabla_0 u|^2\,dV_{g_0}\right).\end{aligned} \tag{2.6}$$

Under the assumption of Theorem 1.1, (2.6) shows that

$$0 \le \int_{M^n} u^{2/(n-2)} |B|^2 \, dV_g$$
$$= \frac{4(n-1)}{n} \left(\int_{M^n} f'(u) u^{n/(n-2)} |\nabla_0 u|^2 \, dV_{g_0} + \frac{4\bar{\lambda}}{n-2} \int_{M^n} u^{-2/(n-2)} |\nabla_0 u|^2 \, dV_{g_0} \right) \le 0,$$

and u must be a constant.

Let $f(u) = u^\alpha$, $\alpha \le 0$, we get Theorem 1.2 holds. The proof of theorems is completed finally.

ACKNOWLEDGMENTS. The research of authors is partially supported by NSFC Project No. 10471108 and No. 10631020 of China. They also thank Prof. Juncheng Wei for bringing their attention to the paper [5]. Authors also would like to express their gratitude to the referee for valuable comments and suggestions.

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