<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Cluster sets of analytic functions in open Riemann surfaces with regular metrics. I</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Kuramochi, Zenjiro</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Mathematical Journal. 11(1) P.83-P.90</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1959</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/9694">https://doi.org/10.18910/9694</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/9694</td>
</tr>
<tr>
<td><strong>rights</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Note</strong></td>
<td></td>
</tr>
</tbody>
</table>

Osaka University Knowledge Archive : OUKA
[https://ir.library.osaka-u.ac.jp/repo/ouka/all/](https://ir.library.osaka-u.ac.jp/repo/ouka/all/)

Osaka University
**Cluster Sets of Analytic Functions in Open Riemann Surfaces with Regular Metrics. I**

By Zenjiro Kuramochi

The purpose of the present paper is to extend the theory of the cluster sets investigated only in case of domains of planar character under some conditions to the case of abstract Riemann surfaces under weaker conditions.

Let $R^* \subseteq R$ be abstract Riemann surfaces and let $\{R_n^*\}$ and $\{R_n\}$ ($R_0^* = R$) be their exhaustions with compact relative boundaries $\partial R_n^*$ and $\partial R_n$ respectively. Let $\{z_i\}$ be a sequence $z_i \in R_*^*$. If any compact surface of $R^*$ has only a finite number of points of $\{z_i\}$, we say that $\{z_i\}$ converges to the ideal boundary of $R^*$.

**Regular metrics of $R^*$.**

1) **Stoilow’s metric.** $R^* - R_0^*$ is composed of a finite number of disjoint non compact surfaces $\{G_i\}$. Let $G_n (n = 1, 2, \ldots)$ be a sequence of non compact surfaces with compact relative boundary such that $G_n \supset G_{n+1} \supset G_{n+2} \cdots$, $\bigcap G_n = 0$. Two sequences $\{G_n\}$ and $\{G_n'\}$ are called equivalent, if and only if, for any given number $m$, there exists a number $n$ such that $G_n \supset G_m'$ and vice versa. We make an ideal boundary point (component) correspond to a class of equivalent sequences and denote the set of all ideal boundary point by $B^*$. A metric is introduced on $R^* + B^*$. It is clear that $R^* + B^*$ and $B^*$ are closed and compact, and that $B^*$ is totally disconnected. The topology induced by this metric is homeomorphic to the original topology in $R^*$.

2) **Martin’s metric.** Let $R^*$ be a Riemann surface with positive boundary. (If $R^*$ is a Riemann surface with null-boundary, $R^* - R_0^*$ has positive boundary. Consider $R^* - R_0^*$ as $R^*$.) Let $G(z; p_i)$ ($p_0$ is a fixed point) be the Green’s function. Let $\{p_i\}$ be a sequence of points of $R^*$ having no point of accumulation in $R^*$. If a sequence $\{K(z; p_i)\}$ ($K(z; p_i) = G(z, p_i)$) converges to a harmonic function in every compact set in $R^*$, $\{p_i\}$ is called fundamental. Two fundamental sequences are called equivalent, if their corresponding $K(z; p_i)$, have the same limit. The class
of all fundamental sequences equivalent to a given one determines an ideal boundary point of $R^*$. The set of all the ideal boundary points of $R^*$ will be denoted by $B^*$ and the set $R^* + B^*$ by $\bar{R}^*$. The domain of definition of $K(z, p)$ may now be extended by writing $K(z, p) = \lim_i K(z, p_i)$ ($z \in R^*, p \in \bar{R}^*$). The distance of two points $p_1$ and $p_2$ of $\bar{R}^*$ is defined by

$$\delta(p_1, p_2) = \sup_{z \in R^* \setminus \bar{R}^*} \left| \frac{K(z, p_1)}{1 + K(z, p_1)} - \frac{K(z, p_2)}{1 + K(z, p_2)} \right|.$$  

It is clear that $\bar{R}^*$ and $B^*$ are closed and compact.

Let $N(z, p)$ be a positive harmonic function in $R^* - R^*$ such that $N(z, p) = 0$ on $\partial R^*$ and $N(z, p)$ has one logarithmic singularity at $p$ and $N(z, p)$ has the minimal Dirichlet integral. We use $N(z, p)$ instead of $K(z, p)$. Then we have another topology which is equivalent in case of a Riemann surface with null-boundary to the topology induced by $K(z, p)$, but it is not the case always. In the above topology $R^* - R^* + B^*$ and $B^*$ are always closed and compact.

3) Green's distance. Let $R^*$ be a Riemann surface (if $R^*$ is a Riemann surface with null-boundary, consider $R^* - R^*$ as $R^*$) with positive boundary and let $G(z, p_0)$ be the Green's function. Let $l$ be a curve in $R^*$. We define the length of $l$ by $\int l d|e^{-G(z, p_0)} + iK(z, p_0)|$, where $h(z, p_0)$ is the conjugate harmonic function of $G(z, p_0)$. For two points $p_1$ and $p_2$ of $R^*$ the distance $\delta(p_1, p_2)$ is defined by the infimum of the length of all curves connecting $p_1$ with $p_2$ in $R^*$. Now all the boundary points are defined by completion of $R^*$ with respect to this metric. It is clear that $B^* + R^*$ and $B^*$ are closed but not always compact. Suppose that a topology induced by $K(z, p)$, $N(z, p)$ or Green's distance is defined only in $B^* + R^* - R^*$. In this case the topology is homeomorphic to the original topology in $R^* - R^*$. We extend the above topology to $R^*$ so that the extended topology is homeomorphic to the original in $R^*$.

4) Teichmüller's metric. Let $\bar{R}$ be a covering surface over $R^*$ with a metric $\delta$ (as a special case, $\bar{R} \subset R^*$). Let $p_1$ and $p_2$ be two points of $\bar{R}$. We define the distance between $p_1$ and $p_2$ by the infimum of diameters of the projections of all curves connecting $p_1$ with $p_2$ in $\bar{R}$ and define all the boundary points $B$ (usually called accessible boundary points).

---

1) Dirichlet integral is taken in a neighbourhood of $p$ with respect to $N(z, p) + \log |z - p|.$

2) We understand by $T_i > T_j$ that $T_i$ is finer than $T_j$. Let $T_1$, $T_2$, $T_3$, and $T_4$ be topologies of Stoilow and of Martin induced by $K(z, p)$ and $N(z, p)$ and Green's distance. Then $T_1 < T_2 < T_3 < T_4.$
by completion of $\mathbb{R}$ with respect to this metric. Then $\mathbb{R} + B$ and $B$ are closed but not always compact.

If the distance $\delta(p_i, p_j) > 0$ for any points $p_i$ and $p_j$ of $\mathbb{R}^* + B^*$ such that there exists a compact domain or a non compact one $G$ with compact relative boundary $\partial G$ with $G \ni p_i : p_i \to G \ni p_j : p_j \to G$, and if the topology induced by this metric is homeomorphic to the original topology in $\mathbb{R}^*$, we say $\delta$ a regular metric. We see easily that the above metrics are regular.

**Exceptional set.** $\mathcal{E}_0$ (set of capacity zero), $\mathcal{E}_{AB}$, $\mathcal{E}_2$ (set of areal measure zero). Let $F$ be a closed set in the $w$-plane. If there exists no non-constant bounded analytic function in $CF^3$, we say $F \subset \mathcal{E}_{AB}$. It is clear that $\mathcal{E}_0 \subset \mathcal{E}_{AB} \subset \mathcal{E}_2$.

In the following $\partial A$ means the boundary of a set $A$ in $\mathbb{R}^* + B^*$ and $\partial A$ means the relative boundary of $A$, i.e. $\partial A = \partial A \cap \mathbb{R}$.

**Negligible set.** $\mathcal{R}_0$ (set of capacity zero)$^3$, $\mathcal{R}_{HB}$, $\mathcal{R}_{AB}$. Let $F$ be a closed set in $\mathbb{R}^* + B^*$. If there exists no non-constant bounded harmonic function which vanishes on $\partial G$ and bounded analytic function with vanishing real part on $\partial G$ in any non compact domain $G$ respectively such that $(\partial G \cap B) \subset F$, we say $F \subset \mathcal{R}_{HB}$ and $F \subset \mathcal{R}_{AB}$ respectively, where $B$ is the boundary of $\mathbb{R}$. It is clear $\mathcal{R}_0 \subset \mathcal{R}_{HB} \subset \mathcal{R}_{AB}$.

Let $R$ be a Riemann surface, $B$ its boundary, and let $\mathbb{R}^*(\supset R)$ be another Riemann surface with boundary $B^*$. We suppose that a regular metric $\delta$ is given on $\mathbb{R}^* + B^*$ such that $\mathbb{R}^* + B^*$ is closed and compact with respect to this metric. Let $z_0$ be a non isolated boundary point of $R$. We denote the part of $R$ contained in $E[z : \delta(z, z_0) < r]$ by $R_r$ and that of $B$ in $E[z : \delta(z, z_0) < r]$ by $B_r$. Let $w = f(z)$ be a one valued meromorphic function in $R$ and let $\overline{W}_r$ be the set of values taken by $w = f(z)$ in $R_r$. Put

$$\lim_{r \to 0} \overline{W}_r = H_K(z_0)^5.$$  

Let $E$ be a negligible set mentioned above which is a closed and totally disconnected set$^5$ on $B$ such that $z_0 \in E$.

---

3) $CA$ means the complementary set of a set $A$.
5) $\bar{A}$ means the closure of a set $A$.
6) In our topology, there exists a continuum $\subseteq \mathcal{E}_0$.

Let $V_r(p)$ be a neighbourhood of $p$ such that $V_r(p) = E[z \in \mathbb{R}^* : \delta(z, p) < r]$. Let $p$ be a point of $F$. If for any given number $r'$, there exists $V_r(p)$ such that $(\partial V_r(p) \cap F) = 0 : r < r'$, we call $F$ a totally disconnected set in $\mathbb{R}^*$.  

---

*Cluster Sets of Analytic Functions*
Let
\[ V_r(B-E) = \sum_{\zeta \in (B-E)} H_R(\zeta) \]
and
\[ \lim_{r \to 0} V_r(B-E) = H_{B-E}(z_0). \]
Then it is clear that \( H_R(z_0) \) and \( H_{B-E}(z_0) \) are closed and that \( H_R(z_0) \supseteq H_{B-E}(z_0) \).

We shall prove

**Theorem 1.** Let \( R^* \) be a Riemann surface with a regular metric \( \delta \) such that \( R^* + B^* \) is closed and compact with respect to \( \delta \) metric. Let \( f(z) \) be a meromorphic function in a Riemann surface \( R \subseteq R^* \). Take \( E \) a closed and totally disconnected set in \( B \) such that \( E \subseteq R_{AB} \) (\( \supseteq R_{HB} \) \( \supseteq R_0 \)). Suppose that \( z_0 \in (E \cap B) \) and that \( z_0 \) is not an isolated point of \( B-E \). Then
\[ H_R(z_0) - H_{B-E}(z_0) \]
is open.

**Theorem 2.** Suppose that \( H_R(z_0) - H_{B-E}(z_0) \) is non empty. Let \( \Omega \) be one of the components of \( H_R(z_0) - H_{B-E}(z_0) \). Then in any small neighbourhood of \( z_0 \), \( f(z) \) takes any value of \( \Omega \) infinitely often except at a set \( F \) of capacity zero for \( E \subseteq R_{HB} \) and \( f(z) \) takes infinitely often except at a set \( F \subseteq \mathbb{C} \) for \( E \subseteq R_{AB} \) and \( f(z) \) takes at least once except at a set \( \subseteq \mathbb{C}_{AB} \)

Up to the present, the theory of cluster sets is investigated only in case of planar domain. This is the case of \( R \) of \( z \)-closed Riemann surface. There are two typical methods to prove Theorem 1. The one is of Beurling-Kunugui-Tsuji and the other is that of Tōki. But in all cases the condition that the domain is planar and \( E \subseteq R_{HB} \) seems to be essential. On the other hand, if Theorem 1 is proved, it is easily seen that Theorem 2 holds for general Riemann surfaces. Theorem 1 is proved by Iversen, Kunugui and Tōki when \( E \) is a single point \( z_0 \) and is proved by Tsuji in case of \( E \subseteq \mathbb{C} \) and by Ohtsuka in case of \( E \subseteq \mathbb{C}_{HB} \) respectively. Further extensions were done by many authors under some additional conditions.

We shall use the following

**Lemma 1.** Let \( G \) be a non compact domain in \( R \) such that \( (\partial G \cap B) \subseteq R_{HB} \) \( \subseteq R_{AB} \) and let \( f(z) \) be an analytic function on \( G \). Suppose that \( f(G) \) is a covering surface over the \( w \)-plane. Then every connected piece \( \psi \) such
that \( \psi \) has no common point with \( f(\partial G) \) over a circle \( K: |w - w_0| < r \) covers \( K \) except at most for a set \( \subset \mathcal{E}_0 (\mathcal{E}_{AB}) \) and \( n(w) = \sup n(w) \leq \infty \) except at an \( F_\sigma \) of capacity zero (totally disconnected \( F_\sigma \) of areal measure zero), where \( n(w) \) is the number of times when \( w \) is covered by \( \psi \). If \( \sup n(w) < \infty \), then the \( F_\sigma \) reduces to a closed set.

Lemma 2. Let \( G \) be a non compact domain in \( \mathbb{R} \) with compact relative boundary \( \partial G \) such that \( (\partial G \cap B) \subset \mathcal{R}_{HB} (\mathcal{R}_{AB}) \), and let \( f(z) \) be a bounded analytic function on \( G \). Then \( \sup n(w) < \infty \). \( n(w) = \sup n(w) \) except at most at a closed set \( \subset \mathcal{E}_0 \) (a closed and totally disconnected set \( \subset \mathcal{E}_0 \)) and \( n(w) \geq 1 \) in \( \Omega \) except on a closed set \( \subset \mathcal{E}_0 (\mathcal{E}_{AB}) \) for \( \Omega \) such that \( \sup n(w) \geq 1 \), where \( \Omega \) is a component of the complementary set of \( f(\partial G) \).

Lemma 3. Let \( \psi \) be a connected piece over a circle \( K \) such that \( \sup n(w) < \infty \) and \( n(w) = \sup n(w) \) except at a closed and totally disconnected set \( F \). Let \( p_1 \) and \( p_2 \) be two boundary points such that projections of \( p_1 \) and \( p_2 \) are both contained in \( K \) and \( \text{proj } p_1 = \text{proj } p_2 \). Then \( \delta (p_1, p_2) > 0 \) for every regular metric.

Let \( D = E[w: n(w) = \sup n(w)] \). Since \( H = E[w: n(w) < \sup n(w)] \) is closed and totally disconnected, we can find neighbourhoods \( V_1 \) and \( V_2 \) of \( \text{proj } p_1 \) and \( \text{proj } p_2 \) such that \( \partial V_i \subset D (i = 1, 2) \) and \( \overline{V}_1 \cap \overline{V}_2 = \emptyset \). Then every connected piece \( \psi \) over \( V_i \) has a compact relative boundary. Hence \( \delta (p_1, p_2) > 0 \).

Lemma 4. Let \( G \) be a non compact domain such that \( (\partial G \cap B) \subset E \subset \mathcal{R}_{AB} \) and that \( \partial G \) is composed of two kinds of analytic curves \( L \) and \( \beta \) \( (L + \beta = \partial G) \), where \( L \) is composed of a finite number of analytic curves \( L_1, L_2, \ldots, L_s \) and every endpoint of \( L_i \) is situated on \( \beta \). Suppose that \( G \) is represented as a covering surface \( \psi \) over a circle \( K: |w - w_0| < r \) by \( f(z) \) and that \( f(\beta) \) lies on \( \partial K \) and \( f(\text{int } L_i) \subset K \), where \( \text{int } L_i = L - \text{end points of } L_i \). If \( \psi \) does not cover any open set \( \emptyset \) in \( K \), then \( \sup n(w) < \infty \).

Proof. Let \( \Omega_1, \Omega_2, \ldots \) be components of \( K - f(L) \). Then \( L \) is compact. We can deform \( L \) slightly so that the number of \{ \( \Omega_i \) \} is finite: \( \Omega_1, \Omega_2, \ldots, \Omega_t \). Clearly there exists a component \( \Omega_0 \) of \{ \( \Omega_i \) \} such that \( \emptyset \subset \Omega_0 \). First we shall show \( n(w) = 0 \) in \( \Omega_0 \). To the contrary, suppose that there exists a connected piece \( \psi' \) over \( \Omega_0 \). Then \( (\partial \psi' \cap \partial G) \subset E \subset \mathcal{R}_{AB} \). Hence by \( f(\partial \psi') \subset \partial \Omega_0 \), \( n(w) = \sup n(w) \geq 1 \) except at most a set \( \emptyset \) by Lemma \( 1 \). This is a contradiction. Hence \( n(w) = 0 \) in \( \Omega_0 \) by \( n(w) = 0 \) in \( \emptyset \). Next let \( \Omega_1 \) be another of \{ \( \Omega_i \) \} such that \( \partial \Omega_0 \cap \partial \Omega_1 = \emptyset \). Let \( w_1 \) and \( v(w_1) \) and \( w_2 \) and \( v(w_2) \) be two points and two neighbourhoods \( (w_1 \in v(w_1) \subset \Omega_0 \) and \( w_2 \in v(w_2) \subset \Omega_0 \) in a neighbourhoods of \( (\partial \Omega_0 \cap \partial \Omega_1) \) so that by a slight deformation of \( L \) \( v(w_1) \) and \( v(w_2) \) may be contained in the same component.
\( \Omega' \), where \( \Omega' \) is a component of \( K - f(L') \), where \( L' \) are curves of \( L \) after a slight deformation. Then we see that \( \sup_{w \in \Omega'} n(w) - \sup_{w \in \Omega'} n(w) \leq \infty \). On the other hand, the number of components is finite. Hence \( \sup_{w \in \Omega'} n(w) \leq \infty \).

Proof of Theorem 1. If the values taken by \( f(z) \) in every neighbourhood of \( z_0 \) is dense in the \( w \)-plane, our assertion is trivial. Hence we suppose that \( f(z) \) is bounded in a neighbourhood of \( z_0 \).

Let \( K_n(z_0) = \overline{E[z \in R^*: |z - z_0| < 1/n]} \). If \( H_{E}(z_0) = H(z_0) \), our assertion is trivial. We suppose \( 0 \leq H_R(z_0) - H_{E}(z_0) \leq w \). We shall show \( w \in \text{int} H_R(z_0) \).

Put \( \sum_{\xi(\in R^*(B)) \cap K_m} H_R(\xi) = M_n \). Then by \( w_0 \notin H_{E}(z_0) \), there exists a number \( n_0 \) such that \( w_0 \notin M_{n_0} \). Put dist \( (w_0, M_{n_0}) = \rho > 0 \). Since \( E \) is closed and totally disconnected, we can find a domain \( D \) in \( \overline{R^*} \) such that \( K_m(z_0) \supset D \supset K_{m+1}(z_0) \) and \( \partial D \cap E = 0 \). Then there exists a number \( \rho_1 \) such that \( |f(z) - w_0| \leq \rho_1 \) for \( z \in \partial D \cap E \). Otherwise, there exists a sequence \( \{z_i\} : f(z_i) \to w_0 \). \( z_i \) tends to a point \( z \in \partial D \cap E \). This implies by \( D \cap E = 0 \) that \( w_0 \in H_R(z) \). This contradicts dist \( (\xi, w_0) = \rho > 0 \). Since \( w_0 \notin H_R(z) \), there exists a sequence \( \{z_i\} \) such that \( \Delta_i \) contains at least one \( z_i \).

Case 1. The number of \( \Delta_i \) is infinite for a certain \( n_0 \). In this case, we shall show that \( \Delta_i \to z_0 \). If it were not so, there are infinite number of \( \Delta_i \), such that \( \Delta_i \) tends to \( B \) and each \( \Delta_i \) has at least one point \( z_i^* \) outside a certain \( K_{m+1}(z_0) \). Let \( D \) be a domain in \( \overline{R^* + B^*} \) such that \( D \subset K_m(z_0) \subset K_0(z_0) \), \( D \supset K_{m+1}(z_0) \) and \( \partial D \cap E = 0 \). Then since \( \{z_i\} \to z_0 \) and \( z_i^* \) is outside \( K_{m+1}(z_0) \), every \( \Delta_i \) intersects \( \partial D \) at \( z_i^* \). Hence there exists a sequence \( \{\xi_i\} \) such that \( \{\xi_i\} \to z_0 \in (K_m(z_0) \cap (B - E), f(z) \to w_0 \). This contradicts dist \( \sum_{\xi(\in R^*(B) \cap K_m} H_R(\xi), w_0 \geq \rho_0 \). Thus \( \{\Delta_i\} \to z_0 \). Now every \( \Delta \) satisfies the condition \( \partial \Delta \cap B \subset E \). Hence by Lemma 1 every \( \Delta \) covers \( K \) except for a closed set \( \subset \mathcal{C}_{AB} \). Hence there is a dense set in \( K \) of points which is covered by \( f(z) \) infinitely many times in every neighbourhood of \( z_0 \). Thus \( w_0 \in \text{int} H_R(z_0) \) and \( B(H_R(z_0)) \subset B(H_{E}(z_0)) \).

Case 2. The number of \( \{\Delta_i\} \) is finite for a certain \( n_0 \). Since \( \lim_{i \to \infty} z_i = z_0 \), there exists at least one \( \Delta \) containing a subsequence \( \{\xi_i\} \) of \( \{z_i\} \). Let \( \Delta \) be one of them. \( \partial \Delta \cap B \subset E \) implies that \( \Delta \) covers \( K \) except for a closed set \( \subset \mathcal{C}_{AB} \). We shall show \( \sup_{w \in \Omega'} n'(w) = \infty \), where \( n'(w) \) is the number of times when \( w \) is covered by \( \Delta \). First we shall show
Cluster Sets of Analytic Functions

89

\( \{ \partial \Delta \cap z_o \} \neq 0 \). Assume \( \partial \Delta \cap z_o = 0 \). Since there exists \( \{ z_i \} \) such that \( z_i \to z_o \), there exists a number \( l \) such that \( K_l(z_o) \subset \text{int} \, \Delta \). On the other hand, since \( z_o \in B - E \) there exists a point \( z^* \in \text{int} \, (z^* \in (B - E)) \). Let \( \{ z^*_n \} \) be a sequence such that \( z^*_n \to z^* \). Then \( \lim f(z^*_n) - w_o < \rho \). This contradicts \( \text{dist} \left( \sum_{\xi \in (B - E) \cap K_{m_0}} H_R(\xi), w \right) \geq \rho_o \). Hence \( \partial \Delta \cap z_o = 0 \). Next assume \( \sup n'(w) \) 

\( < \infty \) in \( K \). Then \( F = E[w : n(w) < \sup n(w)] \) is closed and totally disconnected by Lemma 1. There exists only inner points of \( R \) on \( D = E[w : n(w) = \sup n(w)] \). We can find a neighbourhood \( V(w_o) \) of \( w_o \) such that \( \partial V(w_o) \supset D \) and \( \text{dist} (\partial V(w_o), F) > 0 \). Let \( \Delta \) be the image of a connected piece over \( V(w_o) \) such that \( \Delta \supset \Delta_F \) and it contains a subsequence \( \{ z^*_n \} \) of \( \{ z_i \} \). Then similarly \( \partial \Delta \cap z_o = 0 \). On the other hand, \( \partial V(w_o) \supset D \) implies that \( \partial \Delta \) is compact in \( R \), whence by the regularity of metric dist \( (\partial \Delta, z_o) > 0 \). This is a contradiction. Hence \( \sup n'(w) = \infty \).

We shall show that every point \( w \) of \( K \) is contained in \( H_R(z_o) \). Otherwise suppose that \( H_R(z_o) \supset K \). Then since \( H_R(z_o) \) is closed, there exists an open set \( \Theta \) in \( K \) such that \( f(z) \) does not cover \( \Theta \) for \( z \in (\partial (\Delta \cap K_{m}(z_o)) \) for a certain \( m \). Let \( D \) be a domain such that \( K_{m}(z_o) \supset D \supset K_{m+1}(z_o) \) and \( \partial D \cap E = 0 \). Then \( \Delta \cap \partial D \) does not tend to \( B \) because \( |H_R(\xi) - w_o| > \rho_0 \) : \( \xi \in (B \cap K_{m}(z_o)) \). Hence \( \partial D \cap \partial \Delta \) is closed and compact in \( R \). Then we can deform \( \partial D \) slightly so that \( \partial \Delta \cap D \) is composed of a finite number of analytic curves \( D_1, D_2, \ldots, D_s \). Let \( \Omega \) be a components of \( K \cap (\partial \Delta - \partial \Delta) \). We can deform \( D \) slightly so that \( f(D) \cap \Omega \). Hence as above \( \sup n'(w) = \infty \) in \( \Omega \), where \( n'(w) \) is the number of times when \( w \) is covered by \( D \cap \Delta \). On the other hand, \( f(\Delta \cap D) \) does not cover an open set \( \Theta \). Hence by Lemma 4 \( \sup n'(w) < \infty \) in \( K \). This is a contradiction. Hence the set of values taken by \( f(z) \) in any \( R \cap K_{m}(z_o) \) dense in \( K \). Thus \( K \subset H_R(z_o) \) and \( w_o \in \text{int} \, H_R(z_o) \). From case 1 and 2 we have

9) Cf. Theorem 5 for Lemma 1, Theorem 13 for Lemm 2.

10) \( \bar{\Delta} \) means the closure of \( \Delta \) with respect to \( \delta \) metric.

11) We can construct a new domain \( D^* \) such that \( \bar{\Delta} \cap \partial D^* \cap B = 0 \), \( D \supset D^* \) and \( \partial (D^* \cap \Delta) - \partial \Delta \) is composed of a finite number of analytic curves. Since \( \partial D \) is closed, \( B \cap (\partial D \cap \Delta) = 0 \) and \( \langle \partial D \cap \partial \Delta \cap \partial \Delta \rangle \subset R \). Let \( p \) be a point \( \in (\partial D \cap \Delta) \subset R \). Then there exists a neighbourhood \( v(p) \) of \( p \) such that \( v(p) \) is mapped conformally by \( f(z) \) such that \( f(\partial D \cap v(p)) \) is contained in the set \( E_{|w-w_o| = \rho} \). Hence for sufficiently small \( v(p) \), \( f(v(p) \cap \partial D) \) consists of only one analytic curve \( \alpha \) (may have some number of branches). Hence if there are infinite number of components \( \delta_1, \delta_2, \ldots \) of \( (\partial (\Delta \cap D - \partial D) \cap v(p)) \), then there exists a number \( i_o \) such that \( \delta_i \) has its end points on \( v(p) \). In this case, we replace \( \delta_i \) with \( \delta_i \) by a new curve \( \beta \) such that \( \beta \subset \partial D^* \), where \( D^* \) is a new domain such that \( K_{m+1}(z_o) \subset D^* \subset D \subset K_{m}(z_o) \), \( \bar{\Delta} \cap \partial D^* \cap B = 0 \). But \( \bar{\Delta} \cap \partial D \) is covered by a system of a finite number of neighbourhoods of \( v(p) \) with the property that \( \partial (D^* \cap \Delta) \cap v(p) \) consists of a finite number of analytic curves. Hence \( D^* \) is a new domain such that \( \bar{\Delta} \cap \partial D^* \cap B = 0 \), \( D^* \subset D \) and \( \partial (D^* \cap \Delta) - \partial \Delta \) consists of a finite number of analytic curves \( L_1, L_2, \ldots, L_t \). Put \( D = D^* \).
Proof of Theorem 2. Let $\Omega$ be a component of $H_R(z_0) - H_{NB}(z_0)$. Let $\Delta$ be the inverse image of a connected piece $\psi$ over $\Omega$. Then $\partial \Delta$ does not fall in $\Omega$ and $\sup n(w) = \infty$ as above. Hence we have Theorem 2 by Lemma 1.

Let $R$ be a Riemann surface. Suppose that $R^* + B^*$ is closed and not compact with respect to a regular metric $\delta$. If any compact set of $R^* + B^*$ has only a finite number of points of the sequence $z_1, z_2, \cdots$, we say that $\{z_i\}$ determines a non accessible boundary point (N.A.B.P.). Let $NB$ be the set of all the N.A.B.P.s and let $K_r(z_0) = E|z \in R^* + B^*: \delta(z, z_0) < r|$, where $K_r(z_0)$ is not always compact.

Put $H_R(\xi) = \Lambda(\Sigma T U)$ for $\xi \in R^* + B^*$ and $H_R(\xi)$ for $\xi \in (K_r(z_0) \cap NB)$ the set of all values $w$ such that there exists a divergent sequence $\{z_i\}$: $z_i \in K_r(z_0)$ and $\lim f(z_i) \to w$, where $NB \cap K_r(z_0)$ is a set of N.A.B.P.s such that $z_i \in K_r(z_0)$ and $\{z_i\}$ determines a N.A.B.P. Put $H_{NB+NB-E}(z_0) = \cap H_R(\xi)$, where the summation is over $(B + NB - E) \cap K_r(z_0)$. Then we can prove similarly the following:

**Theorem 1'.** Let $z_0 \in B$ and $E$ be a closed and totally disconnected set on $B$ such that $E \subseteq \mathbb{G}_{AB}$ and $z_0 \in \overline{B-E}$. Then

$$B(H_R(z_0)) \subseteq B(H_{NB+NB-E}(z_0)).$$

**Theorem 2'.** ( = Theorem 2).

As an application we shall consider an extension of Lindelöf’s theorem. Let $R$ be a Riemann surface with compact relative boundary $\partial R$ such that $R$ has only one ideal boundary component $p (\in \mathbb{G}_{AB})$ with finite or infinite genus. Let $L_1, L_2, \cdots, L_t$ be curves in $R$ tending to $p$.

**Theorem 3.** Let $f(z)$ be a bounded analytic function in $R - \sum L_i$. If $f(z)$ converges along $L_i$, then $f(z) = f(z) \cdots = f(z) = w_0$ and $f(z)$ converges to $w_0$ uniformly in $R$ as $z$ tends to $p$.

In fact, $H(p) = w_1 + w_2 + w_3 + \cdots + w_t = B(H\mathbb{L}(p)) \supset B(H_R(p))$ implies that $H(p)$ is one point and $w_1 = w_2 = \cdots = w_t$.

**Remark.** This theorem does not hold for bounded harmonic function in $R$ with infinite genus. Let $U(z)$ be a bounded harmonic function in $R$ (not only in $R - \sum L$). If the harmonic dimension of $p$ is larger than one, $U(z)$ does not converge as $z \to p$.

(Received March 23, 1959)