Kuramochi, Zenjiro
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Cluster Sets of Analytic Functions in Open Riemann Surfaces with Regular Metrics. I

By Zenjiro KURAMOCHI

The purpose of the present paper is to extend the theory of the cluster sets investigated only in case of domains of planar character under some conditions to the case of abstract Riemann surfaces under weaker conditions.

Let $R^*\supseteqq R$ be abstract Riemann surfaces and let $\{R_n^*\}$ and $\{R_n\}$ $(R_0^*=R_0)$ be their exhaustions with compact relative boundaries $\{\partial R_n^*\}$ and $\{\partial R_n\}$ respectively. Let $\{z_i\}$ be a sequence: $z_i\in R^*$. If any compact surface of R^* has only a finite number of points of $\{z_i\}$, we say that $\{z_i\}$ converges to the ideal boundary of R^*.

Regular metrics of R^*.

1) Stoilow's metric. $R^*-R_n^*$ is composed of a finite number of disjoint non compact surfaces $\{G_i\}$. Let G_n $(n=1, 2, \cdots)$ be a sequence of non compact surfaces with compact relative boundary such that $G_n\supset G_{n+1}\supset G_{n+2}\cdots$, $\bigcap_n G_n=0$. Two sequences $\{G_n^i\}$ and $\{G_n^j\}$ are called equivalent, if and only if, for any given number m, there exists a number n such that $G_n^i\supset G_m^j$ and vice versa. We make an ideal boundary point (component) correspond to a class of equivalent sequences and denote the set of all ideal boundary point by B^*. A metric is introduced on R^*+B^*. It is clear that R^*+B^* and B^* are *closed and compact, and that B^* is totally disconnected.* The topology induced by this metric is *homeomorphic* to the original topology in R^*.

2) Martin's metric. Let R^* be a Riemann surface with positive boundary. (If R^* is a Riemann surface with null-boundary, $R^*-R_0^*$ has positive boundary. Consider $R^*-R_0^*$ as R^*). Let $G(z\cdot p_0)$ (p_0 is a fixed point) be the Green's function. Let $\{p_i\}$ be a sequence of points of R^* having no point of accumulation in R^*. If a sequence $\{K(z, p_i)\}\left(K(z, p_i)=\dfrac{G(z, p_i)}{G(p_i, p_0)}\right)$ converges to a harmonic function in every compact set in R^*, $\{p_i\}$ is called fundamental. Two fundamental sequences are called equivalent, if their corresponding $K(z, p_i)_s$ have the same limit. The class of all fundamental sequences equivalent to a given one determines an ideal boundary point of R^*. The set of all the ideal boundary points of R^* will be denoted by B^* and the set R^*+B^* by $\bar{R}^*$. The domain of definition of $K(z, p)$ may now be extended by writting $K(z, p)=\lim_i K(z, p_i)$ $(z \in R^*, p \in \bar{R}^*)$. The distance of two points p_1 and p_2 of $\bar{R}^*$ is defined by

$$\delta(p_1, p_2) = \sup_{z \in R_1^* - R_0^*} \left| \frac{K(z, p_1)}{1+K(z, p_1)} - \frac{K(z, p_2)}{1+K(z, p_2)} \right| .$$

It is clear that $\bar{R}^*$ *and* B^* *are closed and compact.*

Let $N(z, p)$ be a positive harmonic function in $R^*-R_0^*$ such that $N(z, p)=0$ on ∂R_0^* and $N(z, p)$ has one logarithmic singularity at p and $N(z, p)$ has the minimal Dirichlet integral[1]. We use $N(z, p)$ instead of $K(z, p)$. Then we have another topology which is equivalent in case of a Riemann surface with null-boundary to the topology induced by $K(z, p)$, but it is not the case always[2]. In the above topology $R^*-R_0^*+B^*$ and B^* are always closed and compact.

3) **Green's distance.** Let R^* be a Riemann surface (if R^* is a Riemann surface with null-boundary, consider $R^*-R_0^*$ as R^*) with positive boundary and let $G(z, p_0)$ be the Green's function. Let l be a curve in R^*. We define the length of l by $\int_l d|e^{-G(z,p_0)+ih(z,p_0)}|$, where $h(z, p_0)$ is the conjugate harmonic function of $G(z, p_0)$. For two points p_1 and p_2 of R^* the distance $\delta(p_1, p_2)$ is defined by the infinimum of the length of all curves connecting p_1 with p_2 in R^*. Now all the boundary points are defined by completion of R^* with respect to this metric. It is clear that R^*+B^* and B^* are *closed* but *not always compact.* Suppose that a topology induced by $K(z, p)$, $N(z, p)$ or Green's distance is defined only in $B^*+R^*-R_0^*$. In this case the topology is homeomorphic to the original topology in $R^*-R_0^*$. We extend the above topolgy to R^* so that the extended topolgy is homeomorphic to the original in R^*.

4) **Teichmüller's metric.** Let $\tilde{R}$ be a covering surface over R^* with a metric δ (as a special case, $\tilde{R} \subset R^*$). Let p_1 and p_2 be two points of $\tilde{R}$. We define the distance between p_1 and p_2 by the infinimum of diameters of the projections of all curves connecting p_1 with p_2 in $\tilde{R}$ and define all the boundary points B (usually called accessible boundary points)

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1) Dirichlet integral is taken in a neighbourhood of p with respect to $N(z, p)+\log|z-p|$.

2) We understand by $T_i > T_j$ that T_i is finer than T_j. Let T_s, T_k, T_N, and T_g be topologies of Stoilow and of Martin induced by $K(z, p)$ and $N(z, p)$ and Green's distance. Then $T_s < T_g < T_N < T_k$.

by completion of $\tilde{R}$ with respect to this metric. Then $\tilde{R}+B$ and B are *closed* but *not always compact.*

If the distance $\delta(p_1, p_2)>0$ for any points p_1 and p_2 of R^*+B^* such that there exists a compact domain or a non compact one G with compact relative boundary ∂G with $G\ni p_i: p_i\to p_1\in B^*$ and $G\not\ni p_j: p_j\to p_2$, and if the topology induced by this metric is homeomorphic to the original topology in R^*, we say δ a *regular metric.* We see easily that the above metrics are regular.

Exceptional set. $\mathfrak{E}_0$ (*set of capacity zero*), $\mathfrak{E}_{AB}$, $\mathfrak{E}_2$ (*set of areal measure zero*). Let F be a closed set in the w-plane. If there exists no non-constant bounded analytic function in CF[3], we say $F\subset\mathfrak{E}_{AB}$. It is clear that $\mathfrak{E}_0\subset\mathfrak{E}_{AB}\subset\mathfrak{E}_2$.

In the following $\dot{\partial}A$ means the boundary of a set A in R^*+B^* and ∂A means the relative boundary of A, i.e. $\partial A=\dot{\partial}A\cap R$.

Negligible set. $\mathfrak{N}_0$ (*set of capacity zero*)[4], $\mathfrak{N}_{HB}$, $\mathfrak{N}_{AB}$. Let F be a closed set in R^*+B^*. If there exists no non-constant bounded harmonic function which vanishes on ∂G and bounded analytic function with vanishing real part on ∂G in any non compact domain G respectively such that $(\dot{\partial}G\cap B)\subset F$, we say $F\subset\mathfrak{N}_{HB}$ and $F\subset\mathfrak{N}_{AB}$ respectively, where B is the boundary of R. It is clear $\mathfrak{N}_0\subset\mathfrak{N}_{HB}\subset\mathfrak{N}_{AB}$.

Let R be a Riemann surface, B its boundary, and let $R^*(\supset R)$ be another Riemann surface with boundary B^*. We suppose that a regular metric δ is given on R^*+B^* such that R^*+B^* is closed and compact with respect to this metric. Let z_0 be a non isolated boundary point of R. We denote the part of R contained in $E[z: \delta(z, z_0)<r]$ by R_r and that of B in $E[z: \delta(z, z_0)<r]$ by B_r. Let $w=f(z)$ be a one valued meromorphic function in R and let W_r be the set of values taken by $w=f(z)$ in R_r. Put

$$\lim_{r\to 0}\overline{W}_r = H_R(z_0)^{5)} .$$

Let E be a negligible set mentioned above which is a closed and totally disconnected set[6] on B such that $z_0\in E$.

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3) CA means the complementary set of a set A.

4) See Z. Kuramochi: Mass distribution on the ideal boundary, II. Osaka Math. 8, 1956.

5) $\bar{A}$ means the closure of a set A.

6) In our topology, there exists a continuum $\subset\mathfrak{E}_0$.

Let $V_r(p)$ be a neighbourhood of p such that $V_r(p)=E[z\in\overline{R^*}: \delta(z, p)<r]$. Let p be a point of F. If for any given number r', there exists $V_r(p)$ such that $(\partial V_r(p)\cap F)=0: r<r'$, we call F a totally disconnected set in $\overline{R^*}$.

Let $$V_r(B-E) = \sum_{\zeta \in (B_r - E)} H_R(\zeta)$$

and $$\lim_{r \to 0} \overline{V}_r(B-E) = H_{B-E}(z_0)\,.$$

Then it is clear that $H_R(z_0)$ and $H_{B-E}(z_0)$ are closed and that $H_R(z_0) \supset H_{B-E}(z_0)$.

We shall prove

Theorem 1. *Let R^* be a Riemann surface with a regular metric δ such that R^*+B^* is closed and compact with respect to δ metric. Let $f(z)$ be a meromorphic function in a Riemann surface $R \subset R^*$. Take E a closed and totally disconnected set in B such that $E \subset \mathfrak{A}_{AB}$ $(\supset \mathfrak{N}_{HB} \supset \mathfrak{N}_0)$. Suppose that $z_0 \in (E \cap B)$ and that z_0 is not an isolated point of $B-E$. Then*

$$H_R(z_0) - H_{B-E}(z_0) \qquad \textit{is open.}$$

Theorem 2. *Suppose that $H_R(z_0) - H_{B-E}(z_0)$ is non empty. Let Ω be one of the components of $H_R(z_0) - H_{B-E}(z_0)$. Then in any small neighbourhood of z_0, $f(z)$ takes any value of Ω infinitely often except at a set F of capacity zero for $E \subset \mathfrak{N}_{HB}$ and $f(z)$ takes infinitely often except at a set $F \subset \mathfrak{E}_2$ for $E \subset \mathfrak{N}_{AB}$ and $f(z)$ takes at least once except at a set $\subset \mathfrak{E}_{AB}$ for $E \subset \mathfrak{N}_{AB}$.*

Up to the present, the theory of cluster sets is investigated only in case of planar domain. This is the case of R of z-closed Riemann surface. There are two typical methods to prove Theorem 1. The one is of Beurling-Kunugui-Tsuji and the other is that of Tôki. But in all cases the condition that the domain is planar and $E \subset \mathfrak{N}_{HB}$ seems to be essential. On the other hand, if Theorme 1 is proved, it is easily seen that Theorem 2 holds for general Riemann surfaces. Theorem 1 is proved by Iversen, Kunugui and Tôki when E is a single point z_0 and is proved by Tsuji in case of $E \subset \mathfrak{E}_0$ and by Ohtsuka[7] in case of $E \subset \mathfrak{E}_{HB}$ respectively. Further extensions were done by many authors under some additional conditions.

We shall use the following

Lemma 1.[8] *Let G be a non compact domain in R such that $(\dot{\partial} G \cap B) \subset \mathfrak{N}_{HB}(\mathfrak{N}_{AB})$ and let $f(z)$ be an analytic function on G. Suppose that $f(G)$ is a covering surface over the w-plane. Then every connected piece ψ such*

7) F. Iversen: Sur quelques propriétés des fonctions monogènes au voisinage d'un point singulier, Ofv. af Finska Vet-Soc. Förh. 58, 1916.

A. Beurling: Etudes sur un problème de majoration. Thèse de Upsal, 1933.

K. Kunugui: Sur un théorème de MM. Seidel-Beurling. Proc. Acad. Tokyo, 15, 1939.

T. Tôki: On the behaviour of a meromorphic function in the neighbourhood of a transcendental singularity. Proc. Acad. Tokyo, 17, 1941.

M. Tsuji: On the cluser set of a meromorphic function, Proc. Acad. Tokyo, 19, 1943.

M. Ohtsuka: On exceptional values of a meromorphic function, Nagoya. Math. 9. 1955.

8) Z. Kuramochi: Representation of Riemann surfaces: Osaka Math. 11, 1959.

that ψ has no common point with $f(\partial G)$ over a circle K: $|w-w_0|<r$ covers K except at most for a set $\subset \mathfrak{E}_0$ $(\mathfrak{E}_{AB})$ and $n(w)=\sup n(w) \leqq \infty$ except at an F_σ of capacity zero (totally disconnected F_σ of areal measure zero), where $n(w)$ is the number of times when w is covered by ψ. If $\sup n(w)<\infty$, then the F_σ reduces to a closed set.

Lemma 2. *Let G be a non compact domain in R with compact relative boundary ∂G such that $(\dot{\partial} G \cap B) \subset \mathfrak{N}_{HB}(\mathfrak{N}_{AB})$, and let $f(z)$ be a bounded analytic function on G. Then $\sup n(w)<\infty$. $n(w)=\sup n(w)$ except at most at a closed set $\subset \mathfrak{E}_0$ (a closed and totally disconnected set $\subset \mathfrak{E}_2$) and $n(w) \geqq 1$ in Ω except on a closed set $\subset \mathfrak{E}_0(\mathfrak{E}_{AB})$ for Ω such that $\sup n(w) \geqq 1$, where Ω is a component of the complementary set of $f(\partial G)$.*

Lemma 3. *Let ψ be a connected piece over a circle K such that $\sup n(w)<\infty$ and $n(w)=\sup n(w)$ except at a closed and totally disconnected set F. Let p_1 and p_2 be two boundary points such that projections of p_1 and p_2 are both contained in K and proj $p_1 \neq$ proj p_2. Then $\delta(p_1, p_2)>0$ for every regular metric.*

Let $D=E[w: n(w)=\sup n(w)]$. Since $H=E[w: n(w)<\sup n(w)]$ is closed and totally disconnected, we can find neighbourhoods V_1 and V_2 of proj p_1 and proj p_2 such that $\partial V_i \subset D (i=1, 2,)$ and $\overline{V}_1 \cap \overline{V}_2=0$. Then every connected piece ψ over V_i has a compact relative boundary. Hence $\delta(p_1, p_2)>0$.

Lemma 4. *Let G be a non compact domain such that $(\dot{\partial} G \cap B) \subset E \subset \mathfrak{N}_{AB}$ and that ∂G is composed of two kinds of analytic curves L and β $(L+\beta=\partial G)$, where L is composed of a finite number of analytic curves $L_1, L_2, \cdots, L_s$ and every endpoint of L_i is situated on β. Suppose that G is represented as a covering surface ψ over a circle K: $|w-w_0|<r$ by $f(z)$ and that $f(\beta)$ lies on ∂K and $f(\text{int } L_i) \subset K$, where int $L_i=L-$end points of L_i. If ψ does not cover any open set $\mathfrak{G}$ in K, then $\sup\limits_K n(w)<\infty$.*

Proof. Let $\Omega_1, \Omega_2, \cdots$ be components of $K-f(L)$. Then L is compact. We can deform L slightly so that the number of $\{\Omega_i\}$ is finite: $\Omega_1, \Omega_2, \cdots, \Omega_t$. Clearly there exists a component Ω_0 of $\{\Omega_i\}$ such that $\mathfrak{G} \subset \Omega_0$. First *we shall show $n(w)=0$ in Ω_0.* To the contrary, suppose that there exists a connected piece ψ' over Ω_0. Then $(\dot{\partial}\psi' \cap B) \subset E \subset \mathfrak{N}_{AB}$. Hence by $f(\partial \psi') \subset \partial \Omega_0$, $n(w)=\sup n(w) \geqq 1$ except at most a set $\mathfrak{E}_2$ by Lemma 1. This is a contradicrion. Hence $n(w)=0$ in Ω_0 by $n(w)=0$ in $\mathfrak{G}$. Next let Ω_1 be another of $\{\Omega_i\}$ such that $\partial \Omega_0 \cap \partial \Omega_1 \neq 0$. Let w_1 and $v(w_1)$ and w_2 and $v(w_2)$ be two points and two neighbourhoods ($w_1 \in v(w_1) \subset \Omega_0$ and $w_2 \in v(w_2) \subset \Omega_1$) in a neighbourhoods of $(\partial \Omega_0 \cap \partial \Omega_1)$ so that by a slight deformation of L $v(w_1)$ and $v(w_2)$ may be contained in the same component

Ω', where Ω' is a component of $K-f(L')$, where L' are curves of L after a slight deformation. Then we see that $|\sup_{v(w_1)} n(w)-\sup_{v(w_2)} n(w)|<\infty$. On the other hand, the number of components is finite. Hence $\sup_K n(w)<\infty$.

Proof of Theorem 1. If the values taken by $f(z)$ in every neighbourhood of z_0 is dense in the w-plane, our assertion is trivial. Hence we suppose that $f(z)$ is bounded in a neighbourhood of z_0. Put $K_n(z_0)=E[z\in\bar{R}^*: \delta(z, z_0)<1/n]$. If $H_{B-E}(z_0)=H(z_0)$, our assertion is trivial. We suppose $0\neq H_R(z_0)-H_{B-E}(z_0)\ni w_0$. *We shall show* $w_0\in int\ H_R(z_0)$.

Put $\overline{\sum_{\zeta\in((B-E)\cap K_n)} H_R(\zeta)}=M_n$. Then by $w_0\notin H_{B-E}(z_0)$, there exists a number n_0 such that $w_0\notin M_{n_0}$. Put dist $(w_0, M_{n_0})=\rho>0$. Since E is closed and totally disconnected, we can find a domain D in $\bar{R}^*$ such that $K_{n_0}(z_0)\supset D\supset K_{n+1}(z_0)$ and $\dot{\partial}D\cap E=0$ and that $f(z)\neq w_0$ on $\dot{\partial}D\cap R$. Then there exists a number $\rho_1>0$ such that $|f(z)-w_0|\geqq\rho_1>0$ for $z\in(\dot{\partial}D\cap R)$. Otherwise, there exists a sequence $\{z_i\}: f(z_i)\to w_0: z_i\in\dot{\partial}D$. $\{z_i\}$ tends to a point $z'\in(\dot{\partial}D\cap B)$. This implies by $D\cap\dot{\partial}E=0$ that $w_0\in H_R(\zeta)$: $\zeta\in B-E$. This contradicts dist $(w_0, M_{n_0})>\rho_0>0$. Since $w_0\in H_R(z_0)$, there exists a sequence $\{z_i\}$ in R such that $\{z_i\}$ tends to z_0: $f(z_i)\to w_0$. Let K: $|w-w_0|<\rho<\min(\rho_0, \rho_1)$ and consider $f(R)$ over K. Then there exists at least one connected pieces $\psi_1, \psi_2, \cdots$ over K. Let $\Delta_1, \Delta_2, \cdots$ be the inverse images of connected pieces $\{\psi_i\}$ such that $\Delta_i\cap\Delta_j=0$ and that Δ_i contains at least one z_i.

Case 1. The number of $\Delta_{i,s}$ is infinite for a certain n_0. In this case. we shall show that $\{\Delta_i\}\to z_0$. If it were not so, there are infinite number of $\Delta_{i,s}$ such that $\{\Delta_i\}$ tends to B and each Δ_i has at least one point z_i^* outside a certain $K_{m+1}(z_0)(m>n_0)$. Let D be a domain in R^*+B^* such that $D\subset K_m(z_0)\subset K_{n_0}(z_0)$, $D\supset K_{n+1}(z_0)$ and $\dot{\partial}D\cap E\neq 0$. Then since $\{z_i\}\to z_0$ and z_i^* is outside $K_{m+1}(z_0)$, every Δ_i intersects $\dot{\partial}D$ at ζ_i. Hence there exists a subsequence $\{\zeta'_i\}$ of $\{\zeta_i\}$ such that $\{\zeta'_i\}\to\zeta_0\in(K_m(z_0)\cap(B-E)$, $|\lim_i f(\zeta'_i)-w_0|<\rho$. This contradicts dist $(\sum_{(B-E)\cap K_m} H(\zeta), w_0)\geqq\rho_0$. Thus $\{\Delta_i\}$ tends to z_0. Now every Δ satisfies the condition $(\dot{\partial}\Delta\cap B)\subset E$. Hence by Lemma 1 every Δ covers K except for a closed set $\subset\mathfrak{E}_{AB}$. Hence there is a dense set in K of points which is covered by $f(z)$ infinitely many times in every neighbourhood of z_0. Thus $w_0\in int\ H_R(z_0)$ and $B(H_R(z_0))\subset B(H_{B-E}(z_0))$.

Case 2. The number of $\{\Delta_i\}$ is finite for a certain n_0. Since $\lim_i z_i=z_0$, there exists at least one Δ containing a subsequence $\{z'_i\}$ of $\{z_i\}$. Let Δ be one of them. $(\dot{\partial}\Delta\cap B)\subset E$ implies that Δ covers K except for a closed set $\subset\mathfrak{E}_{AB}$. We shall show $\sup n'(w)=\infty$, where $n'(w)$ is the number of times when w is covered by Δ. First we shall show

$(\overline{\dot{\partial}\Delta} \frown z_0) \neq 0$. Assume $\overline{\dot{\partial}\Delta} \frown z_0 = 0$. Since there exists $\{z_i\}$ such that $z_i \to z_0$, there exists a number l such that $K_l(z_0) \subset \operatorname{int} \overline{\Delta}$. On the other hand, since $z_0 \in \overline{B-E}$ there exists a point $z^* \in \operatorname{int} \overline{\Delta}(z^* \in (B-E))$. Let $\{z_i^*\}$ be a sequence such that $z_i^* \to z^*$. Then $|\lim f(z_i^*) - w_0| < \rho$. This contradicts $\operatorname{dist}(\sum_{\zeta \in ((B-E) \frown K_{n_0})} H_R(\zeta), w) \geqq \rho_0$. Hence $\overline{\dot{\partial}\Delta} \frown z_0 \neq 0$. Next assume $\sup n'(w) < \infty$ in K. Then $F = E[w : n(w) < \sup n(w)]$ is closed and totally disco nnected by Lemma 1. There exists only inner points of R on $D = E[w : n(w) = \sup n(w)]$. We can find a neighbourhood $V(w_0)$ of w_0 such that $\partial V(w_0) \subset D$ and $\operatorname{dist}(\partial V(w_0), F) > 0$. Let Δ_V be the image of a connected piece over $V(w_0)$ such that $\Delta \supset \Delta_V$ and it contains a subsequence $\{z_i''\}$ of $\{z_i\}$. Then similarly $\overline{\dot{\partial}\Delta_V} \frown z_0 \neq 0$. On the other hand, $\partial V(w_0) \subset D$ implies that $\dot{\partial}\Delta_V$ is compact in R, whence by the regularity of metric $\operatorname{dist}(\dot{\partial}\Delta_V, z_0) > 0$. This is a contradiction. Hence $\sup_K n'(w) = \infty$. We shall show that every point w of K is contained in $H_R(z_0)$. Otherwise suppose that $H_R(z_0) \subsetneqq K$. Then since $H_R(z_0)$ is closed, there exists an open set $\mathfrak{G}$ in K such that $f(z)$ does not cover $\mathfrak{G}$ for $z \in (\Delta \frown K_m(z_0))$ for a certain m. Let D be a domain such that $K_{n_0}(z_0) \supset K_m(z_0) \supset D \supset K_{m+1}(z_0)$ and $\dot{\partial}D \frown E = 0$. Then $\overline{\Delta} \frown \dot{\partial}D$ does not tend to B because $|H_R(\zeta) - w_0| > \rho_0 : \zeta \in (B \frown K_{n_0}(z_0))$. Hence $\dot{\partial}D \frown \overline{\Delta}$ *is closed and compact* in R. Then we can deform ∂D slightly so that $\partial(\Delta \frown D) - \partial\Delta$ is composed of a finite number of analytic curves $D_1, L_2, \cdots, L_s$[11]. Let Ω be a components of $K - f(L)$: $\sum^s L_i = L$. We can deform L slightly so that $f(L) \not\ni w_0$ and $w_0 \in \Omega$. Hence as above $\sup_\Omega n'(w) = \infty$ in Ω, where $n'(w)$ is the number of times when w is covered by $D \frown \Delta$. On the other hand, $f(\Delta \frown D)$ does not cover an open set $\mathfrak{G}$. Hence by Lemma 4 $\sup n')w) < \infty$ in K. This is a contradiction. Hence the set of values taken by $f(z)$ in any $R \frown K_n(z_0)$ is dense in K. Thus $K \subset H_R(z_0)$ and $w_0 \in \operatorname{int} H_R(z_0)$. From case 1 and 2 we have

9) Cf. Theorem 5 for Lemma 1, Theorem 13 for Lemm 2.

10) $\bar{\varDelta}$ means the closure of $\varDelta$ with respect to δ metric.

11) We can construct a new domain D^* such that $\bar{\varDelta} \cap \dot{\partial} D^* \cap B = 0$, $D \supset D^*$ and $\partial(D^* \cap \varDelta) - \partial\varDelta$ is composed of a finite number of analytic curves. Since $\dot{\partial}D$ is closed, $B \cap (\overline{\dot{\partial}D \cap \bar{\varDelta}}) = 0$ and $(\overline{\dot{\partial}D \cap \partial\varDelta}) \subset R$. Let p be a point $\in (\dot{\partial}D \cap \bar{\varDelta}) \subset R$. Then there exists a neighbourhood $v(p)$ of p such that $v(p)$ is mapped conformally by $f(z)$ such that $f(\partial\varDelta \cap v(p))$ is contained in the set $E[|w - w_0| = \rho]$. Hence for sufficiently small $v(p)$, $f(v(p) \cap \partial\varDelta)$ consists of only one analytic curve α (may have some number of branches). Hence if there are infinite number of components $\beta_1, \beta_2, \cdots$ of $(\partial(\varDelta \cap D) - \partial\varDelta) \cap v(p)$, then there exists a number i_0 such that β_i $(i \geqq i_0)$ has its end points on $\alpha \cap v(p)$. In this case, we replace β_i $(i \geqq i_0)$ by a new curve β such that $\beta \subset \partial D^*$, where D^* is a new domain such that $K_{m+1}(z_0) \subset D^* \subset D \subset K_m(z_0)$, $\bar{\varDelta} \cap \dot{\partial} D^* \cap B = 0$. But $\bar{\varDelta} \cap \dot{\partial} D$ is covered by a system of a finite number of neighbourhoods of $v(p)$ with the property that $(\partial D^* \cap \bar{\varDelta}) \cap v(p)$ consists of a finite number of analytic curves. Hence D^* is a new domain such that $\bar{\varDelta} \cap \partial D^* \cap B = 0$, $D^* \subset D$ and $\partial(D^* \cap \varDelta) - \partial\varDelta)$ consists of a finite number of analytic curves $L_1, L_2, \cdots L_s$. Put $D = D^*$.

$$B(H_R(z_0)) \subset B(H_{B-E}(z_0))\,.$$

Proof of Theorem 2. Let Ω be a component of $H_R(z_0)-H_{B-E}(z_0)$. Let Δ be the inverse image of a connected piece ψ over Ω. Then $\partial\Delta$ does not fall in Ω and $\sup n(w)=\infty$ as above. Hence we have Theorem 2 by Lemma 1.

Let R be a Riemann surface. Suppose that R^*+B^* is closed and not compact with respect to a regular metric δ. If any compact set of R^*+B^* has only a finite number of points of the sequence $z_1, z_2, \cdots$, we say that $\{z_i\}$ determines a non accessible boundary point (*N.A.B.P.*). Let *NB* be the set of all the *N.A.B.P.s* and let $K_r(z_0)=E[z\in R^*+B^* : \delta(z, z_0)<r]$, where $K_r(z_0)$ is not always compact.

Put $H_R(\zeta)=\bigcap_{r>0}(\overline{\sum f(z)})$ for $\zeta\in R^*+B^*$ and $H_R(\zeta)$ for $\zeta\in(K_r(z_0)\cap N.B)$ the set of all values w such that there exists a divergent sequence $\{z_i\}$: $z_i\in K_r(z_0)$ and $\lim_i f(z_i)\to w$, where $NB\cap K_r(z_0)$ is a set of *N.A.B.P.s* such that $z_i\in K_r(z_0)$ and $\{z_i\}$ determines a *N.A.B.P.* Put $H_{NB+B-E}(z_0)=\bigcap_{r>0}\overline{\sum H_R(\zeta)}$, where the summation is over $(B+NB-E)\cap K_r(z_0)$. Then we can prove similarly the following:

Theorem 1′. *Let $z_0\in B$ and E be a closed and totally disconnected set on B such that $E\subset\mathfrak{G}_{AB}$ and $z_0\in\overline{B-E}$. Then*

$$B(H_R(z_0)) \subset B(H_{NB+B-E}(z_0))\,.$$

Theorem 2′. (=**Theorem 2**).

As an application we shall consider an extension of Lindelöf's theorem. Let R be a Riemann surface with compact relative boundary ∂R such that R has only one ideal boundary component p ($\in\mathfrak{G}_{AB}$) with finite or infinite genus. Let $L_1, L_2, \cdots, L_t$ be curves in R tending to p.

Theorem 3. *Let $f(z)$ be a bounded analytic function in $R-\sum^t L_i$. If $f(z)$ converges along L_i, then $\underset{\substack{z\to z_0\\ z\in L_1}}{f(z)}=\underset{\substack{z\to z_0\\ z\in L_2}}{f(z)}\cdots=\underset{\substack{z\to z_0\\ z\in L_t}}{f(z)}=w_0$ and $f(z)$ converges to w_0 uniformly in R as z tends to p.*

In fact, $\underset{L_1+L_2+\cdots+L_t}{H(p)}=w_1+w_2+w_3+\cdots+w_t=B(H_{\Sigma L}(p))\supset B(H_R(p))$ implies that $H(p)$ is one point and $w_1=w_2\cdots w_t$.

REMARK. This theorem does not hold for bounded harmonic function in R with infinite genus. Let $U(z)$ be a bounded harmonic function in R (not only in $R-\sum^t L$). If the harmonic dimension of p is larger than one, $U(z)$ does not converge as $z\to p$.

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