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THE MAXIMAL QUOTIENT RING OF A LEFT H-RING

Dedicated to Professor Hiroyuki Tachikawa on his sixtieth birthday

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In [2], M. Harada has introduced two new artinian rings which are closely related to QF-rings; one is a left artinian ring whose non-small left module contains a non-zero injective submodule and the other is a left artinian ring whose non-cosmall left module contains a non-zero projective summand. K. Oshiro called the first ring a *left H-ring* and the second one a *left co-H-ring* ([3]). However, later in [5], he showed that a ring R is a left H-ring if and only if it is a right co-H-ring. QF-rings and Nakayama (artinian serial) rings are left and right H-rings ([3]). As the maximal quotient rings of Nakayama rings are Nakayama, it is natural to ask whether the maximal quotient rings of left H-rings are left H-rings. In this note, we show that this problem is affirmative, by determining the structure of the maximal quotient rings of left H-rings.

1. Preliminaries

Throughout this paper, we assume that all rings R considered are associative rings with identity and all R-modules are unital. Let M be a R-module. We use $J(M)$ and $S(M)$ to denote its Jacobson radical and its socle, respectively.

Definition [3]. A module is *non-small* if it is not a small submodule of its injective hull. We say that a ring R is a *left H-ring* if R is a left artinian ring satisfying the condition that every non-small left R-module contains a non-zero injective submodule.

We note that a left H-ring is also right artinian by [7, Th. 3]. In [5], for a left H-ring R, K. Oshiro gave the following theorem, by using M. Harada's results of [2, Th. 3.6.]: a ring R is a left H-ring if and only if it is left artinian and its complete set E of orthogonal primitive idempotents is arranged as $E=\{e_{11}, \cdots, e_{1n(1)}, \cdots, e_{m1}, \cdots, e_{mn(m)}\}$ for which

(1) each $e_{i1}R$ is injective,

(2) for each i, $e_{ik-1}R\cong e_{ik}R$ or $J(e_{ik-1}R)\cong e_{ik}R$ for $k=2, \cdots, n(i)$, and

(3) $e_{ik}R\not\cong e_{jt}R$ if $i\neq j$.

As a left H-ring is a QF-3 ring by [4], the maximal left quotient ring and the maximal right quotient ring of a left H-ring coincide by [9, Th. 1.4]. From now on, let Q be the maximal quotient ring of a left H-ring R. We shall study the structure of Q. Since maximal quotient rings and left H-rings are Morita-invariant [7], in order to investigate the problem whether Q is a left H-ring or not, we may restrict our attention to basic left H-rings. Therefore, hereafter, we assume that R is a basic left H-ring and E is a complete set of orghogonal primitive idempotents of R. Then E is arranged as $E=\{e_{11}, \cdots, e_{1n(1)}, \cdots, e_{m1}, \cdots, e_{mn(m)}\}$ for which

(1) each $e_{i1}R$ is injective,

(2) for each i, $J(e_{ik-1}R)\cong e_{ik}R$ for $k=2, \cdots, n(i)$.

Definition [10, p. 153]. A primitive idempotent e is called *S-primitive* if the simple module $eR/eJ(R)$ is isomorphic to a minimal right ideal.

We shall use the H.H. Storrer's characterization of the maximal quotient ring of a perfect ring [10].

Since each $e_{i1}R$ $(i=1, \cdots, m)$ is injective, there exists a unique g_i in E such that $(e_{i1}R; Rg_i)$ is an *injective pair*, that is, $S(e_{i1}R)\cong g_iR/J(g_iR)$ and $S(Rg_i)\cong Re_{i1}/J(Re_{i1})$ (cf. K.R. Fuller [1, Th. 3.1]). Each pair $\{e_{i1}, g_i\}$ $(i=1, \cdots, m)$ is very important for studying left H-rings.

Now we shall determine all S-primitive idempotents in E. Let e be an idempotent in E. It is known that e is S-primitive if and only if $S(R_R)e\neq 0$ [10, Lemma 2.3]. Since $S(R_R)=\oplus_{i=1,k=1}^{m,n(m)} S(e_{ik}R)$, $S(e_{ik}R)\not\cong S(e_{jt}R)$ for $i\neq j$ and $S(e_{ik}R)\cong S(e_{it}R)$, we have $S(R_R)e\neq 0$ if and only if $S(e_{ik}R)e\neq 0$ for a unique i. Therefore e is an S-primitive idempotent if and only if $e=g_i$ for some i. Then $E'=\{g_1, \cdots, g_m\}$ is the set of all S-primitive idempotents in E. Put $g=g_1+\cdots+g_m$ and $D=RgR$. Storrer has shown that $D=RgR$ is the minimal dense ideal of R and Q is isomorphic to $\mathrm{Hom}_R(D_R, D_R)=\mathrm{Hom}_R(D_R, R_R)$ by [10, Prop. 1.2 and Th. 2.5]. Since R is a two-sided artinian ring, Q is a left artinian ring by [10, Prop. 3.1].

Lemma 1. *For each e in E, e is also a primitive idempotent in Q. Therefore $S(eQ)$ is a simple Q-module.*

Proof. Since eR is a uniform right ideal, eQ is also a uniform rihgt ideal of Q by [30, Prop. 4.4]. Thus eQ is indecomposable.

By the above lemma, we know that $E=\{e_{11}, \cdots, e_{1n(1)}, \cdots, e_{m1}, \cdots, e_{mn(m)}\}$ is also a complete set of orghogonal primitive idempotents of Q. We shall prove that Q is left H-ring by showing that E satisfies the conditions (1), (2) and (3) of left H-rings. We again note that left H-rings are also right artinian by

[7, Th. 3] and the maximal quotient ring Q of R is a left artinian ring.

Proposition 2. *In the maximal quotient ring Q, $(e_{i1}Q; Qg_i)$ is an injective pair for $i=1, \cdots, m$. Consequently $e_{i1}Q$ and Qg_i are injective Q-modules.*

Proof. By assumption, let $\phi: g_iR \to S(e_{i1}R)$ be an epimorphism. ϕ extends uniquely to a Q-homomorphism $\phi^*: g_iQ \to S(e_{i1}R)Q$ by [10, Prop. 4.3]. Since $S(e_{i1}R)Q = S(e_{i1}Q)$, ϕ^* is also an epimorphism and hence $g_iQ/J(g_iQ) \cong S(e_{i1}Q)$. Since Q is the maximal left quotient ring of R, we have symmetrically that $Qe_{i1}/J(Qe_{i1}) \cong S(Qg_i)$. By [1, Th. 3.1], $(e_{i1}Q; Qg_i)$ is an injective pair for $i=1, \cdots, m$.

Next we shall study isomorphisms among the indecomposable right ideals $e_{ik}Q$. Let f_1, f_2 be idempotents in E and we assume that there exists a monomorphism $\theta: f_1R \to f_2R$ such that $\operatorname{Im}\theta = J(f_2R)$. Then by [10, Prop. 4], θ can be uniquely extended to a Q-homomorphism $\theta^*: f_1Q \to f_2Q$. We shall prove the following result.

Proposition 3. (1) *If f_2 is not S-primitive, then the extension $\theta^*: f_1Q \to f_2Q$ is an isomorphism.*

(2) *If f_2 is S-primitive, then $\theta^*: f_1Q \to f_2Q$ is a monomorphism such that $\operatorname{Im}\theta^* = J(f_2Q)$.*

Proof. From $0 \to f_1R \xrightarrow{\theta} f_2R \to M \to 0$, where $M = f_2R/J(f_2R)$, we have the following exact sequence

$$0 \to f_1Q = \operatorname{Hom}(D, f_1R) \xrightarrow{\theta^*} f_2Q = \operatorname{Hom}(D, f_2R) \to \operatorname{Hom}(D, M).$$

(1) It is sufficient to prove that $\operatorname{Hom}(D, M) = 0$. We assume that there exists a non-zero homomorphism $\phi: D \to M$. Since $D = R(g_1 + \cdots + g_m)R$ by [10, Th. 2.5], there exist some i and some $x \in R$ such that $xg_iR \not\subset \operatorname{Ker}\phi$. Then $g_iR/J(g_iR) \cong M$. Therefore $g_iR \cong f_2R$. This contradicts that f_2 is not S-primitive. Consequently we have that $\operatorname{Hom}(D, M) = 0$, and so θ^* is an isomorphism.

(2) First we shall show that $\operatorname{Im}\theta^* \neq f_2Q$. Since f_2 is S-primitive, we have that $f_2R \subset D$ and so $D = f_2R \oplus (D \cap (1-f_2)R)$. Therefore the projection $\alpha: D \to f_2R$ is not contained in $\operatorname{Im}\theta^* \subseteq \operatorname{Hom}(D, J(f_2R))$. For any $\phi \in J(f_2Q)$, ϕ is not an epimorphism as R-homomorphism. In fact, we shall show that any epimorphism $\alpha: D \to f_2R$ generates f_2Q. Let β be any homomorphism $D \to f_2R$ and $\alpha': f_2R \to D$ the split homomorphism of α. Then we have $\beta = \alpha\alpha'\beta$. Therefore any $\phi \in J(f_2Q)$ is contained in $\operatorname{Im}\theta^*$ and so $\operatorname{Im}\theta^* = J(f_2Q)$, because $J(f_2Q)$ is the unique maximal submodule of f_2Q.

Now we shall prove our main theorem.

Theorem 4. *Let R be a left H-ring. Then the maximal quotient ring Q of R is also an H-ring.*

Proof. Let $E=\{e_{11}, \cdots, e_{in(1)}, \cdots, e_{m1}, \cdots, e_{mn(m)}\}$ be a complete set of orthogonal primitive idempotents of R such that

(1) each $e_{i1}R$ is injective,

(2) for each i, $J(e_{ik-1}R)\cong e_{ik}R$ for $k=2, \cdots, n(i)$.

We have already known that Q is a left artinian ring and E is also a complete set of orthogonal primitive idempotents of Q. By Proposition 2, each $e_{i1}Q$ is an injective Q-module and by Proposition 3, $e_{ik}Q\cong e_{ik-1}Q$ or $e_{ik}Q\cong J(e_{ik-1}Q)$ $k=2, \cdots, n(i)$ for each i. We shall show that $e_{ik}Q\not\cong e_{jt}Q$ if $i\neq j$. If $e_{ik}Q\cong e_{jt}Q$ for some $i\neq j$, k, t, then $S(e_{ik}Q)\cong S(e_{jt}Q)$. Since $S(e_{ik}Q)=S(e_{ik}R)Q$ and $S(e_{jt}Q)=S(e_{jt}R)Q$, we have $S(e_{ik}R)\cong S(e_{jt}R)$ as R-modules by [10, Th. 4.5]. This contradicts the assumption of E.

We recall that g_i is the element of E such that $(e_{i1}R; Rg_i)$ is an injective pair for $i=1, \cdots, m$. Here we define two mappings

$\sigma\colon \{1, \cdots, m\}\to\{1, \cdots, m\}$

$\rho\colon \{1, \cdots, m\}\to\{1, \cdots, n(1)\}\cup\cdots\cup\{1, \cdots, n(m)\}$

by the rule $\sigma(i)=k$ and $\rho(i)=t$ if $g_i=e_{kt}$. We note that $\{\sigma(1), \cdots, \sigma(m)\}\subseteq\{1, \cdots, m\}$ and $1\leqq\rho(i)\leqq n(\sigma(i))$.

Here we shall define a special left H-ring.

Definition [7, p. 94]. A left H-ring is *Type* $(*)$ if $\{\sigma(1), \cdots, \sigma(m)\}$ is a permutation of $\{1, \cdots, m\}$ and $\rho(i)=n(\sigma(i))$ for all $i=1, \cdots, m$.

Cororally. *Let R be a left H-ring. Then the maximal quotient ring Q of R is a QF-ring if and only if R is Type* $(*)$.

Proof. It is easy by Proposition 3.

Example. Let T be a local QF-ring, $J=J(T)$ and $S=S(T)$.

Put $V=\begin{pmatrix} T & T & T \\ J & T & T \\ J & J & T \end{pmatrix}$ and $W=\begin{pmatrix} 0 & 0 & S \\ 0 & 0 & S \\ 0 & 0 & S \end{pmatrix}$. The factor ring $R=V/W$ is a left H-ring such that e_1R is injective, $J(e_1R)\cong e_2R$ and $J(e_2R)\cong e_3R$, where e_i is the matrix such that its (i, i)-position is 1 and all other entries are zero. R is represented as follows: $\begin{pmatrix} T & T & \tilde{T} \\ J & T & \tilde{T} \\ J & J & \tilde{T} \end{pmatrix}$ where $\tilde{T}=T/S$. Since $(e_1R; Re_2)$ is injective pair by [8, § 2], the minimal dense ideal is Re_2R. Therefore the maximal quotient ring Q of R is a left H-ring such that e_1Q is an injective module, $e_1Q\cong e_2Q$ and $J(e_2Q)\cong e_3Q$. Since $e_1Q/J(e_1Q)\cong S(e_1Q)$, we have that $\mathrm{Hom}_Q(e_1Q, J(e_1Q))\cong$

$J(e_1Qe_1)$, $\mathrm{Hom}_Q(J(e_1Q), e_1Q) \cong e_1Qe_1/S(e_1Qe_1)$, $\mathrm{Hom}_Q(J(e_1Q), J(e_1Q)) \cong e_1Qe_1/S(e_1Qe_1)$. Moreover, since $e_1Qe_1 = e_1Re_1 = T$ by [10, Lemma 4.2], Q is represented as a matrix ring $\begin{pmatrix} T & T & \tilde{T} \\ T & T & \tilde{T} \\ J & J & \tilde{T} \end{pmatrix}$.

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