<table>
<thead>
<tr>
<th>Title</th>
<th>The maximal quotient ring of a left H-ring</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kado, Jiro</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 27(2) P.247–P.251</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1990</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/9705">https://doi.org/10.18910/9705</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/9705</td>
</tr>
<tr>
<td>rights</td>
<td></td>
</tr>
</tbody>
</table>
THE MAXIMAL QUOTIENT RING OF A LEFT H-RING

Dedicated to Professor Hiroyuki Tachikawa on his sixtieth birthday

JIRO KADO

(Received March 20, 1989)

In [2], M. Harada has introduced two new artinian rings which are closely related to QF-rings; one is a left artinian ring whose non-small left module contains a non-zero injective submodule and the other is a left artinian ring whose non-cosmall left module contains a non-zero projective summand. K. Oshiro called the first ring a left H-ring and the second one a left co-H-ring ([3]). However, later in [5], he showed that a ring $R$ is a left $H$-ring if and only if it is a right co-$H$-ring. QF-rings and Nakayama (artinian serial) rings are left and right $H$-rings ([3]). As the maximal quotient rings of Nakayama rings are Nakayama, it is natural to ask whether the maximal quotient rings of left $H$-rings are left $H$-rings. In this note, we show that this problem is affirmative, by determining the structure of the maximal quotient rings of left $H$-rings.

1. Preliminaries

Throughout this paper, we assume that all rings $R$ considered are associative rings with identity and all $R$-modules are unital. Let $M$ be a $R$-module. We use $J(M)$ and $S(M)$ to denote its Jacobson radical and its socle, respectively.

Definition [3]. A module is non-small if it is not a small submodule of its injective hull. We say that a ring $R$ is a left $H$-ring if $R$ is a left artinian ring satisfying the condition that every non-small left $R$-module contains a non-zero injective submodule.

We note that a left $H$-ring is also right artinian by [7, Th. 3]. In [5], for a left $H$-ring $R$, K. Oshiro gave the following theorem, by using M. Harada’s results of [2, Th. 3.6.]: a ring $R$ is a left $H$-ring if and only if it is left artinian and its complete set $E$ of orthogonal primitive idempotents is arranged as $E=\{e_{11}, \ldots, e_{1(n(1))}, \ldots, e_{m1}, \ldots, e_{mn(m)}\}$ for which

1. each $e_{ii}R$ is injective,
2. for each $i$, $e_{ik-1}R \cong e_{ik}R$ or $J(e_{ik-1}R) \cong e_{ik}R$ for $k=2, \ldots, n(i)$, and
3. $e_{ik}R \cong e_{ij}R$ if $i \neq j$. 


As a left $H$-ring is a $QF$-3 ring by [4], the maximal left quotient ring and the maximal right quotient ring of a left $H$-ring coincide by [9, Th. 1.4]. From now on, let $Q$ be the maximal left quotient ring of a left $H$-ring $R$. We shall study the structure of $Q$. Since maximal quotient rings and left $H$-rings are Morita-invariant [7], in order to investigate the problem whether $Q$ is a left $H$-ring or not, we may restrict our attention to basic left $H$-rings. Therefore, hereafter, we assume that $R$ is a basic left $H$-ring and $E$ is a complete set of orthogonal primitive idempotents of $R$. Then $E$ is arranged as $E = \{e_{i_1}, \ldots, e_{in(i)}, \ldots, e_m, \ldots, e_{mn(m)}\}$ for which

1. each $e_{i_1}R$ is injective,
2. for each $i$, $f(e_{i_1-1}R) \cong e_{i_1}R$ for $k = 2, \ldots, n(i)$.

Definition [10, p. 153]. A primitive idempotent $e$ is called $S$-primitive if the simple module $eR/e\f(R)$ is isomorphic to a minimal right ideal.

We shall use the H.H. Storrer’s characterization of the maximal quotient ring of a perfect ring [10].

Since each $e_{i_1}R$ $(i = 1, \ldots, m)$ is injective, there exists a unique $g_i$ in $E$ such that $(e_{i_1}R; Rg_i)$ is an injective pair, that is, $S(e_{i_1}R) \cong g_iR/f(g_iR)$ and $S(Rg_i) \cong Rg_i/f(Rg_i)$ (cf. K.R. Fuller [1, Th. 3.1]). Each pair $\{e_{i_1}, g_i\}$ $(i = 1, \ldots, m)$ is very important for studying left $H$-rings.

Now we shall determine all $S$-primitive idempotents in $E$. Let $e$ be an idempotent in $E$. It is known that $e$ is $S$-primitive if and only if $S(R e) \neq 0$ [10, Lemma 2.3]. Since $S(R e) = \bigoplus_{i=1}^{n_g} S(e_{i_1}R)$, $S(e_{i_1}R) \cong S(e_{i_1}R)$ for $i \neq j$ and $S(e_{i_1}R) \cong S(e_{i_1}R)$, we have $S(R e) \neq 0$ if and only if $S(e_{i_1}R) \neq 0$ for a unique $i$. Therefore $e$ is an $S$-primitive idempotent if and only if $e = g_i$ for some $i$. Then $E' = \{g_1, \ldots, g_m\}$ is the set of all $S$-primitive idempotents in $E$. Put $g = g_1 + \cdots + g_m$ and $D = RgR$. Storrer has shown that $D = RgR$ is the minimal dense ideal of $R$ and $Q$ is isomorphic to $\text{Hom}_R(D_R, D_R)$ by [10, Prop. 1.2 and Th. 2.5]. Since $R$ is a two-sided artinian ring, $Q$ is a left artinian ring by [10, Prop. 3.1].

**Lemma 1.** For each $e$ in $E$, $e$ is also a primitive idempotent in $Q$. Therefore $S(eQ)$ is a simple $Q$-module.

Proof. Since $eR$ is a uniform right ideal, $eQ$ is also a uniform right ideal of $Q$ by [30, Prop. 4.4]. Thus $eQ$ is indecomposable.

By the above lemma, we know that $E = \{e_{1_1}, \ldots, e_{1n(1)}, \ldots, e_{m_1}, \ldots, e_{mn(m)}\}$ is also a complete set of orthogonal primitive idempotents of $Q$. We shall prove that $Q$ is left $H$-ring by showing that $E$ satisfies the conditions (1), (2) and (3) of left $H$-rings. We again note that left $H$-rings are also right artinian by
Proposition 2. In the maximal quotient ring $Q$, $(e_iQ; Qg_i)$ is an injective pair for $i=1, \ldots, m$. Consequently $e_iQ$ and $Qg_i$ are injective $Q$-modules.

Proof. By assumption, let $\phi: g_1R \to S(e_iR)$ be an epimorphism. $\phi$ extends uniquely to a $Q$-homomorphism $\phi^*: g_1Q \to S(e_iQ)$ by [10, Prop. 4.3]. Since $S(e_iR)Q = S(e_iQ)$, $\phi^*$ is also an epimorphism and hence $g_1Q/J(g_1Q) \cong S(e_iQ)$. Since $Q$ is the maximal left quotient ring of $R$, we have symmetrically that $Qe_i/J(Qe_i) \cong S(Qg_i)$. By [1, Th. 3.1], $(e_iQ; Qg_i)$ is an injective pair for $i=1, \ldots, m$.

Next we study isomorphisms among the indecomposable right ideals $e_iQ$. Let $f_1, f_2$ be idempotents in $E$ and we assume that there exists a monomorphism $\theta: f_1R \to f_2R$ such that $\text{Im } \theta = J(f_2R)$. Then by [10, Prop. 4], $\theta$ can be uniquely extended to a $Q$-homomorphism $\theta^*: f_1Q \to f_2Q$. We shall prove the following result.

Proposition 3. (1) If $f_2$ is not $S$-primitive, then the extension $\theta^*: f_1Q \to f_2Q$ is an isomorphism.

(2) If $f_2$ is $S$-primitive, then $\theta^*: f_1Q \to f_2Q$ is a monomorphism such that $\text{Im } \theta^* = J(f_2Q)$.

Proof. From $0 \to f_1R \to f_2R \to M \to 0$, where $M = f_2R/J(f_2R)$, we have the following exact sequence

$$0 \to f_1Q = \text{Hom}(D, f_1R) \to f_2Q = \text{Hom}(D, f_2R) \to \text{Hom}(D, M).$$

(1) It is sufficient to prove that $\text{Hom}(D, M) = 0$. We assume that there exists a non-zero homomorphism $\phi: D \to M$. Since $D = R(g_1 + \cdots + g_m)R$ by [10, Th. 2.5], there exist some $i$ and some $x \in R$ such that $xg_iR \subseteq \text{Ker } \phi$. Then $g_iR/J(g_iR) \cong M$. Therefore $g_iR \cong f_iR$. This contradicts that $f_2$ is not $S$-primitive. Consequently we have that $\text{Hom}(D, M) = 0$, and so $\theta^*$ is an isomorphism.

(2) First we shall show that $\text{Im } \theta^* = f_2Q$. Since $f_2$ is $S$-primitive, we have that $f_2R \subseteq D$ and so $D = f_2R \oplus (D \cap (1-f_2)R)$. Therefore the projection $\alpha: D \to f_2R$ is not contained in $\text{Im } \theta^* \subseteq \text{Hom}(D, J(f_2R))$. For any $\phi \in J(f_2Q)$, $\phi$ is not an epimorphism as $R$-homomorphism. In fact, we shall show that any epimorphism $\alpha: D \to f_2R$ generates $f_2Q$. Let $\beta$ be any homomorphism $D \to f_2R$ and $\alpha': f_2R \to D$ the split homomorphism of $\alpha$. Then we have $\beta = \alpha \alpha' \beta$. Therefore any $\phi \in J(f_2Q)$ is contained in $\text{Im } \theta^*$ and so $\text{Im } \theta^* = J(f_2Q)$, because $J(f_2Q)$ is the unique maximal submodule of $f_2Q$.

Now we shall prove our main theorem.
Theorem 4. Let \( R \) be a left \( H \)-ring. Then the maximal quotient ring \( Q \) of \( R \) is also an \( H \)-ring.

Proof. Let \( E = \{e_{i1}, \ldots, e_{iv(i)}, \ldots, e_{mn}, \ldots, e_{m(m)}\} \) be a complete set of orthogonal primitive idempotents of \( R \) such that

1. each \( e_{ii}R \) is injective,
2. for each \( i, J(e_{ik}R) \approx e_{ik}R \) for \( k = 2, \ldots, n(i) \).

We have already known that \( Q \) is a left artinian ring and \( E \) is also a complete set of orthogonal primitive idempotents of \( Q \). By Proposition 2, each \( e_{ii}Q \) is an injective \( Q \)-module and by Proposition 3, \( e_{ik}Q \approx e_{ik}Q \) or \( e_{ik}Q \approx J(e_{ik}R) \) \( k = 2, \ldots, n(i) \) for each \( i \). We shall show that \( e_{ik}Q \approx e_{ij}Q \) for some \( i \neq j \). If \( e_{ik}Q \approx e_{ij}Q \) for some \( i \neq j, k, t \), then \( S(e_{ik}Q) \approx S(e_{ij}Q) \). Since \( S(e_{ik}Q) = S(e_{ik}R)Q \) and \( S(e_{ij}Q) = S(e_{ij}R)Q \), we have \( S(e_{ik}R) \approx S(e_{ij}R) \) as \( R \)-modules by [10, Th. 4.5]. This contradicts the assumption of \( E \).

We recall that \( g_i \) is the element of \( E \) such that \( (e_{ii}R; Rg_i) \) is an injective pair for \( i = 1, \ldots, m \). Here we define two mappings

\[
\sigma: \{1, \ldots, m\} \to \{1, \ldots, m\} \\
\rho: \{1, \ldots, m\} \to \{1, \ldots, n(1)\} \cup \cdots \cup \{1, \ldots, n(m)\}
\]

by the rule \( \sigma(i) = k \) and \( \rho(i) = t \) if \( g_i = e_{kt} \). We note that \( \{\sigma(1), \ldots, \sigma(m)\} \subseteq \{1, \ldots, m\} \) and \( 1 \leq \rho(i) \leq n(\sigma(i)) \).

Here we shall define a special left \( H \)-ring.

Definition [7, p. 94]. A left \( H \)-ring is Type \( (*) \) if \( \{\sigma(1), \ldots, \sigma(m)\} \) is a permutation of \( \{1, \ldots, m\} \) and \( \rho(i) = n(\sigma(i)) \) for all \( i = 1, \ldots, m \).

Corollary. Let \( R \) be a left \( H \)-ring. Then the maximal quotient ring \( Q \) of \( R \) is a \( QF \)-ring if and only if \( R \) is Type \( (*) \).

Proof. It is easy by Proposition 3.

Example. Let \( T \) be a local \( QF \)-ring, \( J = J(T) \) and \( S = S(T) \).

Put \( V = \begin{pmatrix} T & T & T \\ J & T & T \end{pmatrix} \) and \( W = \begin{pmatrix} 0 & 0 & S \\ 0 & 0 & S \end{pmatrix} \). The factor ring \( R = V/W \) is a left \( H \)-ring such that \( e_1R \) is injective, \( J(e_1R) \approx e_2R \) and \( J(e_2R) \approx e_3R \), where \( e_i \) is the matrix such that its \((i, i)\)-position is 1 and all other entries are zero. \( R \) is represented as follows:

\[
\begin{pmatrix}
T & T & T \\
J & T & T \\
J & J & T
\end{pmatrix}
\]

where \( \bar{T} = T/S \). Since \((e_1R; Re_2)\) is injective pair by [8, § 2], the minimal dense ideal is \( Re_2R \). Therefore the maximal quotient ring \( Q \) of \( R \) is a left \( H \)-ring such that \( e_1Q \) is an injective module, \( e_1Q \approx e_2Q \) and \( J(e_2Q) \approx e_3Q \). Since \( e_1Q/J(e_1Q) \approx S(e_1Q) \), we have that \( \text{Hom}_Q(e_1Q, J(e_1Q)) \approx \)
\[ f(e_iQe_i), \text{Hom}_Q(f(e_iQ), e_iQ) = e_iQe_i/\text{S}(e_iQe_i), \text{Hom}_Q(f(e_iQ), f(e_iQ)) = e_iQe_i/\text{S}(e_iQe_i). \]

Moreover, since \( e_iQe_i = e_iRe_i = T \) by [10, Lemma 4.2], \( Q \) is represented as a matrix ring \[
\begin{pmatrix}
T & T & \hat{T} \\
J & J & \hat{T}
\end{pmatrix}
\]

References


Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku
Osaka, Japan