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<th><strong>Title</strong></th>
<th>The maximal quotient ring of a left H-ring</th>
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Osaka University
THE MAXIMAL QUOTIENT RING OF A LEFT H-RING

Dedicated to Professor Hiroyuki Tachikawa on his sixtieth birthday

JIRO KADO

(Received March 20, 1989)

In [2], M. Harada has introduced two new artinian rings which are closely related to QF-rings; one is a left artinian ring whose non-small left module contains a non-zero injective submodule and the other is a left artinian ring whose non-cosmall left module contains a non-zero projective summand. K. Oshiro called the first ring a left H-ring and the second one a left co-H-ring ([3]). However, later in [5], he showed that a ring $R$ is a left H-ring if and only if it is a right co-H-ring. QF-rings and Nakayama (artinian serial) rings are left and right H-rings ([3]). As the maximal quotient rings of Nakayama rings are Nakayama, it is natural to ask whether the maximal quotient rings of left H-rings are left H-rings. In this note, we show that this problem is affirmative, by determining the structure of the maximal quotient rings of left H-rings.

1. Preliminaries

Throughout this paper, we assume that all rings $R$ considered are associative rings with identity and all $R$-modules are unital. Let $M$ be a $R$-module. We use $J(M)$ and $S(M)$ to denote its Jacobson radical and its socle, respectively.

Definition [3]. A module is non-small if it is not a small submodule of its injective hull. We say that a ring $R$ is a left H-ring if $R$ is a left artinian ring satisfying the condition that every non-small left $R$-module contains a non-zero injective submodule.

We note that a left H-ring is also right artinian by [7, Th. 3]. In [5], for a left H-ring $R$, K. Oshiro gave the following theorem, by using M. Harada’s results of [2, Th. 3.6.]: a ring $R$ is a left H-ring if and only if it is left artinian and its complete set $E$ of orthogonal primitive idempotents is arranged as $E=\{e_{1i}, \ldots, e_{i_{n(i)}}, \ldots, e_{m1}, \ldots, e_{mn(n)}\}$ for which

1. each $e_{ii}R$ is injective,
2. for each $i$, $e_{ik}R \cong e_{ik}R$ or $J(e_{ik-1}R) \cong e_{ik}R$ for $k=2, \ldots, n(i)$, and
3. $e_{ik}R \cong e_{ij}R$ if $i \neq j$. 

As a left $H$-ring is a QF-3 ring by [4], the maximal left quotient ring and the maximal right quotient ring of a left $H$-ring coincide by [9, Th. 1.4]. From now on, let $Q$ be the maximal left quotient ring of a left $H$-ring $R$. We shall study the structure of $Q$. Since maximal quotient rings and left $H$-rings are Morita-invariant [7], in order to investigate the problem whether $Q$ is a left $H$-ring or not, we may restrict our attention to basic left $H$-rings. Therefore, hereafter, we assume that $R$ is a basic left $H$-ring and $E$ is a complete set of orthogonal primitive idempotents of $R$. Then $E$ is arranged as $E = \{e_{11}, \cdots, e_{1n(1)}, \cdots, e_{m1}, \cdots, e_{mn(m)}\}$ for which

1. each $e_{ii}R$ is injective,
2. for each $i$, $J(e_{ik}R) = e_{ik}R$ for $k=2, \cdots, n(i)$.

Definition [10, p. 153]. A primitive idempotent $e$ is called $S$-primitive if the simple module $eR/eJ(R)$ is isomorphic to a minimal right ideal.

We shall use the H.H. Storrer's characterization of the maximal quotient ring of a perfect ring [10].

Since each $e_{ii}R (i=1, \cdots, m)$ is injective, there exists a unique $g_i$ in $E$ such that $(e_{ii}R; Rg_i)$ is an injective pair, that is, $S(e_{ii}R) \cong g_iR/J(g_iR)$ and $S(Rg_i) \cong Re_{ii}/J(Re_{ii})$ (cf. K.R. Fuller [1, Th. 3.1]). Each pair $\{e_{ii}, g_i\} (i=1, \cdots, m)$ is very important for studying left $H$-rings.

Now we shall determine all $S$-primitive idempotents in $E$. Let $e$ be an idempotent in $E$. It is known that $e$ is $S$-primitive if and only if $S(Rg)e \neq 0$ [10, Lemma 2.3]. Since $S(Rg) = \bigoplus_{i, j} S(e_{ij}R)$, $S(e_{ii}R) \cong S(e_{ij}R)$ for $i \neq j$, and $S(e_{ii}R) = S(e_{ii}R)$, we have $S(Rg)e = 0$ if and only if $S(e_{ii}R)e = 0$ for a unique $i$. Therefore $e$ is an $S$-primitive idempotent if and only if $e = g_i$ for some $i$. Then $E' = \{g_1, \cdots, g_m\}$ is the set of all $S$-primitive idempotents in $E$. Put $g = g_1 + \cdots + g_m$ and $D = RgR$. Storrer has shown that $D = RgR$ is the minimal dense ideal of $R$ and $Q$ is isomorphic to $\text{Hom}_{R}(D_R, D_R) = \text{Hom}_{R}(D_R, R_R)$ by [10, Prop. 1.2 and Th. 2.5]. Since $R$ is a two-sided artinian ring, $Q$ is a left artinian ring by [10, Prop. 3.1].

**Lemma 1.** For each $e$ in $E$, $e$ is also a primitive idempotent in $Q$. Therefore $S(eQ)$ is a simple $Q$-module.

Proof. Since $eR$ is a uniform right ideal, $eQ$ is also a uniform right ideal of $Q$ by [30, Prop. 4.4]. Thus $eQ$ is indecomposable.

By the above lemma, we know that $E = \{e_{11}, \cdots, e_{1n(1)}, \cdots, e_{m1}, \cdots, e_{mn(m)}\}$ is also a complete set of orthogonal primitive idempotents of $Q$. We shall prove that $Q$ is left $H$-ring by showing that $E$ satisfies the conditions (1), (2) and (3) of left $H$-rings. We again note that left $H$-rings are also right artinian by
[7, Th. 3] and the maximal quotient ring \( Q \) of \( R \) is a left artinian ring.

**Proposition 2.** In the maximal quotient ring \( Q \), \((e^n_i Q; Qg_i)\) is an injective pair for \( i=1, \cdots, m \). Consequently \( e^n_i Q \) and \( Qg_i \) are injective \( Q \)-modules.

Proof. By assumption, let \( \phi: g_i R \rightarrow S(e^n_i R) \) be an epimorphism. \( \phi \) extends uniquely to a \( Q \)-homomorphism \( \phi^*: g_i Q \rightarrow S(e^n_i Q) \) by [10, Prop. 4.3]. Since \( S(e^n_i R)Q = S(e^n_i Q) \), \( \phi^* \) is also an epimorphism and hence \( g_i Q \cap J(g_i Q) = S(e^n_i Q) \). Since \( Q \) is the maximal left quotient ring of \( R \), we have symmetrically that \( Qe^n_i / J(Qe^n_i) = S(Qg_i) \). By [1, Th. 3.1], \((e^n_i Q; Qg_i)\) is an injective pair for \( i=1, \cdots, m \).

Next we shall study isomorphisms among the indecomposable right ideals \( e^n_i Q \). Let \( f_1, f_2 \) be idempotents in \( E \) and we assume that there exists a monomorphism \( \theta: f_1 R \rightarrow f_2 R \) such that \( \text{Im} \theta = J(f_2 R) \). Then by [10, Prop. 4], \( \theta \) can be uniquely extended to a \( Q \)-homomorphism \( \theta^*: f_1 Q \rightarrow f_2 Q \). We shall prove the following result.

**Proposition 3.** (1) If \( f_2 \) is not \( S \)-primitive, then the extension \( \theta^*: f_1 Q \rightarrow f_2 Q \) is an isomorphism.

(2) If \( f_2 \) is \( S \)-primitive, then \( \theta^*: f_1 Q \rightarrow f_2 Q \) is a monomorphism such that \( \text{Im} \theta^* = J(f_2 Q) \).

Proof. From \( 0 \rightarrow f_1 R \rightarrow f_2 R \rightarrow M \rightarrow 0 \), where \( M = f_2 R / J(f_2 R) \), we have the following exact sequence

\[
0 \rightarrow f_1 Q \rightarrow \text{Hom}(D, f_1 R) \rightarrow f_2 Q \rightarrow \text{Hom}(D, f_2 R) \rightarrow \text{Hom}(D, M) \rightarrow 0.
\]

(1) It is sufficient to prove that \( \text{Hom}(D, M) = 0 \). We assume that there exists a non-zero homomorphism \( \phi: D \rightarrow M \). Since \( D = R(g_1 + \cdots + g_m)R \) by [10, Th. 2.5], there exist some \( i \) and some \( x \in R \) such that \( xg_i R \subseteq \text{Ker} \phi \). Then \( g_i R \cap J(g_i R) = M \). Therefore \( g_i R = f_2 R \). This contradicts that \( f_2 \) is not \( S \)-primitive. Consequently we have that \( \text{Hom}(D, M) = 0 \), and so \( \theta^* \) is an isomorphism.

(2) First we shall show that \( \text{Im} \theta^* = f_2 Q \). Since \( f_2 \) is \( S \)-primitive, we have that \( f_2 R \subseteq D \) and so \( D = f_2 R \oplus (D \cap (1-f_2) R) \). Therefore the projection \( \alpha: D \rightarrow f_2 R \) is not contained in \( \text{Im} \theta^* \subseteq \text{Hom}(D, J(f_2 R)) \). For any \( \phi \in J(f_2 Q) \), \( \phi \) is not an epimorphism as \( R \)-homomorphism. In fact, we shall show that any epimorphism \( \alpha: D \rightarrow f_2 R \) generates \( f_2 Q \). Let \( \beta \) be any homomorphism \( D \rightarrow f_2 R \) and \( \alpha': f_2 R \rightarrow D \) the split homomorphism of \( \alpha \). Then we have \( \beta = \alpha \alpha' \beta \). Therefore any \( \phi \in J(f_2 Q) \) is contained in \( \text{Im} \theta^* \) and so \( \text{Im} \theta^* = J(f_2 Q) \), because \( J(f_2 Q) \) is the unique maximal submodule of \( f_2 Q \).

Now we shall prove our main theorem.
Theorem 4. Let \( R \) be a left \( H \)-ring. Then the maximal quotient ring \( Q \) of \( R \) is also an \( H \)-ring.

Proof. Let \( E = \{ e_{1}, \ldots, e_{i}, \ldots, e_{m} \} \) be a complete set of orthogonal primitive idempotents of \( R \) such that

1. each \( e_{i} R \) is injective,
2. for each \( i \), \( (e_{ik} R) = e_{ik} R \) for \( k = 2, \ldots, n(i) \).

We have already known that \( Q \) is a left artinian ring and \( E \) is also a complete set of orthogonal primitive idempotents of \( Q \). By Proposition 2, each \( e_{ik} Q \) is an injective \( Q \)-module and by Proposition 3, \( e_{ik} Q \approx e_{ik} Q \) or \( e_{ik} Q \approx (e_{ik} Q) k = 2, \ldots, n(i) \) for each \( i \). We shall show that \( e_{ik} Q \approx e_{ik} Q \) for some \( i \neq j, k, t \) if \( e_{ik} Q \approx e_{ik} Q \) for some \( i \neq j, k, t \) then \( S(e_{ik} Q) = S(e_{ik} Q) \) as \( Q \)-modules by [10, Th. 4.5]. This contradicts the assumption of \( E \).

We recall that \( g_{i} \) is the element of \( E \) such that \( (e_{i} R, R_{g_{i}}) \) is an injective pair for \( i = 1, \ldots, m \). Here we define two mappings

\[ \sigma: \{1, \ldots, m\} \rightarrow \{1, \ldots, m\} \]
\[ \rho: \{1, \ldots, m\} \rightarrow \{1, \ldots, n(1)\} \cup \cdots \cup \{1, \ldots, n(m)\} \]

by the rule \( \sigma(i) = k \) and \( \rho(i) = t \) if \( g_{i} = e_{it} \). We note that \( \{\sigma(1), \ldots, \sigma(m)\} \subseteq \{1, \ldots, m\} \) and \( 1 \leq \rho(i) \leq n(\sigma(i)) \).

Here we shall define a special left \( H \)-ring.

Definition [7, p. 94]. A left \( H \)-ring is Type \((\ast)\) if \( \{\sigma(1), \ldots, \sigma(m)\} \) is a permutation of \( \{1, \ldots, m\} \) and \( \rho(i) = n(\sigma(i)) \) for all \( i = 1, \ldots, m \).

Corollary. Let \( R \) be a left \( H \)-ring. Then the maximal quotient ring \( Q \) of \( R \) is a \( QF \)-ring if and only if \( R \) is Type \((\ast)\).

Proof. It is easy by Proposition 3.

Example. Let \( T \) be a local \( QF \)-ring, \( J = J(T) \) and \( S = S(T) \).

\[
\begin{pmatrix}
T & T & T \\
J & T & T \\
J & J & T
\end{pmatrix}
\]

Put \( V = \begin{pmatrix} 0 & 0 & S \\ 0 & 0 & S \end{pmatrix} \) and \( W = \begin{pmatrix} 0 & 0 & S \\ 0 & 0 & S \end{pmatrix} \). The factor ring \( R = V/W \) is a left \( H \)-ring such that \( e_{i} R \) is injective, \( J(e_{i} R) \approx e_{2} R \) and \( J(e_{2} R) \approx e_{3} R \), where \( e_{i} \) is the matrix such that its \((i, i)\)-position is 1 and all other entries are zero. \( R \) is represented as follows:

\[
\begin{pmatrix}
T & T & T \\
J & T & T \\
J & J & T
\end{pmatrix}
\]

where \( T = T/\bar{S} \). Since \( (e_{i} R, R_{e_{2}}) \) is injective pair by [8, § 2], the minimal dense ideal is \( R_{e_{2}} R \). Therefore the maximal quotient ring \( Q \) of \( R \) is a left \( H \)-ring such that \( e_{i} Q \) is an injective module, \( e_{1} Q \approx e_{2} Q \) and \( J(e_{2} Q) \approx e_{3} Q \). Since \( e_{1} Q/J(e_{1} Q) \approx S(e_{1} Q) \), we have that \( \Hom_{Q}(e_{1} Q, J(e_{1} Q)) \approx \)
$\text{Maximal Quotient Ring}$

\[ J(e_i Q e_i), \text{Hom}_Q(J(e_i Q), e_i Q) = e_i Q e_i / S(e_i Q e_i), \text{Hom}_Q(J(e_i Q), J(e_i Q)) = e_i Q e_i / S(e_i Q e_i). \]

Moreover, since $e_i Q e_i = e_i Re_i = T$ by [10, Lemma 4.2], $Q$ is represented as a matrix ring $\begin{pmatrix} T & T & \hat{T} \\ T & T & \hat{T} \\ J & J & \hat{T} \end{pmatrix}$.

References


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