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Osaka University
This is a continuation of the previous paper with the same title which will be referred to as [I]. Throughout the paper, $A$ denotes a (left and right) artinian ring with identity 1, $J$ its Jacobson radical and unless otherwise stated, all modules are (unital and) finitely generated.

Let $n$ be any natural number. Then we say that $A$ is of right $n$-th local (resp. colocal) type in case for every indecomposable right $A$-module $M$, the $n$-th top $\text{top}^n M := M/\text{ann}_A J^n$ (resp. the $n$-th socle $\text{soc}^n M := \text{ann}_A J^n$ of $M$) of $M$ is indecomposable.

In this paper, we first examine an artinian ring which is of both left and right $n$-th local type (in this case the artinian ring is said to be of two-sided $n$-th local type or simply $n$-th local type) and give some necessary and sufficient conditions to be of this type, in particular for an algebra, we characterize this type by a structure of $A$ (2.5). Note that this type of rings include the class of serial rings ([4]). Next, we come back to the case $n=2$ and restrict our interest to the case where $A$ is an algebra over an algebraically closed field $k$, and give some further necessary conditions for $A$ to be of right 2nd local type (3.4). (It is shown in [I, Example 2] that the necessary conditions stated in [I, Theorem 1] are not sufficient for $A$ to be of right 2nd local type.) These conditions contain the list of all possible “shapes” of indecomposable projective right $A$-modules. (That of indecomposable projective left $A$-modules follows directly from [I, Theorem 1].) As an application, we give some necessary and sufficient conditions for a left serial algebra over an algebraically closed field to be of right 2nd local type (4.1). It should be noted that by [I, Theorem 1], an algebra over an algebraically closed field which is of right 2nd local type is left serial if every indecomposable projective left $A$-module $P$ is of height $\geq 4$ (i.e. $J^3 P \neq 0$). We remark that these theorems remain valid also in the case where the base field $k$ is a splitting field for $A$. The last section is devoted to some examples.

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1. Preliminaries

The notation and terminology used throughout this paper is the same as that used in [I]. We may and will assume that $A$ is a basic ring. For the convenience of readers, we quote some propositions from [I] and [6] which are frequently used in the sequel.

**Theorem 1.1** ([I, Theorem 1]). Let $A$ be a ring with selfduality which is of right 2nd local type and $e$ in $\text{pi}(A)$. Then

1. $J^2e$ is a uniserial waist in $Ae$ if $J^2e \neq 0$,
2. $eJ^m$ is a direct sum of local modules for every $m \geq 2$,
3. for each local direct summand $L$ of $eJ^2$, $LJ^2$ is uniserial (thus $eJ^4$ is a direct sum of uniserial modules). Further if $A$ is an algebra, we have
4. $Ae$ is uniserial if $h(Ae) \geq 5$.
In particular if the base field $k$ is, in addition, an algebraically closed field, then
5. $Ae$ is uniserial if $h(Ae) \geq 4$,
and then
6. $eJ^2$ is a direct sum of uniserial modules.

We denote by $1_M$ the identity map of $M$ for any $A$-module $M$.

**Lemma 1.2** ([6, Lemma 1.1]). Let $M_1$, $M_2$ and $T$ be submodules of a left $A$-module $M$ such that $M=M_1+M_2$ and $T=M_1 \cap M_2$. If $T'$ is a submodule of $T$ and $\phi: M_1 \rightarrow M_2$ is an extension of $1_{T'}$, then putting $M_1':=(M_1)(1_{M_1}-\varphi)$ the following hold.

1. $M=M_1'+M_2$.
2. $M_1' \cap M_2=(T)(1_{T'}-\varphi)$.
3. The epimorphism $(1_{M_1}-\varphi): M_1 \rightarrow M_1'$ induces epimorphisms $M_1/T' \rightarrow M_1'$ and $T/T' \rightarrow M_1' \cap M_2$, in particular $|M_1' \cap M_2| \leq |T|-|T'|$.

**Lemma 1.3** ([6, Lemma 1.2]). Let $M_1$, $M_2$ and $T$ be submodules of a left $A$-module $M$ such that $M=M_1+M_2$ and $T=M_1 \cap M_2$. Then

1. $1_T$ is extendable to a homomorphism $M_1 \rightarrow M_2$ iff $M=M_1' \oplus M_2$ for some submodule $M_1'$ of $M$.
2. $1_T$ is not extendable to any homomorphism $U \rightarrow M_2$ for any submodule $U$ of $M_1$ with $T \subseteq U$ iff $\text{soc}M=\text{soc}M_2$.

2. Artinian rings of $n$-th local type

In this section, we give some necessary and sufficient conditions for an artinian ring $A$ to be of $n$-th local type for any natural number $n$.

**Lemma 2.1.** Let $n$ be any natural number and $(E): 0 \rightarrow S \xrightarrow{\alpha} L_1 \oplus L_2 \xrightarrow{\beta} M$
→ 0 be an exact sequence of right $A$-modules such that $\alpha_1 S \leq L_1$ and $\alpha_2 S \leq L_2 J^*$ in $L_2$ such that $\alpha = (\alpha_1, \alpha_2)^T$. Then $\top^* M$ is decomposable.

Proof. The sequence $(E)$ induces the following exact sequence:

$$0 \to \pi (S) \to \top^* L_1 \oplus \top^* L_2 \to \top^* M \to 0$$

where $\pi: L_1 \oplus L_2 \to \top^* L_1 \oplus \top^* L_2$ is the canonical projection. Also, we have $\text{Im } \alpha' \leq \top^* L_1$ by the assumption. Hence $\top^* M = ((\top^* L_1) / \text{Im } \alpha') \oplus \top^* L_2$ is decomposable.

Lemma 2.2. Let $(\alpha, D) = (\alpha_i)^T_{i=1} : eA/eJ \to \bigoplus_{i=1}^n f_i A/I_i$ be a homomorphism where $e, f_i$ are in $\text{pi}(A), I_i \subseteq f_i J$ and $\alpha_i$ is a left multiplication by an element $u_i$ in $f_i A$ for each $i=1, \ldots, n$. Then $(\alpha, D)$ is $j$-fusible $(j=1, \ldots, n)$ iff there are some $a_i$ in $f_i A f_j$ for each $i \neq j$ and there is some $b$ in $I e$ such that $u_j = \sum a_i u_i + b$ and $a_i I_i \leq I_j$ for each $i \neq j$. In particular when $(\alpha, D)$ is $j$-fusible, we have $A u_j \leq \sum_{i \neq j} A u_i$ if $I_\varepsilon \leq u_j$; and $A u_j \leq \sum_{i \neq j} A u_i + I e$ if $I_j = f_j I$ for some ideal $I$ of $A$.

Proof. $(\alpha, D)$ is $j$-fusible iff we have a commutative diagram

$$\begin{array}{ccc}
eq \alpha_{i=1}^j & \oplus & f_i A / I_i \\
eq \alpha_{i=1}^j & \oplus & f_i A / I_i \\
eq {\phi_i}_{i \neq j} & \oplus & f_i A / I_j \\
eq {\phi_i}_{i \neq j} & \oplus & f_i A / I_j \\
eq eA / eJ & \to & eA / eJ
\end{array}$$

for some homomorphism $\phi_i : f_i A / I_i \to f_i A / I_j$ which are left multiplications by some elements $a_i$ in $f_i A f_j$ for all $i \neq j$ iff $u_j = \sum a_i u_i + b$ and $a_i I_i \leq I_j$ for some $a_i$ in $f_i A f_j$ and $b$ in $I_j$ (consequently $b$ in $I \varepsilon$) only if $u_j \subseteq \sum_{i \neq j} A u_i + I e$ only if $A u_j \subseteq \sum_{i \neq j} A u_i + I e$ (if $I_j = f_j I$ for some ideal $I$ of $A$).

In case $I_\varepsilon \leq u_j$, we have $A u_j \subseteq \sum_{i \neq j} A u_i$ since $A u_j \subseteq \sum_{i \neq j} A u_i + Ju_j$ and $Ju_j$ is small in $A u_j$.

Lemma 2.3 (cf. [1, Theorem 3.2]). Let $n$ be any natural number. Then the condition

(2R) $\alpha = (\alpha_1, \alpha_2)^T : S \to L_1 \oplus L_2$ is fusible if $S$ is a simple right $A$-module, $L_i$ are local right $A$-modules and $\alpha_1 S \leq L_1 J$, $\alpha_2 S \leq L_2 J^*$.

implies the following

(2R)' Let $(\alpha, D) = (\alpha_1, \alpha_2)^T : T \to L_1 \oplus L_2$ be a homomorphism of right $A$-modules such that $L_1$ is local, $L_2$ is local and colocal of height $> n$ and $h(L_1) \leq h(L_2)$; and $\alpha_i T \leq L_i J$ for each $i=1, 2$ and $\alpha_i$ is monic. Then $(\alpha, D)$ is 2-fusible.

Proof. Assume that (2R) holds and hypothesis of (2R)' is satisfied. Then
noting that $\alpha$ is monic since $\alpha_1$ is, we have an exact sequence

\[(E) \quad 0 \to T \xrightarrow{\alpha} L_1 \oplus L_2 \xrightarrow{\beta=(\beta_1, \beta_2)} M \to 0\]

which does not split since $\alpha T \leq (L_1 \oplus L_2)J$. We have only to show that $\beta_2 L_2$ is a direct summand of $M$ by [1, Proposition 1.2]. Note that $\beta_3$ is monic since $\alpha_1$ is. Then putting $M_1:=\beta_1(L_1)$, $M_2:=\beta_2 L_2$ and $U:=\beta_2 \alpha_2(T)$, we have $M = M_1 + M_2$ and $U = M_1 \cap M_2$. Also, $h(M_2) > n$, $M_1$ are local modules and $M_2$ is, in addition, colocal. Further $U \leq M_1 J$ and $h(M_1) \leq h(M_2)$. Take any simple submodule $S_A \leq U$ and consider the map $\varphi = (\varphi_1, \varphi_2)^T : S \to M_1 \oplus M_2$ where each $\varphi_i$ is the inclusion map $S \to M_i$. Then $\varphi_1 S \leq U \leq M_1 J$ and $\varphi_2 S \leq \text{soc} M_2 \leq M_2 J^*$ since $M_2$ is colocal of height $> n$. Hence $\varphi : S \to M_1 \oplus M_2$ is fusible by (2R). If it is 1-fusible, we have a commutative diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\varphi_2} & M_2 \\
\downarrow & & \downarrow \\
S & \xrightarrow{\varphi_1} & M_1
\end{array}
\]

for some $\psi : M_2 \to M_1$. Then $\psi$ is monic since $M_2$ is colocal. Therefore $\psi$ is an isomorphism since $h(M_1) \leq h(M_2)$ and both $M_1$ and $M_2$ are local. Accordingly, we may assume that $\varphi : S \to M_1 \oplus M_2$ is 2-fusible. Thus there is a homomorphism $\varphi : M_1 \to M_2$ such that the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\varphi_1} & M_1 \\
\downarrow & & \downarrow \\
S & \xrightarrow{\varphi_2} & M_2
\end{array}
\]

is commutative. Then $M = M_1 + M_2 = M'_1 + M_2$ where $M'_1 := (1 - \psi)(M_1)$ by (1.2). Also, $|M'_1 \cap M_2| < |U|$, $M'_1$ is local, $M'_1 \cap M_2 \leq M'_1 J$ and $h(M'_1) \leq h(M_2)$. Hence iterating this argument, we obtain $M = M'_1 \oplus M_2$ for some $0 \neq M'_1 \leq M$. Thus $\beta_2 L_2 = M_2$ is a direct summand of $M$. //

2.4. Here, we do not assume that every module is finitely generated.

**Definition.** Let $A$ be a ring and $M$ a right (or left) $A$-module. Then $M$ is called to be a **highest** module in case $h(M) = h(A_A) (=h(A_A))$.

**Proposition 2.4.1.** The following statements for a ring $A$ are equivalent:

1. Every highest right $A$-module is local if it is colocal.
2. Every highest indecomposable projective right $A$-module is injective.
3. Every highest indecomposable injective right $A$-module is projective.
4. Every highest indecomposable right $A$-module is projective and injective.
Proof. (1)⇒(2). Let \( M_A \) be highest indecomposable projective and \( E_A \) an injective hull of \( M \). Then since \( M \) is local, \( M \) is colocal by (1) and hence \( E \) is colocal. Then again by (1), \( E \) is local. Therefore \( M=E \) since \( M \leq E \) and \( h(M)=h(E) \).

(2)⇒(4). Let \( M_A \) be highest indecomposable and \( \pi: \bigoplus_{i \in I} P_i \to M \) a projective cover of \( M \) with each \( P_i \) indecomposable. Then \( \pi \) canonically yields the epimorphism \( \pi': \bigoplus_{i \in I} P_i/K_i \to M \) where \( K_i:=\ker(\pi|P_i) \). Since \( M \) is highest, \( P_i/K_i \) is highest for some \( i \) in \( I \). Accordingly, \( P_i \) is highest and \( K_i=0 \) by (2). So \( P_i \) is isomorphic to a direct summand of \( M \) since \( P_i \) is injective. Hence \( P_i \cong M \) since \( M \) is indecomposable. As a consequence, \( M \) is projective as well as injective by (2).

(3)⇒(4). Let \( M_A \) be highest indecomposable and \( \sigma: M \to \bigoplus_{i \in I} E_i \) an injective hull of \( M \) with each \( E_i \) indecomposable. Put \( M_i:=\text{Im} \sigma_i \) for all \( i \) in \( I \) where \( \sigma=(\sigma_i)_{i \in I} \). Then \( \sigma \) induces the monomorphism \( \sigma': M \to \bigoplus_{i \in I} M_i \). Since \( M \) is highest, \( M_i \) is highest for some \( i \in I \). And, \( E_i \) is highest. Then by (3), \( E_i \) is projective, in particular, local. Hence \( E_i=M_i \) and \( \rho_i: M \to E_i \) is an epimorphism. Thus \( E_i \) is isomorphic to a direct summand of \( M \) since \( E_i \) is projective. Therefore \( M \cong E_i \) since \( M \) is indecomposable. As a consequence, \( M \) is injective and projective.

The implications (4)⇒(i) are obvious for \( i=1, 2 \) and 3. //

**Proposition 2.4.2.** Let \( A \) be a ring with self-duality satisfying the following condition:

\[(*) \text{ (R)} \quad \text{Every highest indecomposable projective right } A\text{-module is colocal; and} \]

\n
\[(*) \text{ (L)} \quad \text{Every highest indecomposable projective left } A\text{-module is colocal.} \]

Then all the conditions (1)–(4) in (2.4.1) and their left side versions hold.

Proof. \((*) \) implies that every highest local right (left) \( A \)-module is projective and hence colocal. Then by self-duality of \( A \), \((*) \) implies (1) in (2.4.1) and its left side version. //

**Remark.** In case \( A \) is an algebra, (2.4.1) remains valid also under the assumption that every module in consideration is finitely generated. For, the injective hull of every simple right \( A \)-module is finitely generated in this case.

**Theorem 2.5.** Let \( A \) be a ring and \( n \) any natural number. Then the following statements are equivalent:

\[(1) \text{ } A \text{ is of } n\text{-th local type, i.e.} \]

\[(R) \text{ } A \text{ is of right } n\text{-th local type; and} \]
(L) $A$ is of left $n$-th local type.

(2) (R) $\alpha = (\alpha_1, \alpha_2)^T:\mathcal{S} \rightarrow \mathcal{L}_1 \oplus \mathcal{L}_2$ is fusible if $S$ is a simple right $A$-module, $L_i$ are local right $A$-modules and $\alpha_1 S \leq L_1 J$, $\alpha_2 S \leq L_2 J^*$; and

(L) The left side version of (2R).

(3) i) (R) For each $e$ in $\text{pi}(\mathcal{A})$, $eJ^*$ is a uniserial waist in $eA$ if $eJ^* \neq 0$; and

(L) The left side version of (3-iR).

ii) (R) Every monomorphism $S \rightarrow L_2 J^n$ is extendable to a homomorphism $L_1 \rightarrow L_2$ where $L_i$ are local right $A$-modules with $1 < h(L_1) \leq n$ and $S$ is a simple submodule of $Z^q$ and

(L) The left side version of (3-iiR).

(4) (R) Every indecomposable right $A$-module is local if it is of height $> n$; and

(L) The left side version of (4R).

In particular, if $A$ is an algebra, then the above conditions are equivalent to (3-i).

Proof. We show the following implications: (1) $\Rightarrow$ (2) $\Rightarrow$ (3-i), (2, 3-i) $\Rightarrow$ (3-ii), (3) $\Rightarrow$ (2), (2, 3) $\Rightarrow$ (4) $\Rightarrow$ (1) and in case $A$ is an algebra, we show (3-i) $\Rightarrow$ (4). By left-right symmetry, we have only to show (1R) $\Rightarrow$ (2R) $\Rightarrow$ (3-iL), (2R, 3-iiR) $\Rightarrow$ (3-iR), (3-i, 3-iiR) $\Rightarrow$ (2R) and (2R, 3R) $\Rightarrow$ (4R), finally in case $A$ is an algebra, (3-i) $\Rightarrow$ (4R). (Note the implication (4R) $\Rightarrow$ (1R) is trivial.)

(1R) $\Rightarrow$ (2R). By (2.1), top$(\text{Cok} \alpha)$ is decomposable, thus Cok $\alpha$ is decomposable by (1R). Hence $\alpha: S \rightarrow \mathcal{L}_1 \oplus \mathcal{L}_2$ is fusible by [I, Proposition 1.3].

(2R) $\Rightarrow$ (3-iL). It is clear that $J^* e$ is uniserial by the proof of [I, Proposition 2.1]. Suppose $J^* e \neq 0$. Then $J^* e = A \cup S$ for some $0 \neq u_i$ in $f_2 J^* e \setminus f_2 J^{m+1} e$ where $f_i$ is in $\text{pi}(\mathcal{A})$. Let $u_i = f_2 J^* e \setminus f_2 J^{m+1} e$ be any element where $f_i \in \text{pi}(\mathcal{A})$ and $1 \leq n$. Then by (2R), the map $\alpha = (\alpha_1, \alpha_2)^T: eA/ef \rightarrow (f_2 A/ef J^{m+1}) \oplus (f_2 A/ f_2 J^{m+1})$ is fusible where $\alpha_1$ are the left multiplications by $u_i$ since $\alpha_1(eA/ef J^{m+1}) \leq f_2 J^* e \setminus f_2 J^{m+1} e$ and $\alpha_2(eA/ef J^{m+1}) \leq f_2 J^* e \setminus f_2 J^{m+1} e$. In case it is 2-fusible, we have $J^* e = A \cup u_i$ by (2.2) since $J^{m+1} e = J^* e$ by (2.2). If $m < n$, then $A \cup u_i \leq J^* e$ and $u_i \in f_2 J^{m+1} e$, a contradiction. Hence $n \leq m$ and $A \cup u_i \leq J^* e$. As a consequence, we obtain $J^* e \leq A \cup u_i \leq J^* e$. Now let $x X$ be any submodule of $A e$. We show $J^* e \leq X$ if $X \leq J^* e$. Obviously, we may assume $X \leq J^* e$. Suppose $X \leq J^* e$. Then there is an element $x$ in $X \setminus J^* e$. Here we may assume that $x = f x$ for some $f$ in $\text{pi}(\mathcal{A})$ since $f x \in J^* e$ for all $f$ in $\text{pi}(\mathcal{A})$, then $x \in J^* e$, a contradiction. Then by the above, we obtain $J^* e \leq A x$ since $A x \leq J^* e$. Hence $J^* e \leq A x \leq X$. Thus $J^* e$ is a waist in $A e$.

(2R, 3-iiR) $\Rightarrow$ (3-iiR). Let $L_1$ and $L_2$ be local right $A$-modules with $1 < h(L_1) \leq n$, $S$ a simple submodule of $L_1$ and $\alpha_2: S \rightarrow L_2 J^*$ any monomorphism. Note that in this case, $n < h(L_2)$ since $0 \neq \alpha_2 S \leq L_2 J^*$. By (2R), we have that $(\alpha, D) = (\alpha_1, \alpha_2)^T: S \rightarrow L_1 \oplus L_2$ is fusible where $\alpha_1: S \rightarrow L_1$ is the inclusion map.
Assume that \((\alpha, D)\) is 1-fusible. Then noting that \(L_2\) is colocal, \(L_2\) is embedded into \(L_1\). But this is impossible since \(h(L_2) \leq n < h(L_3)\). Hence \((\alpha, D)\) is 2-fusible. Thus \(\alpha_2\) is extendable to a homomorphism \(L \to L_2\).

\((3-i, 3-iiR) \Rightarrow (2R)\). We may assume that \(S = eA|e|, L_1 = fA|X, L_2 = gA|gJ^{r+1}\) (by 3-iR) for some \(e, f, g\) in \(\text{pi}(A)\), \(n \leq r\) and \(X \leq fA\), further \(\alpha_1\) and \(\alpha_2\) are nonzero maps given by left multiplications by some \(u \in fJ^s\backslash fJ^{s+1}e = \phi\) (for \(1 \leq m\)) and \(v \in gJ^t\backslash gJ^{t+1} = \phi\), respectively.

In case \(m < n\). We show \(h(L_i) \leq n\). Suppose \(h := h(L_i) > n\). Then \(0 \neq fJ^{h-1} \subseteq fJ^s\) and \(fJ^{h-1} \subseteq X\), hence \(X \subseteq fJ^{s+1}\), that is \(X = fJ^s\) for some \(s \geq h\). Hence \(X = fJ^h\) since \(h(fA|X) = h\). Therefore \(u \in fJ^{h-1} \subseteq fJ^s\) since \(0 \neq u, S\) is simple. Thus \(m = h - 1 \geq n\), a contradiction. (Note that \(m\) is uniquely determined by \(u\).) As a consequence, \(\alpha : S \to L_1 \oplus L_2\) is 2-fusible by (3-iiR).

In case \(n \leq m\). \(Av = J^v\) is a uniserial waist in \(Ae\) by (3-iL). Hence \(Au \leq Av\) or \(Av \leq Au\). Note since \(n \leq m\), it holds that \(Au = J^{s*}\) and \(uA = fJ^s\) by (3-i). Then \(u \in X \geq uJ\) implies that \(fJ^{s+1} \subseteq X\) and \(fJ^s \subseteq X\). Hence \(X = fJ^{s+1}\) s\(h(fA|X) = h\). Therefore \(u \in fJ^{h-1} \subseteq fJ^s\) since \(0 \neq u, S\) is simple. Thus \(m = h - 1 \geq n\), a contradiction. (Note that \(m\) is uniquely determined by \(u\).)

As a consequence, \(\alpha : S \to L_1 \oplus L_2\) is 2-fusible by (3-iiR).

(2R, 3R) \Rightarrow (4R). By (2.3), (2R)' in (2.3) holds. Let \(M\) be any right \(A\)-module of height \(n\). We show that \(M\) is decomposed into local right \(A\)-modules of height \(>n\) and indecomposable right \(A\)-modules of height \(\leq n\) by induction on \(m := |\text{top } M|\).

We may assume that \(2 \leq m\) since it is obvious in case \(m = 1\). Let \(M = \bigoplus_{i=1}^m L_i\) be an irredundant sum of local modules \(L_i\). By the hypothesis of induction, we have \(M = L_1 + \bigoplus_{i=2}^m L_i\) for some \(r \leq m\) such that \(M_i\) is a local module of height \(> n\) or an indecomposable module of height \(\leq n\). We may assume that \(h(L_i) \leq h(M_i)\) for each \(i = 2, \cdots, m\). Put \(T := L_1 \cap \bigoplus_{i=2}^m L_i\). Again, we may assume \(T \neq 0\). Putting \(\alpha_i : T \to L_1\) and \(\theta : T \to \bigoplus_{i=2}^m L_i\) the inclusion maps, \(\alpha_j := \pi_j \theta\) where \(\pi_j : \bigoplus_{i=2}^m L_i \to L_j\) is the canonical projection for each \(j = 2, \cdots, r\) and \(\alpha := (\alpha_i)_{i=1}^r\), we have an exact sequence:

\[
(E) \quad 0 \to T \xrightarrow{\alpha} L_1 + \bigoplus_{i=2}^m L_i \xrightarrow{\theta} M \to 0.
\]

By the hypothesis of induction, we have only to show that \(M\) is decomposable. To this end, it is sufficient to show that \(\alpha : T \to L_1 \oplus M_2 \oplus \cdots \oplus M_r\) is fusible. Note that \(\alpha_i\) is monic. Since \(n < h(M_i)\), we have \(n < h(M_i)\) for some \(i = 2, \cdots, r\),
say \( i=r \). Then \( M_r \) is local and colocal by (3-i). Further since the sum \( M = \sum_{i=1}^{r} L_i \) is irredundant, we have \( T \leq L_i J \) and hence \( \pi_i T \leq M_i J \). Accordingly, \((\alpha_1, \alpha_2)^T : T \to L_i \oplus M \) is 2-fusible by (2R)'. Thus there is a homomorphism \( \gamma : L_i \to M \) such that \( \alpha_r = \gamma \alpha_i \). Hence \( \alpha : T \to L_i \oplus M_2 \oplus \cdots \oplus M_r \) is \( r \)-fusible. In fact, putting \( \delta :\alpha = (\gamma, 0, 0, \cdots, 0) : L_i \oplus M_2 \oplus \cdots \oplus M_{r-1} \to M_r \), we have \( \delta(\alpha_i)_{i=r} = \gamma \alpha_i = \alpha_r \).

(3-i) \( \Rightarrow \) (4R) in case \( A \) is an algebra. We may assume that \( n < h(A_A) \). Let \( M \) be any indecomposable right \( A \)-module of height \( h > n \). Note that \( A/J^h \) satisfies (*) in (2.4.2) and has selfduality. Then applying (2.4.2) to the ring \( A/J^h \), we obtain that \( M^h \) is projective, that is, \( M_A \cong eA/eJ^h \) for some \( e \) in \( \pi(A) \). Thus \( M \) is local. //

**Remark.** Theorem 2.5 is a generalization of Nakayama [4, Theorem 17].

3. **Further necessary conditions**

3.1. Throughout this section, the base field \( k \) is algebraically closed when \( A \) is assumed to be an algebra. Here, we investigate further necessary conditions for an algebra \( A \) to be of right \( 2 \)-local type and determine all the “shapes’’ of indecomposable projective right \( A \)-modules.

**Lemma 3.1.1.** Let \( A \) be an artinian ring, \( C \) and \( L \) be right \( A \)-submodules of a right \( A \)-module \( M \) and let \( C \leq M \) for a natural number \( h \). Then \( (C+L)/L \leq (M/L)J^h \). In particular, \( h(M) = h(M/L) \) iff \( M \cong M/J^h \).

Proof. Clear. //

**Lemma 3.1.2.** Let \( A \) be an artinian ring of right \( 2 \)-local type, \( L_1, L_2 \) be local right \( A \)-modules and \( S \) a simple right \( A \)-module. Then any monomorphism \( \alpha=(\alpha_1, \alpha_2)^T : S \to M \) is fusible if \( \alpha_1 S \leq L_1 J \) and \( \alpha_2 S \leq L_2 J \).

Proof. Clear from the implication (1R) \( \Rightarrow \) (2R) in Theorem 2.5. //

**Remark.** In the above, if further \( \alpha_1 S \leq L_1 J \) holds, then the conclusion remains valid under a weaker assumption that the \( L_i \) are indecomposable by [1, Proposition 2.5.a].

**Lemma 3.1.3.** Let \( A \) be an algebra of right \( 2 \)-local type, \( M \) a quasi-projective local right \( A \)-module and \( L_1 \) and \( L_2 \) be simple right \( A \)-submodules of \( M \) such that \( 3 \leq h(M/L_2) = h(M/L_1) \) \( (=k) \). If there exist simple right \( A \)-modules \( S_1 \leq (M/L_1)J_{k-1} \) such that \( S_1 \cong S_2 \), then we have \( L_1 = L_2 \).

Proof. Let \( S \) be a simple right \( A \)-module and \( \alpha_i : S \to S_i \) be isomorphisms. Then by (3.1.2), \( (\alpha_1, \alpha_2)^T : S \to M/L_1 \oplus M/L_2 \) is fusible, say 2-fusible. Thus
we have a commutative diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\alpha_1} & M/L_1 \\
\downarrow & & \downarrow \beta \\
S & \xrightarrow{\alpha_2} & M/L_2
\end{array}
\]

for some homomorphism \( \beta: M/L_2 \to M/L_2 \). \( S \subseteq \text{Ker} \beta \) yields \( h(\text{Coim} \ \beta)=h \) by (3.1.1). Then \( \beta \) is an epimorphism since \( M/L_2 \) is local of height \( h \). Accordingly, \( \beta \) is an isomorphism since \( |M/L_1|=|M/L_2| \). Quasi-projectivity of \( M \) implies that \( \beta \) is liftable to an automorphism \( \gamma \) of \( M \). And, the projective cover of \( M \) is of the form \( eA \) for some \( e \) in \( \pi_1(A) \) since \( M \) is local. Therefore by projectivity of \( eA, \gamma \) is liftable to an automorphism \( \delta \) of \( eA \) which is a left multiplication by an element \( t \) in \( eAe/eJe \). Then the restriction isomorphism \( \gamma': L_1 \to L_2 \) of \( \gamma \) is given by the left multiplication by \( t:=t+eJe \) in \( eAe/eJe \). But since we assume that the base field \( k \) is algebraically closed, \( \gamma' \) is also given by a right multiplication by an element \( t' \) in \( k \). As a consequence, \( L_2=L_1t' =L_1. \) //

**Proposition 3.1.4.** Let \( A \) be an algebra of right 2nd local type. Then for every quasi-projective local right \( A \)-module \( M \) of height \( \geq 3 \), we have \( |\text{soc} \ M| \leq 2 \). Further if \( e \) is in \( \pi_1(A) \), \( 3 \leq t \) and \( eJ^{t-1}eJ^t=S_1 \oplus S_2 \) with each \( S_i \) simple, then \( S_1 \approx S_2 \).

**Proof.** Put \( h:=h(M) \geq 3. \) Then there exists a simple right \( A \)-module \( S \leq M/J^{h-1}. \) Assume that \( |\text{soc} \ M| \geq 3 \). Then there are simple right \( A \)-modules \( S_1 \) and \( S_2 \) such that \( S \oplus S_1 \oplus S_2 \) is a direct summand of \( \text{soc} \ M. \) Put \( M_i:=M/S_i \) for each \( i \). Then \( h(M_1)=h(M_2)=h(M) \geq 3 \) by (3.1.1). Also, \( M_1J^{h-1}=(S \oplus S_1)/S_1 \approx (S \oplus S_2)/S_2 \leq M_2J^{h-1}. \) Hence by (3.1.3), we have \( S_1=S_2 \), a contradiction. Therefore we must have \( |\text{soc} \ M| \leq 2. \) Next, suppose \( eJ^{t-1}eJ^t=S_1 \oplus S_2 \) with each \( S_i \) simple but \( S_1 \not\approx S_2 \). Then putting \( M_i:=M/S_i \) for each \( i=1,2 \) where \( M:=eA/eJ^t \) is quasi-projective local, similarly we have \( h(M_1)=h(M_2) =t \geq 3 \) and \( M_1J^{t-1}=(S_1 \oplus S_2)/S_1 \approx (S_1 \oplus S_2)/S_2 \leq M_2J^{t-1}. \) Then \( S_1=S_2 \) by (3.1.3), a contradiction. //

**Corollary 3.1.5.** Let \( A \) be an algebra of right 2nd local type and \( e \) in \( \pi_1(A) \). Then \( eJ^2 \) is a direct sum of at most two uniserial modules.

**Proof.** Clear from (1.1; 6) and (3.1.4).

**Remark.** [I, Example 2] was a counter example of sufficiency of the necessary conditions stated in (1.1) for \( A \) to be of right 2nd local type. This example does not satisfy the condition stated in (3.1.4).

3.2 In case \( eJ^2 \) is uniserial.
Proposition 3.2.1. Let \( A \) be an algebra of right 2nd local type, \( e \) in \( \text{pi}(A) \). Suppose \( ef^2 \) is uniserial and \( |\text{soc}(eA)| = 1 \). Then \( eA \) is uniserial if \( h(eA) \geq 4 \).

Proof. Let \( D := \text{Hom}_k(?, k) \) be the usual self-duality of \( A \). Then since \( ef \) is colocal and of height \( \geq 3 \), \( D(ef) \) is local and of height \( \geq 3 \). Hence \( D(ef) \) is colocal by (1.1), thus \( ef \) is local. Therefore \( eA \) is uniserial since \( ef^2 \) is uniserial. \( \square \)

Lemma 3.2.2. Let \( A \) be a ring, \( M \) a module and \( S \) a submodule of \( M \). Then \( S \) is a semisimple direct summand of \( M \) iff \( MJ \cap S = 0 \). In particular, \( \text{soc} M \leq MJ \) iff \( M \) has no simple direct summand.

Proof. \( (\Rightarrow) \). If \( M = S \oplus X \) for some \( X \leq M \) with \( S \) semisimple, then \( MJ = XJ \) and \( S \cap MJ = S \cap XJ = 0 \).

\( (\Leftarrow) \). Suppose that \( MJ \cap S = 0 \). Then \( S = (S + MJ)/MJ \) and \( M/MJ = (S + MJ)/MJ \oplus X/MJ \) for some \( X \leq M \) with \( MJ \leq X \leq M \). Therefore \( S \) is semisimple and \( M = S + MJ + X = S + X \) since \( MJ \) is small in \( M \). Further \( (S + MJ) \cap X \leq MJ \) implies that \( S \cap X \leq MJ \) and \( S \cap X = S \cap (S \cap X) \leq S \cap MJ = 0 \). Thus \( M = S \oplus X \). \( \square \)

Proposition 3.2.3. Let \( A \) be an algebra of right 2nd local type, \( e \) in \( \text{pi}(A) \). Suppose that \( ef^2 \) is uniserial and \( |\text{soc}(eA)| = 2 \). Then \( ef = X \oplus Y \) for some \( X \leq ef \) such that \( X \) is simple; and \( Y \) is a uniserial module (in case \( h(eA) \geq 4 \)) or a colocal module of height 2 (in case \( h(eA) = 3 \)).

Proof. By assumption, we have \( \text{soc}(ef) = \text{soc}(eA) \leq ef^2 \). Hence by (3.2.2), it follows from \( |\text{soc}(ef)| = 2 \) that \( ef = X \oplus Y \) with \( X \) simple and \( Y \) colocal.

In case \( h(eA) = 3 \), \( y(Y) = h(ef) = 1 \) since \( X \) is simple. Thus \( Y \) is colocal and of height 2.

In case \( h(eA) \geq 4 \). Since \( X \) is simple, \( ef^2 = YJ \). Thus \( YJ \) is uniserial. Further \( h(Y) \geq 3 \) since \( 0 \neq eJ^2 = YJ^2 \). Hence \( Y \) is local by the same argument as in the proof of (3.2.1). Therefore \( Y \) is uniserial. \( \square \)

3.3. In case \( ef^2 \) is a direct sum of two uniserial modules.

Lemma 3.3.1. Let \( A \) be a ring, \( M \) a right \( A \)-module and \( L \) a right \( A \)-submodule of \( M \). Then the following statements are equivalent:

1. \( Lf = L \cap MJ \).
2. \( M = L + X \) and \( |\text{top} M| = |\text{top} L| + |\text{top} X| \) for some \( X \leq M \).
3. Every sum \( L = \sum L_i \) with each \( L_i \) local and \( m = |\text{top} L| \) can be extended to a sum \( M = \sum L_i + \sum X_i \) with each \( X_i \) local and \( m + n = |\text{top} M| \).

In particular, if both \( L \) and \( M \) are modules of height 2 and without simple
direct summands, then all the conditions (1)–(3) hold.

Proof. Let \( \pi: M \to \text{top } M \) be the canonical projection.

(1) \(\Rightarrow\) (2). We have \( \text{top } M = \pi(L) \oplus \pi(X') \) for some \( X' \subseteq M \). Let \( \pi: P \to \pi(X') \) be a projective cover of \( \pi(X') \). Then \( \pi = \pi q \) for a homomorphism \( q: P \to X' \). Put \( X := \text{Im } q \). Then \( M = L + X' = L + X + MJ = L + X \).

Also, \( |\pi(X')| = |\text{top } P| = |\text{top } X| \) for \( \text{Ker } q \leq \text{Ker } p = PJ \). Further \( |\pi(L)| = |\text{top } L| \) by (1). Hence \( |\text{top } M| = |\text{top } L| + |\text{top } X| \).

(2) \(\Rightarrow\) (1). Noting that \( YJ \subseteq Y \cap MJ \) for every \( Y \subseteq M \), we have \( |\text{top } M| = |\pi(L) + \pi(X)| = |\pi(L)| + |\pi(X)| \leq |\text{top } L| + |\text{top } X| = |\text{top } M| \). Thus \( |\pi(L)| = |\text{top } L| \) i.e. \( |\pi L| = |L \cap MJ| \). Hence \( LJ = L \cap MJ \).

If both \( L \) and \( M \) are modules of height 2 and without simple direct summands, then \( LJ = \text{soc } L = L \cap \text{soc } M = L \cap MJ \) by (3.2.2). Hence (1)–(3) hold. //

Proposition 3.3.2. Let \( A \) be an algebra of right 2nd local type and \( e \) in \( \pi(\text{soc } A) \). If \( ej^2 \) is a direct sum of two uniserial right \( A \)-modules, then so is \( ej \).

Proof. Assume that the hypothesis of the proposition is satisfied and put \( P := eA/ej^3 \). Then \( P \) is a quasi-projective local right \( A \)-module of height 3. By (3.1.4) and the assumption that \( ej^2 \) is a direct sum of two uniserial modules, we have \( 2 = |PJ^2| \leq |\text{soc } PJ| = |\text{soc } P| \leq 2 \). Hence \( \text{soc } PJ = PJ^2 \) i.e. \( PJ \) has no simple direct summand by (3.2.2). In particular, if \( PJ = \sum_{i=1}^n L_i \) is an irredundant sum of local modules \( L_i \), then \( h(L_i) = 2 \) for all \( i \) and every partial sum \( \sum_{i \in I} L_i \) with \( I \subseteq \{1, \cdots, n\} \) has no simple direct summand. We claim the following:

(a) Every colocal submodule of \( PJ \) is uniserial.

(b) Let \( PJ = \sum_{i=1}^n L_i \) be any irredundant sum of local modules \( L_i \). If \( L_i \) and \( L_j \) are colocal for some \( i \neq j \) in \( \{1, \cdots, n\} \), then \( \text{soc } L_i \neq \text{soc } L_j \).

Proof of (a). Let \( L \) be a colocal submodule of \( PJ \). If \( L \) is simple, then the assertion is trivial. So we may assume that \( h(L) = 2 \). Put \( S := \text{soc } L \) \( (=LJ) \). Then \( PJ^2 \geq LJ = S \) and \( PJ^2 = S \oplus T \) for some simple module \( T \) by assumption. Also, \( S \cong T \) by (3.1.4). This implies that \( P/S \) is quasi-projective. Hence \( |\text{soc } P/S| \leq 2 \) by (3.1.4). It follows from \( L \cap PJ^2 = S \) that \( 2 \geq |\text{soc } P/S| \geq |(L/S) \oplus (PJ^2/S)| = |L/S| + 1 \). Hence \( L/S \) is simple i.e. \( L \) is uniserial.

Proof of (b). Suppose that \( \text{soc } L_i = \text{soc } L_j = S \). Then since \( L_i + L_j \) has no simple direct summand, \( \text{soc } (L_i + L_j)/(L_i + L_j)J = L_iJ + L_jJ = S \) is simple. Hence \( L_i + L_j \) is uniserial by (a). Thus \( L_i = L_j \).

Now we come back to the proof of the proposition. We have \( \text{soc } P = \text{soc }
We next show that \( P/J \) is a direct sum of two uniserial modules. Otherwise, by (b), \( P/J \) has a local submodule \( L \) of height 2 which is not colocal i.e. \( \text{soc } L = S \oplus T \). Define injections \( \alpha_1: S \to L/T \) and \( \alpha_2: S \to P \) in the obvious way. Then by (3.1.2), \( (\alpha, D) = (\alpha_1, \alpha_2) : S \to (L/T) \oplus P \) is fusible. If it is 1-fusible, then \( \varphi \alpha_2 = \alpha_1 \) for some \( \varphi: P \to L/T \) and \( 2 = h(L/T) \geq h(\text{Im } \varphi) = h(\text{Coi} \text{m } \varphi) = 3 \) by (3.1.1). This contradiction shows that \( (\alpha, D) \) is 2-fusible i.e. \( \varphi \alpha_1 = \alpha_2 \) for some \( \varphi: L/T \to P/J \) (note that \( h(L/T) = 2 \) and \( \text{soc}^2 P = P/J \) where \( \varphi \) is monic since \( \varphi \) does not vanish the simple socle of \( L/T \). Accordingly, \( P/J \) has a uniserial submodule \( L_1 \) of height 2 and with \( \text{soc } L_1 = S \). Similarly, \( P/J \) has a uniserial submodule \( L_2 \) of height 2 and with \( \text{soc } L_2 = T \). By (3.3.1), there is an irredundant sum \( P/J = \sum_{i=1}^{n} L_i \) of local modules \( L_i \). \( M \leq L_1 \oplus L_2 \) implies that \( 3 \leq n \) and \( \text{soc } L_i = S \oplus T \) for all \( i \geq 3 \) by (b). Applying the same argument as above to \( L := L_2 \), we have \( \varphi \alpha_1 = \alpha_2 \) for some \( \varphi: L/T \to P/J \). Put \( N := \varphi(L/T) \). If \( N \leq L \), then \( N = L_1 \) is not colocal by (a) and then \( \text{soc } L_1 \leq \text{soc } (N + L_1) \). Hence by (1.3; 2), there is a map \( \eta: N \to L_1 \) such that \( \eta \alpha_2 = \theta_1 \) where \( \theta_2: S \to L_1 \) is the inclusion map. If \( N = L_1 \), putting \( \eta = 1_{L_1} \) we also have \( \eta \alpha_2 = \theta_2 \). Let \( \pi: L \to L/T \) be the canonical projection, \( \theta_1: S \to L \) the inclusion map and put \( \lambda := \eta \varphi \pi \). Then we have a commutative diagram:

\[
\begin{array}{c}
L \xrightarrow{\pi} L/T \xrightarrow{\varphi} N \xrightarrow{\eta} L_1 \\
\theta_1 \downarrow \quad \alpha_1 \downarrow \quad \alpha_2 \downarrow \quad \theta_2 \\
S \quad S \quad S \quad S.
\end{array}
\]

Hence \( \lambda \theta_1 = \theta_2 \) and \( L_2 + L_1 = L_2 \oplus L_1 \) where \( L_2 := (L_2 - \lambda)(L_2) \approx L/S \). Also, \( \text{soc } L_2 = T \) and \( h(L_2) = 2 \). Therefore \( P/J = (L_1 + L_3) + L_2 + \cdots + L_n = (L_1 + L_2) + L_2 + \cdots + L_n \). This contradicts (b). As a consequence, \( P/J = L_1 \oplus L_2 \). Now we have that \( eJ \) is decomposable since \( \text{top}(eJ) = eJ \) is decomposable. Also, \( |\text{soc } (eJ)| = 2 \) since \( 2 = |\text{soc } (eJ^2)| \leq |\text{soc } (eJ)| \leq |\text{soc } (eA)| \leq 2 \). Hence \( eJ = X \oplus Y \) with both \( X \) and \( Y \) colocal. Then both \( XJ \) and \( YJ \) are uniserial since \( eJ^2 = XJ \oplus YJ \). Further \( 2 = |\text{top } P/J| = |eJ| = |eJ^2| = |\text{top } X \oplus \text{top } Y| \) yields that both \( X \) and \( Y \) are local thus uniserial. 

Summarizing the above propositions and (1.1), we obtain the following

**Theorem 3.4.** Let \( A \) be an algebra (over an algebraically closed field) which is of right 2nd local type and let \( e \) be in \( \text{pi}(A) \). Then

(L) (1) \( J^2 e \) is a uniserial waist in \( Ae \) if \( J^2 e \neq 0 \).

(2) \( Ae \) is uniserial if \( h(Ae) \geq 4 \).

(3) Therefore the structure of \( Ae \) is one of the following:
(Lₐ) $Ae$ is uniserial.
(Lₛ) $h(Ae)=2$ and $Ae$ is not uniserial.
(L₅) $h(Ae)=3$ and $Ae$ is colocal but not uniserial.

(R) (1) $eJ^2$ is a direct sum of at most two uniserial right $A$-modules.
(2) $|\text{soc } L| \leq 2$ for each quasi-projective local right $A$-module $L$ of height $\geq 3$.

(3) If $eJ^{t-1}/eJ^t=S_1 \oplus S_2$ with each $S_i$ a simple right $A$-module, then $S_i \cong S_2$ for each $t \geq 3$.

(4) The structure of $eA$ is one of the following:

(R₁) $eA$ is uniserial.
(R₂) $h(eA)=2$, $eA$ is not uniserial and $|\text{soc } (eA)| \geq 3$.
(R₃) $h(eA)=3$, $eA$ is colocal but not uniserial.
(R₄) $h(eA)=3$, $eJ=X \oplus Y$ where $X_A$ is simple and $Y_A$ is a colocal but not uniserial module of height 2.
(R₅) $eJ$ is a direct sum of two (nonzero) uniserial right $A$-modules.

REMARK. (1) All these types actually appear in examples (see Example 1 in section 5).
(2) In general, the conditions in (3.4) are not sufficient for algebras to be of right 2nd local type (see Example 4 in section 5).

4. Left serial algebras of right 2nd local type

Throughout this section, our ring $A$ is a left serial ring and the base field $k$ is algebraically closed when $A$ is considered as an algebra. We know by (1.1) that most of algebras of right 2nd local type is left serial. So in this section, we examine the left serial case and show that in this case the necessary conditions obtained in section 3 are sufficient for an algebra $A$ to be of right 2nd local type modifying the proof of Sumioka [6, Proposition 3.8]. Namely, we show:

Theorem 4.1. Let $A$ be a left serial algebra over an algebraically closed field $k$. Then the following statements are equivalent:

(1) $A$ is of right 2nd local type.
(2) $eJ$ is a direct sum of at most two uniserial modules if $h(eA) \geq 3$ for each $e$ in $\text{pi}(A)$.
(3) Every indecomposable right $A$-module is local if it is of height $\geq 3$.

REMARK. This theorem is similar to [8, Proposition 4.4].

4.2. We quote the following definitions and propositions concerning a left serial ring $A$ from Sumioka [6].

Let $A$ be a left serial ring, $L$ a uniserial left $A$-module of length $n$ and
put $L_i := \text{soc} L$ and $D_i(L) := \text{End}_A(\text{top } L_i)$ for each $i=1, \ldots, n$. Then $D_i(L)$ are division rings. For $n \geq i \geq j \geq 1$, any element $\varphi_i$ in $D_i(L)$ is induced by an endomorphism $\varphi_i$ of $L_i$ since $L_i$ is quasi-projective, and $\varphi_i$ induces an element $\varphi_j$ in $D_j(L)$. We define a map $\lambda_{ij} : D_j(L) \to D_i(L)$ by $(\varphi_j) \lambda_{ij} \varphi_i$. Then as easily seen, $\lambda_{ij}$ are well-defined and ring monomorphisms with $\lambda_{ij} \lambda_{ji} = \lambda_{ii}$ for all $i$, $j$ and $l$ with $n \geq i \geq j \geq l \geq 1$. Hence by the maps $\lambda_{ij}$, we can regard the sequence $D_1(L), D_2(L), \ldots, D_n(L)$ as a descending chain of division rings.

**Lemma 4.2.1 ([6, Lemma 3.1])**. Let $A$ be a left serial ring. Then the following conditions are equivalent for a uniserial module $L$ of length $n$ and a natural number $r \leq n$:

1. $D_i(L) = D_r(L)$.
2. Every automorphism $\alpha$ of $\text{soc } L$ is extendable to an automorphism of $L$ if $\alpha$ is extendable to an automorphism of $\text{soc } L$.

Let $A$ be a left serial ring, $S$ a simple left $A$-module and $L$ a uniserial left $A$-module of length $\geq 2$. We denote by $c(S)$ the number of isomorphism classes of uniserial left $A$-modules of length $2$ whose socles are isomorphic to $S$ and put $m(L) := \dim D_i(L)_{D_2(L)}$. Then we have

**Lemma 4.2.2 ([6, Lemma 3.3])**. Let $A$ be a left serial ring and $e$ in $\pi(A)$. Then $|eJ|eJ^2| \leq 2$ iff $c(\text{soc } L) + m(L) \leq 3$ for every uniserial left $A$-module $L$ of length $\geq 2$ and with $\text{soc } L \cong A(eJJ)e$.

**Lemma 4.2.3 ([6, Lemma 4.3])**. Let $A$ be a left serial ring, $e$ in $\pi(A)$ and $r \geq 1$. Then the following conditions are equivalent:

1. For any uniserial modules $L_1$ and $L_2$ such that $\text{soc } L_1 \cong A(eJ)e$ and $r \leq |L_1| \leq |L_2|$, any isomorphism $\alpha : \text{soc } L_1 \to \text{soc } L_2$ is extendable to a homomorphism $L_1 \to L_2$ if $\alpha$ is extendable to a homomorphism $\text{soc } L_1 \to \text{soc } L_2$.
2. $eJ^{r-1}$ is a direct sum of uniserial modules.

**Definition.** Let $A$ be a left serial ring. Then we say that a simple left $A$-module $S$ is of $V$-type in case $S \cong A(eJ)e$ for some $e$ in $\pi(A)$ such that $eJ$ is a direct sum of at most two uniserial right $A$-modules.

**Lemma 4.3.** Let $A$ be a left serial algebra which satisfies the condition (2) in (4.1) and $S$ a simple left $A$-module. If $S \cong \text{soc } L$ for some left $A$-module $L$ of height $\geq 3$, then $S$ is of $V$-type.

Proof. Let $S \cong A(eJ)e$ where $e$ is in $\pi(A)$. By the selfduality $D := \text{Hom}_k (\cdot, \cdot)$, we have $eA \cong D(E(A(eJ)e)) \cong D(E(L))$ where $E(\cdot)$ denotes the injective hull of $(-)$. This implies that $h(eA) \geq h(L) \geq 3$. Hence the assertion follows from the condition (2) in (4.1).

**Lemma 4.4.** Let $A$ be a left serial ring and let $L_1$ and $L_2$ be uniserial left
\[A\text{-modules such that } 2 \leq |L_1| \leq |L_2| \text{ and soc } L_1 \text{ is of V-type. Then every isomorphism } \alpha : \text{soc } L_1 \rightarrow \text{soc } L_2 \text{ is extendable to a homomorphism } L_1 \rightarrow L_2 \text{ if } \alpha \text{ is extendable to a homomorphism } \text{soc}^2 L_1 \rightarrow \text{soc}^2 L_2.\]

Proof. Clear from (4.2.3). //

Let \( A \) be a left serial algebra. Then since we assume that the base field \( k \) is algebraically closed, we have \( D_1(L)=D_2(L)=\cdots=D_n(L)(=k) \) and hence \( m(L)=1 \) for every uniserial \( A \)-module \( L \). Following the terminology of [7] or [6], all simple \( A \)-modules are of first kind in this case. This is equivalent to say that \( ej/ej^2 \) is (zero or) square-free (i.e. a direct sum of pairwise nonisomorphic simple right \( A \)-modules) for each \( e \) in \( \text{pi}(A) \).

**Lemma 4.5** (Cf. [6, Lemma 3.4]). Let \( A \) be a left serial algebra and let \( L_1 \) and \( L_2 \) be uniserial modules such that \( 2 \leq |L_1| \leq |L_2| \) and \( S:=\text{soc } L_1 \bowtie \text{soc } L_2 \) is of V-type. If \( \text{soc}^2 L_1 \bowtie \text{soc}^2 L_2 \), then any isomorphism \( \alpha : \text{soc } L_1 \rightarrow \text{soc } L_2 \) is extendable to a monomorphism \( L_1 \rightarrow L_2 \).

Proof. Clear from (4.2.1) and (4.4). //

**Lemma 4.6.** Let \( A \) be a left serial ring and \( L_1 \) a uniserial left \( A \)-module of length \( \geq 2 \) and \( \alpha_i : S \rightarrow L_1 \) a homomorphism for each \( i=1, \ldots, n \) where \( S \) is a simple left \( A \)-module of V-type and \( n \geq 3 \). If \( 0 \rightarrow S \rightarrow \bigoplus_{i=1}^n L_i \rightarrow M \rightarrow 0 \) is an exact sequence with \( \alpha=(\alpha_i)_{i=1}^n \), then \( M \) is decomposable.

Proof. This follows from (4.4) and the proof of [6, Lemma 3.5]. //

**Lemma 4.7.** Let \( A \) be a left serial algebra satisfying the condition (2) in (4.1) and let \( M \) be a left \( A \)-module. Then \( M \) is indecomposable with \( |\text{top } M| = 2 \) iff \( M=L_1+L_2 \) for some uniserial left \( A \)-modules \( L_1 \) with \( 2 \leq |L_1| \leq |L_2| \) such that \( S:=L_1 \cap L_2 \) is simple and the identity map \( 1_S \) of \( S \) is not extendable to any homomorphism \( \text{soc}^2 L_1 \rightarrow \text{soc}^2 L_2 \). Moreover in this case \( M \) is colocal.

Proof. \((\Leftarrow)\) and \( S=\text{soc } M \) follow immediately from (1.3).

\((\Rightarrow)\). It is clear that \( h(M) \geq 2 \). If \( h(M)=2 \), the assertion is obvious. Therefore we may assume that \( h(M) \geq 3 \). Then \( M=L_1+L_2 \) for some uniserial modules \( L_1 \) and \( L_2 \) such that \( L_1 \cap L_2 \neq 0 \) and \( 2 \leq |L_1| \leq |L_2| \geq 3 \). Thus \( (\text{soc } L_1 \bowtie) \text{soc } L_2 \) is of V-type by (4.3). By (4.4), the rest of the proof is quite similar to that of [6, Proposition 3.6]. //

**Corollary 4.7.1.** Let \( A \) be a left serial ring and \( M \) a colocal left \( A \)-module such that \( \text{soc } M \) is of V-type and \( |\text{top } M| = 2 \). Then \( M=L_1+L_2 \) for some uniserial left \( A \)-modules \( L_1 \) with \( 2 \leq |L_1| \leq |L_2| \) such that \( S:=\text{soc } M=L_1 \cap L_2 \) and \( 1_S \) is not extendable to any homomorphism \( \text{soc}^2 L_1 \rightarrow \text{soc}^2 L_2 \). //
Remark. In the above, we have \( h(M)=h(L_2)=|L_2| \). So the number \( s(M):=\min \{|L_1|, |L_2|\}=|M|-h(M)+1 \) is uniquely determined by \( M \). Further we define \( s(L):=|L| \) for every uniserial left \( A \)-module \( L \).

**Lemma 4.8.** Let \( A \) be a left serial algebra and let \( L \) be a uniserial left \( A \)-module of length \( \geq 2 \) and \( M \) a colocal left \( A \)-module such that \( \text{soc} \ M \) is of \( V \)-type and \( |\text{top} \ M|=2 \). If \( |L|\leq s(M) \), then any isomorphism \( \alpha: \text{soc} \ L\rightarrow \text{soc} \ M \) is extendable to a homomorphism \( L\rightarrow M \).

Proof. This is a simple modification of the proof of the case (i) of [6, Lemma 3.7]. Put \( S:=\text{soc} \ M \). By Corollary 4.7.1, \( M=L_1+L_2 \) for some uniserial left \( A \)-modules \( L_i \) with \( 2\leq |L_1|\leq |L_2| \) such that \( L_1\cap L_2=S \) and \( 1_S \) is not extendable to any homomorphism \( \text{soc}^2 L_1\rightarrow \text{soc}^2 L_2 \). \( |L|\leq s(M) \) yields \( |L|\leq |L_1|\leq |L_2| \). Since \( 1_S \) is not extendable to any homomorphism \( L_1\rightarrow L_2 \), we have \( \text{soc}^2 L_i\cong \text{soc}^2 L_2 \) by (4.5) and the fact that \( S \) is of \( V \)-type. By (4.2.2), \( c(S)\leq 2 \) since \( m(S)=1 \). Accordingly, \( \text{soc}^2 \L \cong \text{soc}^2 L_1 \) or \( \text{soc}^2 \L \cong \text{soc}^2 L_2 \). Hence by (4.5), we infer that \( \alpha \) is extendable to a homomorphism \( L\rightarrow L_i \) for some \( i=1, 2 \) and thus to a homomorphism \( L\rightarrow M \). \( \Box \)

**Proposition 4.9.** Let \( A \) be a left serial ring, \( S \) a simple left \( A \)-module of \( V \)-type and \( L_i \) colocal left \( A \)-modules with \( |\text{top} \ L_i|=2 \) for all \( i=1, \ldots, n \), in particular let \( L_i \) be uniserial and \( |L_i|\leq s(L_i) \) for all \( i=2, \ldots, n \). Assume that a sequence

\[
0 \rightarrow S \xrightarrow{\alpha} \bigoplus_{i=1}^n L_i \xrightarrow{\beta} M \rightarrow 0
\]

is exact. Then \( M \) is decomposable if \( |\text{top} \bigoplus_{i=1}^n L_i|\geq 3 \).

Proof. Put \( \alpha:=(\alpha_i)_\mathbb{1} \). Then we may assume that \( \alpha_i\neq 0 \) for all \( i \). If \( |\text{top} \ L_j|=2 \) for some \( j=2, \ldots, n \), then by (4.8), \( (\alpha_i, \alpha_j): S\rightarrow L_i\oplus L_j \) is \( 2 \)-fusible. Thus \( \alpha: S\rightarrow \bigoplus_{i=1}^n L_i \) is \( j \)-fusible by the same argument as in the proof of the implication \((2R, 3R)\Rightarrow(4R)\) in Theorem 2.5. Therefore \( M \) is decomposable. So we may assume that \( L_i \) are uniserial for all \( i=1, \ldots, n \). Then by (4.6), \( M \) is decomposable. \( \Box \)

**Lemma 4.10.** Let \( A \) be a left serial ring and \( M \) an indecomposable left \( A \)-module of height 2. Then \( \text{soc} \ M \) is homogeneous (i.e. a direct sum of copies of one simple left \( A \)-module).

Proof. Let \( M=\sum_{i=1}^n L_i \) be an irredundant sum of uniserial left \( A \)-modules. Then since \( M \) is indecomposable of height 2, every partial sum \( L=\sum_{i\in I} L_i \) with \( I\subseteq \{1, \ldots, n\} \) has no simple direct summand and hence \( \text{soc} \ L=JL \) by (3.2.2).
Assume that soc $M$ is not homogeneous and put $soc\ M=\bigoplus_{i=1}^{m}S_{i}^{(t_{i})}$ with each $S_{i}\cong S_{j}$ if $i\neq j$. Then $2\leq m$. Put $I_{j}:=\{i\in\{1,\ \cdots,\ n\} \mid soc\ L_{i}\cong S_{j}\}$ and $M_{j}:=\sum_{i\in I_{j}}L_{i}$ for each $j=1,\ \cdots,\ m$. Then for any $I\subseteq\{1,\ \cdots,\ m\}$, we have $soc\left(\sum_{j\in\mathcal{J}}M_{j}\right)\cong\sum_{j\in\mathcal{J}}\sum_{i\in I_{j}}L_{i}\cong\sum_{i\in I_{j}}soc\ L_{i}\cong\bigoplus_{i\in I_{j}}S_{i}^{(t_{i})}$ for some natural numbers $t_{j}$. In particular, $soc\ M_{j}\cong S_{j}^{(t_{j})}$ and $soc\left(\sum_{j\in\mathcal{J}}M_{j}\right)\cong\bigoplus_{j\in\mathcal{J}}S_{j}^{(t_{j})}$. Hence $M_{j}\cap\left(\sum_{j\in\mathcal{J}}M_{j}\right)=0$, that is, $M=M_{1}\oplus\left(\sum_{j=2}^{m}M_{j}\right)$ is decomposable, a contradiction. \\

The following corollary is of interest comparing with Kawada algebras ([3], [5]).

**Corollary 4.11.** Let $A$ be a left serial algebra of right 2nd local type and $M$ an indecomposable right $A$-module. Then $top\ M$ is homogeneous.

Proof. Clear from (4.11) noting that $top\ M\cong D(soc\ DM)=D(soc\ (soc^{2}DM))$ and the fact that $soc^{2}DM\cong D\ top^{2}M$ is indecomposable left $A$-module of height 2 (we may assume that $h(M)\geq 2$ since the assertion is trivial in case $h(M)=1$) where $D$ is a selfduality of $A$.

**Lemma 4.12.** Let $A$ be a left serial ring and $M$ an indecomposable left $A$-module of height 2 such that $soc\ M\cong S^{(r)}$ where $S$ is a simple left $A$-module of $V$-type. Then $M$ is colocal and $|top\ M|\leq 2$.

Proof. It is sufficient to prove that $M$ can be decomposed into colocal left $A$-modules $M_{i}$ with $|top\ M_{i}|\leq 2$ for any left $A$-module $M$ of height 2 such that $soc\ M\cong S^{(r)}$ where $S$ is a simple left $A$-module of $V$-type. We prove this by induction on $n:=|top\ M|$.

If $n=1$ or 2, the assertion is clear from (1.3). So we may assume that $n\geq 3$. By the hypothesis of induction, we have only to show that $M$ is decomposable. Hence we may assume that for any irredundant sum expression $M=\sum_{i=1}^{n}L_{i}$ (with each $L_{i}$ uniserial) of $M$, $|L_{i}|=h(L_{i})=2$ for all $i$. Let $M=\sum_{i=1}^{n}L_{i}$ be an irredundant sum of uniserial left $A$-modules $L_{i}$. Then again by the hypothesis of induction, $M=L_{1}+(\bigoplus_{i=2}^{m}M_{i})$ for some colocal left $A$-modules $M_{j}$ with $|top\ M_{j}|\leq 2$. Also, we have $2=|L_{i}|=s(M_{i})$ for all $i=2,\ \cdots,\ m$. Putting $S:=L_{1}\cap(\bigoplus_{i=2}^{m}M_{i})$ and $\pi_{j}:\bigoplus_{i=2}^{m}M_{i}\to M_{j}$ the canonical projections, we have an exact sequence

$$0\to S\xrightarrow{\alpha}L_{1}\oplus(\bigoplus_{i=2}^{m}M_{i})\xrightarrow{\beta}M\to 0$$

where $\alpha=(\alpha_{i})_{i=1}^{m},\ \beta=(\beta_{i})_{i=1}^{m},\ \alpha_{i}:=-1_{S},\ \alpha_{j}:=-1_{S}\pi_{j}$ for all $j=2,\ \cdots,\ m$ and $\beta_{i}$
are the inclusion maps. If $S=0$, then clearly $M$ is decomposable. Thus we may assume that $S$ is simple since $L_1$ is a uniserial left $A$-module of length 2. Then by (4.9), $M$ is decomposable for $S$ is of V-type. //


(3)⇒(1). Trivial.

(1)⇒(2). Noting that every colocal right $A$-module is uniserial since $A$ is a left serial algebra, we see that the cases (R$_3$) and (R$_4$) in Theorem 3.4 do not occur. Hence the condition (2) holds.

(2)⇒(3). By self-duality of $A$, the condition (3) is equivalent to the following:

(3)' Every indecomposable left $A$-module is colocal if it is of height $\geq 3$.

We show the implication (2)⇒(3)'. Assume the condition (2) holds. Let $M$ be any left $A$-module of height $\geq 3$. Then in order to verify the condition (3)', it is sufficient to show that $M$ is decomposed into (a): colocal left $A$-modules $M_i$ of height $\geq 3$ with $|\text{top } M_i| \leq 2$; and (b): indecomposable left $A$-modules of height $\leq 2$. We prove this by induction on $n:=|\text{top } M|$. It is clear in case $n=1$ or 2 by (4.7). Now assume $n \geq 3$. Let $M=\sum_{i=1}^s L_i$ be an irredundant sum of local left $A$-modules $L_i$. Then by the hypothesis of induction, $M=L_1+\left(\sum_{i=2}^s M_i\right)$ such that $M_i$ is of (a)-type of the above for $i=2$, $\cdots$, $r$ and is of (b)-type for $i=r+1$, $\cdots$, $s$. We may assume that $|L_1| \leq s(M_i)$ for each $i=2$, $\cdots$, $r$ if $r=s$; and $|L_1| \leq 2$ if $r<s$. Note that $1<r$ since $h(M) \geq 3$. Again, by the hypothesis of induction, we have only to show that $M$ is decomposable. Therefore we may assume that $T:=L_1 \cap \left(\sum_{i=2}^s M_i\right)$ is not zero and $|L_1| \geq 2$. Let $\pi_j: \sum_{i=2}^s M_i \to M_j$ be the canonical projection for each $j=2$, $\cdots$, $s$. Then we have an exact sequence

(E) \quad 0 \to T \stackrel{\alpha}{\to} L_1 \oplus \left(\sum_{i=2}^s M_i\right) \stackrel{\beta}{\to} M \to 0

where $\alpha=(\alpha_i), \beta=(\beta_i), \alpha_i=-1, \alpha_j=1, \beta_i$ for each $j=2$, $\cdots$, $s$ and $\beta_i$ are the inclusion maps. We divide the argument into two cases: (i) $|L_1| \geq 3$; and (ii) $|L_1| = 2$.

(i) In case $|L_1| \geq 3$. It follows $r=s$ and $\text{soc } L_1$ is of V-type by (4.3). If $T$ is not simple, $\alpha_i$ is extendable to a homomorphism $L_i \to M_j$ for each $j=2$, $\cdots$, $s$ by (4.4) and (4.8) since $\text{soc } L_i \leq T$. Then there is a uniserial left $A$-module $L'_i$ such that $|L_i'| < |L_i|, M=L_i'+\left(\sum_{i=2}^s M_i\right)$ and $|L_i' \cap \left(\sum_{i=2}^s M_i\right)| < |T|$ by (1.2). Iterating this argument, we come to the case (ii) or the case (i) with $T$ simple. Hence we may assume that $T=\text{soc } L_1$ is simple of V-type since $|L_1|
3. Then by (4.9), $M$ is decomposable for $n \geq 3$.

(ii) In case $|L_i| = 2$. Clearly, $T = \text{soc } L_1$ is simple. If $\alpha_j = 0$ for some $j = 2, \ldots, s$ in (E), then $M_j$ is a direct summand of $M$. Hence we may assume that $\alpha_j \neq 0$ for any $j = 2, \ldots, s$. Then $T \cong \text{soc } M_i$ for every $i = 2, \ldots, r$ and $T$ is of $V$-type since $h(M_i) \geq 3$. If $r = s$, then by (4.9), $M$ is decomposable. Thus we may assume that $r < s$. Since for each $i = r + 1, \ldots, s$, $\alpha_i \neq 0$ and $\text{soc } M_i$ is homogeneous by (4.10), we have $\text{soc } M_i \cong T^{(\alpha_i)}$ for some $r_i$. Then by (4.12), $M_i$ is colocal and $|\text{top } M_i| \leq 2$ since $T$ is of $V$-type. Hence $M$ is decomposable by (4.9).

REMARK. (1) The implication $(2) \Rightarrow (3)$ in (4.1) is still true in the case where the base field $k$ is not algebraically closed by [7] or [6, Lemma 3.7].

(2) Considering the results of [I, Theorem 2], (2.5) and (4.1), it is of interest to characterize those algebras having the following property for any fixed natural number $n$:

Every indecomposable right module is local if it is of height $> n$.

5. Examples

In this section, we give some examples using bounden quiver algebras over an algebraically closed field $k$ as in [I]. (See [2] for details concerning bounden quiver algebras.) For a vertex $i$ of a bounden quiver, we denote by $e_i$ the primitive idempotent corresponding to the vertex $i$.

EXAMPLE 1. Algebras of right 2nd local type which have an $e$ in $\pi(A)$ of type $(L_i)$ for some $1 \leq i \leq 3$ or of type $(R_j)$ for some $1 \leq j \leq 5$ in Theorem 3.4.

$(L_1)$ and $(R_1)$: Take serial rings.

$(L_2)$ and $(R_2)$: Let $A$ be the algebra defined by the following bounden quiver

\[ \begin{array}{cccccc}
1 & 2 & 3 \\
\alpha_1 & \alpha_2 \\
4 & \alpha_3 \\
5 & \beta_1 & \beta_2 \\
6 \end{array} \]

Then clearly, $A$ is of right 2nd local type since $J^2 = 0$. Also, $Ae_1$ is of type $(L_2)$ and $e_4A$ is of type $(R_2)$.

$(L_3)$ and $(R_3)$: Take the algebra $A$ defined in [I, Example 1], namely

\[ \begin{array}{c}
\alpha \\
1 \\
\beta \\
2 \\
\gamma \\
3 \end{array} \]

Then clearly, $A$ is of right 2nd local type since $J^2 = 0$. Also, $Ae_1$ is of type $(L_2)$ and $e_4A$ is of type $(R_2)$.
Then as verified in \([I]\), \(A\) is of right 2nd local type and \(Ae_i\) is of type \((L_i)\) and 
\(e_iA\) is of type \((R_i)\).

\((R_4)\): Let \(A\) be the algebra defined by the following

\[
\begin{array}{c}
\beta \\
\gamma \\
\delta \\
\end{array} \\
\begin{array}{c}
3 \\
5 \\
1 \\
2 \\
\end{array} \\
\begin{array}{c}
\delta \\
\gamma \\
\beta \\
\end{array}
\]

\[\beta \delta = \gamma \epsilon .\]

Then computing the Auslander-Reiten quiver (see \([2]\)) of \(A\), we see that \(A\) has the following property:

Every indecomposable right \(A\)-module is local if it is of height \(\geq 3\).

In particular, \(A\) is of right 2nd local type and \(e_iA\) is of type \((R_i)\).

\((R_5)\): Let \(A\) be the following quiver algebra of type \(A_5\)

\[1 \rightarrow 2 \rightarrow 3 \leftarrow 4 \leftarrow 5 .\]

Then by \([8, \text{Proposition 4.4}]\) or \([6]\), \(A\) is of right (1st) local type and hence 2nd local type. Also, \(e_sA\) is of type \((R_s)\).

**Example 2.** A left serial algebra of right 2nd local type which is not of right (1st) local type.

Let \(A\) be the following bounden quiver algebra

\[
\begin{array}{c}
\alpha \\
\beta \\
\delta \\
\gamma \\
\epsilon \\
\end{array} \\
\begin{array}{c}
5 \\
3 \\
2 \\
1 \\
6 \\
\end{array} \\
\begin{array}{c}
\delta \\
\gamma \\
\beta \\
\epsilon \\
\alpha \\
\end{array}
\]

\[\beta \delta = \beta \epsilon = 0 .\]

Then \(A\) is left serial and also by Theorem 4.1, \(A\) is of right 2nd local type. In fact, \(e_iA\) is simple for each \(i=1, 4, 5, 6\); and \(e_sA\) is of height 2 and \(e_sA\) is of height 3 having type \((R_s)\). But by \([8, \text{Proposition 4.4}]\), \(A\) is not of right (1st) local type since \(e_sA\) is of type \((R_s)\).

**Example 3.** An algebra of right 2nd local type having an indecomposable right \(A\)-module of height \(\geq 3\) that is not local.

Let \(A\) be the algebra defined by the following quiver of type \(A_5\)

\[1 \leftarrow 2 \rightarrow 3 \leftarrow 4 \leftarrow 5 .\]

Then as easily seen, \(A\) is of right 2nd local type and the following indecomposable right \(A\)-module is not local but it is of height 3:

\[
k \rightarrow k \leftarrow k \rightarrow k \rightarrow k .
\]
Example 4. The conditions in Theorem 3.4 are not sufficient for algebras to be of right 2nd local type in general.

Let $A$ be the following quiver algebra of type $D_6$

\[
\begin{array}{cccccc}
6 \\
\uparrow \\
1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 5.
\end{array}
\]

Then as easily verified, $A$ satisfies all the conditions in (3.4). But it is not of right 2nd local type. For instance, let $M$ be the right $A$-module corresponding to the following $k$-representation of $\mathcal{O}^o$

\[
\begin{array}{cccc}
 k[0] \\
 \downarrow^{1} \\
 k \oplus k \oplus k \oplus k \oplus k \oplus k \oplus k = (1,1).
\end{array}
\]

Then $M$ is indecomposable but $\text{top}^2M$ is decomposable:

\[
\begin{array}{cccc}
 0 \\
 \downarrow \\
 \text{top}^2M = (k \rightarrow \kappa \leftarrow k \rightarrow k \rightarrow 0) \\
 \kappa \\
 \downarrow \\
 \kappa \oplus (0 \rightarrow \kappa \leftarrow \kappa \rightarrow \kappa \rightarrow 0).
\end{array}
\]

References

[7] H. Tachikawa: *On rings for which every indecomposable right module has a unique
maximal submodule, Math. Z. 71 (1959), 200–222.


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