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ON THE RADICALS OF Γ -RINGS

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1. Introduction

N. Nobusawa [1] introduced the notion of a Γ -ring, more general than a ring, and proved analogues of the Wedderburn-Artin theorems for simple Γ -rings and for semi-simple Γ -rings; Barnes [2] obtained analogues of the classical Noether-Lasker theorems concerning primary representations of ideals for Γ -rings; Luh [3, 4] gave a generalization of the Jacobson structure theorem for primitive Γ -rings having minimum one-sided ideals, and obtained several other structure theorems for simple Γ -rings; Coppage-Luh [5] introduced the notions of Jacobson radical, Levitzki nil radical, nil radical and strongly nilpotent radical for Γ -rings and Barnes' [2] prime radical was studied further. Also, inclusion relations for these radicals were obtained, and it was shown that the radicals all coincide in the case of a Γ -ring which satisfies the descending chain condition on one-sided ideals.

In this paper the notions of semi-prime ideals are extended to Γ -rings, and it is shown that all of the following conditions are equivalent: (1) Q is a semi-prime ideal. (2) Q^c is an n -system. (3) The Γ -residue class ring M/Q contains no non-zero strongly nilpotent ideals. (4) The prime radical $P(Q)$ of the ideal Q coincides with Q . Also, the following characterization of $P(M)$ is obtained. $P(M)$ is a semi-prime ideal which is contained in every semi-prime ideal in M . Let R be the right operator ring of a Γ -ring M . For $P \subseteq R$ and for $Q \subseteq M$ we define $P^* = \{x \in M : [\Gamma, x] \subseteq P\}$ and $Q^{**} = \{\sum_i [\alpha_i, x_i] \in R : M(\sum_i [\alpha_i, x_i]) \subseteq Q\}$.

In [5] the following theorem was proved. If $P(M)$ is the prime radical of the right operator ring R of the Γ -ring M , then $P(M) = P(R)^*$.

We show the following result dual to the above theorem, $P(R) = P(M)^{*'*}$. As a result, it is obtained that $P(M)^{*'*} = P(M)$ and $P(R)^{**'} = P(R)$. The similar properties hold for the Levitzki nil radical and Jacobson radical. Also, some radical properties are considered.

2. Preliminaries

Let M and Γ be additive abelian groups. If for all $a, b, c \in M$, and $\alpha, \beta \in \Gamma$,

the following conditions are satisfied, (1) $aab \in M$ (2) $a(\alpha + \beta)b = aab + a\beta b$, $a\alpha(b+c) = aab + a\alpha c$, $(a+b)\alpha c = a\alpha c + b\alpha c$ (3) $(aab)\beta c = a\alpha(b\beta c)$, then following Barnes [2], M is called a Γ -ring. If these conditions are strengthened to, (1') $aab \in M$, $\alpha a\beta \in \Gamma$ (2') same as (2) (3') $(aab)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ (4) $xyy = 0$ for all $x, y \in M$ implies $y = 0$, then M is called a Γ -ring in the sense of Nobusawa [1]. If A and B are subsets of a Γ -ring M and $\Theta \subseteq \Gamma$, we denote $A\Theta B$, the subset of M consisting of all finite sums of the form $\sum_i a_i \alpha_i b_i$, where $a_i \in A$, $b_i \in B$, and $\alpha_i \in \Theta$. For singleton subsets we abbreviate this notation, for example, $\{a\}\Theta B = a\Theta B$. A right (left) ideal of a Γ -ring M is an additive subgroup I of M such that $I\Gamma M \subseteq I$ ($M\Gamma I \subseteq I$). If I is both a right ideal and a left ideal, then we say that I is an ideal, or a two-sided ideal of M . For each a of a Γ -ring M , the smallest right ideal containing a is called the principal right ideal generated by a and is denoted by $|a\rangle$. Similarly we define $\langle a|$ and $\langle a\rangle$, the principal left and two-sided (respectively) ideals generated by a . Let I be an ideal of a Γ -ring M . If for each $a+I$, $b+I$ in the factor group M/I , and each $\gamma \in \Gamma$, we define $(a+I)\gamma(b+I) = a\gamma b + I$, then M/I is a Γ -ring which we shall call the Γ -residue class ring of M with respect to I . Let M be a Γ -ring and F the free abelian group generated by $\Gamma \times M$. Then

$$A = \{\sum_i n_i (\gamma_i, x_i) \in F : a \in M \Rightarrow \sum_i n_i a \gamma_i x_i = 0\}$$

is a subgroup of F . Let $R = F/A$, the factor group, and denote the coset $(\gamma, x) + A$ by $[\gamma, x]$. It can be verified easily that $[\alpha, x] + [\alpha, y] = [\alpha, x+y]$ and $[\alpha, x] + [\beta, x] = [\alpha+\beta, x]$ for all $\alpha, \beta \in \Gamma$ and $x, y \in M$. We define a multiplication in R by

$$\sum_i [\alpha_i, x_i] \sum_j [\beta_j, y_j] = \sum_{i,j} [\alpha_i, x_i \beta_j y_j].$$

If we define a composition on $M \times R$ into M by $a \sum_i [\alpha_i, x_i] = \sum_i a \alpha_i x_i$ for $a \in M$, $\sum_i [\alpha_i, x_i] \in R$, then M is a right R -module, and we call R the right operator ring of the Γ -ring M . Similarly we may construct a left operator ring L of M so that M is a left L -module. If A is a right (left) ideal of R (L), then MA (AM) is an ideal of M . For subsets $N \subseteq M$, $\Phi \subseteq \Gamma$, we denote by $[\Phi, N]$ the set of all finite sums $\sum_i [\gamma_i, x_i]$ in R , where $\gamma_i \in \Phi$, $x_i \in N$, and we denote by $[(\Phi, N)]$ the set of all elements $[\varphi, x]$ in R , where $\varphi \in \Phi$, $x \in N$. Thus, in particular, $R = [\Gamma, M]$. For $P \subseteq R$ we define $P^* = \{a \in M : [\Gamma, a] = [\Gamma, \{a\}] \subseteq P\}$. It then follows that if P is a right (left) ideal of R , then P^* is a right (left) ideal of M . Also, for any collection C of sets in R , $\bigcap_{P \in C} P^* = (\bigcap_{P \in C} P)^*$. For $Q \subseteq M$ we define $Q^{*'} = \{\sum_i [\alpha_i, x_i] \in R : M(\sum_i [\alpha_i, x_i]) \subseteq Q\}$. Then it follows that if Q is a right (left) ideal of M , then $Q^{*'} is a right (left) ideal of R . Also for any collection C of sets in R , $\bigcap_{P \in C} P^{*'} = (\bigcap_{P \in C} P)^*$.$

tion \mathcal{D} of sets in M , $\bigcap_{Q \in \mathcal{D}} Q^* = (\bigcap Q)^*$. For other notions relevant to Γ -rings we refer to [5].

3. Semi-prime ness

Following Barnes [2] an ideal P of a Γ -ring M is prime if for any ideals $A, B \subseteq M$, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. A subset S of M is an m -system in M if $S = \emptyset$ or if $a, b \in S$ implies $\langle a \rangle \Gamma \langle b \rangle \cap S \neq \emptyset$. Barnes [2] has shown that an ideal P is prime if and only if its complement P^c is an m -system. The prime radical $P(A)$ of the ideal A in a Γ -ring M is the set consisting of those elements r of M with the property that every m -system in M which contains r meets A (that is, has nonempty intersection with A). The prime radical of the zero ideal in a Γ -ring M may be called the prime radical of the Γ -ring M which we denote by $P(M)$. Barnes [2] has characterized $P(M)$ as the intersection of all prime ideals of M . We now make the following definition. An ideal Q in a Γ -ring M is said to be a semi-prime ideal if and only if it has the following property: If A is an ideal in M such that $A\Gamma A \subseteq Q$, then $A \subseteq Q$. It is clear that a prime ideal is semi-prime. Moreover, the intersection of any set of semi-prime ideals is a semi-prime ideal. It follows easily by induction that if Q is a semi-prime ideal and A is an ideal and $(A\Gamma)^n A = (A\Gamma A\Gamma \cdots A\Gamma)A \subseteq Q$ for an arbitrary positive integer n , $A \subseteq Q$. Following Coppage-Luh [5] a subset S of M is strongly nilpotent if there exists a positive integer n such that $(ST)^n S = 0$. We state the following theorem whose proof we omit since it can be established by very easy modifications of the proof of Theorem 4.11 in [6].

Theorem 3.1. *An ideal Q in a Γ -ring M is a semi-prime ideal in M if and only if the Γ -residue class ring M/Q contains no nonzero strongly nilpotent ideals.*

The following result is easy to prove.

Theorem 3.2. *If Q is an ideal in a Γ -ring M , the following conditions are equivalent: (1) Q is a semi-prime ideal. (2) If $a \in M$ such that $\langle a \rangle \Gamma \langle a \rangle \subseteq Q$, then $a \in Q$.*

A set N of elements of a Γ -ring M is said to be an n -system if $N = \emptyset$ or if $a \in N$ implies $\langle a \rangle \Gamma \langle a \rangle \cap N \neq \emptyset$. The equivalence of conditions (1) and (2) of Theorem 3.2 assures us that an ideal Q in a Γ -ring M is semi-prime if and only if its complement Q^c is an n -system. By proofs analogous to Lemma 4.14 and Theorem 4.15 in McCoy [6] we have the following results.

Lemma 3.3. *If N is an n -system in a Γ -ring M and $a \in N$, there exists an m -system L such that $a \in L$ and $L \subseteq N$.*

Theorem 3.4. *An ideal Q in a Γ -ring M is a semi-prime ideal in M if and*

only if $P(Q)=Q$.

In view of Barnes' characterization of $P(M)$ as the intersection of all prime ideals of M (Theorem 7 in [2]) we have the following immediate corollary to the preceding theorem.

Corollary 3.5. *If Q is an ideal in a Γ -ring M , then $P(Q)$ is the smallest semi-prime ideal in M which contains Q .*

We have the following characterization of $P(M)$ which follows immediately from Corollary 3.5 and Theorem 7 in [2].

Theorem 3.6. *$P(M)$ is a semi-prime ideal which is contained in every semi-prime ideal in M .*

4. The prime radical

Coppage-Luh [5] have proved the following Lemma 4.1 and Theorem 4.2.

Lemma 4.1. *If P is a prime ideal of R , then P^* is a prime ideal of M .*

Theorem 4.2. *If $P(R)$ is the prime radical of the right operator ring R of the Γ -ring M , then $P(M)=P(R)^*$.*

We prepare the following lemma.

Lemma 4.3. *If Q is a prime ideal of M , then $Q^{* \prime}$ is a prime ideal of R .*

This proof is found in the proof of Theorem 4.1 in [5]. We now prove the following theorem dual to Theorem 4.2.

Theorem 4.4. *If $P(R)$ is the prime radical of the right operator ring R , then $P(R)=P(M)^{*\prime}$.*

Proof. Let P be a prime ideal of R , by Lemma 4.1 P^* is a prime ideal of M . Let us set $P^*=Q$. Then by Lemma 4.3 $Q^{* \prime}$ is a prime ideal of R . Since $Q^{* \prime}=\{r\in R: Mr\subseteq Q\}=\{r\in R: [\Gamma, Mr]\subseteq P\}$, it follows that $RQ^{* \prime}=[\Gamma, M]Q^{* \prime}=[\Gamma, MQ^{* \prime}]\subseteq P$. Thus by the primeness of P , $Q^{* \prime}\subseteq P$. Also, $[\Gamma, MP]=[\Gamma, M]P=RP\subseteq P$. Hence, $P\subseteq Q^{* \prime}$. Therefore $P=Q^{* \prime}$. It follows that $P(R)$, which is the intersection of all prime ideals of R , contains $\bigcap_{Q\in\mathcal{D}} Q^{* \prime}=(\bigcap_{Q\in\mathcal{D}} Q)^{* \prime}$, where \mathcal{D} is a certain collection of prime ideals of M . But $(\bigcap_{Q\in\mathcal{D}} Q)^{* \prime}\supseteq P(M)^{*\prime}$, so we conclude that $P(R)\supseteq P(M)^{*\prime}$. On the other hand, $P(M)^{*\prime}=(\bigcap_{Q\in\mathcal{D}} Q)^{* \prime}=\bigcap_{Q\in\mathcal{D}} Q^{* \prime}$, where the intersection is taken over all prime ideals of M . Since, by Lemma 4.3 each $Q^{* \prime}$ is a prime ideal of R , and $P(R)$ is the intersection of all prime ideals of R , it follows $P(R)\subseteq P(M)^{*\prime}$. Thus $P(R)=P(M)^{*\prime}$.

The following result is a consequence of Theorem 4.2 and Theorem 4.4.

Theorem 4.5. *If $P(R)$ is the prime radical of the right operator ring R , then $P(M)=P(M)^{**'}$ and $P(R)=P(R)^{**'}$.*

The next theorem follows immediately from previous Theorem 4.4 and Theorem 4.2 in [5].

Theorem 4.6. *If I is an ideal of a Γ -ring M , then $P(I)^{*'}=I^{*'} \cap P(R)$, where $P(I)$ denotes the prime radical of I considered as a Γ -ring.*

5. The Levitzki nil radical

Following Coppage-Luh [5] a subset S of a Γ -ring M is locally nilpotent if for any finite set $F \subseteq S$ and any finite set $\Phi \subseteq \Gamma$, there exists a positive integer n such that $(F\Phi)^n F = 0$. Also the Levitzki nil radical of M is the sum of all locally nilpotent ideals of M and is denoted by $\mathcal{L}(M)$.

Coppage-Luh [5] have proved the next theorem.

Theorem 5.1. *If M is a Γ -ring then $\mathcal{L}(M)=\mathcal{L}(R)^*$, where $\mathcal{L}(R)$ is the Levitzki nil radical of the right operator ring R of M .*

We know the following result whose proof will be found in Jacobson [7], p. 163.

Theorem 5.2. *$\mathcal{L}(R)$ is a locally nilpotent ideal.*

We prove the following two lemmas.

Lemma 5.3. *If J is a locally nilpotent ideal of a Γ -ring M , then $J^{*'} is a locally nilpotent ideal of R , where R is the right operator ring of M .$*

Proof. A finite subset of $J^{*''}$ is a subset of $[\phi_1, F_1]$, where ϕ_1 is a finite subset of Γ and F_1 is a finite subset of M . Since $M[\phi_1, F_1] \subseteq J$, $M\phi_1 F_1 \subseteq J$. Thus $F_1\phi_1 F_1$ is a finite subset of J . Since J is locally nilpotent, $(F_1\phi_1 F_1)^n F_1\phi_1 F_1 = 0$ for some n . Hence, $(F_1\phi_1)^{2n+1} F_1 = 0$. Thus $[\phi_1, F_1]^{2n+2} = [\phi_1, (F_1\phi_1)^{2n+1} F_1] = [\phi_1, 0] = 0$. Then $J^{*''}$ is locally nilpotent.

Lemma 5.4. *If M is a Γ -ring then $\mathcal{L}(R) \subseteq \mathcal{L}(M)^{*''}$, where $\mathcal{L}(R)$ is the Levitzki nil radical of the right operator ring R of M .*

Proof. Since $\mathcal{L}(R)$ is an ideal by Theorem 5.2., we have $R\mathcal{L}(R) \subseteq \mathcal{L}(R)$, so $[\Gamma, M]\mathcal{L}(R) \subseteq \mathcal{L}(R)$, that is, $[\Gamma, M\mathcal{L}(R)] \subseteq \mathcal{L}(R)$. Thus, $M\mathcal{L}(R) \subseteq \mathcal{L}(R)^*$. By Theorem 5.1., we get $M\mathcal{L}(R) \subseteq \mathcal{L}(M)$. Therefore $\mathcal{L}(R) \subseteq \mathcal{L}(M)^{*''}$.

Theorem 5.5. *If M is a Γ -ring then $\mathcal{L}(M)^{*'} = \mathcal{L}(R)$, where $\mathcal{L}(R)$ is the*

Levitzki nil radical of the right operator ring R of M .

Proof. In view of Lemma 5.3 and the fact that $\mathcal{L}(M)$ is a locally nilpotent ideal (Theorem 7.1 in [5]), we know that $\mathcal{L}(M)^{* \prime}$ is a locally nilpotent ideal of R . By the definition of $\mathcal{L}(R)$, $\mathcal{L}(M)^{* \prime} \subseteq \mathcal{L}(R)$. On the other hand by Lemma 5.4, $\mathcal{L}(R) \subseteq \mathcal{L}(M)^{* \prime}$. Hence $\mathcal{L}(M)^{* \prime} = \mathcal{L}(R)$.

The next theorem follows from Theorem 5.1 and Theorem 5.5.

Theorem 5.6. *If $\mathcal{L}(R)$ is the Levitzki nil radical of the right operator ring R of a Γ -ring M , then $\mathcal{L}(M)^{* \prime *} = \mathcal{L}(M)$ and $\mathcal{L}(R)^{** *} = \mathcal{L}(R)$.*

The next theorem follows immediately from previous Theorem 5.5 and Theorem 7.3 in [5].

Theorem 5.7. *If I is an ideal of a Γ -ring M , then $\mathcal{L}(I)^{* \prime} = I^{* \prime} \cap \mathcal{L}(R)$.*

6. The Jacobson radical

Following Coppage-Luh [5] an element a of a Γ -ring M is right quasi-regular (abbreviated *rqr*) if, for any $\gamma \in \Gamma$, there exists $\eta_i \in \Gamma$, $x_i \in M$, $i=1, 2, \dots, n$ such that

$$x\gamma a + \sum_{i=1}^n x\eta_i x_i - \sum_{i=1}^n x\gamma a\eta_i x_i = 0 \quad \text{for all } x \in M.$$

A subset S of M is *rqr* if every element in S is *rqr*. $\mathcal{J}(M) = \{a \in M : \langle a \rangle \text{ is rqr}\}$ is the right Jacobson radical of M . Coppage-Luh [5] have shown the following Lemma 6.1 and Theorem 6.2.

Lemma 6.1. *An element a of a Γ -ring M is *rqr* if and only if, for all $\gamma \in \Gamma$, $[\gamma, a]$ is *rqr* in the right operator ring R of M .*

Theorem 6.2. *If M is a Γ -ring then $\mathcal{J}(M) = \mathcal{J}(R)^*$, where $\mathcal{J}(R)$ denotes the Jacobson radical of the right operator ring R of M .*

We prove the following theorem dual to Theorem 6.2.

Theorem 6.3. *If M is a Γ -ring, then $\mathcal{J}(R) = \mathcal{J}(M)^{* \prime}$.*

Proof. If $\sum_i [\gamma_i, x_i] \in \mathcal{J}(M)^{* \prime}$, then for all $x \in M$

$$x(\sum_i [\gamma_i, x_i]) = \sum_i x\gamma_i x_i \in \mathcal{J}(M).$$

By Theorem 6.2, for all $\gamma \in \Gamma$

$$[\gamma, \sum_i x\gamma_i x_i] = [\gamma, x] \sum_i [\gamma_i, x_i] \in \mathcal{J}(R).$$

Hence $R \sum_i [\gamma_i, x_i]$ is *rqr*. By the definition of $\mathcal{J}(R)$, $\sum_i [\gamma_i, x_i] \in \mathcal{J}(R)$ and then

$\mathcal{J}(M)^{*'} \subseteq \mathcal{J}(R)$. If $\sum_i [\delta_i, x_i] \in \mathcal{J}(R)$, for all $\gamma \in \Gamma$ and all $x \in M$

$$[\gamma, x] \sum_i [\delta_i, x_i] = [\gamma, \sum x \delta_i x_i] \in \mathcal{J}(R).$$

Thus

$$\langle [\gamma, \sum x \delta_i x_i] \rangle = \{[\gamma, a] \in R : a \in \langle \sum x \delta_i x_i \rangle\} \subseteq \mathcal{J}(R).$$

Hence, $[\gamma, a]$ is *rqr* for all $\gamma \in \Gamma$ and for $a \in \langle \sum x \delta_i x_i \rangle$. By Lemma 6.1 $\langle \sum x \delta_i x_i \rangle$ is *rqr* and then $x(\sum [\delta_i, x_i]) = \sum x \delta_i x_i \in \mathcal{J}(M)$, that is, $M(\sum [\delta_i, x_i]) \subseteq \mathcal{J}(M)$. Therefore $\sum [\delta_i, x_i] \in \mathcal{J}(M)^{*'}$. Then $\mathcal{J}(R) \subseteq \mathcal{J}(M)^{*'}$. Thus the proof is completed.

By Theorem 6.2 and Theorem 6.3 we have the following theorem.

Theorem 6.4. $\mathcal{J}(M) = \mathcal{J}(M)^{*'*}$ and $\mathcal{J}(R) = \mathcal{J}(R)^{**'}$.

The next result follows from Theorem 6.3 and Theorem 8.4 in [5].

Theorem 6.5. If I is an ideal in a Γ -ring M , $\mathcal{J}(I)^{*'} = I^{*'} \cap \mathcal{J}(R)$.

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