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ON A KRULL ORDER

Dedicated to Professor Gorô Azumaya on his 60th birthday

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Let R be a ring with $1(\neq 0)$, τ an automorphism of R , and D a τ -derivation of R (i.e. $D(ab) = D(a)\tau(b) + a \cdot D(b)$ for all $a, b \in R$). Then a skew polynomial ring $A = R[t; \tau, D] = R \oplus tR \oplus t^2R \oplus \cdots$ is well defined by $at = t\tau(a) + D(a)$ ($a \in R$). Then if R is a two-sided simple ring, every ideal of A is invertible. On the other hand, as is well known, a (commutative) polynomial ring over a Krull domain is also a Krull domain. Furthermore, if R is a (non-commutative) Krull order in the sense of Marubayashi, then so is $R[t]$ ([11]). This is the case when $\tau = id$ and $D = 0$. In this paper we define a new "Krull order", and prove the following. If R is a Krull order then A is also a Krull order. Further we obtain some results on the structure of the group of reflexive fractional ideals of A . Any two-sided simple ring is a Krull order in our sense. In the case when R is a prime Goldie ring, R is a Krull order if and only if R is a maximal order and the ascending chain condition on integral reflexive ideals holds.

As a matter of fact, we prove main results in a more general situation. Namely we take some "positively filtered ring" instead of $R[t; \tau, D]$. By virtue of this, for example, if M is an invertible R -bimodule over a Krull order R then the tensor ring $T(M)$ is a Krull order. We believe this generalization is proper for this kind of study. However, if we assume R to be a prime Goldie ring, arguments may become more brief. But this excludes the case when R is a two-sided simple ring from our study. As is seen in §1, we take, as a starting point, the set of ideals which have trivial dual modules. This may be a feature of our study on Krull orders. Main results are analogous to those on a polynomial ring over a unique factorization domain.

For the completeness of this paper, we need some arguments on the construction of a positively filtered ring. But we postpone these until the forthcoming paper. However the case when $A = R[t; \tau, D]$ is treated in 4. Appendix. In all that follows, all rings are associative, but not necessarily commutative. Every ring has $1(\neq 0)$, which is preserved by homomorphisms, inherited by subrings and acts as the identity operator on modules.

1. Preliminary results

Let A, B be rings. If M is a left (resp. right) A -module, we write ${}_A M$

(resp. M_A). If N is a left A -, right B -bimodule we write ${}_A N_B$, and we briefly call N an A - B -module.

Let Q be a ring, and M an additive submodule of Q . We define the *left order* of M (in Q) as $O_l(M) = \{x \in Q : xM \subseteq M\}$. Similarly we define the *right order* of M as $O_r(M) = \{x \in Q : Mx \subseteq M\}$. Then, $\{x \in Q : MxM \subseteq M\} = \{x \in Q : Mx \subseteq O_l(M)\} = \{x \in Q : xM \subseteq O_r(M)\}$, which is denoted by M^{-1} . Evidently M^{-1} is an $O_r(M)$ - $O_l(M)$ -submodule, $M^{-1}M$ is an ideal of $O_r(M)$, and MM^{-1} is an ideal of $O_l(M)$. Let R be a subring of Q . By $T(Q; R)$ (abbr. $T(R)$) we denote the set of all ideals I satisfying the following conditions.

- (i) I is faithful as a left R -module as well as a right R -module.
- (ii) If $xI \subseteq R$ or $Ix \subseteq R$ ($x \in Q$) then $x \in R$.

Evidently $T(R)$ satisfies the following.

- (i) $R \in T(R)$.
- (ii) If $I \in T(R)$, and I' is an ideal of R such that $I \subseteq I'$ then $I' \in T(R)$.
- (iii) If $I_1, I_2 \in T(R)$ then $I_1 I_2 \in T(R)$, and so $I_1 \cap I_2 \in T(R)$ (by (ii)).
- (iv) If $I \in T(R)$ then $O_l(I) = R = O_r(I)$. Therefore if $xI = 0$ or $Ix = 0$ ($x \in Q$) then $x = 0$.

Proposition 1.1. *Let A, B be subrings of Q , and M an A - B -submodule of Q . Then the following conditions are equivalent.*

- (1) *There are B - A -submodules M', M'' of Q such that $MM' \in T(A)$, $M''M \in T(B)$.*
- (2) *$MM^{-1} \in T(A)$, and $M^{-1}M \in T(B)$.*
- (3) *$O_l(M) = O_r(MM^{-1}) = A$, and $O_r(M) = O_l(M^{-1}M) = B$. Further ${}_A M$, M_B , MM_A^{-1} , and ${}_B M^{-1}M$ are faithful modules.*
- (4) *$O_l(M) = O_r(M^{-1}) = A$, and $O_r(M) = O_l(M^{-1}) = B$. Further ${}_A M$, M_B , M_A^{-1} , and ${}_B M^{-1}$ are faithful modules.*

Proof. The implication (2) \Rightarrow (1) is trivial, and it is easy to see that (2) \Rightarrow (3), (3) \Rightarrow (4). (1) \Rightarrow (2) Evidently $O_l(M) = A$, and $O_r(M) = B$. Therefore $M' \subseteq M^{-1}$, and $M'' \subseteq M^{-1}$. Hence $MM' \subseteq MM^{-1}$, and $M''M \subseteq M^{-1}M$. Thus we obtain (2). (4) \Rightarrow (2) If $M^{-1}My \subseteq B$ then $M^{-1}MyM^{-1} \subseteq M^{-1}$, hence $MyM^{-1} \subseteq O_r(M^{-1}) = A$. Therefore $yM^{-1} \subseteq M^{-1}$, so $y \in O_l(M^{-1}) = B$. On the other hand, if $zM^{-1}M \subseteq B$ then $zM^{-1} \subseteq M^{-1}$, hence $z \in O_l(M^{-1}) = B$. If $bM^{-1}M = 0$ ($b \in B$) then $bM^{-1} \subseteq O_l(M) = A$, and so $bM^{-1} = 0$. Hence $b = 0$. Thus ${}_B M^{-1}M$ is faithful. Similarly $M^{-1}M_B$ is faithful. Hence $M^{-1}M \in T(B)$. Symmetrically we have $MM^{-1} \in T(A)$. This completes the proof.

Let A, B be subrings of Q . By $F(Q; A, B)$ (abbr. $F(A, B)$) we denote the set of all A - B -submodules M satisfying the condition (1) of Proposition 1.1. We put $F(Q) = \bigcup_{A, B} F(Q; A, B)$, where A, B run through all subrings of Q . In the sequel, if $M \in F(Q; A, B)$ then we write ${}_A M_B \in F(Q)$, conveniently. Note that $T(Q; A) \subseteq F(Q; A, A)$, and that if ${}_A M_B \in F(Q)$ then $xM = 0$ or $Mx = 0$ ($x \in$

Q) implies $x=0$.

Proposition 1.2. *Let ${}_A M_B, {}_B N_C \in F(Q)$.*

(i) ${}_B M_A^{-1} \in F(Q)$, and $MIM^{-1} \in T(A)$ for any $I \in T(B)$.

(ii) ${}_A MN_C \in F(Q)$.

Proof. (i) It follows from Proposition 1.1 that ${}_B M_A^{-1} \in F(Q)$. Let $I \in T(B)$. If $MIM^{-1}x \subseteq A$ ($x \in Q$) then $IM^{-1}x \subseteq M^{-1}$, and so $IM^{-1}xM \subseteq M^{-1}M \subseteq B$. Therefore $M^{-1}xM \subseteq B$, hence $M^{-1}x \subseteq M^{-1}$. Thus $x \in O_r(M^{-1}) = A$. On the other hand, $yMIM^{-1} \subseteq A$ implies that $M^{-1}yMIM^{-1}M \subseteq B$, and so $M^{-1}yM \subseteq B$. Hence $M^{-1}y \subseteq M^{-1}$, and therefore $y \in A$. Thus $MIM^{-1} \in T(A)$. (ii) If $xMN \subseteq MN$ then $M^{-1}xMNN^{-1} \subseteq M^{-1}MNN^{-1} \subseteq B$, and so $M^{-1}xM \subseteq B$. Then $x \in A$ as in (i). Thus $O_l(MN) = A$, and similarly $O_r(MN) = C$. Now, $MNN^{-1}M^{-1}MN \subseteq MM^{-1}MN \subseteq MN$, and so $N^{-1}M^{-1} \subseteq (MM)^{-1}$. Therefore $MNN^{-1}M^{-1} \subseteq (MN)(MN)^{-1}$, and $N^{-1}M^{-1}MN \subseteq (MN)^{-1}(MN)$. Since $NN^{-1} \in T(B)$ and $M^{-1}M \in T(B)$, it follows from (i) that $(MN)(MN)^{-1} \in T(A)$ and $(MN)^{-1}(MN) \in T(C)$. By Proposition 1.1, we have ${}_A MN_C \in F(Q)$.

If ${}_A M_B \in F(Q)$ then ${}_B M_A^{-1} \in F(Q)$, and so ${}_A (M^{-1})_B^{-1} \in F(Q)$. Since $MM^{-1} \subseteq A$ we have $M \subseteq (M^{-1})^{-1}$. Then $M^{-1} \supseteq ((M^{-1})^{-1})^{-1}$. On the other hand, $M^{-1} \subseteq ((M^{-1})^{-1})^{-1}$. Hence $M^{-1} = ((M^{-1})^{-1})^{-1}$. We put $M^* = (M^{-1})^{-1}$. Then $M \subseteq M^* = M^{**}$ for any $M \in F(Q)$.

Proposition 1.3. *For any ${}_A M_B \in F(Q)$, $M^* = \{x \in Q: Ix \subseteq M \text{ for some } I \in T(A)\} = \{x \in Q: xJ \subseteq M \text{ for some } J \in T(B)\}$.*

Proof. If $x \in M^*$ then $M^{-1}x \subseteq B$, and so $MM^{-1}x \subseteq M$, where $MM^{-1} \in T(A)$. Conversely if $Ix \subseteq M$ for some $I \in T(A)$, then $IxM^{-1} \subseteq MM^{-1} \subseteq A$, so $xM^{-1} \subseteq A$. Hence $x \in (M^{-1})^{-1} = M^*$. Symmetrically we obtain the latter half.

Evidently, for any subring A or Q , $T(Q; A) = \{I \in F(Q; A, A): I^* = A\}$.

Proposition 1.4. *Let ${}_A M_B, {}_B N_C \in F(Q)$. Then $(MN)^{-1} = (N^{-1}M^{-1})^*$, and $(M^*N)^* = (MN)^* = (MN^*)^*$.*

Proof. Since $N^{-1}M^{-1} \subseteq (MN)^{-1}$, we have $(N^{-1}M^{-1})^* \subseteq ((MN)^{-1})^* = (MN)^{-1}$. On the other hand, $x \in (MN)^{-1}$ implies that $MNx \subseteq A$, and so $Nx \subseteq M^{-1}$. Then $N^{-1}Nx \subseteq N^{-1}M^{-1}$, hence $x \in (N^{-1}M^{-1})^*$, because of $N^{-1}N \in T(C)$. Thus $(MN)^{-1} = (N^{-1}M^{-1})^*$. Using this, $(MN)^* = ((N^{-1}M^{-1})^*)^{-1} = (N^{-1}M^{-1})^{-1}$. As $(M^*)^{-1} = M^{-1}$, we have $(M^*N)^* = (N^{-1}M^{-1})^{-1} = (MN)^*$. Similarly $(MN^*)^* = (N^{-1}M^{-1})^{-1} = (MN)^*$.

If ${}_A M_B \in F(Q)$ and $M^* = M$, we call M a reflexive A - B -submodule of Q . By $F^*(Q; A, B)$ (abbr. $F^*(A, B)$) we denote the set of all reflexive A - B -submodules of Q , and we put $F^*(Q) = \bigcup_{A, B} F^*(Q; A, B)$, where A, B run through all subrings of Q . By $F_i(Q; A)$ (abbr. $F_i(A)$) we denote $\{M \in F(Q; A, A): M \subseteq A\}$, and we denote $F_i(Q; A) \cap F^*(Q; A, A)$ by $F_i^*(Q; A)$ (abbr. $F_i^*(A)$). If $I \in$

$F_i(A)$ (resp. $I \in F_i^*(A)$) we call I an *integral ideal* (resp. *reflexive ideal*) of A .

Let ${}_A M_B, {}_B N_C \in F^*(Q)$. We define $M \circ N$ by $(MN)^*$. Then, from Proposition 1.2 and Proposition 1.4, we have the following.

Theorem 1.5. *The set of all reflexive submodules of Q , $F^*(Q)$ is a Brandt groupoid. The set of identities of $F^*(Q)$ is the set of all subrings of Q .*

Let A, B be subrings of Q , and ${}_A M_B$ an A - B -submodule of Q . If there are B - A -submodules M', M'' of Q such that $MM' = A$ and $M''M = B$, we call M an invertible A - B -submodule of Q . Then it is easily seen that ${}_A M_B \in F^*(Q; A, B)$ and $M^{-1} = M' = M''$. Here we note the following

Proposition 1.6. *Let ${}_A M_B, {}_B N_C \in F^*(Q)$. If ${}_A M_B$ or ${}_B N_C$ is an invertible submodule then $M \circ N = MN$.*

Proof. We first assume that ${}_B N_C$ is invertible. If $xMN \subseteq C$ then $xM \subseteq N^{-1}$, so $NxM \subseteq NN^{-1} = B$. Therefore $Nx \subseteq M^{-1}$, and so $x \in N^{-1}M^{-1}$. Thus $(MN)^{-1} = N^{-1}M^{-1}$. Similarly $(MN)^{-1} = N^{-1}M^{-1}$, when ${}_A M_B$ is invertible. Hence $M \circ N = (N^{-1}M^{-1})^{-1} = M^*N^* = MN$, when ${}_A M_B$ or ${}_B N_C$ is invertible (cf. Proposition 1.4).

REMARK. Let ${}_A M_B$ be invertible in Q . Then $Q \otimes {}_A M \xrightarrow{\sim} Q$, $q \otimes m \mapsto qm$ ($q \in Q, m \in M$) (, and symmetrically $M \otimes {}_B Q \xrightarrow{\sim} Q$). In fact, if $1 = \sum_i m'_i m_i$ ($m'_i \in M^{-1}, m_i \in M$) then the inverse of the homomorphism $Q \otimes {}_A M \rightarrow Q$ is given by the map $q \mapsto \sum_i qm'_i \otimes m_i$ ($q \in Q$). As is well known, M is an invertible A - B -bimodule, that is, M_B is finitely generated, projective, and a generator, and $A \xrightarrow{\sim} \text{End}_B(M)$ by the map induced by ${}_A M$ (cf. [3]).

Let A, B be subrings of Q . If there exists an A - B -submodule $M \in F^*(Q; A, B)$ we write $A \sim B$ (in Q). Then “ \sim ” is an equivalence relation on the subrings of Q .

If $O_l(I) = O_r(I) = A$ holds for any ideal I of A such that both ${}_A I$ and I_A are faithful, we say that A is *maximal* in Q .

Proposition 1.7. *For any subring A of Q , the following conditions are equivalent:*

- (1) A is maximal in Q .
- (2) ${}_A I_A \in F(Q; A, A)$ for every ideal I of A such that both ${}_A I$ and I_A are faithful.

Proof. The implication (2) \Rightarrow (1) is trivial, and (1) \Rightarrow (2) follows from Proposition 1.1 (3).

Proposition 1.8. *Let ${}_A U_B \in F^*(Q; A, B)$.*

- (i) *If A is maximal in Q then so is B .*
- (ii) *There is a group isomorphism $F^*(Q; A, A) \xrightarrow{\sim} F^*(Q; B, B)$, $M \mapsto (U^{-1}MU)^*$*

$= U^{-1} \circ M \circ U \ (M \in F^*(Q; A, A)).$

(iii) *If A is a prime ring then so is B .*

Proof. (i) Let I' be an ideal of B such that ${}_B I', I'_B$ are faithful. Put $I = UI'U^{-1}$. It is easy to see that both ${}_A I$ and I_A are faithful. Therefore, by assumption, $O_l(I) = O_r(I) = A$. It $xI' \subseteq I'$ then $UxU^{-1}I = UxU^{-1}UI'U^{-1} \subseteq UxI'U^{-1} \subseteq UI'U^{-1} = I$, and so $UxU^{-1} \subseteq O_l(I) = A$. Then $xU^{-1} \subseteq U^{-1}$, so $xU^{-1}U \subseteq U^{-1}U$. Hence $x \in B$. Thus $O_l(I') = B$. Similarly $O_r(I') = B$. Hence B is maximal in Q . (ii) This follows from Theorem 1.5. (iii) Let I, J be ideals of B , and assume that $IJ = 0$. Then $UIU^{-1} \cdot UJU^{-1} = 0$, and so $UIU^{-1} = 0$ or $UJU^{-1} = 0$. If $UIU^{-1} = 0$ then $UI = 0$, so $I = 0$. Hence B is a prime ring.

Proposition 1.9. *Let A, B be subrings of Q such that $A \sim B$ in Q , and assume that A is a prime ring and is maximal in Q . Let M be an A - B -submodule of Q . Assume that there are elements u, v of Q such that $0 \neq uM \subseteq B$ and $0 \neq Mv \subseteq A$. Then ${}_A M_B \in F(Q; A, B)$.*

Proof. By Proposition 1.8, B is a prime ring, and is maximal in Q . Since BuM and MvA are non-zero ideals of B and A respectively, we have $O_r(M) = B$ and $O_l(M) = A$. Since $M^{-1} \ni u, v, M^{-1}M$ and MM^{-1} are non-zero ideals of B and A , respectively. Then, by Proposition 1.1 (3), $M \in F(Q; A, B)$.

Now we define a Krull subring of Q . A subring A of Q is said to be a *Krull subring* of Q if A is maximal in Q and the ascending chain condition on reflexive ideals of A holds. The following proposition follows from Proposition 1.8.

Proposition 1.10. *Let A, B be subrings of Q such that $A \sim B$ in Q . If A is a Krull subring of Q then so is B .*

Let A be any subring of Q . Let $P \in F^*(Q; A)$, and let $P \neq A$. Then P is said to be *irreducible* if $P = I_1 \circ I_2 \ (I_1, I_2 \in F^*(Q; A))$ implies that $P = I_1$ or $P = I_2$, and P is said to be *maximal* if $P \subseteq I' \in F^*(Q; A)$ implies that $I' = A$. Assume that P is maximal in $F^*(Q; A)$, and let $P = I_1 \circ I_2$. Then $P = (I_1 I_2)^* \subseteq I_i^* = I_i \ (i=1,2)$, hence $P = I_i$ or $I_i = A$. Therefore P is irreducible. Conversely, if P is irreducible then P is maximal. Thus "maximal" and "irreducible" are equivalent.

Assume that A is maximal in Q , and let P be irreducible in $F^*(Q; A)$. If $IJ \subseteq P$ for some ideals I, J of A then $(I+P)(J+P) \subseteq P$. If $I \not\subseteq P$ and $J \not\subseteq P$ then $I+P, J+P \in T(Q; A)$ by Proposition 1.7, so that $(I+P)(J+P) \in T(Q; A)$. Then have a contradiction $P \in T(Q; A)$. Hence P is a prime ideal. Conversely if $P \in F^*(Q; A)$ is a (proper) prime ideal then P is irreducible. Therefore, as is well known, if P, P' are irreducible in $F^*(A)$ then $P \circ P' = P' \circ P$. Then, in the usual way, we have the following.

Proposition 1.11. *Let A be a Krull subring of Q . Then any irreducible re-*

flexive ideal of A is a prime ideal, and $F_i^*(Q; A)$ is commutative. Any element of $F_i^*(Q; A)$ is uniquely represented as a product of irreducible elements of $F_i^*(Q; A)$.

Proposition 1.12. *Let A be a Krull subring of Q , and let ${}_A M_B \in F(Q; A, B)$. Assume that A is a prime ring. Then any non-zero A - B -submodule of M belongs to $F(Q; A, B)$, and there are elements x_1, \dots, x_r of M such that $M^* = (\sum_{i=1, \dots, r} Ax_i B)^*$.*

Proof. By Proposition 1.8, B is a prime ring and is maximal in Q . Let M_0 be a non-zero A - B -submodule of M . Then, since $M^{-1}M_0$ and M_0M^{-1} are non-zero ideals of B and A respectively, we have $M_0 \in F(Q; A, B)$, by virtue of Proposition 1.9. Now let $0 \neq x_1 \in M$. Then $Ax_1B \in F(Q; A, B)$, and $(Ax_1B)^* \subseteq M^*$. If $(Ax_1B)^* \subsetneq M^*$ then there is an element $x_2 \in M$ with $x_2 \notin (Ax_1B)^*$. If $(Ax_1B + Ax_2B)^* \subsetneq M^*$, then $(Ax_1B + Ax_2B)^* \subsetneq (Ax_1B + Ax_2B + Ax_3B)^*$ for some $x_3 \in M$. Continuing this process we obtain $x_1, \dots, x_r \in M$ such that $M^* = (\sum_i Ax_i B)^*$, because ACC holds on $\{N \in F^*(Q; A, B) : N \subseteq M^*\}$. (In fact, $N \subseteq M^*$ means $N \circ (M^*)^{-1} \subseteq A$, and conversely.)

Proposition 1.13. *Let Q' be any overring of Q , and A a prime subring of Q . Assume that, for any non-zero ideal I of A , $IQ = QI = Q$ holds. Then $T(Q; A) = T(Q'; A)$, and $F(Q; A, A) = \{M \in F(Q'; A, A) : M \subseteq Q, MQ = QM = Q\}$. Therefore $F_i(Q; A) = F_i(Q'; A)$, and $F_i^*(Q; A) = F_i^*(Q'; A)$.*

Proof. Evidently $T(Q; A) \supseteq T(Q'; A)$. Let $I \in T(Q; A)$, and let $Ix \subseteq A$ ($x \in Q'$). Then $Qx = QIx \subseteq QA = Q$, so $x \in Q$. Hence $x \in A$. Similarly $yI \subseteq A$ ($y \in Q'$) implies that $y \in A$. Thus $I \in T(Q'; A)$. Let $M \in F(Q; A, A)$, and put $M' = \{x \in Q : MxM \subseteq M\}$. Then $MM', M'M \in T(Q; A) = T(Q'; A)$. Then, by Proposition 1.1 (1), we have $M \in F(Q'; A, A)$. Furthermore, $Q \supseteq MQ \supseteq MM'Q = Q$, and so $MQ = Q$. Similarly $QM = Q$. Conversely, let $N \in F(Q'; A, A)$, $N \subseteq Q$, and $NQ = QN = Q$. If $zN \subseteq A$ ($z \in Q'$) then $zQ = zNQ \subseteq AQ = Q$, and so $z \in Q$. Hence $N \in F(Q; A, A)$. The remainder is obvious.

Corollary. *Assume the same assumptions as in Proposition 1.13. If A is maximal in Q (resp. a Krull subring of Q) then A is maximal in Q' (resp. a Krull subring of Q'), and conversely.*

Proof. This follows from Proposition 1.7 and Proposition 1.13.

Let A be a subring of Q . By $S(Q; A)$ (abbr. $S(A)$) we denote $\cup I^{-1}$, where I runs through reflexive ideals of A . Evidently $S(Q; A)$ is a subring containing A . We call $S(Q; A)$ the *Asano overring* of A in Q .

Proposition 1.14. *Let A be a prime Krull subring of Q . Assume that $I \cdot S(Q; A) = S(Q; A)I = S(Q; A)$ for any non-zero ideal I of A . Then any irreducible reflexive ideal of A is a (non-zero) minimal prime ideal of A , and con-*

versely (cf. [11]).

Proof. Let $P \in F_i^*(Q; A)$ be irreducible. Then P is a prime ideal. If there exists a non-zero prime ideal P' of A such that $P' \not\subseteq P$. Then $(P'P^{-1})P \subseteq P'$ implies that $P'P^{-1} \subseteq P'$. Then we have a contradiction $P^{-1} \subseteq A$. Hence P is minimal in the set of all non-zero prime ideals of A . Conversely, let P be a minimal prime ideal. Since $P \cdot S(Q; A) = S(Q; A) \ni 1$, there are reflexive ideals I_1, \dots, I_r of A such that $I_1 \cdots I_r \subseteq P$. Then $I_i \subseteq P$ for some i . Hence some irreducible component P'' of I_i is contained in P . Then, by the minimality of P , we have $P'' = P$. This completes the proof.

Note that, in the above case, A is a Krull subring of $S(Q; A)$, and $S(Q; A)$ is a left and right Utumi's quotient ring of A .

Proposition 1.15. *Let A be a prime subring of Q , and assume that A is maximal in Q . Let M be a non-zero left A -submodule of Q . Put $O_i(M) = B$ and $M' = \{x \in Q : Mx \subseteq A\}$.*

- (α) *If $M'M \in T(B)$ then $M \in F(Q; A, B)$.*
- (β) *Assume that M satisfies the following conditions:*
 - (i) *$xM' \neq 0$ for any non-zero $x \in M$.*
 - (ii) *M_B is faithful.*
 - (iii) *$\{y \in Q : yM' \subseteq A\} = M$.*

Then $M \in F^(Q; A, B)$ (, and conversely). (Cf. [6].)*

Proof. (α) As $M'M \in T(B)$, we have $MM'M \neq 0$, so $MM' \neq 0$. Hence $MM' \in F_i(Q; A)$, and so $O_i(M) = A$. Therefore $M' = M^{-1}$. If $MM'x \subseteq A$ then $M'x \subseteq M'$, so $MM'x \subseteq MM'$. Hence $x \in A$. If $yMM' \subseteq A$, then $MM'yMM' \subseteq MM'$, and so $MM'y \subseteq A$. Hence $y \in A$. Thus $MM' \in T(A)$. Hence $M \in F(Q; A, B)$. (β) Since MM' is a non-zero ideal of A , we have $MM' \in F_i(A)$, and $M' = M^{-1}$. If $xM' \subseteq M'$ then $MxM' \subseteq MM' \subseteq A$, hence $Mx \subseteq M$ by (iii). Therefore $x \in B$. If $xM' = 0$ then $x \in M$, hence $x = 0$ by (i). Thus $O_i(M') = B$, and ${}_B M'$ is faithful. Therefore (4) of Proposition 1.1 holds. Hence $M \in F^*(Q; A, B)$, by (iii).

2. A positively filtered ring over a Krull order

Let R be a subring of a ring Q . If R, Q satisfy the following conditions we call R a *Krull order* of Q .

- (i) R is a Krull subring of Q .
- (ii) Q is a left and right quotient ring of R .
- (iii) $IQ = QI = Q$ for any non-zero ideal I of R .

REMARK. If R is a prime Goldie ring, and Q is the maximal quotient ring of R then (ii), (iii) hold. Evidently every two-sided simple ring is a Krull order of itself.

Let R be a Krull order of Q . Let M be a non-zero R - R -submodule of Q . Then $M \cap R \neq 0$, and so $Q(M \cap R) = Q = (M \cap R)Q$. Therefore $QM = Q = MQ$. Hence Q is a simple R - Q -module as well as a simple Q - R -module. In particular, Q is a two-sided simple ring. Let $M \in F(Q; R, R)$. Then $QM = Q \ni 1$, so that $I \subseteq M$ for some dense left ideal I of R . Then $IR \subseteq M$, and so $0 \neq IR \cdot M^{-1} \subseteq R$. Put $IR \cdot M^{-1} = J$. Then $R \supseteq IR \cdot M^{-1}M = JM$, hence $M \subseteq J^{-1}$. Since $(IR)^* \circ M^{-1} = J^*$ we have $M^* = (IR)^* \circ J^{-1}$. Conversely, let N be a non-zero R - R -submodule of Q such that $N \subseteq J_1^{-1}$ for some non-zero ideal J_1 of R . Then, by Proposition 1.12, $N \in F(Q; R, R)$. Summing up, we have

Proposition 2.1. *Let R be a Krull order of Q .*

- (i) *Both ${}_Q Q_R$ and ${}_R Q_Q$ are simple.*
- (ii) *For a non-zero R - R -submodule N of Q , $N \in F(Q; R, R)$ if and only if $N \subseteq I^{-1}$ for some non-zero ideal I of R .*
- (iii) *$F^*(Q; R, R) = \{I \circ J^{-1} : I, J \in F^*(Q; R)\}$, which is an abelian group.*

For any ring A we denote by $Q_l(A)$ (resp. $Q_r(A)$) the left (resp. right) maximal quotient ring of A . Further we put $Q(A) = Q_l(A) \cap Q_r(A)$, more precisely, $Q(A) = \{x \in Q_r(A) : Iv \subseteq A \text{ for some dense left ideal } I\}$. By Corollary of Proposition 1.13, if R is a Krull order of Q , then R is a Krull order of $Q(R)$ ($\supseteq Q$).

In the remainder of this paper we assume the followings: R is a Krull order of Q . X is Q - Q -module containing Q , as a Q - Q -submodule, and such that X/Q is an invertible Q - Q -module. Y is an R - R -submodule of X containing R , such that Y/R is an invertible R - R -module, and such that $X = Q \otimes_R Y = Y \otimes_R Q$. $Q\langle X \rangle$ is an overring of Q satisfying the following conditions:

- (i) $Q\langle X \rangle \supseteq X$ as a Q - Q -submodule, and $Q\langle X \rangle = \bigcup_{i \geq 0} X^i$, where $X^0 = Q$.
- (ii) For any integer $i \geq 1$, the canonical map

$$(X/Q) \otimes_Q \cdots \otimes_Q (X/Q) \text{ (} i\text{-times)} \rightarrow X^i/X^{i-1},$$

$(x_1 + Q) \otimes \cdots \otimes (x_i + Q) \mapsto x_1 \cdots x_i + X^{i-1}$ is an isomorphism (cf. [13]).

We put $R\langle Y \rangle = \bigcup_{i \geq 0} Y^i$, where $Y^0 = R$. If $i < 0$ then we put $X^i = Y^i = 0$. Evidently $Q \otimes_R (Y/R) \xrightarrow{\sim} X/Q$. $q \otimes (y + R) \mapsto qy + Q$, and $(Y/R) \otimes_R Q \xrightarrow{\sim} X/Q$, $(y + R) \otimes q \mapsto yq + Q$. Therefore $Q \otimes_R (\otimes_R^i (Y/R)) \xrightarrow{\sim} \otimes_Q^i (X/Q)$ as Q - R -modules, and $(\otimes_R^i (Y/R)) \otimes_R Q \xrightarrow{\sim} \otimes_Q^i (X/Q)$ as R - Q -modules, where $\otimes_R^i (Y/R) = (Y/R) \otimes_R \cdots \otimes_R (Y/R)$ (i -times). For any $i \geq 1$, the following diagram is commutative:

$$\begin{array}{ccc} \otimes_R^i (Y/R) & \xrightarrow{\beta} & \otimes_Q^i (X/Q) \\ \alpha \downarrow & & \downarrow \approx \\ Y^i/Y^{i-1} & \xrightarrow{\delta} & X^i/X^{i-1} \end{array}$$

Since ${}^i_R \otimes_R (Y/R)$ is projective, the canonical map ${}^i \otimes_R (Y/R) \rightarrow Q \otimes_R ({}^i \otimes_R (Y/R))$ ($\simeq {}^i \otimes_Q (X/Q)$) is a monomorphism, so that α is an isomorphism. Therefore δ is a monomorphism, that is, $Y^i \cap X^{i-1} = Y^{i-1}$. In particular, $Y \cap Q = R$. Using the diagram

$$\begin{array}{ccc} Q \otimes_R Y^i & \xrightarrow{\varepsilon_i} & X^i \\ \downarrow & & \downarrow \\ Q \otimes_R (Y^i/Y^{i-1}) & \xrightarrow{\approx} & X^i/X^{i-1}, \end{array}$$

by induction on i , we can prove that each ε_i is an isomorphism. Therefore $Q \otimes_R R \langle Y \rangle = Q \langle X \rangle$, and symmetrically $R \langle Y \rangle \otimes_R Q = Q \langle X \rangle$. We put $\bar{Q} = Q \langle X \rangle$ and $\bar{R} = R \langle Y \rangle$.

REMARK. Let $Q = Q(R)$, and let Y be an R - R -module containing R , as an R - R -submodule, and such that Y/R is an invertible R - R -module. Then, X , $Q \langle X \rangle$, and $R \langle Y \rangle$ as above exist, and those are uniquely determined by $Y \supseteq R$. The proof is given in §4, in the case when $Y/R_R \simeq R_R$.

First we prove the following

Theorem 2.2. *If R is a Krull order then $R \langle Y \rangle$ is also a Krull order.*

We need many lemmas.

Lemma 2.3. *For any integer $i \geq 1$, there is a one to one correspondence from the set of all R - R -submodules of Q to the set of all R - R -submodules of X^i/X^{i-1} , such that $M \mapsto (MY^i + X^{i-1})/X^{i-1}$.*

Proof. This follows from [12; Proposition 3.3 and its proof].

Corollary 1. *For any integer $i \geq 1$, X^i/X^{i-1} is a simple Q - R -module as well as a simple R - Q -module.*

Proof. This follows from the fact that ${}_Q Q_R, {}_R Q_Q$ are simple.

Corollary 2. *For any integer $i \geq 1$, there is a one to one correspondence $M \mapsto M^i$ from the set of all R - R -submodules of Q to itself, which is defined by $M^i Y^i + X^{i-1} = Y^i M + X^{i-1}$. (Note that this map is multiplicative.)*

Lemma 2.4. *Let M be an R - Q -submodule of X^r ($r \geq 1$) such that $X^{r-1} \oplus M = X^r$. Then $QM \subseteq M$.*

Proof. Any y in QM is written as a sum $y = y_1 + y_2$ ($y_1 \in X^{r-1}, y_2 \in M$), and $Iy \subseteq M$ for some dense left ideal I of R . Then, for any $a \in I, ay_1 = ay - ay_2 \in X^{r-1} \cap M = \{0\}$. Hence $Iy_1 = 0$. Since ${}_Q X^{r-1}$ is projective, we have $y_1 = 0$. Thus

$y=y_2 \in M$.

Lemma 2.5. *Let A be an $R\text{-}\bar{Q}$ -submodule of \bar{Q} . Then $QA \subseteq A$.*

Proof. We may assume that $0 \neq A \neq \bar{Q}$. Then, since ${}_R Q_Q$ is simple, we have $Q \cap A = 0$. Therefore there exists an integer r such that $X^{r-1} \cap A = 0$ and $X^r \cap A \neq 0$. Since X^r/X^{r-1} is a simple R - Q -module, we have $X^{r-1} \oplus (X^r \cap A) = X^r$, hence $\bar{Q} = X^{r-1} \oplus ((X^r \cap A) \otimes_{\bar{Q}} \bar{Q})$ by [13; Corollary 1 of Proposition 1]. Then $A = A \cap \bar{Q} = X^{r-1} \cap A + (X^r \cap A) \otimes_{\bar{Q}} \bar{Q} = (X^r \cap A) \otimes_{\bar{Q}} \bar{Q}$. By Lemma 2.4, $Q(X^r \cap A) \subseteq X^r \cap A$, and so $QA \subseteq A$.

Corollary. *If A is an ideal of \bar{R} then $QA = AQ$ (, so that QA is an ideal of \bar{Q}).*

Proof. Noting that $\bar{Q} = Q\bar{R} = \bar{R}Q$, AQ is an $R\text{-}\bar{Q}$ -submodule. Hence $QA \subseteq AQ$. Symmetrically we obtain $AQ \subseteq QA$.

The following is well known, but we give its proof for completeness.

Lemma 2.6. *Let B be a ring, and I an ideal of B . Then the following conditions are equivalent:*

- (1) *I is an invertible B - B -module.*
- (2) *I is invertible in $Q(B)$.*

Proof. The implication (2) \Rightarrow (1) is well known. (1) \Rightarrow (2) Put $\{a \in Q_r(B) : aI \subseteq B\} = I'$. Then, since I is a dense right ideal, $I' \xrightarrow{\sim} \text{Hom}(I_B, B_B)$ canonically (cf. [16]). Since I_B is a generator, we have $I'I = B$. Since I_B is finitely generated and projective, we have $II' = B$. Then, since I is a dense left ideal, $I' \subseteq Q_l(B)$, and so $I' \subseteq Q(B)$. Thus I is invertible in $Q(B)$.

Lemma 2.7. *Every non-zero ideal of \bar{Q} is invertible. (Cf. [14; Examples].)*

Proof. Let A be any non-zero ideal of \bar{Q} . We may assume that $A \neq \bar{Q}$. Then there is an integer $r \geq 1$ such that $X^{r-1} \cap A = 0$ and $X^r \cap A \neq 0$. Put $M = X^r \cap A$. Then, as in the proof of Lemma 2.5, $X^{r-1} \oplus M = X^r$, and $A = M \otimes_{\bar{Q}} \bar{Q} = \bar{Q} \otimes_{\bar{Q}} M$. Since $M \xrightarrow{\sim} X^r/X^{r-1}$, M is an invertible Q - Q -module. Then it is easily seen that $\bar{Q} \xrightarrow{\sim} \text{End}(A_{\bar{Q}})$ by the map induced by ${}_q A$, so that ${}_q A_{\bar{q}}$ is invertible, because $A_{\bar{q}} = M \otimes_{\bar{Q}} \bar{Q}_{\bar{q}}$ is finitely generated, projective, and a generator (cf. [12; Lemma 3.1]).

If every non-zero ideal of a ring B is invertible, B is said to be an Asano order. Noting Lemma 1.6, an Asano order is a Krull order. A Krull order R is an Asano order if and only if $T(Q(R); R) = \{R\}$.

Lemma 2.8. (i) $S(\bar{R}) \subseteq S(\bar{Q}) \subseteq Q(\bar{R}) = Q(\bar{Q})$. (ii) *For any non-zero ideal A of \bar{R} , $A \cdot S(\bar{Q}) = S(\bar{Q})A = S(\bar{Q})$. Therefore \bar{R} is a prime ring.*

Proof. Since ${}_q \bar{Q}$ is projective, $\{x \in \bar{Q} : Ix = 0\} = 0$ for any dense left

ideal I of R . Then, as $Q\bar{R}=\bar{Q}$, we have $\bar{Q}\subseteq Q_i(\bar{R})$. Symmetrically $\bar{Q}\subseteq Q_r(\bar{R})$, and hence $\bar{Q}\subseteq Q(\bar{R})$. Thus $Q(\bar{R})=Q(\bar{Q})$. Since $AQ(=AQ)$ is a non-zero ideal of \bar{Q} , we have $S(\bar{Q})A=S(\bar{Q})QA=S(\bar{Q})$. Similarly $A\cdot S(\bar{Q})=S(\bar{Q})$. Therefore \bar{R} is a prime ring, and $A^{-1}\subseteq S(\bar{Q})$. Hence $S(\bar{R})\subseteq S(\bar{Q})$.

In virtue of Propositions 1.13 and 2.8, the notations $T(R)$, $F_i(R)$, $F_i^*(R)$, $T(\bar{R})$, $F_i(\bar{R})$, and $F_i^*(\bar{R})$ do not produce ambiguity.

By ρ_i we denote the correspondence $M\mapsto M'$ given in Corollary 2 of Lemma 2.3. Then $\rho_i(M)Y^i+X^{i-1}=Y^iM+X^{i-1}$, and if $M\subseteq R$ then $\rho_i(M)Y^i+Y^{i-1}=Y^iM+Y^{i-1}$, because of $X^{i-1}\cap Y^i=Y^{i-1}$. Further, note that $\rho_i(M')$ $\rho_i(M'')=\rho_i(M'M'')$ for any M', M'' . Put $\rho_1=\rho$. Then it is easy to verify that $\rho_i=\rho^i$ for all $i\geq 1$.

For any R - R -submodule M of $Q(\bar{R})$, we put $M^*=\{x\in Q(\bar{R}): xI\subseteq M \text{ for some } I\in T(R)\}$. Note that $R^*=R$ and $\bar{Q}^*=\bar{Q}$.

Lemma 2.9. (i) $\rho(T(R))=T(R)$. (ii) For any R - R -submodule M of Q , $\rho(M^*)=(\rho(M))^*$ holds. Therefore $\rho(F_i^*(R))=F_i^*(R)$.

Proof. (i) For any ideal I of R and any $x\in Q$, $I\cdot RxR\subseteq R$ (or $RxR\cdot I\subseteq R$) if and only if $\rho(I)\rho(RxR)\subseteq R$ (or $\rho(RxR)\rho(I)\subseteq R$), because of $\rho(R)=R$. Therefore we obtain (i). (ii) If $x\in M^*$ then $xI\subseteq M$ for some $I\in T(R)$. Then $\rho(RxR)\rho(I)\subseteq\rho(M)$, and so $\rho(RxR)\subseteq\rho(M)^*$ by (i). Thus $\rho(M^*)\subseteq(\rho(M))^*$. Similarly $\rho^{-1}(M^*)\subseteq(\rho^{-1}(M))^*$. Then $\rho^{-1}((\rho(M))^*)\subseteq M^*$, whence $(\rho(M))^*\subseteq\rho(M^*)$.

Lemma 2.10. \bar{R} is maximal in $Q(\bar{R})$.

Proof. Let A be any non-zero ideal of \bar{R} , and let $yA\subseteq A$ ($y\in Q(\bar{R})$). Then $yAQ\in AQ$, and so $y\in\bar{Q}$, because AQ is an invertible ideal of \bar{Q} . Put $W=\{x\in\bar{Q}: xA\subseteq A\}$. Then W is an \bar{R} - \bar{R} -submodule containing \bar{R} . For any $i\geq 0$, there exists a unique R - R -submodule W_i of Q such that $(W\cap X^i)+X^{i-1}=W_iY^i+X^{i-1}$, by Lemma 2.3. Similarly, for A , $(A\cap X^i)+X^{i-1}=A_iY^i+X^{i-1}$, where A_i is an R - R -submodule of Q . Since $W\supseteq\bar{R}$, we have $W\cap X^i\supseteq Y^i$, and so $W_i\supseteq R$. Since $A\subseteq\bar{R}$, we have $A\cap X^i=A\cap Y^i$, and so $A_i\subseteq R$. It is easy to verify that $W_j\cdot\rho^j(A_i)Y^{i+j}\subseteq A_{i+j}Y^{i+j}+X^{i+j-1}$ for all $i, j\geq 0$. Therefore $W_j\cdot\rho^j(A_i)\subseteq A_{i+j}$ for all $i, j\geq 0$. Noting that $A_0\subseteq A_1\subseteq A_2\subseteq\cdots$, we put $I=\bigcup_{i\geq 0}A_i$. Then I is a non-zero ideal of R , and $W_j\rho^j(I)\subseteq I$ for all $j\geq 0$. By Lemma 2.9, $\rho^j(I^*)=(\rho^j(I))^*$, and so $W_j\cdot\rho^j(I^*)\subseteq I^*$. By Lemma 2.9 (ii), the number of irreducible components of $\rho^j(I^*)$ is equal to the one of I^* . As $R\subseteq W_j$, we have $\rho^j(I^*)\subseteq I^*$, hence $\rho^j(I^*)=I^*$. Thereby $W_j\subseteq R$, and so $W\cap X^j\subseteq Y^j+X^{j-1}$. Noting that $W\supseteq\bar{R}$, we obtain $W\cap X^j=Y^j+W\cap X^{j-1}$ for all $j\geq 0$. Now $W\cap Q=W_0\subseteq R$, hence $W\cap X^j\subseteq Y^j$ for all $j\geq 0$. Thus $W\subseteq\bar{R}$, as required. Similarly $Ax\subseteq A$ implies that $x\in\bar{R}$.

Lemma 2.11. For any $i\geq 1$, $(Y^i)^*=Y^i$, and $\bar{R}^*=\bar{R}$.

Proof. Let f be any right R -homomorphism from \bar{R} to R . Extend f to a right Q -homomorphism \bar{f} from $\bar{Q} = \bar{R} \otimes_R Q$ to Q . If $y \in \bar{R}^*$ then $yI \subseteq \bar{R}$ for some $I \in T(R)$, and so $y \in \bar{Q}$. Then $\bar{f}(y)I \subseteq R$, and so $\bar{f}(y) \in R$ for any f . If (f_λ, u_λ) ($\lambda \in \Lambda$) is a projective coordinate system for \bar{R}_R , then so is $(\bar{f}_\lambda, u_\lambda)$ ($\lambda \in \Lambda$) for \bar{Q}_Q . Therefore $y = \sum_\lambda u_\lambda \bar{f}_\lambda(y) \in \bar{R}$. Hence $\bar{R}^* = \bar{R}$. If $x \in (Y^i)^*$ then $xJ \subseteq Y^i$ for some $J \in T(R)$. Then, as $JQ = Q$, we have $x \in X^i$. Hence $x \in X^i \cap \bar{R}^* = X^i \cap \bar{R} = Y^i$.

Lemma 2.12. *Let A be any reflexive \bar{R} - \bar{R} -submodule of $Q(\bar{R})$. Then $A^* = A$.*

Proof. If $xI \subseteq A$ for some $I \in T(R)$, then $A^{-1}xI \subseteq A^{-1}A \subseteq \bar{R}$. Using Lemma 2.11, $A^{-1}x \subseteq \bar{R}$. Therefore $x \in (A^{-1})^{-1} = A$.

Lemma 2.13. *Let A be any non-zero \bar{R} - R -submodule of \bar{R} . Then there exists a finitely generated \bar{R} - R -submodule A_0 of A such that $A \subseteq \bigcup_{j \geq 0} \beta^j(A_0)$, where $\beta(M) = M^*$ for any R - R -submodule M of $Q(\bar{R})$.*

Proof. For any $i \geq 0$, Y^i/Y^{i-1} is an invertible R - R -bimodule. Therefore there exists a unique ideal A_i of R such that $(A \cap Y^i) + Y^{i-1} = Y^i A_i + Y^{i-1}$. In particular, $A \cap R = A_0$. Since $Y(A \cap Y^i) \subseteq A \cap Y^{i+1}$, we have an ascending chain $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$. If $A_i = 0$ then $A \cap Y^i \subseteq Y^{i-1}$, and so $A_k \neq 0$ for some k . Then $A_k^* \subseteq A_{k+1}^* \subseteq \dots$, which are reflexive ideals of R . Therefore, for some integer $m \geq k$, $A_m^* = A_{m+1}^* = \dots$. By Proposition 1.12, $A_m^* = (\sum_{j=1, \dots, i} R z_j R)^*$ for some $z_1, \dots, z_t \in A_m$. Noting that ${}_R Y^m$ is finitely generated, we have that $\sum_j Y^m z_j R \subseteq \sum_i R b_i R + Y^{m-1}$ for some $b_1, \dots, b_s \in A \cap Y^m$. Let $n \geq m$. Then $A \cap Y^n \subseteq Y^n A_m^* + Y^{n-1} = Y^n A_m^* + Y^{n-1} \subseteq Y^{n-m} (\sum_j Y^m z_j R)^* + Y^{n-1}$. Therefore, if $a \in A \cap Y^n$ then $aJ \subseteq Y^{n-m} (\sum_j Y^m z_j R) + Y^{n-1}$ for some $J \in T(R)$, and so $aJ \subseteq \sum_i Y^{n-m} b_i R + Y^{n-1}$. Then $aJ \subseteq \sum_i Y^{n-m} b_i R + A \cap Y^{n-1}$. Thus $A \cap Y^n \subseteq (\sum_i Y^{n-m} b_i R + A \cap Y^{n-1})^*$ for any $n \geq m$. By induction we obtain $A \cap Y^n \subseteq \beta^{n-m+1} (\sum_i Y^{n-m} b_i R + A \cap Y^{m-1})$ ($n \geq m$). However, from the above proof, this holds whenever $A_n \neq 0$ and $A_n^* = \dots = A_n^*$. Therefore, for any $n \geq 0$ with $A_n \neq 0$, $A \cap Y^n \subseteq (\sum_i R c_i R + A \cap Y^{n-1})^*$ for some $c_1, \dots, c_r \in A \cap Y^n$. On the other hand, if $A_n = 0$ then $0 = A_0 = \dots = A_n$, and so $A \cap Y^n = 0$. Hence there exists a finitely generated R - R -submodule W of $A \cap Y^{m-1}$ such that $A \cap Y^{m-1} \subseteq \beta^m(W)$. Then, for any $n \geq m$, $A \cap Y^n \subseteq \beta^{n-m+1} (\sum_i Y^{n-m} b_i R + \beta^m(W)) \subseteq \beta^{n+1} (\sum_i Y^{n-m} b_i R + W)$. This completes the proof.

Now we can complete the proof of Theorem 2.2 with the following

Lemma 2.14. *The ascending chain condition on reflexive ideals of \bar{R} holds.*

Proof. Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ be an ascending chain of reflexive ideals of \bar{R} . Put $A = \bigcup_i A_i$. Then, by Lemma 2.13, $A \subseteq \bigcup_{j \geq 0} \beta^j(A')$ for some finitely generated \bar{R} - R -submodule A' of A . Then $A' \subseteq A_i$ for some i . By Lemma

2.12, $\beta(A_i) = A_i$, and so $\beta^j(A') \subseteq A_i$ for all j . Hence $A = A_i$.

Next we proceed to the proof of the following

Theorem 2.15. *For any non-zero ideal A of $Q\langle X \rangle$, $A \cap R\langle Y \rangle$ is a reflexive ideal of $R\langle Y \rangle$.*

Lemma 2.16. *The following conditions are equivalent.*

- (1) *For any non-zero ideal A of \bar{Q} , $A \cap \bar{R}$ is a reflexive ideal of \bar{R} .*
- (2) *For any $B \in T(\bar{R})$, $QB = \bar{Q}$ holds.*
- (3) *For any non-zero ideal C of \bar{R} , $(QC)^{-1} = C^{-1}Q = QC^{-1}$ holds.*

Proof. (1) \Rightarrow (2) If $QB \not\subseteq \bar{Q}$ then $B \subseteq QB \cap \bar{R} \not\subseteq \bar{R}$, and $QB \cap \bar{R}$ is a reflexive ideal, a contradiction. (2) \Rightarrow (3) From $CQ = QC$, we have $C^{-1}CQC^{-1} = C^{-1}QCC^{-1}$. Then, by assumption, $QC^{-1} = C^{-1}Q$. Hence $(QC)^{-1} = C^{-1}Q = QC^{-1}$. (3) \Rightarrow (1) Let $C \in T(\bar{R})$. Then $QC = \bar{Q}$, because of $C^{-1} = \bar{R}$. Now, put $A \cap \bar{R} = A'$. If $Cx \subseteq A' (x \in \bar{R})$, then $\bar{Q}x = QCx \subseteq A$, and so $x \in A \cap \bar{R} = A'$. Similarly $yC \subseteq A'$ implies that $y \in A'$. Hence A' is a reflexive ideal, by Proposition 1.3.

REMARK 1. The condition (2) is equivalent to that $B \cap R \neq 0$ for any $B \in T(\bar{R})$.

REMARK 2. If C is an ideal of \bar{R} such that $C \cap R \in T(R)$, then $C \in T(\bar{R})$. In fact, if $xC \in \bar{R}$ then $x(C \cap R) \subseteq \bar{R}$, and so $x \in \bar{R}$, by Lemma 2.11.

Lemma 2.17. *For any $I \in F(Q; R, R)$, $(\bar{R}I^{-1})^* = \bar{R}I^{-1}$ holds.*

Proof. The proof is similar to the one of Lemma 2.11.

Let M be a monic Q - Q -submodule of degree n (i.e. $X^{n-1} \oplus M = X^n$). Then, by [13; Corollary 1 of Proposition 1], $X^{n+m} = X^{n-1} \oplus (X^m \otimes_Q M)$ for any $m \geq 0$. Therefore $X^m \otimes_Q M \simeq X^{n+m}/X^{n-1}$ as Q - Q -bimodules, canonically. Since $Y^{n+m} \cap X^{n-1} = Y^{n-1}$, Y^{n+m}/Y^{n-1} is canonically embedded in X^{n+m}/X^{n-1} , and $Q \otimes_R (Y^{n+m}/Y^{n-1}) \simeq X^{n+m}/X^{n-1}$. Hence there exists a unique R - R -submodule V_m of $X^m \otimes_Q M$ such that the following diagram is commutative:

$$\begin{array}{ccc} X^m \otimes_Q M & \xrightarrow{\approx} & X^{n+m}/X^{n-1} \\ \uparrow & & \uparrow \\ V_m & \xrightarrow{\approx} & Y^{n+m}/Y^{n-1} \end{array}$$

Namely, $V_m + X^{n-1} = Y^{n+m} + X^{n-1}$. Then $Q \otimes_R V_m = X^m \otimes_Q M$, and $V_m = X^m M \cap (Y^{n+m} + X^{n-1})$. Therefore $V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots$, where $V_0 = M \cap (Y^n + X^{n-1})$. By [13; Corollary 1 of Proposition 1], $Q = X^{n-1} \oplus (\bar{Q} \otimes_Q M)$. Put $A = \bar{Q} \otimes_Q M$. Then $A \cap (\bar{R} + X^{n-1}) = \bigcup_i V_i$, and $A = \bigcup_{m \geq 0} (X^m \otimes_Q M) = \bigcup_{m \geq 0} (Q \otimes_R V_m) = Q \otimes_R V$, where $V = \bigcup_i V_i$. By Lemma 2.13, $A \cap \bar{R} \subseteq \bigcup_{j \geq 0} \beta^j(A')$ for some finitely

generated \bar{R} - R -submodule A' of $A \cap \bar{R}$. However, by virtue of Lemma 2.11, $\beta(A \cap \bar{R}) = A \cap \bar{R}$, whence $A \cap \bar{R} = \bigcup_{j \geq 0} \beta^j(A')$. As ${}_{\bar{R}}A'_R$ is finitely generated, $A' \subseteq \bar{R}V_s$ for some s . Now we assume that M is invertible in $Q(\bar{R})$. Then, since V_0 is an invertible R - R -module and $Q \otimes_R V_0 = M = V_0 \otimes_R Q$, we know that ${}_R V_{0R}$ is invertible in $Q(\bar{R})$. In this situation, we need the following

Lemma 2.18. *For any R - R -submodule W of \bar{Q} , $W^*V_0^{-1} = (WV_0^{-1})^*$ holds.*

Proof. By virtue of Proposition 1.2, there is a one to one mapping $I \mapsto V_0^{-1}IV_0$ from $T(R)$ onto itself. Let x be in $W^*V_0^{-1}$. Then $xV_0 \subseteq W^*$. Since V_{0R} is finitely generated, $xV_0I \subseteq W$ for some $I \in T(R)$. Then $xV_0IV_0^{-1} \subseteq WV_0^{-1}$, and so $x \in (WV_0^{-1})^*$. Similarly we can prove that $(WV_0^{-1})^* \subseteq W^*V_0^{-1}$.

We still assume that M is a monic Q - Q -submodule which is invertible in $Q(\bar{R})$, and notations are the same as before. Since $V_s \subseteq A = \bar{Q}M = \bar{Q}V_0$, we have $V_sV_0^{-1} \subseteq \bar{Q}$. Since both ${}_R V_s$ and ${}_R V_0^{-1}$ are finitely generated, ${}_R V_sV_0^{-1}$ is also finitely generated, and so $V_sV_0^{-1}I \subseteq \bar{R}$ for some non-zero ideal I of R , because of $\bar{Q} = \bar{R}Q$. Then, as $A' \subseteq \bar{R}V_s$, we have $A'V_0^{-1}I \subseteq \bar{R}$, and so $A'V_0^{-1}II^{-1} \subseteq \bar{R}I^{-1}$. Then, by Lemma 2.18 and 2.17, $\beta^j(A')V_0^{-1} = \beta^j(A'V_0^{-1}) \subseteq \bar{R}I^{-1}$ for all $j \geq 0$. Hence, as $A \cap \bar{R} = \bigcup_{j \geq 0} \beta^j(A')$, we obtain $(A \cap \bar{R})V_0^{-1}I \subseteq \bar{R}$. Put $N = \{x \in Q(\bar{R}) : (A \cap \bar{R})x \subseteq \bar{R}\}$ and $N' = \{y \in Q(\bar{R}) : Ay \subseteq \bar{Q}\}$. Evidently $N' = V_0^{-1}\bar{Q}$, and $V_0^{-1}I \subseteq N$ implies that $N' \subseteq N\bar{Q}$. Next, let us prove that $N\bar{Q} \subseteq N'$. Since ${}_R V_0$ is finitely generated, there exists a non-zero ideal I' such that $V_0I' \subseteq \bar{R}$. Then $V_0I' = I''V_0$ for some non-zero ideal I'' of R , for ${}_R V_{0R}$ is invertible. Therefore $A = \bar{Q}V_0 = \bar{Q}I''V_0 = \bar{Q}V_0I' \subseteq \bar{Q}(A \cap \bar{R})$, whence $A = \bar{Q}(A \cap \bar{R})$. Hence $N \subseteq N'$. Thus $N' = N\bar{Q}$. Finally, $zN \subseteq \bar{R}$ implies $zN' = zN\bar{Q} \subseteq \bar{Q}$, and so $z \in \bar{Q}V_0 = A$. Since $\bar{R} \subseteq N$, we have $z \in \bar{R}$. Hence $z \in A \cap \bar{R}$. Therefore a left \bar{R} -submodule $A \cap \bar{R}$ satisfies (β) of Proposition 1.15. Thus we have the following

Proposition 2.19. *Let M be a monic Q - Q -submodule which is invertible in $Q(\bar{R})$. Put $A = \bar{Q}M$, $A^{-1} = \{x \in Q(\bar{R}) : Ax \subseteq \bar{Q}\}$, and $(A \cap \bar{R})^{-1} = \{x \in Q(\bar{R}) : (A \cap \bar{R})x \subseteq \bar{R}\}$. Then $A = Q(A \cap \bar{R})$, and $A^{-1} = (A \cap \bar{R})^{-1}\bar{Q}$. Further, $A \cap \bar{R} \in F^*(Q(\bar{R}); \bar{R}, B)$, where $B = O_r(A \cap \bar{R})$.*

Evidently Theorem 2.15 follows from Proposition 2.16, Proposition 2.19 above and Lemma 20 below. (Cf. the proof of Lemma 2.7).

Lemma 2.20. *Let A be any non-zero ideal of \bar{Q} . Then $A = \bar{Q}M = M\bar{Q}$ for some monic Q - Q -submodule M . Such a M is uniquely determined by A , and is invertible in $S(\bar{Q})$.*

Proof. The first half follows from the proof of Lemma 2.7. Since $A = M \otimes_Q \bar{Q}$, any right Q -homomorphism from M to Q can be extended to a right \bar{Q} -homomorphism from A to \bar{Q} . Since A is invertible, this is given by a left multiplication of an element of A^{-1} . Therefore if we put $M' = \{x \in A^{-1} :$

$xM \subseteq Q\}$, then $M'M = Q$, because M_Q is a generator. Symmetrically $MM'' = Q$ for some Q - Q -submodule M'' of A^{-1} . Hence M is invertible in $S(\bar{Q})$. Let N be any monic Q - Q -submodule with $A = \bar{Q}N$. Let $\deg N = r$. Then $\bar{Q} = X^{r-1} \oplus A$, and $N = A \cap X^r$, by [13; Corollary 1 of Proposition 1]. Hence N is uniquely determined by A .

In all that follows we denote $F^*(Q(\bar{Q}); \bar{Q}, \bar{Q})$, $F(Q(\bar{Q}); \bar{Q}, \bar{Q})$, $F^*(Q(\bar{R}); \bar{R}, \bar{R})$, and $F(Q(\bar{R}), \bar{R}, \bar{R})$ by $F^*\{\bar{Q}\}$, $F\{\bar{Q}\}$, $F^*\{\bar{R}\}$, and $F\{\bar{R}\}$, respectively. Similarly we denote $F^*(Q; R, R)$ and $F(Q; R, R)$ by $F^*\{R\}$ and $F\{R\}$, respectively (cf. Proposition 1.13).

Let $M \in F\{\bar{R}\}$. Then $MI \subseteq \bar{R}$ for some $I \in F_i(\bar{R})$, by Proposition 2.1. Using Corollary of Lemma 2.5, $QMI = MIQ = MQI$, and so $QM = MQ$, for QI is invertible. Symmetrically $QM = MQ$, whence $MQ = QM$. Let $x \in Q(M^{-1})^{-1}$. Then $xM \subseteq QM$ for some $C \in T(\bar{R})$. Since $CQ = \bar{Q}$, we have $x \in QMQ = MQ$. Thus $QM = Q(M^{-1})^{-1}$. Therefore a group homomorphism ψ from $F^*\{\bar{R}\}$ to $F^*\{\bar{Q}\}$ is well defined by $\psi(M) = QM$. Let A, B be non-zero ideals of \bar{Q} . Then $AB \cap \bar{R} \supseteq (A \cap \bar{R}) \circ (B \cap \bar{R})$. Since $AB \cap \bar{R} \subseteq B \cap \bar{R}$, we have $(AB \cap \bar{R})(B \cap \bar{R})^{-1} \subseteq \bar{R}$. By Proposition 2.19, $B^{-1} \supseteq (B \cap \bar{R})^{-1}$, and so $(AB \cap \bar{R})(B \cap \bar{R})^{-1} \subseteq A \cap \bar{R}$. Therefore $AB \cap \bar{R} \subseteq (A \cap \bar{R}) \circ (B \cap \bar{R})$. Hence $AB \cap \bar{R} = (A \cap \bar{R}) \circ (B \cap \bar{R})$. Then a group homomorphism ϕ from $F^*\{\bar{Q}\}$ to $F^*\{\bar{R}\}$ is well defined by $\phi(AB^{-1}) = (A \cap \bar{R}) \circ (B \cap \bar{R})^{-1}$. Because of Proposition 2.19, $\psi\phi = id$. Hence $F^*\{\bar{R}\} \simeq \text{Im } \phi \times \text{Ker } \psi$, and $F^*\{\bar{Q}\} \simeq \text{Im } \phi$. Let I, J be in $F_i^*(\bar{R})$. If $IQ \subseteq JQ$ then $1 \in \bar{Q} \subseteq I^{-1}JQ$, and so $G \subseteq I^{-1}J$ for some $G \in F_i(R)$. Then $(\bar{R}G\bar{R})^* \subseteq I^{-1} \circ J$. Therefore $I^{-1} \circ J \in \text{Ker } \psi$ if and only if $(\bar{R}G\bar{R})^* \subseteq I^{-1} \circ J \subseteq ((\bar{R}F\bar{R})^*)^{-1}$ for some $F, G \in F_i(R)$. In particular, $J \in \text{Ker } \psi$ if and only if $J \cap \bar{R} \neq 0$. In this case, $J \cap \bar{R} \in F_i^*(\bar{R})$, by Lemma 2.12. Let $P' \in F_i^*(\bar{Q})$ be irreducible. Then, by Corollary of Lemma 2.5, $P' \cap \bar{R}$ is a prime ideal, so that $P' \cap \bar{R}$ is irreducible in $F_i^*(\bar{R})$, and $Q(P' \cap \bar{R}) = P'$ by Proposition 2.19. Conversely, if $P \in F_i^*(\bar{R})$ is irreducible and $QP \neq \bar{Q}$ then, by the maximality of P in $F_i^*(\bar{R})$, we have $QP \cap \bar{R} = P$, and QP is maximal. Let $J \in F_i^*(\bar{R})$, and $J = P_1 \circ \dots \circ P_r$, where each P_i is irreducible in $F_i^*(\bar{R})$. Then $QJ \cap \bar{R} = (QP_1 \cap \bar{R}) \circ \dots \circ (QP_r \cap \bar{R})$, and each $QP_i \cap \bar{R}$ is either P_i or \bar{R} . Let I', I'' be in $F_i^*(\bar{R})$. Then, $I' \circ I''^{-1} \in \text{Ker } \psi \Leftrightarrow QI' = QI'' \Leftrightarrow QI' \cap \bar{R} = QI'' \cap \bar{R}$. Therefore $\text{Ker } \psi = \coprod (P)$, where P ranges over all irreducible reflexive ideals P such that $P \cap \bar{R} \neq 0$ (or equivalently, $QP = \bar{Q}$), and (P) denotes the infinite cyclic group generated by P .

Lemma 2.21. (i) Let $I \in F\{R\}$, and assume that $I\bar{R} = \bar{R}I$. Then $\bar{R}I \in F\{\bar{R}\}$, $(\bar{R}I)^{-1} = \bar{R}I^{-1} = I^{-1}\bar{R}$, and $\bar{R}I \cap X^i = IY^i = Y^iI$ for all $i \geq 0$. Therefore, $I \in F^*\{R\}$ then $\bar{R}I \in F^*\{\bar{R}\}$.

(ii) Let $J \in F_i^*(R)$ be irreducible, and assume that $JY = YJ$. Then, if $aRb \subseteq \bar{R}J$ ($a, b \in \bar{R}$) then $a \in \bar{R}J$ or $b \in \bar{R}J$. Therefore $\bar{R}J$ is irreducible in $F_i^*(\bar{R})$.

Proof. (i) Since $0 \neq \bar{R}I \cdot I^{-1}\bar{R} \subseteq \bar{R}$ and $0 \neq \bar{R}I^{-1} \cdot I\bar{R}$, we have $\bar{R}I \in F\{\bar{R}\}$, by Proposition 1.9. Let $x \in (I\bar{R})^{-1}$. Then $xI \subseteq \bar{R}$, and so $xII^{-1} \subseteq \bar{R}I^{-1}$. By Lemma 2.17, $x \in \bar{R}I^{-1}$. Hence $(I\bar{R})^{-1} = \bar{R}I^{-1}$, and symmetrically $(\bar{R}I)^{-1} = I^{-1}\bar{R}$. Since Y^{i+1}/Y_R^i is projective, $Y^{i+1} = Y^i \oplus W$ for some right R -submodule W of Y^{i+1} . Then $\bar{R} = Y^i \oplus (W \otimes_R \bar{R})$, by [13; Proposition 1]. Then $\bar{Q} = \bar{R} \otimes_R Q = (Y^i \otimes_R Q) \oplus (W \otimes_R \bar{R} \otimes_R Q) = X^i \oplus (W \otimes_R \bar{Q})$, and $\bar{R}I = Y^i I \oplus W\bar{R}I$. Hence $X^i \cap \bar{R}I = Y^i I$, and symmetrically $X^i \cap I\bar{R} = IY^i$. (ii) By (i), $J\bar{R} \in F^*(\bar{R})$. Let B, C be R - R -submodules of \bar{R} such that $BC \subseteq J\bar{R}$. Then, as $(B+J\bar{R})(C+J\bar{R}) \subseteq J\bar{R}$, we may assume that $B, C \supseteq J\bar{R}$. For any integer $i \geq 1$, there are ideals B_i, C_i of R such that $(B \cap Y^i) + Y^{i-1} = B_i Y^i + Y^{i-1}$, $(C \cap Y^i) + Y^{i-1} = C_i Y^i + Y^{i-1}$, because each Y^i/Y^{i-1} is an invertible R - R -bimodule. Then, as $J\bar{R} \cap Y^{i+j} = JY^{i+j}$, we have $B_j \cdot \rho^j(C_i) \subseteq J$ for all i , where ρ is the one as before. Now, assume that $B \not\supseteq J\bar{R}$. Then $B_j \not\supseteq J$ for some j , so that $\rho^j(C_i) \subseteq J$. Then $C_i \subseteq \rho^{-j}(J) = J$ for all i . Noting that $C_0 = C \cap R \subseteq J$, this implies that $C \subseteq J\bar{R}$. This completes the proof.

Here we consider the following condition.

(#) For any $I \in F^*(Q; R)$, $IY = YI$.

Lemma 2.22. *Assume that the condition (#) holds. Let $P \in F^*(\bar{R})$. Then P is an irreducible ideal such that $P \cap R \neq 0$ if and only if $P = I\bar{R}$ for some irreducible reflexive ideal I of R .*

Proof. The "if" part follows from Lemma 2.21. Conversely, let $P \in F^*(\bar{R})$ be irreducible, and let $P \cap R \neq 0$. Then $P \cap R \in F^*(R)$. If $IJ \subseteq P \cap R$ for some $I, J \in F_i(R)$, then $I^*J^* \subseteq P \cap R$, because of $P \cap R \in F^*(R)$. Then $I^*\bar{R} \cdot J^*\bar{R} \subseteq P$, whence $I^* \subseteq P$ or $J^* \subseteq P$, because P is a prime ideal. Hence $P \cap R$ is a prime ideal of R . Then, by Lemma 2.21, $(P \cap R)\bar{R}$ is irreducible. Hence $(P \cap R)\bar{R} = P$.

Assume that the condition (#) holds. Let I, J be in $F^*\{R\}$. Then $(\bar{R}I) \circ (\bar{R}J) = ((\bar{R}I \cdot \bar{R}J)^{-1})^{-1} = ((\bar{R}IJ)^{-1})^{-1} = \bar{R}(I \circ J)$, by Lemma 2.21. Therefore the mapping $\theta: I \mapsto \bar{R}I$ is a homomorphism from $F^*\{R\}$ to $\text{Ker } \psi$. Evidently $I \subseteq \bar{R}I \cap Q$. Let $I = F \circ G^{-1}$ ($F, G \in F^*(R)$). Then $(\bar{R}I \cap Q)G \subseteq \bar{R}F \cap Q = \bar{R}F \cap R = F$, because R_R is a direct summand of \bar{R}_R . Therefore $(\bar{R}I \cap Q)GG^{-1} \subseteq FG^{-1}$, and so $\bar{R}I \cap Q \subseteq F \circ G^{-1} = I$. Hence $I = \bar{R}I \cap Q$. On the other hand, all irreducible $P \in F^*(\bar{R})$ with $P \cap R \neq 0$ generate $\text{Ker } \psi$. Therefore, by Lemma 2.22, θ is an isomorphism from $F^*\{R\}$ to $\text{Ker } \psi$. Thus we obtain the following

Theorem 2.23. *Assume that the condition (#) holds. Then $\theta: F^*\{R\} \xrightarrow{\sim} \text{Ker } \psi$, $I \mapsto \bar{R}I$, as groups. Further, $\bar{R}I \cap Q = I$ for all $I \in F^*\{R\}$.*

Proposition 2.24. *Assume that the condition (#) holds. If $I \cdot S(R) = S(R)I = S(R)$ for all $I \in F_i(R)$, then $A \cdot S(\bar{R}) = S(\bar{R})A = S(\bar{R})$ for all $A \in F_i(\bar{R})$. (Cf.*

Proposition 1.14.)

Proof. From (#), it follows that $S(R) \subseteq S(\bar{R})$. Let $A \in F_i(\bar{R})$. Then $AA^{-1} \cap R \neq 0$, because of Lemma 2.16. Therefore $S(R) \subseteq AA^{-1}S(R) \subseteq A \cdot S(\bar{R})$, hence $A \cdot S(\bar{R}) = S(\bar{R})$. Symmetrically $S(\bar{R})A = S(\bar{R})$.

3. In this section, we study further on reflexive R - R -submodules of $Q(\bar{R})$. For any additive submodules V, W of $Q(\bar{R})$, we put $(V \cdot W) = \{x \in Q(\bar{R}) : xW \subseteq V\}$, and $(W \cdot V) = \{x \in Q(\bar{R}) : Wx \subseteq V\}$.

Proposition 3.1. (i) If $N \in F(Q(\bar{R}); R, R)$ and $N \subseteq \bar{R}$, then $QN = NQ$.

(ii) Let $N \in F(Q(\bar{R}); R, R)$, and assume that $QN = NQ$. Then QN is an invertible Q - Q -submodule of $Q(\bar{R})$, $(QN)^{-1} = QN^{-1} = N^{-1}Q$, $QN^* = N^*Q$, and $\bar{R}N^* = (\bar{R}N)^*$. Furthermore, $(\bar{R}N \cdot R) = N^{-1}\bar{R}$, and $(\bar{R} \cdot N^{-1}\bar{R}) = \bar{R}N^*$.

(iii) Let M be a Q - Q -submodule of Q , and assume that M is invertible in $Q(\bar{R})$. Then $M \cap \bar{R} \in F^*(Q(\bar{R}); R, R)$, and $Q(M \cap \bar{R}) = M = (M \cap \bar{R})Q$. Further there is an invertible R - R -submodule V_0 of $Q(\bar{R})$ such that $V_0^{-1}(M \cap \bar{R})$, $(M \cap \bar{R})V_0^{-1} \in F^*\{R\}$ and $QV_0 = M = V_0Q$.

Proof. (i) First we prove that ${}_0QN_R$ is simple. Let U be any non-zero Q - R -submodule of QN . Then $Q = QNN^{-1} \supseteq UN^{-1} \neq 0$, and so $Q = UN^{-1}$, because ${}_0Q_R$ is simple. Then $QN = UN^{-1}N \subseteq U$, whence $U = QN$. Thus ${}_0QN_R$ is simple. Then there is an integer $n \geq 0$ such that $QN \cap X^{n-1} = 0$ and $QN \cap X^n \neq 0$. By making use of Corollary 1 of Lemma 2.3, we have $X^{n-1} \oplus QN = X^n$. Then, by Lemma 2.4, $QN \supseteq NQ$. Symmetrically $QN \subseteq NQ$, whence $QN = NQ$, as desired. (ii) $QN = NQ$ yields $N^{-1}Q = N^{-1}QNN^{-1} = N^{-1}NQN^{-1} = QN^{-1}$, and so $N^{-1}Q = QN^{-1}$. Therefore ${}_0QN_Q$ is invertible in $Q(\bar{R})$, and $(QN)^{-1} = N^{-1}Q = QN^{-1}$. Hence $QN = ((QN)^{-1})^{-1} = N^*Q = QN^*$. Now, $\bar{R} \otimes_R QN = \bar{R} \otimes_R Q \otimes_Q QN = \bar{Q} \otimes_Q QN \simeq \bar{Q} \cdot QN = \bar{R}QN$ (cf. Remark to Proposition 1.6), and therefore any right R -homomorphism f from \bar{R} to R can be extended to a right Q -homomorphism \bar{f} from $\bar{R}QN$ to QN . Then, for any $x \in (\bar{R}N)^*$, we can see that $\bar{f}(x) \in N^*$, whence it follows that $x \in \bar{R}N^*$, because \bar{R}_R is projective. (Cf. the proof of Lemma 2.11.) Since $\bar{R}N^* \subseteq (\bar{R}N)^*$ is evident we have $\bar{R}N^* = (\bar{R}N)^*$. Symmetrically, $Jy \subseteq N\bar{R}$ ($J \in T(R)$) implies that $y \in N^*\bar{R}$. Let $\bar{R}Nz \subseteq \bar{R}$. Then $N^{-1}Nz \subseteq N^{-1}\bar{R}$, and so $z \in N^{-1}\bar{R}$, because of $N^{-1} = (N^{-1})^*$. If $uN^{-1}\bar{R} \subseteq \bar{R}$ then $uN^{-1}N \subseteq \bar{R}N$, whence $u \in (\bar{R}N)^* = \bar{R}N^*$. This completes the proof of (ii). (iii) Since ${}_0Q_Q$ is simple, an invertible Q - Q -module M is also simple. Then, as in the proof of (i), $X^{n-1} \oplus M = X^n$ for some $n \geq 0$. Then $M \simeq X^n/X^{n-1}$, canonically. Let V_0 be as in Lemma 2.18. Then $M = Q \otimes_R V_0 = V_0 \otimes_R Q$, and ${}_R V_{0R}$ is invertible in $Q(\bar{R})$. Put $N = M \cap \bar{R}$. Then $N \neq 0$, for \bar{R}_R is essential in \bar{Q}_R . Put $I = \{x \in Q : V_0x \subseteq N\}$. Then $N = V_0 \otimes_R I$, because V_0 is invertible. Since V_{0R} is finitely generated, $JV_0 \subseteq \bar{R}$ for some non-zero ideal J of R . Put $\text{Hom}(\bar{R}_R, R_R)(JV_0) = J'$. Then, since \bar{R}_R is projective, J' is a

non-zero ideal of R . Noting that $\bar{R} \otimes_R Q = \bar{Q}$, we have $J'I \subseteq R$. Therefore, by Proposition 2.1, $I \in F\{R\}$. If $zI' \subseteq I(z \in Q, I' \in T(R))$ then $V_0 z I' \subseteq V_0 I = N$, and so $V_0 z \subseteq M \cap \bar{R} = N$, that is, $z \in I$. Thus $I \in F^*\{R\}$, and hence $N = V_0 I = V_0 \circ I \in F^*(Q(\bar{R}); R, R)$. Further, $NQ = V_0 I Q = V_0 Q = M$. Likewise $QN = M$. It is evident that $I = V_0^{-1}N$. Symmetrically $NV_0^{-1} \in F^*\{R\}$.

Let $N' \in F^*(Q(\bar{R}); R, R)$, and assume that $N' \subseteq \bar{R}$. Put $QN' \cap \bar{R} = N$. Then $N \in F^*(Q(\bar{R}); R, R)$. Therefore if we put $J = N' \circ N^{-1}$, then $J \in F^*(R)$, and $N' = J \circ N$. Evidently $QN \cap \bar{R} = N$. Further, as in (iii) above, $N = IV_0$, where $I \in F^*\{R\}$. Therefore $N' = (J \circ I)V_0$, where $J \circ I \in F^*\{R\}$, and V_0 is an invertible R - R -submodule of $Q(\bar{R})$ with $V_0 Q = QV_0 = QN'$.

Proposition 3.2. *Let $U \in F(Q(\bar{R}); R, R)$, and suppose that $\bar{R}U = U\bar{R}$ and $QU = UQ$.*

(i) $\bar{R}U \in F(Q(\bar{R}); \bar{R}, \bar{R})$, $(\bar{R}U)^{-1} = \bar{R}U^{-1} = U^{-1}\bar{R}$, and $QU^{-1} = U^{-1}Q$. Therefore $((\bar{R}U)^{-1})^{-1} = \bar{R}U^* = U^*\bar{R}$, and $QU^* = U^*Q$.

(ii) QU is written as a product $QU = M_2 M_1^{-1}$ with monic Q - Q -submodules M_i such that $\bar{Q}M_i = M_i \bar{Q}$ ($i=1, 2$).

(iii) $U^*Y = YU^*$.

Proof. (i), (ii) Put $M = QU$. Then, by assumption, $\bar{Q}M = M\bar{Q}$. By Proposition 3.1, $U^{-1}Q = QU^{-1} = M^{-1}$, and hence $\bar{Q}M \in F^*\{Q\}$, because of $\bar{Q}M^{-1} = M^{-1}\bar{Q} = (\bar{Q}M)^{-1}$. Therefore $\bar{Q}M = (\bar{Q}M_2)^{-1}(\bar{Q}M_1)$ for some monic Q - Q -submodules M_i such that $\bar{Q}M_i = M_i \bar{Q}$ ($i=1, 2$), by Lemma 2.20. Since $(\bar{Q}M_2)^{-1} = \bar{Q}M_2^{-1} = M_2^{-1}\bar{Q}$, we have $\bar{Q}M = \bar{Q}M_2^{-1}M_1$ and so $\bar{Q}M_2^{-1}M_1 M^{-1} = \bar{Q}$. Then $M_2^{-1}M_1 M^{-1}$ is a monic Q - Q -submodule, and so $M_2^{-1}M_1 M^{-1} = Q$, by [13; Corollary 1 of Proposition 1]. Hence $M = M_2^{-1}M_1$. As $\bar{R}U = U\bar{R}$, we have $U^{-1}\bar{R}U U^{-1} = U^{-1}U\bar{R}U^{-1}$, whence $U^{-1}\bar{R} = \bar{R}U^{-1}$ by Proposition 3.1 (ii). Since $UU^{-1} \in T(R)$, it follows from Remark 2 of Lemma 2.16 that $RU \cdot U^{-1}\bar{R} \in T(\bar{R})$. Similarly $\bar{R}U^{-1} \cdot U\bar{R} \in T(\bar{R})$. Hence $\bar{R}U \in F\{\bar{R}\}$. The remainder follows from Proposition 3.1 (ii). (iii) By (i), we may assume that $U = U^*$. Since $\bar{Q}M_i = M_i \bar{Q}$, it follows from [13; Corollary 1 of Proposition 1] that $XM_i = X^{n_i+1} \cap \bar{Q}M_i = X^{n_i+1} \cap M_i \bar{Q} = M_i X$, where $n_i = \deg M_i$ ($i=1, 2$). Then, as $M = M_2^{-1}M_1$, we have $XM = MX$. Since $U^{-1} \subseteq M^{-1}$, $UYU^{-1} \subseteq MXM^{-1} = X$, and so $UYU^{-1} \subseteq X \cap \bar{R} = Y$. Then $UYU^{-1}U \subseteq YU$. Now, $XM = X \otimes_Q M = Y \otimes_R M$, so that any right R -homomorphism from Y to R can be extended to a right Q -homomorphism from XM to M . Then, since Y_R is projective, we have $(YU)^* = YU$. Therefore $UY \subseteq YU$, and symmetrically $YU \subseteq UY$. Thus $YU = UY$. (Cf. the proof of Lemma 2.11.)

Theorem 3.3. *Assume that the condition (#) holds. Let M be a monic Q - Q -submodule of $Q\langle X \rangle$ such that $Q\langle X \rangle M = MQ\langle X \rangle$, and let $N = M \cap R\langle Y \rangle$. Then M is invertible in $S(Q\langle X \rangle)$, $N \in F^*(Q(\bar{R}); R, R)$, $M = QN = NQ$, and $Q\langle X \rangle M \cap R\langle Y \rangle = R\langle Y \rangle N = NR\langle Y \rangle$.*

Proof. By Lemma 2.20 and Proposition 3.1, M is invertible in $S(\bar{Q})$, $M=QN=NQ$, and $N \in F^*(Q(\bar{R}); R, R)$. Put $A=\bar{Q}M \cap \bar{R}$ and $\bar{R}N=B$. Then $A \supseteq B$, and $QB=BQ=\bar{Q}M=QA=AQ$. By Proposition 2.16 and Theorem 2.15, $(QA)^{-1}=QA^{-1}=A^{-1}Q$. Therefore $QA^{-1}B=QA^{-1} \cdot QB=(QA)^{-1}QA=B$, hence $I \subseteq A^{-1}B$ for some $I \in F_i(R)$. Then $AI \subseteq B$, so $AI^* \subseteq B^*=B$ by Proposition 3.1. Therefore if we put $I=\{x \in R: Ax \subseteq B\}$ then $I=I^*$. Assume that $I \neq R$. Then $I \subseteq P$ for some irreducible $P \in F_i^*(R)$. Put $B'=(B \cdot \bar{R})$. Then, by Proposition 3.1 (ii), $B'=N^{-1}\bar{R}$, and $BB'AI \subseteq AI \subseteq \bar{R}P$. Now $AI \cdot \bar{R}B'=AIB' \subseteq BB' \subseteq \bar{R}$, and so $\bar{R}B' \subseteq (AI)^{-1}$. Then, by Proposition 1.11, $\bar{R}B' \in F\{\bar{R}\}$, and so $\bar{R}B' \cdot AI \subseteq \bar{R}$ by virtue of the commutativity of $F^*\{\bar{R}\}$. Then, by Lemma 2.21 (ii), $B \subseteq \bar{R}P$ or $B'AI \subseteq \bar{R}P$. However, if $B \subseteq \bar{R}P$ then $NP^{-1} \subseteq \bar{R} \cap M=N$, so $P^{-1} \subseteq R$, a contradiction. On the other hand, if $B'AI \subseteq \bar{R}P$ then $\bar{R}B'AI \subseteq \bar{R}$, and so $\bar{R}B' \cdot A \cdot \bar{R}(I \circ P^{-1}) \subseteq \bar{R}$. Therefore $A \cdot \bar{R}(I \circ P^{-1}) \cdot \bar{R}B' \subseteq \bar{R}$, hence $A(I \circ P^{-1}) \subseteq (\bar{R} \cdot B')=B$ by Propositions 3.1 and 3.2. This is a contradiction. Thus $I=R$. Hence $A=B$, that is, $\bar{Q}M \cap \bar{R}=\bar{R}(M \cap \bar{R})$. Symmetrically $M\bar{Q} \cap \bar{R}=(M \cap \bar{R})\bar{R}$. This complete the proof.

Theorem 3.4. *Assume that the condition (#) holds. If every reflexive ideal of R is invertible then so is $R \langle Y \rangle$.*

Proof. Let A be any reflexive ideal of \bar{R} . Then A can be written as $A=(I\bar{R}) \circ (B \cap \bar{R})$, where $I \in F_i^*(R)$, and $B=QA=AQ$ (cf. Theorem 2.23). By assumption, $I\bar{R}$ is invertible. On the other hand, $B=\bar{Q}M=M\bar{Q}$ for some monic Q - Q -submodule M , by Lemma 2.20. Put $M \cap \bar{R}=N$. Then $B \cap \bar{R}=\bar{R}N=N\bar{R}$ by Theorem 3.3. By Proposition 3.1 (iii), N is written as a product $N=JV_0$, where $J \in F^*\{R\}$, and V_0 is an invertible R - R -submodule of $Q(\bar{R})$. By Propositions 2.1 and 1.6, J is invertible, hence so is N . Then $B \cap \bar{R}$ is invertible. In fact, $(B \cap \bar{R})^{-1}=N^{-1}\bar{R}=\bar{R}N^{-1}$. Thus A is invertible.

Theorem 3.5. *Assume that the condition (#) holds. Put $\bar{S}=\{N \in F^*(Q(\bar{R}); R, R): QN=NQ, \bar{R}N=N\bar{R}\}$. Then $\lambda: \bar{S} \xrightarrow{\sim} F^*\{\bar{R}\}$ as group, where $\lambda(N)=\bar{R}N$.*

Proof. By Proposition 3.2, λ is well defined, and is a group homomorphism. If $\bar{R}N=\bar{R}$ then $N, N^{-1} \subseteq \bar{R}$. On the other hand, $\bar{Q} \cdot QN=\bar{Q}$, and so $QN=Q$, as in the proof of Proposition 3.2. Hence $N, N^{-1} \subseteq Q$. Therefore $N, N^{-1} \subseteq \bar{R} \cap Q=R$. Thus $N=R$. Let $A \in F_i^*(\bar{R})$. Then $A=(\bar{R}I) \circ (\bar{R}N)$, where $I \in F_i^*(R)$, and N is as in Theorem 3.3. Therefore $\text{Im } \lambda \supseteq F_i^*(\bar{R})$, and so $\text{Im } \lambda=F^*\{\bar{R}\}$, because of Proposition 2.1.

Assume that the condition (#) holds. Evidently $\lambda(N) \subseteq \bar{R}$ if and only if $N \subseteq \bar{R}$, so that λ induces a semi-group isomorphism from $S=\{N \in \bar{S}: N \subseteq \bar{R}\}$ to $F^*(\bar{R})$. Further, by Theorem 3.3, $S_p=\{N \in S: QN \cap \bar{R}=N\}$ is isomorphic to $\{A \in F_i^*(\bar{R}): QA \cap \bar{R}=A\}$. Therefore $\bar{S}_p=\{N_1 \circ N_2^{-1}: N_1, N_2 \in$

$S_p\} \cong \text{Im } \phi (\cong F^*\{\bar{Q}\})$ as group. Hence the direct product $F^*\{\bar{R}\} = \text{Im } \phi \times \text{Ker } \psi$ induces the direct product $\bar{S} = \bar{S}_p \times F^*\{R\}$. Let $N \in S_p$. Then N is written as a product $N = V_0 I$, where $I \in F^*\{R\}$, and V_0 is an invertible R - R -submodule of $Q(\bar{R})$ such that $QV_0 = V_0 Q$. Then $\bar{R}N = N\bar{R} = V_0 I \bar{R} = V_0 \bar{R} I$, and so $\bar{R}V_0 I I^{-1} = V_0 \bar{R} I I^{-1}$. Hence $V_0 \bar{R} \subseteq (\bar{R}V_0)^* = \bar{R}V_0$ by Proposition 3.1. Symmetrically $\bar{R}V_0 \subseteq V_0 \bar{R}$, whence $V_0 \bar{R} = \bar{R}V_0$. Therefore \bar{S} is generated by $F^*\{R\}$ and the subgroup of all invertible R - R -submodules V of $Q(\bar{R})$ with $QV = VQ$, $\bar{R}V = V\bar{R}$.

Finally we note the following

Lemma 3.6. *If R is a prime Goldie ring and $Q = Q(R)$, then any monic Q - Q -submodule is invertible in $Q(\bar{R})$.*

Proof. Let M be a monic Q - Q -submodule of degree n . We may assume that $n \geq 1$. Then, since $MQ = M \otimes_Q \bar{Q}$, any right Q -homomorphism f from M to Q can be extended to a right \bar{Q} -homomorphism \bar{f} from $M\bar{Q}$ to \bar{Q} . Since $Q(\bar{R})_{\bar{Q}}$ is injective (cf. §4. Appendix), \bar{f} is given by a left multiplication of an element of $Q(\bar{R})$. Since M_Q is a generator, if we put $M' = \{x \in Q(\bar{R}) : xM \subseteq Q\}$ then $M'M = Q$. Symmetrically $MM'' = Q$ for some Q - Q -submodule M'' of $Q(\bar{R})$. Hence ${}_Q M_Q$ is invertible in $Q(\bar{R})$.

4. Appendix

Lemma 4.1 *If ${}_R R$ is Noetherian then so is ${}_{\bar{R}} \bar{R}$.*

Proof. It suffices to prove that any left ideal of \bar{R} is finitely generated. Let I be any left ideal of \bar{R} . For any integer $n \geq 0$, Y^n/Y^{n-1} is an invertible R - R -bimodule, and hence there exists a unique left ideal I_n of R such that $I \cap Y^n + Y^{n-1} = Y^n I_n + Y^{n-1}$. Then $I_0 = I \cap R \subseteq I_1 \subseteq I_2 \subseteq \dots$. Therefore, $I_m = I_{m+1} = \dots$ for some m . Put $J = I_m$. Since ${}_R J$ and ${}_R Y^m$ are finitely generated, ${}_R Y^m J$ is also finitely generated, so that $Y^m J \subseteq \sum_i R a_i + Y^{m-1}$ for some a_1, \dots, a_r of $I \cap Y^m$. Then, for any $n \geq m$, $I \cap Y^n \subseteq Y^n J + Y^{n-1} \subseteq \sum_i Y^{n-m} a_i + Y^{n-1}$, and so $I \cap Y^n = \sum_i Y^{n-m} a_i + I \cap Y^{n-1}$. Therefore $I \cap Y^n \subseteq \sum_i \bar{R} a_i + I \cap Y^{m-1}$ for all $n \geq m$. Hence $I = \sum_i \bar{R} a_i + I \cap Y^{m-1}$. Since ${}_R I \cap Y^{m-1}$ is finitely generated, ${}_{\bar{R}} I$ is finitely generated.

If R is a prime Goldie ring and $Q = Q(R)$, then \bar{Q} is a prime Goldie ring, by Lemma 4.1. Hence, as is well known, $Q(\bar{Q})_{\bar{Q}}$ is injective.

In the sequel, R is any ring. Let σ, τ be automorphisms of R , and D an endomorphism of R as an additive group. If $D(xy) = \sigma(x)D(y) + D(x)\tau(y)$ for all $x, y \in R$, then D is said to be a (σ, τ) -derivation of R ([5]). If $\sigma = id_R$, D is called a τ -derivation. Let I be a dense right ideal of R , and f a right R -homomorphism from I to $Q_r(R)$. Then, as is well known, there exists a unique element b of $Q_r(R)$ such that $f(x) = bx$ for all $x \in I$ (cf. [16]). Let ν

be any automorphism of R . Then ν is uniquely extended to an automorphism of $Q_r(R)$, and symmetrically of $Q_l(R)$. And these induce the same automorphism of $Q(R)$. Therefore we denote these automorphisms by ν , too.

Lemma 4.2. *Let τ be an automorphism of R , and g an additive homomorphism from a dense right ideal I to $Q_r(R)$ such that $g(xa) = g(x)\tau(a)$ for all $x \in I$, $a \in R$. Then there exists a unique element b of $Q_r(R)$ such that $g(x) = b \cdot \tau(x)$ for all $x \in I$.*

Proof. Put $h = g\tau^{-1}$. Then h is a right R -homomorphism from a dense right ideal $\tau(I)$ to $Q_r(R)$. Hence there exists a unique element b of $Q_r(R)$ such that $h(\tau(x)) = b \cdot \tau(x)$ for all $x \in I$.

Lemma 4.3. *Let D be a (σ, τ) -derivation of R . Then D is uniquely extended to a (σ, τ) -derivation of $Q_r(R)$, and symmetrically of $Q_l(R)$. And these induce the same (σ, τ) -derivation of $Q(R)$.*

Proof. Let $b \in Q_r(R)$, and let I be a dense right ideal of R such that $bI \subseteq R$. A map g from I to $Q_r(R)$ is defined by $g(x) = D(bx) - \sigma(b)D(x)$ ($x \in I$). Then g is as in Lemma 4.2, whence there exists a unique $b' \in Q_r(R)$ such that $g(x) = b' \cdot \tau(x)$ for all $x \in I$. Note that b' does not depend on the choice of I . Put $D'(b) = b'$. Then D' is a unique (σ, τ) -derivation of $Q_r(R)$ such that $D'|R = D$. Similarly D is uniquely extended to a (σ, τ) -derivation D'' of $Q_l(R)$, and it is easy to verify that $D'|Q(R) = D''|Q(R)$.

We denote D' , D'' , and $D'|Q(R)$ by D , too.

Let D be a τ -derivation of R , and put $Q = Q(R)$. By Lemma 4.2, the skew polynomial ring $R[t; \tau, D]$ defined by $at = t\tau(a) + D(a)$ ($a \in R$) is a subring of the skew polynomial ring $Q[t; \tau, D]$. Put $Y = R + tR$ and $X = Q + tQ$. Then, for any $i \geq 1$, $Y^i = R + tR + \dots + t^i R$, and $X^i = Q + tQ + \dots + t^i Q$. It is easy to see that these satisfy the conditions in §2.

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