Title: On semi-primary abelian categories
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Let $C$ be an abelian category with exact direct limits, namely cocomplete $C_3$-category ([5], p. 83).

In this note we always assume that $C$ contains a generator $U$, and hence $C$ is locally small by [5], p. 71. In [2], Gabriel and Popesco have given a characterization of $U$ being projective and small by using the concept of localization in [1]. We shall give another proof without localization in the section 1.

In the section 2, we shall define a function $\varphi$ of $C$ into itself, which is analogous to the radical of semi-primary ring.

We shall show that $C$ has such a function when the endomorphism ring $[U, U]$ is a semi-primary ring, and we shall give some criteria by means of $\varphi$ that $U$ is small and projective.

In the section 3, we shall add some remarks in the previous author’s work on category of triangular matrices, [4].

In this note we shall freely make use of concepts in categories from [5].

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1. Preliminary results

In this section we shall summarize all results which we need in the following sections.

Almost all results in this section have been proved in [2] and [6] by using the concepts of localization in [1]. However, we shall give here another approach to them by means of rather homological method.

Let $C$ be an abelian cocomplete $C_3$-category ([5], p. 81) and $U$ an object of $C$. Let $A=\langle U, U \rangle$. By mod $A$ we mean the category of $A$-right modules. Let $T:C\rightarrow$ mod $A; T(V)=[U, V]$ for any $V \in C$ be the functor of $C$ into mod $A$. In this case we can define a coadjoint $S$ of $T$ such that $S(M)=M \otimes U$ by [5], p. 143, namely $\eta: [M, T(V)] \approx [S(M), V]$. Furthermore, we have natural transformation $\varphi_V: ST(V) \rightarrow V$ and $\varphi_M: M \rightarrow TS(M)$, (see[5], pp 118–119).

**Theorem 0** (Gabriel and Popesco [2]). Let $C$, $U$ and $A$ be as above. Then the following statements are equivalent:
1) $U$ is a generator.
2) $T$ is a completely faithful (namely, full and faithful).

3) $\psi_V$ is isomorphic for all $V \in C$ and $S$ is an exact functor.

Proof. 1) $\rightarrow$ 2). See [2] or [5] in which we do not need the concept of localization.

3) $\rightarrow$ 2). $[ST(V), T(V')] \cong [T(V), T(V')]$ and $[ST(V), V'] \cong [V, V']$ for $V, V' \in C$.

2) $\rightarrow$ 3). $[ST(V), V'] \cong [T(V), T(V')] \cong [V, V']$. Hence, $[ST(V), ]$ and $[V, ]$ give the equivalent functors. Therefore, $\psi_V = \eta^{-1} \alpha^{-1} I_V$ is isomorphic.

Thus, it remains to show that $S$ is exact. First, we show that if $M \in \text{mod } A$ is contained in a free module $F$, then $0 \rightarrow S(M) \rightarrow S(F)$ is exact. In order that, we assume first that $M$ is finitely generated, say $M = (m_1, m_2, \ldots, m_n)$ and hence we may assume that $F$ is also finitely generated. Then we have a commutative diagram

$$
0 \leftarrow M \leftarrow \sum_{i=1}^{n} A v_{\beta_i} \leftarrow \kappa A = \sum_{\alpha \in k} A w_{\alpha} \\
F = \sum_{i=1}^{m} A u_{a_i} = \sum_{i=1}^{m} A u_{a_i},
$$

where $u_{a_i}, v_{\beta_i}$ and $w_{\alpha}$ are free bases and $i$ is the inclusion map, $f$ is a natural mapping such that $f(v_{\beta_i}) = m_i, \alpha = i_f$, and $K = \ker f$.

Operating $S$ on the above 1) we obtain commutative exact diagram:

$$
0 \leftarrow M \leftarrow \sum_{i=1}^{n} A v_{\beta_i} \leftarrow \kappa A = \sum_{\alpha \in k} A w_{\alpha} \\
F = \sum_{i=1}^{m} A u_{a_i} = \sum_{i=1}^{m} A u_{a_i},
$$

where $V = \text{im } (\kappa U \rightarrow \sum_{i=1}^{n} U)$ and $K' = \ker S(\alpha)$.

It is clear that there exists the inclusion map $i_i$ of $V$ into $K'$. Operating again $T$ on 2) we have

$$
0 \leftarrow 0 \leftarrow T(K') \leftarrow T(i_i) \leftarrow T(V) \leftarrow T(\kappa U) \leftarrow \kappa A
$$

3) $\sum \oplus A v_{\beta_i} \leftarrow T(i_i) \leftarrow T(\kappa U) \leftarrow \kappa A \leftarrow \phi_{KA}$
where the vertical line is exact and $T(i_0), T(i_1)$ are inclusions. Since $K$ is also $\ker \alpha$, there exists a unique isomorphism $\theta$ such that

$$
\begin{array}{cc}
T(K') & \theta \to K \\
\downarrow i_s & \\
\sum \oplus A_{\beta_i} & \\
\end{array}
$$

is commutative. Let $a \in T(K')$ and put $k = \theta a$. Then $T(i_0)T(\beta)\varphi_{KA}w_k = i_1a = i_0 \theta a = i_0 k = i_s T(i_0)T(\beta)\varphi_{KA}w_k$ by the naturality of $\varphi$. Put $b = T(\beta)\varphi_{f_k} \in T(V)$, $i_s a = i_0 T(i_0)b$. Since $i_s$ is injective, $a = T(i_0)b$. Hence, $T(i_0)$ is isomorphic. Since $T$ is faithful, $i_0$ is isomorphic by [5], p. 56. Therefore, $0 \to S(M) \to S(F)$ is exact from (2). Next, let $M$ be any submodule of free $A$-module $F$: $0 \to M \to F$. Then $M$ is a direct limit of the family of finitely generated $A$-submodules $M_{\alpha i}$; $M = \lim \lim M_{\alpha i}$. Since $S$ is colimit and exact preserving by [5], p. 85 and p. 55, $0 \to S(M) = \lim S(M_{\alpha i}) \to S(F)$ is exact from the first argument. Hence, $\text{Tor}^1(M, U) = 0$ for all $M \in \text{mod } A$, ([5], p. 112, § 8), which implies that $S$ is exact.

From now on we fix a generator $t$ in $C$ and $A = [U, U]$. Then for any subobject $U'$ in $U$ it is clear that $[U, U']$ is identified to a right ideal in $A$, and we shall denote it by $r_U U'$ or $r$. By $KU$ we mean the image of $f: \sum_{k \in K} U_k \to U$ defined by $f(U_k) = kU$ for any subset $K$ in $A$. We note from the definitions that $r_{U'} U = ST(U')$. Then we have from [5], p. 71.

**Lemma 1.** For any subobject $U'$ in $U$ we have $U' = r_{U} U$.

**Lemma 2.** Let $U$ be a generator in $C$ and $r_1, r_2$ right ideals in $A$. Then we have

1) $(r_1 + r_2) U = r_1 U \cup r_2 U$.
2) $(r_1 \cap r_2) U = r_1 U \cap r_2 U$.

Proof. 1) is trivial from the definition.

2) We have the following row exact and commutative diagrams:

$$
\begin{array}{cc}
0 & \to r_1 U \to r_1 U \cup r_2 U \to (r_1 U \cup r_2 U)/r_2 U \to 0 \\
\uparrow & \uparrow & \uparrow \approx \\
0 & \to r_1 U \cap r_2 U \to r_1 U \cap r_2 U \to 0 \\
\uparrow & \uparrow \approx \\
0 & \to (r_1 \cap r_2) U \to r_1 U \cap (r_1 \cap r_2) U, \text{ and} \\
\uparrow & \uparrow \\
0 & \to r_2 \to r_2 \cup r_1 \to (r_2 \cup r_1)/r_2 U \to 0 \\
\uparrow & \uparrow \approx \\
0 & \to r_1 \cap r_2 \to r_1 \cap r_2 \to 0
\end{array}
$$

Since $S$ is an exact functor, we obtain $(r_1 \cap r_2) U = r_1 U \cap r_2 U$ from 4) by operating $S$ on 5).
The following proposition is an immediate consequence of [6], Prop. 1.1 and [5], p. 104. However, we shall prove it without localization.

**Proposition 3.** Let $C$, $U$ and $A$ be as above and $U$ a generator. Then the following statements are equivalent.

1) $S(\_)= \otimes U$ is an equivalent functor.
2) $T(\_)=\left[ U, \_ \right]$ and $S(\_)$ give a one-to-one correspondence between right ideals and subobjects in $U$.
3) For any maximal right ideal $r$ in $A$ $S(A|r) \neq 0$.
4) $U$ is projective and small in $C$.

**Proof.** 1) $\Rightarrow$ 2) $\Rightarrow$ 3) are trivial.
4) $\Rightarrow$ 1) is proved in [5], p. 104.
3) $\Rightarrow$ 4) It is clear from 3) that for any non-zero $A$-module $M$, $S(M)=M \otimes U \neq 0$, since $S$ is exact by Theorem 0. Let $V_1 \xrightarrow{\alpha} V_2 \rightarrow 0$ be exact in $C$ and $T(V_1) \rightarrow T(V_2) \rightarrow T(K) \rightarrow 0$ be exact in mod $A$.
Since $S$ is exact, $ST(V_1)=V_1 \xrightarrow{\alpha} ST(V_2)=V_2 \rightarrow S(K) \rightarrow 0$ is exact. Hence, $S(K)=0$, which means $K=0$ from the above. Therefore, $T$ is exact and hence, $U$ is projective. Finally we shall show that $U$ is small. Let $f: U \rightarrow \sum_{i \in I} V_i$ be a morphism in $C$, where $V_i$'s are any objects in $C$.

Put $U_j=f^{-1}(\sum_{k \in J} V_k)$, where $J$ is a finite set of $I$. Since $C$ is $C_5$-category, $U=\cup U_j$ by [5] p. 83. Then $A=\cup r_j$ by Lemma 2 and 3), where $r_j=[U, U_j]$.

Put $1=\sum_{i \in I} f_i$, $f_i \in r_{j_i}$. Then $U=\cup_{i \in I} U_{j_i}$, which implies $im f \subset \sum_{i \in I} \sum_{j \in J_i} V_j$.

An object $V$ in $C$ is called minimal if there exist no proper subobjects in $V$. If $V'$ is a direct sum of minimal sub-objects, then $V'$ is called semi-simple. We note that some properties of semi-simple modules are valid in $C$.

**Lemma 4.** For any artinian and noetherian object $V$, $[V, V]$ is a semi-primary ring.

It is well known in mod $A$, and its proof is valid in $C$.

**2. Semi-primary category $C$**

Let $C$ be an abelian category mentioned in the section 1. We shall consider a function $\varphi$ of object in $C$ into itself which is similar to the radical of a ring.

I. $\varphi(C)$ is a subobject in $C$ for any $C$ in $C$ such that $C/\varphi(C)$ is semi-simple.

II. $C=\varphi(C)$ if and only if $C=0$.

III. If $C/C'$ is semi-simple for some subobject $C'$ in $C$, then $C' \supset \varphi(C)$.

Let $\varphi$, $\varphi_1$ be functions in $C$ satisfying I and II. We note in this case that every non-zero object contains a maximal subobject. If $\varphi_1(C) \supset \varphi_2(C)$ for all
C ∈ C, then we shall say φ₂ is smaller than φ₁. Furthermore, if φ₂ satisfies III and C is locally small, then φ₂ is a unique minimal function among those satisfying I and II, since φ₂(C) = ∩ D, where D runs all maximal subobjects in C. In this case φ₂ is a functor which satisfies the following commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\varphi(C) & \downarrow{\varphi(f)} & \varphi(C') \\
\end{array}
\]

where \(f \in C\) and \(i, i'\) are inclusions and \(\varphi(f)\) is defined as follows: Let \(V\) be a maximal subobject in \(C'\) then \(f^{-1}(V) = C\) or \(C/f^{-1}(V) \cong C'/V\) ([5], pp. 22–24), and hence \(f(\varphi(C)) \subset V\), which implies \(\text{im}(f|\varphi(C)) \subset \varphi(C')\). Conversely, if \(\varphi\) satisfying I, II induces a functor in \(C\) satisfying 6), then \(\varphi\) satisfies III. In fact, let \(V \neq 0\) in \(C\), then \(V\) contains a maximal subobject \(V_0\). The commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{f} & V/V_0 \\
\varphi(V) & \xrightarrow{\varphi(f)} & \varphi(V/V_0) = 0 \\
\end{array}
\]

shows \(\varphi(V) \subset V_0\).

We put \(\varphi'(U) = \varphi(U)\), \(\varphi^i(U) = \varphi(\varphi^{i-1}(U))\).

**Lemma 5.** Let \(U\) be a generator of \(C\). If \(\varphi^i\) is defined in \(U\) such that \(\varphi^n(U) = 0\) for some \(n\) and satisfies I, II (resp. I, II and III), then \(\varphi\) induces a function \(\varphi\) in \(C\) such that \(\varphi\) satisfies I, II (reps. I, II and III).

**Proof.** First, we define \(\overline{\varphi}(\varphi^i(U)) = \varphi^{i+1}(U)\) for all \(i\). Let \(V\) be any object in \(C\) which is different from any \(\varphi^i(U)\), and \(g: \sum_{\{i, j\} \neq f} U_f \to V\) the canonical morphism defined by \(f: U_f \to V\). We assume that \(\text{im}(g|\varphi(U)) = V\) and \(\text{im}(\sum \varphi^{i+1}(U)) = V\). Then define \(\varphi(V) = \text{im}(g|\sum \varphi^{i+1}(U))\). It is clear that \(V/\varphi(V)\) is semi-simple and that \(V/\varphi(V) \neq 0\) if \(V \neq 0\). Next, we assume \(\varphi\) satisfies III for \(U\). Let \(V_0\) be a maximal subobject in \(V\), then \(f^{-1}(V_0) \supset \varphi(U)\). Therefore, \(\varphi(V) \subset V_0\).

**Definition.** Let \(V\) be an object in \(C\). If \([V, V]\) is a semi-primary ring, \(V\) is called a semi-primary object.

From Lemma 4, every artinian and noetherian object is semi-primary.

**Proposition 6.** Let \(U\) be a projective, small generator in an abelian \(C_\pi\)-category. Then \(U\) is semi-primary if and only if a function \(\varphi\) in \(U\) satisfying I, II and III is defined and \(U/\varphi(U)\) is a directsum of finite many of simple objects and \(\varphi^n(U) = 0\) for some \(n\).
Proof. It is clear from Theorem 0 and Proposition 3. We note here that \( \varphi'(U) = S(n'U) \), where \( n' \) is the radical of \( [U, U] \).

The main purpose of this section is to study some structure of \( C_3 \)-category with semi-primary generator.

**Theorem 7.** Let \( C \) be an abelian \( C_3 \)-category with semi-primary generator \( U \). Then we can define a function \( \varphi \) in \( C \) which satisfies I and II and \( U/\varphi(U) \) is a finite directsum of simple subobjects and \( \varphi''(U) = 0 \) for some \( n \).

Proof. Let \( A = [U, U] \) and \( n \) the radical of \( A \). Put \( U_i = n'^i U \) for all \( i \). It is clear that \( U_i \supset U_{i+1} \). Put \( \tau_i = [U, U_i] \). Then \( \tau_{i+1} = n' \cap \tau_{i+1} \).

Then \( \tau_{i+1} U = U_i \cap U_{i+1} = U_{i+1} \) by Lemma 2. Since \( n'/n'^i \) is semi-simple, so is \( n'/\tau_{i+1} \), say \( n'/\tau_{i+1} = \bigoplus \tau_{a_i}/\pi \), \( n' \supset \tau_{a_i} \supset \tau_{i+1} \), and \( \tau_{a_i} \) is simple. Put \( U_{a_i} = \tau_{a_i} U \).

If \( U_{a_i} = U_{i+1} \), \( \tau_{a_i} \supset n' \cap \tau_{i+1} \), which is a contradiction. Hence, \( U_i \supset U_{a_i} \supset U_{i+1} \).

We shall show that \( U_{a_i}/U_{i+1} \) is simple. Let \( V \) be a subobject such that \( U_{a_i} \supset V \). Then \( \tau_{a_i} \supset V \). In fact if \( \tau_{a_i} \supset V \), \( \tau_{a_i} \supset \tau_{i+1} \), and hence, \( U_{a_i} \supset V \). Therefore, \( V = \tau_{a_i} U = \tau_{a_i} U_{a_i} \). Since \( n' = U \cup U_{a_i} \), \( U_i = U_{a_i} \). On the other hand, \( \tau_{a_i} \cup \tau_{a_j} = \tau_{i+1} \). Hence, \( U_{a_i} \cap U_{a_j} = U_{i+1} \).

Since \( C \) is \( C_3 \)-category, \( U_i/U_{i+1} \approx \bigoplus A_{a_i}/U_{i+1} \) is semi-simple. We define \( \varphi'(U) = U_i \). Then \( U/\varphi(U) \) is a finite directsum of simple subobjects from the above, and \( \varphi''(U) = 0 \) if \( n'' = 0 \). Then we can define a function \( \varphi \) in \( C \) from Lemma 5.

Let \( V_0 \) be a subobject in \( V \) such that \( V_0 + V' = V \) implies \( V = V' \) for any subobject \( V' \) in \( V \). \( V_0 \) is called negligible. By \([U:U_i]\) we mean the number of simple components in \( U/U_i \).

**Theorem 8.** Let \( C \) be an abelian \( C_3 \)-category with semi-primary generator \( U \). Then the following conditions are equivalent.

1) \( U \) is projective and small.
2) \([A:n]=[U: \varphi(U)]\), where \( \varphi(U) = nU \), \( A = [U, U] \) and \( n \) is the radical of \( A \).
3) \( \varphi(U) \) is negligible in \( U \).
4) \( \varphi \) satisfies the condition III.
5) \( T: C \to \text{mod} A \) is preserving minimal objects.

Proof. If \( U \) is projective and small, then \( C \) is equivalent to \( \text{mod} A \) by Proposition 3. Hence, 2), 3), 4) and 5) are trivial. We assume 2). We put \( a = [U, nU] \). If we restrict the argument in the proof of Theorem 7 to the case of \( i = 1 \), we get \([A:a] = [U: nU] = n \). Hence, \( a = n \). For every maximal right ideal \( r \), \( r = \sum \tau_{a_i}/n \), which implies \( U \subseteq \bigcup \tau_{a_i} U = U \). Hence, we obtain 1) from Proposition 3.
3) Let α be as above. We assume αφn. Then there exists a right ideal b properly containing n such that a/π=(eA+n)/π. Let e be an idempotent element in A such that b/π=(eA+n)/π. Since bφn, bUφnU=aU. Hence, U=(a+b)U=bU. Put U_0=eU. Then U_0+nU=(eA+n)U=bU=U. Therefore, U_0=U by 3). Hence, e=1, which is a contradiction.

4) If nφα, we obtain the fact U=U_0+nU and U_0φU. Since U/U_0 contains a maximal object from Theorem 7, there exists a maximal subobject VφU. Therefore, U_0φU.

5) If nφα, then there exists a maximal subobject V in U such that U=V+nU. Since 0→[U, V]→[U, U]→[U, U/V] is exact and [U, U/V] is minimal, r_V=[U, V] is a maximal right ideal, and hence r_Vφn, which is a contradiction.

It is clear that there are many examples in which semi-primary generators are not projective.

**Corollary 1.** Let U be a semi-primary generator in C. If A/n is a simple rings, U is projective and small, where A=[U, U] and n is its radical.

Proof. Let α=[U, nU]. Since U nU, and α is a two-sided ideal, α=n.

**Corollary 2.** Let B be a semi-primary ring and U be a semi-primary generator in the category of B-right modules. Then n_A Uφn_B. n_A U=U n_B if and only if U is a finitely generated and projective, where, A=[U, U] and n_A (resp. n_B) is the radical of A (resp. B).

Proof. Let φ(U)=U n_B. Then φ is a functor in mod B satisfying I, II and III. Hence, n_A UφU n_B by Theorem 7. If n_A U=U n_B, a function φ' defined in the proof of Theorem 7 satisfies III. Hence, U is projective and small. The converse is trivial.

**Example.** We shall show that there exists a generator U such that φ^i are defined in U satisfying the following conditions: U/φ(U) is a finite directsum of simple object, φ^i satisfies I, II and III for all i and φ^i(U)=0 for some n, however U is not semi-primary.

Let k be a field and K=k(x). Let A=(k 0

\[ k \]

\[ k \]

be a tri-angular matrix ring. Then A is semi-primary with radical n. We define φ(U)=U n in mod A. Put τ=(0 0

\[ k[x] \]

0). Then τ is a right ideal in A. Then [A/τ, A/τ]≈(k 0

\[ k(x)/k[x] \]

k[0]) is not semi-primary. U=A A/τ is the desired generator.

3. Abelian category of commutative diagram

We recall the definition of abelian category of commutative diagram over abelian categories C_i (see [4]).
Let $I=(1, 2, \ldots, n)$ be a finite linear ordered set and $\{C_i\}_{i \in I}$ a family of abelian categories. We assume that there are given cokernel preserving functors $T_{ij}: C_i \to C_j$ for $i<j$. Furthermore, we assume:

(*) There exist natural transformations

$$\psi_{ijk}: T_{jk}T_{ij} \to T_{ik} \quad \text{for all } i<j<k,$$

and

(**) For any $i<j<k<l$ and $V$ in $C_i$

$$T_{hi}T_{jk}T_{ij}(V) \xrightarrow{T_{hi}(\psi_{ijk})} T_{hi}T_{ih}(V) \xrightarrow{T_{ij}(\psi_{ijk})} T_{ih}(V)$$

is commutative.

We call a family of morphism $d_{ij}: T_{ij}(V_i) \to V_j$ an arrow for $V_i \in C_i$, $V_j \in C_j$ and for all $i<j$, when the diagrams

$$T_{jk}T_{ij}(V_i) \xrightarrow{T_{jk}(d_{ij})} T_{jk}(V_j)$$

are commutative.

We define a commutative diagram $[I, C_i]$ as follows; Its objects consist of set \{Vi\}_{i \in I} with arrows \{d_{ij}\} and morphisms consist of set \{(fi)\}_{i \in I}; f_i: V_i \to V_i'$ in $C_i$ such that $d_{ij}T_{ij}(f_i) = f_jd_{ij}$.

**Lemma 9.** Let $T_{ij}$ be functors satisfying (**) Then the natural transformation of $T_{in-1}T_{in-2} \cdots T_{i_{i_0}} \to T_{i_{i_0}}$ does not depend on any choice of combination of $T_{in-1}, \ldots, T_{i_{i_0}}$.

Proof. We can prove the lemma by using induction on the number of functors and naturality of $\psi_{ijk}$. Namely, every natural transformation is equal to $T_{in-1}T_{in-2} \cdots T_{i_{i_0}}(f_i) \to T_{in-1}T_{in-2} \cdots T_{i_{i_0}}$(**).

**Theorem 10.** Let $C_i$ be abelian category with projective class $\xi_i$. Then every $[I, C_i]$ with $T_{ij}$ is imbedding in a good category $[I, C_i]$ with $T_{ij}'$.

Proof. We shall define new functors $T_{ij}'$.
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Then it is clear that $T'_{ij}$ are cokernel preserving and $\psi'_{ijk}=I_{C_k}$ and (** is trivial. Furthermore, there exist unique natural transformations $\phi_{ij}: T'_{ij}\to T_{ij}$ by Lemma 9. Put $C=[I, C_i]$ with $T_{ij}$ and $C'=[I, C_i]$ with $T'_{ij}$. We define a function $F$ of $C$ into $C'$ as follows: For $V=(V_i)$ with arrows $d_{ij}$ in $C$ we put $F(V)=(V_i)$ with the following arrows $d'_{ij}$:

$$
\begin{align*}
    d'_{ii+1} &= d_{ii+1} \\
    d'_{ij} &= d_{ij}\phi_{ij}^T_{ij} \\
    \text{for } i+1<j.
\end{align*}
$$

We have to show that $d'_{ij}$ satisfies (**). We have a diagram for $i<j<k$ and $V_t\in C_t$

$$
\begin{array}{c}
T'_{jk}T'_{ij}(V_i) \\
\downarrow{\psi'_{ijk}} \\
T'_{ik}(V_i)
\end{array}
\begin{array}{c}
\xrightarrow{T'_{jk}(\phi)} \\
\downarrow{\phi} \\
\xrightarrow{d'_{ik}} \\
\text{III}
\end{array}
\begin{array}{c}
T'_{jk}(V_j) \\
\downarrow{T(d_{ij})} \\
\downarrow{d_{jk}} \\
\text{II}
\end{array}
\begin{array}{c}
T_{jk}(V_j) \\
\downarrow{\phi} \\
\downarrow{T_{ik}(\phi)} \\
\text{I}
\end{array}
\begin{array}{c}
T_{jk}(V_i) \\
\xrightarrow{T_{jk}(\phi)} \\
\downarrow{T'_{jk}(\phi)} \\
\xrightarrow{T'_{jk}(\phi)} \\
V_k
\end{array}
$$

I is commutative by Lemma 9, II is commutative by naturality of $\phi$ and so is III by (**). Hence, $d'_{ij}$ satisfies (**). Define $F((f_i))=(f_i)$ for morphism $(f_i)$ in $C$. Then we can similarly show that $F$ is a functor. It is clear that $F$ is an imbedding functor. Since $\psi'_{ijk}=I_{C_k}$, $K'(P_i)=0$ in (*) of [4], Lemma 3.7. Hence, $C'$ is good by [4], Lemma 3.7.

If every objects in $C$ are projective, $C$ is called a semi-simple category.

**Corollary.** Let $C_i$ be a semi-simple abelian category. Then $[I, C_i]$ is imbedding in an abelian hereditary category, (cf. [3], Theorem 5).

**Proof.** It is clear from Theorem 10 and [4], Theorem 3.12.

Finally, we note that if $C_i$ have functor $\varphi_i$ satisfying I, II and III and $T_{ij}(\varphi_i(V_i))\subseteq \varphi_jT_{ij}(V_i)$ on $V=(V_i)$ in $[I, C_i]$. Then

$$
\varphi(V) = (\varphi(V_1), \varphi(V_2) \cup d_{i_2}(V_2), \ldots, \varphi(V_j) \cup \bigcup_{i<j} d_{ij}(V_i), \ldots)
$$

is a functor on $[I, C_i]$ satisfying I, II and III. If $\varphi_{nm}=0$ for all $t$ then $\varphi^{nm}=0$.

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References


