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Osaka University
ON SEMI-PRIMARY ABELIAN CATEGORIES
Dedicated to Professor Atuo Komatu for his 60th birthday

MANABU HARADA

(Received June 7, 1968)

Let $C$ be an abelian category with exact direct limits, namely cocomplete $C_3$-category ([5], p. 83).

In this note we always assume that $C$ contains a generator $U$, and hence $C$ is locally small by [5], p. 71. In [2], Gabriel and Popesco have given a characterization of $U$ being projective and small by using the concept of localization in [1]. We shall give another proof without localization in the section 1.

In the section 2, we shall define a function $\varphi$ of $C$ into itself, which is analogous to the radical of semi-primary ring.

We shall show that $C$ has such a function when the endomorphism ring $[U,U]$ is a semi-primary ring, and we shall give some criteria by means of $\varphi$ that $U$ is small and projective.

In the section 3, we shall add some remarks in the previous author's work on category of tri-angular matrices, [4].

In this note we shall freely make use of concepts in categories from [5].

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1. Preliminary results

In this section we shall summarize all results which we need in the following sections.

Almost all results in this section have been proved in [2] and [6] by using the concepts of localization in [1]. However, we shall give here another approach to them by means of rather homological method.

Let $C$ be an abelian cocomplete $C_3$-category ([5], p. 81) and $U$ an object of $C$. Let $A=[U,U]$. By mod $A$ we mean the category of $A$-right modules. Let $T:C\to \text{mod} A$; $T(V)=[U,V]$ for any $V\in C$ be the functor of $C$ into mod $A$. In this case we can define a coadjoint $S$ of $T$ such that $S(M)=M\otimes A U$ by [5], p. 143, namely $\eta: [M, T(V)]\cong [S(M), V]$. Furthermore, we have natural transformation $\psi_V: ST(V)\to V$ and $\varphi_M: M\to TS(M)$, (see [5], pp 118-119).

Theorem 0 (Gabriel and Popesco [2]). Let $C$, $U$ and $A$ be as above. Then the following statements are equivalent:
1) $U$ is a generator.
2) $T$ is a completely faithful (namely, full and faithful).

3) $\psi_V$ is isomorphic for all $V \in C$ and $S$ is an exact functor.

Proof. 1) $\Rightarrow$ 2). See [2] or [5] in which we do not need the concept of localization.

3) $\Rightarrow$ 2). $[ST(V), V'] \simeq [T(V), T(V')]$ and $[ST(V), V'] \simeq [V, V']$ for $V, V' \in C$.

2) $\Rightarrow$ 3). $[ST(V), V'] \simeq [T(V), T(V')] \simeq [V, V']$. Hence, $[ST(V), \ ]$ and $[V, \ ]$ give the equivalent functors. Therefore, $\psi_V = \eta^{-1} \alpha^{-1} I_V$ is isomorphic.

Thus, it remains to show that $S$ is exact. First, we show that if $M \in \text{mod} A$ is contained in a free module $F$, then $0 \rightarrow S(M) \rightarrow S(F)$ is exact. In order that, we assume first that $M$ is finitely generated, say $M = (m_1, m_2, \ldots, m_n)$ and hence we may assume that $F$ is also finitely generated. Then we have a commutative diagram

$$
\begin{array}{c}
0 \leftarrow M \leftarrow \bigoplus_{i=1}^{n} A v_{\beta_i} \leftarrow \kappa A = \bigoplus_{a} A w_{\alpha}
\end{array}
$$

where $u_{a_i}, v_{\beta_i}$ and $w_{\alpha}$ are free bases and $i$ is the inclusion map, $f$ is a natural mapping such that $f(v_{\beta_i}) = m_i$, $\alpha = \text{id}$, and $K = \ker f$.

Operating $S$ on the above 1) we obtain commutative exact diagram:

$$
\begin{array}{c}
0 \leftarrow S(M) \leftarrow S(F) \leftarrow \bigoplus_{i=1}^{n} U \leftarrow V
\end{array}
$$

where $V = \text{im} (\kappa U \overset{\beta}{\rightarrow} \bigoplus_{i=1}^{n} U)$ and $K' = \ker S(\alpha)$.

It is clear that there exists the inclusion map $i_i$ of $V$ into $K'$. Operating again $T$ on 2) we have

$$
\begin{array}{c}
\sum \bigoplus A v_{\beta_i} \leftarrow T(V) \leftarrow T(\bigoplus U) \leftarrow T(\kappa U) \leftarrow \kappa A
\end{array}
$$
where the vertical line is exact and $T(i_1), T(i_2)$ are inclusions. Since $K$ is also $\ker \alpha$, there exists a unique isomorphism $\theta$ such that

$$
\begin{array}{c}
T(K') \\
\downarrow \theta \\
\sum \oplus A_{\beta_i}
\end{array}
\quad \begin{array}{c}
i_2 \\
\downarrow i_i
\end{array}
\quad \begin{array}{c}
i_1
\end{array}
\quad \begin{array}{c}
i
\end{array}

is commutative. Let $a \in T(K')$ and put $k = \theta a$. Then $T(i_1)T(\beta)\varphi w_k = i_1 a = i_1 \theta a = i_1 k = i_1 T(i_2)T(\beta)\varphi w_k$ by the naturality of $\varphi$. Put $b = T(\beta)\varphi w_k \in T(V)$, $i_3 a = i_3 T(i_1)b$. Since $i_3$ is injective, $a = T(i_1)b$. Hence, $T(i_1)$ is isomorphic. Since $T$ is faithful, $i_1$ is isomorphic by [5], p. 56. Therefore, $0 \to S(M) \to S(F)$ is exact from 2). Next, let $M$ be any submodule of free $A$-module $F$: $0 \to M \to F$. Then $M$ is a direct limit of the family of finitely generated $A$-submodules $M_{a_i}$; $M = \lim M_{a_i}$. Since $S$ is colimit and exact preserving by [5], p. 85 and p. 55, $0 \to S(M) = \lim S(M_{a_i}) \to S(F)$ is exact from the first argument. Hence, Tor$^1(M, U) = 0$ for all $M \in \text{mod } A$, ([5], p. 112, § 8), which implies that $S$ is exact.

From now on we fix a generator $t$ in $C$ and $A = [U, U]$. Then for any subobject $U'$ in $U$ it is clear that $[U, U']$ is identified to a right ideal in $A$, and we shall denote it by $r_{U'}$ or $r$. By $KU$ we mean the image of $f': \sum U_k \to U$ defined by $f(U_k) = kU$ for any subset $K$ in $A$. We note from the definitions that $r_{U'}U = ST(U')$. Then we have from [5], p. 71.

**Lemma 1.** For any subobject $U'$ in $U$ we have $U' = r_{U'}U$.

**Lemma 2.** Let $U$ be a generator in $C$ and $r_1$, $r_2$ right ideals in $A$. Then we have

1) $(r_1 + r_2)U = r_1 U \cup r_2 U$.
2) $(r_1 \cap r_2)U = r_1 U \cap r_2 U$.

**Proof.** 1) is trivial from the definition.

2) We have the following row exact and commutative diagrams:

$$
\begin{array}{c}
0 \to r_1 U \\
\uparrow \quad \uparrow \approx
\end{array}
\begin{array}{c}
r_1 U \cup r_2 U \\
\uparrow \approx
\end{array}
\begin{array}{c}
(r_1 U \cup r_2 U) / r_2 U \to 0
\end{array}
\begin{array}{c}
0 \to r_1 U \cap r_2 U \\
\uparrow \approx
\end{array}
\begin{array}{c}
r_1 U / (r_1 \cap r_2)U, \quad \text{and}
\end{array}
\begin{array}{c}
0 \to r_2 \cup r_1 \\
\uparrow \approx
\end{array}
\begin{array}{c}
(r_2 \cup r_1) / r_2 \to 0
\end{array}
\begin{array}{c}
0 \to r_1 \cap r_2 \\
\uparrow \approx
\end{array}
\begin{array}{c}
r_1 / r_1 \cap r_2 \to 0
\end{array}

4) $0 \to r_1 U \cap r_2 U \to r_1 U \to r_1 U / r_1 U \cap r_2 U \to 0$

5) $0 \to r_1 \cap r_2 \to r_1 \cap r_1 / r_1 \cap r_2 \to 0$

Since $S$ is an exact functor, we obtain $(r_1 \cap r_2)U = r_1 U \cap r_2 U$ from 4) by operating $S$ on 5).
The following proposition is an immediate consequence of [6], Prop. 1.1 and [5], p. 104. However, we shall prove it without localization.

**Proposition 3.** Let $C$, $U$ and $A$ be as above and $U$ a generator. Then the following statements are equivalent.

1) $S(\cdot) = \otimes U$ is an equivalent functor.

2) $T(\cdot) = [U, \cdot]$ and $S(\cdot)$ give a one-to-one correspondence between right ideals and subobjects in $U$.

3) For any maximal right ideal $r$ in $A$ $S(A/r) \neq 0$.

4) $U$ is projective and small in $C$.

**Proof.** 1) $\rightarrow$ 2) $\rightarrow$ 3) are trivial.

4) $\rightarrow$ 1) is proved in [5], p. 104.

3) $\rightarrow$ 4) It is clear from 3) that for any non-zero $A$-module $M$, $S(M) = M \otimes U \neq 0$, since $S$ is exact by Theorem 0. Let $V_1 \rightarrow V_2 \rightarrow 0$ be exact in $C$ and $T(V_1) \rightarrow T(V_2) \rightarrow K \rightarrow 0$ be exact in mod $A$. Since $S$ is exact, $ST(V_1) = V_1 \rightarrow ST(V_2) = V_2 \rightarrow S(K) = 0$ is exact. Hence, $S(K) = 0$, which means $K = 0$ from the above. Therefore, $T$ is exact and hence, $U$ is projective. Finally we shall show that $U$ is small. Let $f: U \rightarrow \sum_{i \in I} V_i$ be a morphism in $C$, where $V_i$'s are any objects in $C$.

Put $U_j = f^{-1}(\sum_{k \in J} V_k)$, where $J$ is a finite set of $I$. Since $C$ is $C_3$-category, $U = \cup U_j$ by [5] p. 83. Then $A = \cup r_j$ by Lemma 2 and 3), where $r_j = [U, U_j]$.

Put $1 = \sum_{i \in I} f_i$, $f_i \in r_{f_i}$. Then $U = \bigcup_{i=1}^{J} U_{j_i}$, which implies $\text{im } f = \sum_{i=1}^{J} \sum_{j \in J} V_i$.

An object $V$ in $C$ is called minimal if there exist no proper subobjects in $V$. If $V'$ is a direct sum of minimal sub-objects, then $V'$ is called semi-simple. We note that some properties of semi-simple modules are valid in $C$.

**Lemma 4.** For any artinian and noetherian object $V$, $[V, V]$ is a semi-primary ring.

It is well known in mod $A$, and its proof is valid in $C$.

**2. Semi-primary category $C$**

Let $C$ be an abelian category mentioned in the section 1. We shall consider a function $\phi$ of object in $C$ into itself which is similar to the radical of a ring.

I. $\phi(C)$ is a subobject in $C$ for any $C$ in $C$ such that $C/\phi(C)$ is semi-simple.

II. $C = \phi(C)$ if and only if $C = 0$.

III. If $C/C'$ is semi-simple for some subobject $C'$ in $C$, then $C' \supset \phi(C)$.

Let $\phi, \phi_1$ be functions in $C$ satisfying I and II. We note in this case that every non-zero object contains a maximal subobject. If $\phi_1(C) \supset \phi_2(C)$ for all
$C \in C$, then we shall say $\varphi_2$ is smaller than $\varphi_1$. Furthermore, if $\varphi_2$ satisfies III and $C$ is locally small, then $\varphi_2$ is a unique minimal function among those satisfying I and II, since $\varphi_2(C) = \bigcap D$, where $D$ runs all maximal subobjects in $C$. In this case $\varphi_2$ is a functor which satisfies the following commutative diagram

$$
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\uparrow \varphi(C) & \quad & \uparrow \varphi(C') \\
\varphi(f) & \xrightarrow{\varphi(i)} & \varphi(i')
\end{array}
$$

where $f \in C$ and $i, i'$ are inclusions and $\varphi(f)$ is defined as follows: Let $V$ be a maximal subobject in $C'$ then $f^{-1}(V) = C$ or $C|f^{-1}(V) \cong C'|V$ ([5], pp. 22-24), and hence $f(\varphi(C)) \subset V$, which implies $\text{im}(f|\varphi(C)) \subset \varphi(C')$. Conversely, if $\varphi$ satisfying I, II induces a functor in $C$ satisfying 6), then $\varphi$ satisfies III. In fact, let $V \neq 0$ in $C$, then $V$ contains a maximal subobject $V_0$. The commutative diagram

$$
\begin{array}{ccc}
V & \longrightarrow & V/V_0 \\
\uparrow \varphi(V) & \quad & \uparrow \varphi(V/V_0) = 0
\end{array}
$$

shows $\varphi(V) \leq V_0$.

We put $\varphi^i(U) = \varphi(U)$, $\varphi^i(U) = \varphi(\varphi^{i-1}(U))$.

**Lemma 5.** Let $U$ be a generator of $C$. If $\varphi^i$ is defined in $U$ such that $\varphi^n(U) = 0$ for some $n$ and satisfies I, II (resp. I, II and III), then $\varphi$ induces a function $\Phi$ in $C$ such that $\Phi$ satisfies I, II (rep. I, II and III).

**Proof.** First, we define $\bar{\varphi}(\varphi^i(U)) = \varphi^{i+1}(U)$ for all $i$. Let $V$ be any object in $C$ which is different from any $\varphi^i(U)$, and $g: \bigoplus_{\{V, \varphi^i(U)\}} U \rightarrow V$ the canonical morphism defined by $f: U \rightarrow V$. We assume that $\text{im}(g|\bigoplus \varphi^i(U)) = V$ and $\text{im}(g|\bigoplus \varphi^{i+1}(U)) \neq V$. Then define $\varphi(V) = \text{im}(g|\bigoplus \varphi^{i+1}(U))$. It is clear that $V/\varphi(V)$ is semi-simple and that $V/\varphi(V) \neq 0$ if $V \neq 0$. Next, we assume $\varphi$ satisfies III for $U$. Let $V_0$ be a maximal subobject in $V$, then $f^{-1}(V_0) \supset \varphi(U)$. Therefore, $\varphi(V) \subseteq V_0$.

**Definition.** Let $V$ be an object in $C$. If $[V, V]$ is a semi-primary ring, $V$ is called a semi-primary object.

From Lemma 4, every artinian and noetherian object is semi-primary.

**Proposition 6.** Let $U$ be a projective, small generator in an abelian $C_\gamma$-category. Then $U$ is semi-primary if and only if a function $\varphi$ in $U$ satisfying I, II and III is defined and $U/\varphi(U)$ is a directsum of finite many of simple objects and $\varphi^n(U) = 0$ for some $n$. 
Proof. It is clear from Theorem 0 and Proposition 3. We note here that 
\( \varphi(U) = S(n'U) \), where \( n \) is the radical of \([U, U]\).

The main purpose of this section is to study some structure of \( C \)-category with semi-primary generator.

**Theorem 7.** Let \( C \) be an abelian \( C \)-category with semi-primary generator \( U \). Then we can define a function \( \varphi \) in \( C \) which satisfies I and II and \( U/\varphi(U) \) is a finite direct sum of simple subobjects and \( \varphi^n(U) = 0 \) for some \( n \).

Proof. Let \( A = \langle [U, U] \rangle \) and \( n \) the radical of \( A \). Put \( U_i = n'^i U \) for all \( i \).

It is clear that \( U_i U_i \supseteq U_{i+1} \). Let \( r_i = \langle [U, U] \rangle \). Then \( r_i U_{i+1} = U_{i+1} \) by Lemma 2. Since \( n'/n'^{i+1} \) is semi-simple, so is \( n'/r_i \), say \( n'/r_i = \bigoplus \theta_i n_\alpha \), \( n' \supseteq r_{i+1} \supseteq r_{i+1} \), and \( r_{i+1} \) is simple. Put \( U_{i+1} = r_{i+1} U \).

If \( U_{i+1} = r_{i+1} U \), \( U_{i+1} \subset n' U_i \), which is a contradiction. Hence, \( U_i U_i \supseteq U_{i+1} \).

We shall show that \( U_{i+1} \) is simple. Let \( V \) be a subobject such that \( U_{i+1} \supseteq V \). Then \( V = r_{i+1} U \), in fact if \( r_{i+1} U \supseteq V_\alpha \), \( r_{i+1} U \supseteq V_\alpha \), and hence, \( U_{i+1} = r_{i+1} U \). Therefore, \( V = r_{i+1} U \). Since \( n'/r_i \) is semi-simple, we obtain \( \sum U_{i+1} \) is semi-simple. We define \( \varphi(U) = U_{i+1} \). Then \( U_\alpha \supseteq \varphi(U) \) is a finite direct sum of simple subobjects from the above, and \( \varphi(U) = 0 \) if \( n' = 0 \). Then we can define a function \( \varphi \) in \( C \) from Lemma 5.

Let \( V_0 \) be a subobject in \( V \) such that \( V_0 V' \) implies \( V = V' \) for any subobject \( V' \) in \( V \). \( V_0 \) is called negligible. By \( [U : U] \) we mean the number of simple components in \( U_\alpha \).

**Theorem 8.** Let \( C \) be an abelian \( C \)-category with semi-primary generator \( U \). Then the following conditions are equivalent.

1) \( U \) is projective and small.
2) \( [A : \alpha] = [U : \varphi(U)] \), where \( \varphi(U) = nU \), \( A = [U, U] \) and \( n \) is the radical of \( A \).
3) \( \varphi(U) \) is negligible in \( U \).
4) \( \varphi \) satisfies the condition III.
5) \( T : C \to \text{mod} A \) is preserving minimal objects.

Proof. If \( U \) is projective and small, then \( C \) is equivalent to \( \text{mod} A \) by Proposition 3. Hence, 2) 3) 4) and 5) are trivial. We assume 2). We put \( a = [U, U] \). If we restrict the argument in the proof of Theorem 7 to the case of \( i = 1 \), we get \( [A : a] = [U : nU] \), \( = n \). Hence, \( a = n \). For every maximal right ideal \( r \), \( r/n = \sum \theta_i r_{i+1} / n \), which implies \( U = \bigcup_{i=1}^{n-1} r_{i+1} U = rU \). Hence, we obtain 1) from Proposition 3.
3) Let \( \alpha \) be as above. We assume \( \alpha \psi n \). Then there exists a right ideal \( b \) properly containing \( n \) such that \( a/n \oplus b/n = A/n \). Let \( e \) be an idempotent element in \( A \) such that \( b/n = (eA + n)/n \). Since \( b \supseteq n \), \( bU \supseteq nU = aU \). Hence, \( U = (a + b)U = bU \). Put \( U_0 = eU \). Then \( U_0 + nU = (eA + n)U = bU = U \). Therefore, \( U_0 = U \) by 3. Hence, \( e = I_\alpha \), which is a contradiction.

4) If \( n \neq \alpha \), we obtain the fact \( U = U_0 + nU \) and \( U_0 \neq U \). Since \( U/U_0 \) contains a maximal object from Theorem 7, there exists a maximal subobject \( V \supset U_0 \). Therefore, \( V \supset U \).

5) If \( n \neq \alpha \), then there exists a maximal subobject \( V \) in \( U \) such that \( U = V + nU \). Since \( 0 \rightarrow [U, V] \rightarrow [U, U] \rightarrow [U, U/V] \) is exact and \( [U, U/V] \) is minimal, \( \tau_V = [U, V] \) is a maximal right ideal, and hence \( \tau_V \supset n \), which is a contradiction.

It is clear that there are many examples in which semi-primary generators are not projective.

**Corollary 1.** Let \( U \) be a semi-primary generator in \( C \). If \( A/n \) is a simple rings, \( U \) is projective and small, where \( A = [U, U] \) and \( n \) is its radical.

**Proof.** Let \( a = [U, nU] \). Since \( U = nU \), and \( a \) is a two-sided ideal, \( a = n \).

**Corollary 2.** Let \( B \) be a semi-primary ring and \( U \) be a semi-primary generator in the category of \( B \)-right modules. Then \( n_A U \supset U n_B \). \( n_A U = U n_B \) if and only if \( U \) is a finitely generated and projective, where \( A = [U, U] \) and \( n \) (resp. \( n_B \)) is the radical of \( A \) (resp. \( B \)).

**Proof.** Let \( \varphi(U) = U n_B \). Then \( \varphi \) is a functor in \( \text{mod} \ B \) satisfying I, II and III. Hence, \( n_A U \supset U n_B \) by Theorem 7. If \( n_A U = U n_B \), a function \( \varphi' \) defined in the proof of Theorem 7 satisfies III. Hence, \( U \) is projective and small. The converse is trivial.

**Example.** We shall show that there exists a generator \( U \) such that \( \varphi^i \) are defined in \( U \) satisfying the following conditions: \( U/\varphi(U) \) is a finite direct sum of simple object, \( \varphi^i \) satisfies I, II and III for all \( i \) and \( \varphi^i(U) = 0 \) for some \( n_i \), however \( U \) is not semi-primary.

Let \( k \) be a field and \( K = k(x) \). Let \( A = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix} \) be a tri-angular matrix ring. Then \( A \) is semi-primary with radical \( n \). We define \( \varphi(U) = U n \) in \( \text{mod} A \). Put \( \tau = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \). Then \( \tau \) is a right ideal in \( A \). Then \( [A/\tau, A/\tau] \approx \begin{pmatrix} k & 0 \\ k[x]/k[x] & k[x] \end{pmatrix} \) is not semi-primary. \( U = A \oplus A/\tau \) is the desired generator.

3. Abelian category of commutative diagram

We recall the definition of abelian category of commutative diagram over abelian categories \( C_i \) (see [4]).
Let $I=(1, 2, \ldots, n)$ be a finite linear ordered set and $\{C_i\}_{i \in I}$ a family of abelian categories. We assume that there are given cokernel preserving functors $T_{ij}: C_i \to C_j$ for $i<j$. Furthermore, we assume:

(*) There exist natural transformations

$$\psi_{ijk}: T_{jk}T_{ij} \to T_{ik} \quad \text{for all } i<j<k,$$

and

(**) For any $i<j<k<l$ and $V$ in $C_i$

$$\begin{align*}
T_{kl}T_{jk}T_{ij}(V) & \xrightarrow{T_{kl}(\psi)} T_{kl}T_{ik}(V) \\
T_{jl}T_{jk}(V) & \xrightarrow{\psi_{ijk}} T_{ij}(V)
\end{align*}$$

is commutative.

We call a family of morphism $d_{ij}: T_{ij}(V_i) \to V_j$ an arrow for $V_i \in C_i$, $V_j \in C_j$ and for all $i<j$, when the diagrams

$$\begin{align*}
T_{jk}T_{ij}(V_i) & \xrightarrow{T_{jk}(d_{ij})} T_{jk}(V_j) \\
T_{ik}(V_i) & \xrightarrow{d_{ik}} V_k
\end{align*}$$

are commutative.

We define a commutative diagram $[I, C]$ as follows; Its objects consist of set $\{V_i\}_{i \in I}$ with arrows $\{d_{ij}\}$ and morphisms consist of set $\{(f_i)_{i \in I}; f_i: V_i \to V_i'\}$ in $C_i$ such that $d_{ij}T_{ij}(f_i)=f_jd_{ij}.$

**Lemma 9.** Let $T_{ij}$ be functors satisfying (**) Then the natural transformation of $T_{in-i'n}T_{in-zin-1}\cdots T_{izi} \to T_{i'in}$ does not depend on any choice of combination of $T_{in-i'n}, \cdots, T_{izi}$.

Proof. We can prove the lemma by using induction on the number of functors and naturality of $\psi_{ijk}$. Namely, every natural transformation is equal to $T_{in-i'n}(T_{in-zin-1}(\cdots (T_{izi}T_{izi})\cdots))=T_{i'in}.$

We assume that all $C_i$ have projective class $\xi_i$. We define a functor $S_i: C_i \to [I, C]$ by setting $S_i(V_i)=(0, \ldots, 0, V_i, T_{ii+1}(V_i), \ldots, T_{in}(V_i)).$ Then the projective objects in $[I, C]$ are of the form $\oplus S_i(P_i)$ and their retract, where $P_i$ is $\xi_i$-projective for all $i$, ([4], Prop. 1.2'). If the projective objects in $[I, C]$ are only of the former forms, we call $[I, C]$ a good category of commutative diagram.

**Theorem 10.** Let $C_i$ be abelian category with projective class $\xi_i$. Then every $[I, C]$ with $T_{ij}$ is imbedding in a good category $[I, C]$ with $T'_{ij}$.

Proof. We shall define new functors $T'_{ij}$:
Then it is clear that $T'_{ij}$ are cokernel preserving and $\psi'_{ijk}=I_{Ck}$ and (***) is trivial. Furthermore, there exist unique natural transformations $\psi_{ij}: T'_{ij}\rightarrow T_{ij}$ by Lemma 9. Put $C=[I, C_i]$ with $T_{ij}$ and $C'=[I, C_i]$ with $T'_{ij}$. We define a function $F$ of $C$ into $C'$ as follows: For $V=(V_i)$ with arrows $d_{ij}$ in $C$ we put $F(V)=(V_i)$ with the following arrows $d'_{ij}$:

\[ d'_{ii+1}=d_{ii+1}, \]
\[ d'_{ij}=d_{ij}\phi_{ij}T'_{ij} \quad \text{for} \quad i+1<j. \]

We have to show that $d'_{ij}$ satisfies (***) . We have a diagram for $i<j<k$ and $V_i\subseteq C_t$

\[
\begin{array}{ccc}
T'_{jk}T'_{ij}(V_i) & \xrightarrow{T'(\phi)} & T'_{jk}T_{ij}(V_i) \\
\psi'_{ijk} & \downarrow & \phi \\
T_{jk}T_{ij}(V_i) & \xrightarrow{T(d_{ij})} & T_{jk}(V_j) \\
\end{array}
\]

I is commutative by Lemma 9, II is commutative by naturality of $\phi$ and so is III by (***) . Hence, $d'_{ij}$ satisfies (***) . Define $F((f_i))=(f_i)$ for morphism $(f_i)$ in $C$. Then we can similarly show that $F$ is a functor. It is clear that $F$ is an imbedding functor. Since $\psi_{ijk}=I_{Ck}$, $K'(P_i)=0$ in (*) of [4], Lemma 3.7. Hence, $C'$ is good by [4], Lemma 3.7.

If every objects in $C$ are projective, $C$ is called a semi-simple category.

**Corollary.** Let $C_i$ be a semi-simple abelian category. Then $[I, C_i]$ is imbedding in an abelian hereditary category, (cf. [3], Theorem 5).

**Proof.** It is clear from Theorem 10 and [4], Theorem 3.12.

Finally, we note that if $C_i$ have functor $\phi_i$ satisfying I, II and III and $T_{ij}(\phi_i(V_i))\subseteq \phi_jT_{ij}(V_i)$ on $V=(V_i)$ in $[I, C_i]$. Then

\[ \phi(V) = (\phi_i(V_1), \phi_i(V_2)\cup d_{i_1}(V_2), \cdots, \phi_j(V_j) \bigcup_{i<j} d_{ij}(V_i), \cdots) \]

is a functor on $[I, C_i]$ satisfying I, II and III. If $\phi'_{t}=0$ for all $t$ then $\phi'^{nm}=0$. 

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References


