ON SEMI-PRIMARY ABELIAN CATEGORIES

Dedicated to Professor Atuo Komatu for his 60th birthday

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Let $C$ be an abelian category with exact direct limits, namely cocomplete $C_3$-category ([5], p. 83).

In this note we always assume that $C$ contains a generator $U$, and hence $C$ is locally small by [5], p. 71. In [2], Gabriel and Popesco have given a characterization of $U$ being projective and small by using the concept of localization in [1]. We shall give another proof without localization in the section 1.

In the section 2, we shall define a function $\varphi$ of $C$ into itself, which is analogous to the radical of semi-primary ring.

We shall show that $C$ has such a function when the endomorphism ring $[U,U]$ is a semi-primary ring, and we shall give some criteria by means of $\varphi$ that $U$ is small and projective.

In the section 3, we shall add some remarks in the previous author's work on category of tri-angular matrices, [4].

In this note we shall freely make use of concepts in categories from [5].

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1. Preliminary results

In this section we shall summarize all results which we need in the following sections.

Almost all results in this section have been proved in [2] and [6] by using the concepts of localization in [1]. However, we shall give here another approach to them by means of rather homological method.

Let $C$ be an abelian cocomplete $C_3$-category ([5], p. 81) and $U$ an object of $C$. Let $A=[U,U]$. By mod $A$ we mean the category of $A$-right modules. Let $T:C \to \text{mod} A$, $T(V)=[U,V]$ for any $V \in C$ be the functor of $C$ into mod $A$. In this case we can define a coadjoint $S$ of $T$ such that $S(M)=M \otimes_A U$ by [5], p. 143, namely $\eta: [M, T(V)] \approx [S(M), V]_C$. Furthermore, we have natural transformation $\psi_V: ST(V) \to V$ and $\varphi_M: M \to TS(M)$, (see[5], pp 118–119).

**Theorem 0** (Gabriel and Popesco [2]). Let $C, U$ and $A$ be as above. Then the following statements are equivalent:

1) $U$ is a generator.
2) \( T \) is a completely faithful (namely, full and faithful).
3) \( \psi_V \) is isomorphic for all \( V \in C \) and \( S \) is an exact functor.

3)→2). \([ST(V), V']\approx [T(V), T(V')]\) and \([ST(V), V']\approx [V, V']\) for \( V, V' \in C \).
2)→3). \([ST(V), V']\approx [T(V), T(V')]\approx [V, V']\). Hence, \([ST(V), \] and \([V, \] give the equivalent functors. Therefore, \( \psi_V = \eta^{-1} \alpha^{-1} I_V \) is isomorphic.

Thus, it remains to show that \( S \) is exact. First, we show that if \( M \in \text{mod} A \) is contained in a free module \( F \), then \( 0 \rightarrow S(M) \rightarrow S(F) \) is exact. In order that, we assume first that \( M \) is finitely generated, say \( M = (m_1, m_2, \ldots, m_n) \) and hence we may assume that \( F \) is also finitely generated. Then we have a commutative diagram

\[
0 \leftarrow M \overset{f}{\leftarrow} \sum_{i=1}^{n} A v_{\beta_i} \leftarrow \kappa A = \sum_{i=1}^{n} A w_{\alpha_i}
\]

where \( u_{\alpha_i}, v_{\beta_i} \) and \( w_{\alpha_i} \) are free bases and \( i \) is the inclusion map, \( f \) is a natural mapping such that \( f(v_{\beta_i}) = m_i, \alpha = ef, \) and \( K = \ker f \).

Operating \( S \) on the above 1) we obtain commutative exact diagram:

\[
0 \leftarrow S(M) \overset{S(f)}{\leftarrow} \sum_{i=1}^{n} U \leftarrow \iota \leftarrow V
\]

where \( V = \text{im} \left( \sum_{i=1}^{n} U + \beta \right) \) and \( K' = \ker S(\alpha) \).

It is clear that there exists the inclusion map \( i \) of \( V \) into \( K' \). Operating again \( T \) on 2) we have:

\[
\sum \oplus A v_{\beta_i} \leftarrow T(V) \overset{T(\beta)}{\leftarrow} T(U) \overset{\varphi \kappa A}{\leftarrow} \kappa A
\]

where \( \varphi \) is the natural mapping.
where the vertical line is exact and $T(i_1), T(i_2)$ are inclusions. Since $K$ is also $\ker \alpha$, there exists a unique isomorphism $\theta$ such that

$$
\begin{array}{ccc}
T(K') & \xrightarrow{\theta} & K \\
\downarrow i_2 & & \downarrow i_1 \\
\sum \oplus A_{i_j} & & \\
\end{array}
$$

is commutative. Let $a \in T(K')$ and put $k=\theta a$. Then $T(i_1)T(\beta)\varphi_{KA}w_k=i_1a = i_1\theta a = i_1k = i_1T(i_2)T(\beta)\varphi_{KA}w_k$ by the naturality of $\varphi$. Put $b=T(\beta)\varphi w_k \in T(V)$, $i_2a=i_2T(i_1)b$. Since $i_2$ is injective, $a = T(i_1)b$. Hence, $T(i_1)$ is isomorphic. Since $T$ is faithful, $i_2$ is isomorphic by [5], p. 56. Therefore, $0 \to S(M) \to S(F)$ is exact from 2). Next, let $M$ be any submodule of free $A$-module $F$: $0 \to M \to F$. Then $M$ is a direct limit of the family of finitely generated $A$-submodules $M_{a_i}$; $M=\lim M_{a_i}$. Since $S$ is colimit and exact preserving by [5], p. 85 and p. 55, $0 \to S(M)=\lim S(M_{a_i}) \to S(F)$ is exact from the first argument. Hence, $\text{Tor}^1(M, u)=0$ for all $M \in \text{mod } A$, ([5], p. 112, § 8), which implies that $S$ is exact.

From now on we fix a generator $t/u$ in $C$ and $A=[U, U]$. Then for any subobject $U'$ in $U$ it is clear that $[U, U']$ is identified to a right ideal in $A$, and we shall denote it by $r_{U'}$ or $r$. By $KU$ we mean the image of $f: \sum U_k \to U$ defined by $f(U_k)=kU$ for any subset $K$ in $A$. We note from the definitions that $r_{U'}U=ST(U')$. Then we have from [5], p. 71.

**Lemma 1.** For any subobject $U'$ in $U$ we have $U'=r_{U'}U$.

**Lemma 2.** Let $U$ be a generator in $C$ and $r_1, r_2$ right ideals in $A$. Then we have

1) $(r_1+r_2)U=r_1U \cup r_2U$.

2) $(r_1 \cap r_2)U=r_1U \cap r_2U$.

Proof. 1) is trivial from the definition.

2) We have the following row exact and commutative diagrams:

$$
\begin{array}{ccccccc}
0 & \to & r_1U & \to & r_1U \cup r_2U & \to & (r_1U \cup r_2U)/r_2U & \to & 0 \\
\uparrow & & \uparrow \approx & & \uparrow \approx & & \uparrow \approx & & \uparrow \approx \\
0 & \to & r_1U \cap r_2U & \to & r_1U & \to & r_1U/((r_1 \cap r_2)U) & \to & 0 \\
\uparrow & & \uparrow \approx & & \uparrow \approx & & \uparrow \approx & & \uparrow \approx \\
0 & \to & r_2 \cup r_1 & \to & (r_2 \cup r_1)/r_2 & \to & 0 \\
\uparrow & & \uparrow \approx & & \uparrow \approx & & \uparrow \approx & & \uparrow \approx \\
0 & \to & r_1 \cap r_2 & \to & r_1/((r_1 \cap r_2)U) & \to & 0 \\
\end{array}
$$

Since $S$ is an exact functor, we obtain $(r_1 \cap r_2)U=r_1U \cap r_2U$ from 4) by operating $S$ on 5).
The following proposition is an immediate consequence of [6], Prop. 1.1 and [5], p. 104. However, we shall prove it without localization.

**Proposition 3.** Let $C$, $U$ and $A$ be as above and $U$ a generator. Then the following statements are equivalent.
1) $S(\cdot) = \otimes U$ is an equivalent functor.
2) $T(\cdot) = [U, \cdot]$ and $S(\cdot)$ give a one-to-one correspondence between right ideals and subobjects in $U$.
3) For any maximal right ideal $r$ in $A$, $S(A/r) = 0$.
4) $U$ is projective and small in $C$.

Proof. 1) $\rightarrow$ 2) $\rightarrow$ 3) are trivial.
4) $\rightarrow$ 1) is proved in [5], p. 104.
3) $\rightarrow$ 4) It is clear from 3) that for any non-zero $A$-module $M$, $S(M) = M \otimes U$ $= 0$, since $S$ is exact by Theorem 0. Let $V_1 \xrightarrow{\alpha} V_2 \rightarrow 0$ be exact in $C$ and $T(V_1) \rightarrow T(V_2) \rightarrow K \rightarrow 0$ be exact in mod $A$. Since $S$ is exact, $ST(V_1) = V_1 \xrightarrow{\alpha} ST(V_2) = V_2 \rightarrow S(K) \rightarrow 0$ is exact. Hence, $S(K) = 0$, which means $K = 0$ from the above. Therefore, $T$ is exact and hence, $U$ is projective. Finally we shall show that $U$ is small. Let $f: U \rightarrow \sum V_i$ be a morphism in $C$, where $V$'s are any objects in $C$.
Put $U_j = f^{-1}(\sum V_h)$, where $J$ is a finite set of $I$. Since $C$ is $C_3$-category, $U = \bigcup U_j$ by [5] p. 83. Then $A = \bigcup r_j$ by Lemma 2 and 3), where $r_j = [U, U_j]$.
Put $1 = \sum f_i, f_i \in r_{J_i}$. Then $U = \bigcup U_{J_i}$, which implies $\text{im} f \subseteq \sum \sum V_i$.

An object $V$ in $C$ is called minimal if there exist no proper subobjects in $V$. If $V'$ is a direct sum of minimal sub-objects, then $V'$ is called semi-simple. We note that some properties of semi-simple modules are valid in $C$.

**Lemma 4.** For any artinian and noetherian object $V$, $[V, V]$ is a semi-primary ring.

It is well known in mod $A$, and its proof is valid in $C$.

2. Semi-primary category $C$

Let $C$ be an abelian category mentioned in the section 1. We shall consider a function $\varphi$ of object in $C$ into itself which is similar to the radical of a ring.

I. $\varphi(C)$ is a subobject in $C$ for any $C$ in $C$ such that $C/\varphi(C)$ is semi-simple.

II. $C = \varphi(C)$ if and only if $C = 0$.

III. If $C/C'$ is semi-simple for some subobject $C'$ in $C$, then $C \supset \varphi(C)$.

Let $\varphi, \varphi_1$ be functions in $C$ satisfying I and II. We note in this case that every non-zero object contains a maximal subobject. If $\varphi(C) \supset \varphi_1(C)$ for all
Let $U$ be a generator of $C$. If $\phi^i$ is defined in $U$ such that $\phi^{n_i}(U) = 0$ for some $n$ and satisfies I, II (resp. I, II and III), then $\phi$ induces a function $\phi$ in $C$ such that $\phi$ satisfies I, II (resp. I, II and III).

Proof. First, we define $\phi(\phi^i(U)) = \phi^{i+1}(U)$ for all $i$. Let $V$ be any object in $C$ which is different from any $\phi^i(U)$, and $g: \bigoplus_{i \in \mathbb{N}} \bigoplus_{\lambda \in \mathbb{N}} U_{f, \lambda} = V$ the canonical morphism defined by $f: U_f \to V$. We assume that $\text{im}(g| \sum \phi^i(U)) = V$ and $\text{im}((g| \sum \phi^{i+1}(U)) = V$. Then define $\phi(V) = \text{im}(g| \sum \phi^{i+1}(U))$. It is clear that $V/\phi(V)$ is semi-simple and that $V/\phi(V) = 0$ if $V = 0$. Next, we assume $\phi$ satisfies III for $U$. Let $V_0$ be a maximal subobject in $V$, then $f^{-1}(V_0) \ni \phi(U)$. Therefore, $\phi(V) \subseteq V_0$.

**Definition.** Let $V$ be an object in $C$. If $[V, V]$ is a semi-primary ring, $V$ is called a semi-primary object.

From Lemma 4, every artinian and noetherian object is semi-primary.

**Proposition 6.** Let $U$ be a projective, small generator in an abelian $C_1$-category. Then $U$ is semi-primary if and only if a function $\phi$ in $U$ satisfying I, II and III is defined and $U/\phi(U)$ is a direct sum of finite many of simple objects and $\phi^n(U) = 0$ for some $n$. 

C \subseteq C$, then we shall say $\phi_2$ is smaller than $\phi_1$. Furthermore, if $\phi_2$ satisfies III and $C$ is locally small, then $\phi_2$ is a unique minimal function among those satisfying I and II, since $\phi_2(C) = \bigcap D$, where $D$ runs all maximal subobjects in $C$. In this case $\phi_2$ is a functor which satisfies the following commutative diagram

$$
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\downarrow i & & \downarrow i' \\
\phi(C) & \xrightarrow{\phi(f)} & \phi(C')
\end{array}
$$

where $f \in C$ and $i, i'$ are inclusions and $\phi(f)$ is defined as follows: Let $V$ be a maximal subobject in $C'$ then $f^{-1}(V) = C$ or $C|f^{-1}(V) \cong C'/V$ ([5], pp. 22-24), and hence $f(\phi(C)) \subseteq V$, which implies $\text{im}(f| \phi(C)) \subseteq \phi(C')$. Conversely, if $\phi$ satisfying I, II induces a functor in $C$ satisfying 6), then $\phi$ satisfies III. In fact, let $V \neq 0$ in $C$, then $V$ contains a maximal subobject $V_0$. The commutative diagram

$$
\begin{array}{ccc}
V & \longrightarrow & V/V_0 \\
\uparrow & & \uparrow \\
\phi(V) & \longrightarrow & \phi(V/V_0) = 0
\end{array}
$$

shows $\phi(V) \subseteq V_0$.

We put $\phi^i(U) = \phi(U), \phi^i(U) = \phi(\phi^{i-1}(U))$.

**Lemma 5.** Let $U$ be a generator of $C$. If $\phi^i$ is defined in $U$ such that $\phi^{n_i}(U) = 0$ for some $n$ and satisfies I, II (resp. I, II and III), then $\phi$ induces a function $\phi$ in $C$ such that $\phi$ satisfies I, II (resp. I, II and III).
Proof. It is clear from Theorem 0 and Proposition 3. We note here that \( \varphi' = S(n' U) \), where \( n \) is the radical of \([ U, U] \).

The main purpose of this section is to study some structure of \( C_3 \)-category with semi-primary generator.

**Theorem 7.** Let \( C \) be an abelian \( C_3 \)-category with semi-primary generator \( U \). Then we can define a function \( \varphi \) in \( C \) which satisfies I and II and \( U/\varphi(U) \) is a finite directsum of simple subobjects and \( \varphi^n(U) = 0 \) for some \( n \).

Proof. Let \( A = [U, U] \) and \( n \) the radical of \( A \). Put \( U_i = n^n U \) for all \( i \). It is clear that \( U_i \supset U_{i+1} \). Put \( \tau_i = [U, U_i] \). Then \( \tau_{i+1} = \tau_i \cup \tau_i U_{i+1} \) by Lemma 2. Since \( n^n U_{i+1} \) is semi-simple, so is \( \tau_i U_{i+1} \), say \( \tau_i U_{i+1} = \bigoplus \otimes \tau_{aj} \) and \( \tau_{aj} \) is simple. Put \( U_{aj} = \tau_{aj} U \).

If \( U_{aj} = U_{i+1} \), \( \tau_{aj} \subset \tau_i \cup \tau_{i+1} \). Hence, \( U_i U_{i+1} = U_{i+1} \). We shall show that \( U_{aj} U_{i+1} \) is simple. Let \( V \) be a subobject such that \( U_{aj} \supset V \supset \tau_i U_{i+1} \). Then \( \tau_i \supset \tau_{aj} \), in fact if \( \tau_i \supset \tau_{aj} \), \( \tau_i \cap \tau_{aj} = \tau_{i+1} \), and hence, \( \tau_i U_{i+1} = \tau_i U = \tau_i U \). Therefore, \( V = \tau_i U \supset \tau_{aj} U = U_{aj} \). Since \( n^n = \tau_i U_{i+1} U_{ij} = \bigcup \tau_{aj} \cup \tau_{aj} U_{i} = \tau_i U_{i+1} \). On the other hand, \( \tau_{aj} \cup \tau_{aj} U_{i} = \tau_{i+1} \). Hence, \( U_{aj} \cap U_{i+1} = U_{i+1} \). Since \( C \) is \( C_3 \)-category, \( U_{i} U_{i+1} = \bigoplus U_{aj} U_{i+1} \) is semi-simple. We define \( \varphi(U) = U_i \). Then \( U/\varphi(U) \) is a finite directsum of simple subobjects from the above, and \( \varphi^n(U) = 0 \) if \( n^n = 0 \). Then we can define a function \( \varphi \) in \( C \) from Lemma 5.

Let \( V_0 \) be a subobject in \( V \) such that \( V_0 + V' = V \) implies \( V = V' \) for any subobject \( V' \) in \( V \). \( V_0 \) is called negligible. By \([ U : U_i] \) we mean the number of simple components in \( U/U_i \).

**Theorem 8.** Let \( C \) be an abelian \( C_3 \)-category with semi-primary generator \( U \), Then the following conditions are equivalent.

1) \( A \) is projective and small.
2) \([ A : n] = [U : \varphi(U)] \), where \( \varphi(U) = n U, A = [U, U] \) and \( n \) is the radical of \( A \).
3) \( \varphi(U) \) is negligible in \( U \).
4) \( \varphi \) satisfies the condition III.
5) \( T : C \rightarrow \text{mod} A \) is preserving minimal objects.

Proof. If \( U \) is projective and small, then \( C \) is equivalent to \( \text{mod} A \) by Proposition 3. Hence, 2) 3) 4) and 5) are trivial. We assume 2). We put \( a = [U, n U] \). If we restrict the argument in the proof of Theorem 7 to the case of \( i = 1 \), we get \([ A : a] = [U : n U] =. n \). Hence, \( a = n \). For every maximal right ideal \( r, \frac{r}{n} = \sum_{i=1}^{n-1} \otimes \sigma_{aj} U = \tau U \). Hence, we obtain 1) from Proposition 3.
3) Let $\alpha$ be as above. We assume $\alpha \neq n$. Then there exists a right ideal $b$ properly containing $n$ such that $a/n \oplus b/n = A/n$. Let $e$ be an idempotent element in $A$ such that $b/n = (eA+n)/n$. Since $b \supseteq n$, $bU \supseteq nU = aU$. Hence, $U = (a+b)U = bU$. Put $U_\circ = eU$. Then $U_\circ + nU = (eA+n)U = bU = U$. Therefore, $U_\circ = U$ by 3). Hence, $e = I_{A'}$, which is a contradiction.

4) If $n \neq \alpha$, we obtain the fact $U = U_\circ + nU$ and $U_\circ \neq U$. Since $U/U_\circ$ contains a maximal object from Theorem 7, there exists a maximal subobject $V \supset U_\circ$. Therefore, $V \supset U_\circ$.

5) If $n = \alpha$, then there exists a maximal subobject $V$ in $U$ such that $U = V + nU$. Since $0 \rightarrow [U, V] \rightarrow [U, U] \rightarrow [U, U/V]$ is exact and $[U, U/V]$ is minimal, $r_V = [U, V]$ is a maximal right ideal, and hence $r_V \supset n$, which is a contradiction. It is clear that there are many examples in which semi-primary generators are not projective.

**Corollary 1.** Let $U$ be a semi-primary generator in $C$. If $A/n$ is a simple ring, $U$ is projective and small, where $A = [U, U]$ and $n$ is its radical.

Proof. Let $\alpha = [U, nU]$. Since $U \neq nU$, and $\alpha$ is a two-sided ideal, $\alpha = n$.

**Corollary 2.** Let $B$ be a semi-primary ring and $U$ be a semi-primary generator in the category of $B$-right modules. Then $n_A U \supset U n_B$. $n_A U = U n_B$ if and only if $U$ is a finitely generated and projective, where, $A = [U, U]$ and $n_A$ (resp. $n_B$) is the radical of $A$ (resp. $B$).

Proof. Let $\varphi(U) = U n_B$. Then $\varphi$ is a functor in mod $B$ satisfying I, II and III. Hence, $n_A U \supset U n_B$ by Theorem 7. If $n_A U = U n_B$, a function $\varphi'$ defined in the proof of Theorem 7 satisfies III. Hence, $U$ is projective and small. The converse is trivial.

**Example.** We shall show that there exists a generator $U$ such that $\varphi'$ are defined in $U$ satisfying the following conditions: $U / \varphi(U)$ is a finite direct sum of simple object, $\varphi'$ satisfies I, II and III for all $i$ and $\varphi'(U) = 0$ for some $n$, however $U$ is not semi-primary.

Let $k$ be a field and $K = k(x)$. Let $A = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$ be a tri-angular matrix ring. Then $A$ is semi-primary with radical $n$. We define $\varphi(U) = U n$ in mod $A$. Put $r = \begin{pmatrix} 0 & 0 \\ k[x] & 0 \end{pmatrix}$. Then $r$ is a right ideal in $A$. Then $[A/r, A/r] \approx \begin{pmatrix} k & 0 \\ k[x] & k[x] \end{pmatrix}$ is not semi-primary. $U = A \oplus A/r$ is the desired generator.

3. Abelian category of commutative diagram

We recall the definition of abelian category of commutative diagram over abelian categories $C_i$ (see [4]).
Let $I = (1, 2, \ldots, n)$ be a finite linear ordered set and $\{C_i\}_{i \in I}$ a family of abelian categories. We assume that there are given cokernel preserving functors $T_{ij}: C_i \to C_j$ for $i < j$. Furthermore, we assume:

(*) There exist natural transformations $\psi_{ijk}: T_{jk}T_{ij} \to T_{ik}$ for all $i < j < k$, and

(**) For any $i < j < k < l$ and $V$ in $C_i$

\[
\begin{array}{c}
T_{kl}T_{jk}T_{ij}(V) \\
\downarrow \psi_{klj} \\
T_{jl}T_{ij}(V)
\end{array}
\begin{array}{c}
\downarrow \psi_{ijl} \\
T_{il}(V)
\end{array}
\]

is commutative.

We call a family of morphism $d_{ij}: T_{ij}(V_i) \to V_j$ an arrow for $V_i \in C_i$, $V_j \in C_j$ and for all $i < j$, when the diagrams

\[
\begin{array}{c}
T_{jk}T_{ij}(V_i) \\
\downarrow \psi_{ijk} \\
T_{ik}(V_i)
\end{array}
\begin{array}{c}
\downarrow d_{ik} \\
V_k
\end{array}
\]

are commutative.

We define a commutative diagram $[I, C_i]$ as follows; Its objects consist of set $\{V_i\}_{i \in I}$ with arrows $\{d_{ij}\}$ and morphisms consist of set $\{(f_i)_{i \in I}; f_i: V_i \to V_i'\}$ in $C_i$ such that $d_{ij}T_{ij}(f_i) = f_jd_{ij}$.

**Lemma 9.** Let $T_{ij}$ be functors satisfying (**) Then the natural transformation of $T_{tn_{i-1}i}T_{tn_{i-2}i} \cdots T_{ti_{n_2}} \to T_{ti_{n_1}}$ does not depend on any choice of combination of $T_{tn_{i-1}i}, \ldots, T_{ti_{n_2}}$.

Proof. We can prove the lemma by using induction on the number of functors and naturality of $\psi_{ijk}$. Namely, every natural transformation is equal to $T_{tn_{i-1}i}(T_{tn_{i-2}i} \cdots (T_{ti_{n_2}}T_{ti_{n_1}}) \to T_{ti_{n_1}}$.

We assume that all $C_i$ have projective class $\xi_i$. We define a functor $S_i: C_i \to [I, C_i]$ by setting $S_i(V_i) = (0, \ldots, 0, V_i, T_{ii+1}(V_i), \ldots, T_{in}(V_i))$. Then the projective objects in $[I, C_i]$ are of the form $\oplus S_i(P_i)$ and their retract, where $P_i$ is $\xi_i$-projective for all $i$, ([4], Prop. 1.2'). If the projective objects in $[I, C_i]$ are only of the former forms, we call $[I, C_i]$ a good category of commutative diagram.

**Theorem 10.** Let $C_i$ be abelian category with projective class $\xi_i$. Then every $[I, C_i]$ with $T_{ij}$ is imbedding in a good category $[I, C_i]$ with $T'_{ij}$.

Proof. We shall define new functors $T'_{ij}$:
\[ T_{ii+1} = T_{ii+1}, \]
\[ T'_{ij} = T_{j-1} T_{j-2} \cdots T_{ii+1} \text{ for } i+1 < j. \]

Then it is clear that \( T'_{ij} \) are cokernel preserving and \( \psi_{ij} = I_{C_t} \) and (**) is trivial. Furthermore, there exist unique natural transformations \( \phi_{ij}: T'_{ij} \to T_{ij} \) by Lemma 9. Put \( C = [I, C_t] \) with \( T_{ij} \) and \( C' = [I, C_t] \) with \( T'_{ij} \). We define a function \( F \) of \( C \) into \( C' \) as follows: For \( V = (V_i) \) with arrows \( d_{ij} \) in \( C \) we put \( F(V) = (V_i) \) with the following arrows \( d'_{ij} \):

\[ d'_{ii+1} = d_{ii+1}, \]
\[ d'_{ij} = d_{ij} \phi_{ij} T'_{ij} \text{ for } i+1 < j. \]

We have to show that \( d'_{ij} \) satisfies (**). We have a diagram for \( i < j < k \) and \( V_t \subseteq C_t \)

\[ \begin{array}{ccc}
T'_{jk} T'_{ij}(V_i) & \xrightarrow{T'(\phi)} & T'_{jk} T_{ij}(V_i) \\
\psi_{ij} & \downarrow & \phi \\
T_{jk}(V_i) & \xrightarrow{T(d_{ij})} & T_{jk}(V_j) \\
\phi & \downarrow & \phi \\
V_i & \xrightarrow{d_{jk}} & V_j
\end{array} \]

I is commutative by Lemma 9, II is commutative by naturality of \( \phi \) and so is III by (**). Hence, \( d'_{ij} \) satisfies (**). Define \( F((f_i)) = (f_i) \) for morphism \( (f_i) \) in \( C \). Then we can similarly show that \( F \) is a functor. It is clear that \( F \) is an imbedding functor. Since \( \psi_{ij} = I_{C_t}, K^t(P_t) = 0 \) in (*) of [4], Lemma 3.7. Hence, \( C' \) is good by [4], Lemma 3.7.

If every objects in \( C \) are projective, \( C \) is called a semi-simple category.

**Corollary.** Let \( C_t \) be a semi-simple abelian category. Then \([I, C_t]\) is imbedding in an abelian hereditary category, (cf. [3], Theorem 5).

**Proof.** It is clear from Theorem 10 and [4], Theorem 3.12.

Finally, we note that if \( C_t \) have functor \( \varphi_t \) satisfying I, II and III and \( T_{ij}(\varphi_t(V_i)) \subseteq \varphi_t T_{ij}(V_i) \) on \( V = (V_i) \) in \([I, C_t]\). Then

\[ \varphi(V) = (\varphi_t(V_1), \varphi_t(V_2) \cup d_{i_1 j_1}(V_2), \ldots, \varphi_t(V_j) \cup d_{i_j j}(V_i), \ldots) \]

is a functor on \([I, C_t]\) satisfying I, II and III. If \( \varphi_t^m = 0 \) for all \( t \) then \( \varphi^{nm} = 0 \).
References


