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ON SEMI-PRIMARY ABELIAN CATEGORIES

Dedicated to Professor Atuo Komatu for his 60th birthday

MANABU HARADA

(Received June 7, 1968)

Let $C$ be an abelian category with exact direct limits, namely cocomplete $C_3$-category ([5], p. 83).

In this note we always assume that $C$ contains a generator $U$, and hence $C$ is locally small by [5], p. 71. In [2], Gabriel and Popesco have given a characterization of $U$ being projective and small by using the concept of localization in [1]. We shall give another proof without localization in the section 1.

In the section 2, we shall define a function $\varphi$ of $C$ into itself, which is analogous to the radical of semi-primary ring.

We shall show that $C$ has such a function when the endomorphism ring $[U, U]$ is a semi-primary ring, and we shall give some criteria by means of $\varphi$ that $U$ is small and projective.

In the section 3, we shall add some remarks in the previous author's work on category of tri-angular matrices, [4].

In this note we shall freely make use of concepts in categories from [5].

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1. Preliminary results

In this section we shall summarize all results which we need in the following sections.

Almost all results in this section have been proved in [2] and [6] by using the concepts of localization in [1]. However, we shall give here another approach to them by means of rather homological method.

Let $C$ be an abelian cocomplete $C_3$-category ([5], p. 81) and $U$ an object of $C$. Let $A = [U, U]$. By mod $A$ we mean the category of $A$-right modules. Let $T: C \to \text{mod } A$; $T(V) = [U, V]$ for any $V \in C$ be the functor of $C$ into mod $A$. In this case we can define a coadjoint $S$ of $T$ such that $S(M) = M \otimes_A U$ by [5], p. 143, namely $\eta: [M, T(V)] \approx [S(M), V]_C$. Furthermore, we have natural transformation $\psi_V: ST(V) \to V$ and $\varphi_M: M \to TS(M)$, (see [5], pp 118–119).

**Theorem 0** (Gabriel and Popesco [2]). Let $C$, $U$ and $A$ be as above. Then the following statements are equivalent:

1) $U$ is a generator.
2) $T$ is a completely faithful (namely, full and faithful).

3) $\psi_V$ is isomorphic for all $V \in C$ and $S$ is an exact functor.

Proof. 1)$\rightarrow$2). See [2] or [5] in which we do not need the concept of localization.

3)$\rightarrow$2). $[ST(V), V'] \cong [T(V), T(V')]$ and $[ST(V), V'] \cong [V, V']$ for $V, V' \in C$.

2)$\rightarrow$3). $[ST(V), V'] \cong [T(V), T(V')] \cong [V, V']$. Hence, $[ST(V), ]$ and $[V, ]$ give the equivalent functors. Therefore, $\psi_V=\eta^{-1}\alpha^{-1}I_V$ is isomorphic.

Thus, it remains to show that $S$ is exact. First, we show that if $M \in \text{mod} A$ is contained in a free module $F$, then $0 \rightarrow S(M) \rightarrow S(F)$ is exact. In order that, we assume first that $M$ is finitely generated, say $M=(m_1, m_2, \ldots, m_n)$ and hence we may assume that $F$ is also finitely generated. Then we have a commutative diagram

$$
0 \leftarrow M \leftarrow \sum_{i=1}^{n} A v_{\beta_i} \leftarrow \kappa A = \sum_{i \in I} A w_{\kappa_i}
$$

$$
\downarrow i \quad \alpha
$$

$$
\uparrow \alpha \quad \kappa A = \sum_{i \in I} A w_{\kappa_i},
$$

where $u_{\alpha_i}, v_{\beta_i}$ and $w_{\kappa_i}$ are free bases and $i$ is the inclusion map, $f$ is a natural mapping such that $f(v_{\beta_i})=m_i$, $\alpha=if$, and $K=\ker f$.

Operating $S$ on the above 1) we obtain commutative exact diagram:

$$
0 \leftarrow S(M) \leftarrow S(F) \leftarrow \sum_{i=1}^{n} U \leftarrow \sum_{i=1}^{n} A u_{\alpha_i}
$$

where $V=\text{im} (\kappa U \xrightarrow{\beta} \sum_{i=1}^{n} U)$ and $K'=\ker S(\alpha)$.

It is clear that there exists the inclusion map $i_1$ of $V$ into $K'$. Operating again $T$ on 2) we have

$$
\sum \oplus A v_{\beta_i} \xrightarrow{T(i_1)} T(V) \leftarrow \sum \oplus A u_{\alpha_i}
$$

$$
\downarrow \alpha \quad \kappa A
$$

where $V=\text{im} (\kappa U \xrightarrow{\beta} \sum_{i=1}^{n} U)$ and $K'=\ker S(\alpha)$. It is clear that there exists the inclusion map $i_1$ of $V$ into $K'$. Operating again $T$ on 2) we have
where the vertical line is exact and \( T(i_1), T(i_2) \) are inclusions. Since \( K \) is also \( \ker \alpha \), there exists a unique isomorphism \( \theta \) such that

\[
\begin{array}{ccc}
T(K') & \xrightarrow{\theta} & K \\
\downarrow i_\beta & & \downarrow i_\beta \\
\sum \oplus A_{\beta_i} & & \\
\end{array}
\]

is commutative. Let \( a \in T(K') \) and put \( k = \theta a \). Then \( T(i_1)T(\beta)\varphi_K a w_k = i_\beta a = i_\beta k = i_\beta T(i_1)T(\beta)\varphi_K a w_k \) by the naturality of \( \varphi \). Put \( b = T(\beta)\varphi w_k \in T(V) \), \( i_\beta a = i_\beta T(i_1)b \). Since \( i_\beta \) is injective, \( a = T(i_1)b \). Hence, \( T(i_1) \) is isomorphic. Since \( T \) is faithful, \( i_\beta \) is isomorphic by [5], p. 56. Therefore, \( 0 \rightarrow S(M) \rightarrow S(F) \) is exact from 2). Next, let \( M \) be any submodule of free \( A \)-module \( F \); \( 0 \rightarrow M \rightarrow F \). Then \( M \) is a direct limit of the family of finitely generated \( A \)-submodules \( M_{\alpha i} \); \( M = \lim M_{\alpha i} \). Since \( S \) is colimit and exact preserving by [5], p. 85 and p. 55, \( 0 \rightarrow S(M) = \lim S(M_{\alpha i}) \rightarrow S(F) \) is exact from the first argument. Hence, \( \text{Tor}^1(M, U) = 0 \) for all \( M \in \text{mod} A \), ([5], p. 112, § 8), which implies that \( S \) is exact.

From now on we fix a generator \( \mathcal{U} \in \mathcal{C} \) and \( A = [\mathcal{U}, \mathcal{U}] \). Then for any subobject \( \mathcal{U}' \) in \( \mathcal{U} \) it is clear that \([\mathcal{U}, \mathcal{U}']\) is identified to a right ideal in \( A \), and we shall denote it by \( r_{\mathcal{U}'} \) or \( r \). By \( \mathcal{K} \mathcal{U} \) we mean the image of \( f : \sum_{k \in \mathcal{K}} U_k \rightarrow U \) defined by \( f(U_k) = kU \) for any subset \( \mathcal{K} \) in \( A \). We note from the definitions that \( r_{\mathcal{U}'} \mathcal{U} = \mathcal{S}T(\mathcal{U}') \). Then we have from [5], p. 71.

**Lemma 1.** For any subobject \( \mathcal{U}' \) in \( \mathcal{U} \) we have \( \mathcal{U}' = r_{\mathcal{U}'} \mathcal{U} \).

**Lemma 2.** Let \( \mathcal{U} \) be a generator in \( \mathcal{C} \) and \( r_1, r_2 \) right ideals in \( A \). Then we have

1) \( (r_1 + r_2) \mathcal{U} = r_1 \mathcal{U} \cup r_2 \mathcal{U} \).
2) \( (r_1 \cap r_2) \mathcal{U} = r_1 \mathcal{U} \cap r_2 \mathcal{U} \).

Proof. 1) is trivial from the definition.

2) We have the following row exact and commutative diagrams:

\[
\begin{array}{cccc}
0 & \rightarrow & r_1 \mathcal{U} & \rightarrow \mathcal{U} \cup r_2 \mathcal{U} \rightarrow (r_1 \mathcal{U} \cup r_2 \mathcal{U})/r_2 \mathcal{U} \rightarrow 0 \\
\uparrow & & \uparrow \cong & & \uparrow \cong \\
0 & \rightarrow & r_2 \mathcal{U} \cap r_2 \mathcal{U} & \rightarrow \mathcal{U} \cap r_2 \mathcal{U} \rightarrow (r_1 \mathcal{U} \cap r_2 \mathcal{U})/r_2 \mathcal{U} \rightarrow 0 \\
\uparrow & & \uparrow \cong \rightarrow & & \uparrow \cong \rightarrow \\
0 & \rightarrow & (r_1 \cap r_2) \mathcal{U} & \rightarrow \mathcal{U} \cap r_1 \mathcal{U} \rightarrow (r_1 \cap r_2) \mathcal{U} \rightarrow 0 \\
\uparrow & & \uparrow \cong \rightarrow & & \uparrow \cong \rightarrow \\
0 & \rightarrow & r_1 \cap r_2 & \rightarrow r_1 \cap r_2/r_1 \cap r_2 \rightarrow 0 \\
\end{array}
\]

Since \( S \) is an exact functor, we obtain \( (r_1 \cap r_2) \mathcal{U} = r_1 \mathcal{U} \cap r_2 \mathcal{U} \) from 4) by operating \( S \) on 5).
The following proposition is an immediate consequence of [6], Prop. 1.1 and [5], p. 104. However, we shall prove it without localization.

**Proposition 3.** Let $C$, $U$ and $A$ be as above and $U$ a generator. Then the following statements are equivalent.

1) $S(\_)=\otimes U$ is an equivalent functor.

2) $T(\_)[U,\_]$ and $S(\_)$ give a one-to-one correspondence between right ideals and subobjects in $U$.

3) For any maximal right ideal $r$ in $A$ $S(A/r)\cong 0$.

4) $U$ is projective and small in $C$.

**Proof.** 1)$\rightarrow$2)$\rightarrow$3) are trivial.

4)$\rightarrow$1) is proved in [5], p. 104.

3)$\rightarrow$4) It is clear from 3) that for any non-zero $A$-module $M$, $S(M)=M\otimes U \cong 0$, since $S$ is exact by Theorem 0. Let $V_1 \rightarrow V_2 \rightarrow 0$ be exact in $C$ and $T(V_1)\rightarrow T(V_2)\rightarrow K \rightarrow 0$ be exact in mod $A$. Since $S$ is exact, $ST(V_1)\cong V_1 \rightarrow ST(V_2)=V_2 \rightarrow S(K)\rightarrow 0$ is exact. Hence, $S(K)\cong 0$, which means $K=0$ from the above. Therefore, $T$ is exact and hence, $U$ is projective. Finally we shall show that $U$ is small. Let $f:U \rightarrow \bigoplus_{i} V_i$ be a morphism in $C$, where $V_i$'s are any objects in $C$.

Put $U_j=f^{-1}(\bigoplus_{i\in J} V_i)$, where $J$ is a finite set of $I$. Since $C$ is $C_3$-category, $U=\bigcup U_j$ by [5] p. 83. Then $A=\bigcup r_j$ by Lemma 2 and 3), where $r_j=[U, U_j]$. Put $1=\sum_{i} f_i$, $f_i\in r_{f_i}$. Then $U=\bigcup U_{f_i}$, which implies $im f \subset \bigoplus_{i=1}^\infty \bigoplus_{f_i} V_i$.

An object $V$ in $C$ is called minimal if there exist no proper subobjects in $V$. If $V$ is a direct sum of minimal sub-objects, then $V$ is called semi-simple. We note that some properties of semi-simple modules are valid in $C$.

**Lemma 4.** For any artinian and noetherian object $V$, $[V, V]$ is a semi-primary ring.

It is well known in mod $A$, and its proof is valid in $C$.

2. **Semi-primary category $C$**

Let $C$ be an abelian category mentioned in the section 1. We shall consider a function $\varphi$ of object in $C$ into itself which is similar to the radical of a ring.

I. $\varphi(C)$ is a subobject in $C$ for any $C$ in $C$ such that $C/\varphi(C)$ is semi-simple.

II. $C=\varphi(C)$ if and only if $C=0$.

III. If $C/C'$ is semi-simple for some subobject $C'$ in $C$, then $C'\supseteq \varphi(C)$.

Let $\varphi$, $\varphi_1$ be functions in $C$ satisfying I and II. We note in this case that every non-zero object contains a maximal subobject. If $\varphi_1(C)\supseteq \varphi_0(C)$ for all
C ∈ C, then we shall say φ₂ is smaller than φ₁. Furthermore, if φ₂ satisfies III and C is locally small, then φ₂ is a unique minimal function among those satisfying I and II, since φ₂(C) = ∩ D, where D runs all maximal subobjects in C. In this case φ₂ is a functor which satisfies the following commutative diagram

6) $$\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
i & \uparrow & \downarrow \phi(f) \\
\phi(C) & \xrightarrow{i'} & \phi(C')
\end{array}$$

where f ∈ C and i, i' are inclusions and φ(f) is defined as follows: Let V be a maximal subobject in C' then f⁻¹(V) = C or C[f⁻¹(V) ≅ C'/V ([5], pp. 22-24), and hence f(φ(C)) ⊆ V, which implies im (f|φ(C)) ⊆ φ(C'). Conversely, if φ satisfying I, II induces a functor in C satisfying 6), then φ satisfies III. In fact, let V ≠ 0 in C, then V contains a maximal subobject V₀. The commutative diagram

$$\begin{array}{ccc}
V & \longrightarrow & V/V₀ \\
\phi(V) & \longrightarrow & \phi(V/V₀) = 0
\end{array}$$

shows φ(V) ⊆ V₀.

We put φ'(U) = φ(U), φ'(U) = φ(φ⁻¹(U)).

**Lemma 5.** Let U be a generator of C. If φ is defined in U such that φⁿ(U) = 0 for some n and satisfies I, II (resp. I, II and III), then φ induces a function ϕ in C such that ϕ satisfies I, II (reps. I, II and III).

Proof. First, we define ϕ(φᵢ(U)) = φᵢ₊₁(U) for all i. Let V be any object in C which is different from any φ(U), and g: {} \textstyle \bigoplus \sum \oplus U_f → V the canonical morphism defined by f:U_f → V. We assume that im (g|φ(U)) = V and im (g|φ(U) = V. Then define ϕ(V) = im (g|φ(U)). It is clear that V/ϕ(V) is semi-simple and that V/ϕ(V) ≠ 0 if V ≠ 0. Next, we assume φ satisfies III for U. Let V₀ be a maximal subobject in V, then f⁻¹(V₀) ⊆ ϕ(U). Therefore, ϕ(V) ⊆ V₀.

**Definition.** Let V be an object in C. If [V, V] is a semi-primary ring, V is called a semi-primary object.

From Lemma 4, every artinian and noetherian object is semi-primary.

**Proposition 6.** Let U be a projective, small generator in an abelian Cₚ category. Then U is semi-primary if and only if a function φ in U satisfying I, II and III is defined and U/φ(U) is a directsum of finite many of simple objects and φⁿ(U) = 0 for some n.
Proof. It is clear from Theorem 0 and Proposition 3. We note here that 
\( \varphi'(U) = S(n'U) \), where \( n \) is the radical of \([U, U]\).

The main purpose of this section is to study some structure of \( C_3 \)-category with semi-primary generator.

**Theorem 7.** Let \( C \) be an abelian \( C_3 \)-category with semi-primary generator \( U \). Then we can define a function \( \varphi \) in \( C \) which satisfies I and II and \( U/\varphi(U) \) is a finite direct sum of simple subobjects and \( \varphi^n(U) = 0 \) for some \( n \).

Proof. Let \( A = [U, U] \) and \( n \) the radical of \( A \). Put \( U_i = n'U \) for all \( i \).

It is clear that \( U_i \supseteq U_{i+1} \). Put \( \tau_i = [U, U_i] \). Then \( \tau_i \supseteq n' \tau_{i+1} \). \( \tau_{i+1} U = U_i \cap U_{i+1} = U_{i+1} \) by Lemma 2. Since \( n' n'^{-1} \) is semi-simple, so is \( n' / n'^{-1} \), say \( n' / n'^{-1} = \sum \tau_{a_i} \); \( n'' \supseteq \tau_{a_i} \supseteq \tau_{i+1} \), and \( \tau_{a_i} \) is simple. Put \( U_{a_i} = \tau_{a_i} U \).

If \( U_{a_i} = U_{i+1} \), \( \tau_{a_i} U \supseteq \tau_{i+1} \), which is a contradiction. Hence, \( U_i \supseteq U_{a_i} \supseteq U_{i+1} \).

We shall show that \( U_{a_i} / U_{i+1} \) is simple. Let \( V \) be a subobject such that \( U_{a_i} \supseteq V \supseteq U_{i+1} \). Then \( \tau_{V} \supseteq \tau_{a_i} \), in fact if \( \tau_{V} \supseteq \tau_{a_i} \), \( \tau_{V} \cap \tau_{a_i} = \tau_{i+1} \), and hence, \( U_{i+1} = (\tau_{V} \cap \tau_{a_i}) U = V \cap U_{a_i} = V \). Therefore, \( V = \tau_{V} U \cap U_{a_i} = U_{a_i} \). Since \( n' = \sum \tau_{a_i} \), \( U_{i+1} = \sum \tau_{a_i} \). On the other hand, \( \tau_{a_i} \cap U_{i+1} \).

Hence, \( U_{a_i} \cap U_{i+1} \).

Since \( C \) is \( C_3 \)-category, \( U_{i} / U_{i+1} \approx \sum \tau_{a_i} / U_{i+1} \) is semi-simple. We define \( \varphi'(U) = U_i \). Then \( U/\varphi(U) \) is a finite direct sum of simple subobjects from the above, and \( \varphi^n(U) = 0 \) if \( n^n = 0 \). Then we can define a function \( \varphi \) in \( C \) from Lemma 5.

Let \( V_0 \) be a subobject in \( V \) such that \( V_0 + V' = V \) implies \( V = V' \) for any subobject \( V' \) in \( V \). \( V_0 \) is called negligible. By \([U : U_i]\) we mean the number of simple components in \( U/U_i \).

**Theorem 8.** Let \( C \) be an abelian \( C_3 \)-category with semi-primary generator \( U \), Then the following conditions are equivalent.

1) \( U \) is projective and small.

2) \([A : n] = [U : \varphi(U)]\), where \( \varphi(U) = nU \), \( A = [U, U] \) and \( n \) is the radical of \( A \).

3) \( \varphi(U) \) is negligible in \( U \).

4) \( \varphi \) satisfies the condition III.

5) \( T : C \rightarrow \text{mod } A \) is preserving minimal objects.

Proof. If \( U \) is projective and small, then \( C \) is equivalent to \( \text{mod } A \) by Proposition 3. Hence, 2) 3) 4) and 5) are trivial. We assume 2). We put \( a = [U, nU] \). If we restrict the argument in the proof of Theorem 7 to the case of 5), we get \([A : a] = [U : nU], = n \). Hence, \( a = n \). For every maximal right ideal \( r, r/n = \sum_{i=1}^{n-1} \oplus r_{a_i} / n \), which implies \( U \setminus \bigcup_{i=1}^{n-1} r_{a_i} U = r U \). Hence, we obtain 1) from Proposition 3.
3) Let \( \alpha \) be as above. We assume \( \alpha \not\in \pi \). Then there exists a right ideal \( b \) properly containing \( \pi \) such that \( \alpha/\pi = A/\pi \). Let \( e \) be an idempotent element in \( A \) such that \( b/\pi = (eA + \pi)/\pi \). Since \( b \supset \pi \), \( bU \supset \pi U = \alpha U \). Hence, \( U = (\alpha + b)U = bU \). Put \( U_0 = eU \). Then \( U_0 + \pi U = (eA + \pi)U = bU = U \). Therefore, \( U_0 = U \) by 3). Hence, \( e = I_\pi \), which is a contradiction.

4) If \( \pi \not\in \alpha \), we obtain the fact \( U = U_0 + \pi U \) and \( U_0 \not\in U \). Since \( U/U_0 \) contains a maximal object from Theorem 7, there exists a maximal subobject \( V \supset U_0 \).

5) If \( \pi \not\in \alpha \), then there exists a maximal subobject \( V \) in \( U \) such that \( U = V + \pi U \). Since \( 0 \rightarrow [U, V] \rightarrow [U, U] \rightarrow [U, U/V] \) is exact and \( [U, U/V] \) is minimal, \( \tau_V = [U, V] \) is a maximal right ideal, and hence \( \tau_V \supset \pi \), which is a contradiction.

It is clear that there are many examples in which semi-primary generators are not projective.

**Corollary 1.** Let \( U \) be a semi-primary generator in \( \mathcal{C} \). If \( A/\pi \) is a simple rings, \( U \) is projective and small, where \( A = [U, U] \) and \( \pi \) is its radical.

Proof. Let \( \alpha = [U, \pi U] \). Since \( U = \pi U \), and \( \alpha \) is a two-sided ideal, \( \alpha = \pi \).

**Corollary 2.** Let \( B \) be a semi-primary ring and \( U \) be a semi-primary generator in the category of \( B \)-right modules. Then \( \pi_A U \supset \pi_B U \). \( \pi_A U = \pi_B U \) if and only if \( U \) is a finitely generated and projective, where, \( A = [U, U] \) and \( \pi_A \) (resp. \( \pi_B \)) is the radical of \( A \) (resp. \( B \)).

Proof. Let \( \varphi(U) = \pi_B U \). Then \( \varphi \) is a functor in \( \text{mod} \ B \) satisfying I, II and III. Hence, \( \pi_A U \supset \pi_B U \) by Theorem 7. If \( \pi_A U = \pi_B U \), a function \( \varphi' \) defined in the proof of Theorem 7 satisfies III. Hence, \( U \) is projective and small. The converse is trivial.

**Example.** We shall show that there exists a generator \( U \) such that \( \varphi^i \) are defined in \( U \) satisfying the following conditions: \( U/\varphi(U) \) is a finite directsum of simple object, \( \varphi^i \) satisfies I, II and III for all \( i \) and \( \varphi^i(U) = 0 \) for some \( n \), however \( U \) is not semi-primary.

Let \( k \) be a field and \( K = k(x) \). Let \( A = \left( \begin{array}{c} k \\ k \end{array} \right) \) be a tri-angular matrix ring.

Then \( A \) is semi-primary with radical \( \pi \). We define \( \varphi(U) = \pi U \) in \( \text{mod} \ A \). Put \( \tau = \left( \begin{array}{c} 0 \\ k[x] \end{array} \right) \). Then \( \tau \) is a right ideal in \( A \). Then \( [A/\tau, A/\tau] \approx \left( \begin{array}{c} k \\ k(x)/k[x] \end{array} \right) \) is not semi-primary. \( U = A \oplus A/\tau \) is the desired generator.

3. Abelian category of commutative diagram

We recall the definition of abelian category of commutative diagram over abelian categories \( C_i \) (see [4]).
Let $I = \{1, 2, \ldots, n\}$ be a finite linear ordered set and $\{C_i\}_{i \in I}$ a family of abelian categories. We assume that there are given cokernel preserving functors $T_{ij} : C_i \to C_j$ for $i < j$. Furthermore, we assume:

(*) There exist natural transformations 

$$\psi_{ijk} : T_{jk}T_{ij} \to T_{ik}$$

for all $i < j < k$, and

(**) For any $i < j < k < l$ and $V$ in $C_i$

$$\begin{array}{ccc}
T_{ki} & & T_{ki} \\
\downarrow \psi_{kli} & & \downarrow \psi_{kli} \\
T_{ji} & & T_{ji}
\end{array}$$

is commutative.

We call a family of morphisms $d_{ij} : T_{ij}(V_i) \to V_j$ an arrow for $V_i \in C_i$, $V_j \in C_j$ and for all $i < j$, when the diagrams

$$\begin{array}{ccc}
T_{jk}(V_i) & & T_{jk}(V_j) \\
\downarrow \psi_{ij} & & \downarrow \psi_{ij} \\
T_{ik}(V_i) & & T_{ik}(V_i)
\end{array}$$

are commutative.

We define a commutative diagram $[I, C]$ as follows; Its objects consist of set $\{V_i\}_{i \in I}$ with arrows $\{d_{ij}\}$ and morphisms consist of set $\{(f_i)_{i \in I} ; f_i : V_i \to V_{i'}\}$ in $C_i$ such that $d_{ij}T_{ij}(f_i) = f_jd_{ij}$.

Lemma 9. Let $T_{ij}$ be functors satisfying (**). Then the natural transformation of $T_{i_{n-1}i_n}T_{i_{n-2}i_{n-1}} \cdots T_{i_1i_2} \to T_{i_1i_n}$ does not depend on any choice of combination of $T_{i_{n-1}i_n}, \ldots, T_{i_1i_2}$.

Proof. We can prove the lemma by using induction on the number of functors and naturality of $\psi_{ijk}$. Namely, every natural transformation is equal to $T_{i_{n-1}i_n}(T_{i_{n-2}i_{n-1}}(\cdots (T_{i_2i_1}) \to T_{i_1i_n}$.

We assume that all $C_i$ have projective class $\xi_i$. We define a functor $S_i : C_i \to [I, C]$ by setting $S_i(V_i) = \{0, \ldots, 0, V_i, T_{ii+1}(V_i), \ldots, T_{in}(V_i)\}$. Then the projective objects in $[I, C_i]$ are of the form $\oplus S_i(P_i)$ and their retract, where $P_i$ is $\xi_i$-projective for all $i$, ([4], Prop. 1.2'). If the projective objects in $[I, C_i]$ are only of the former forms, we call $[I, C_i]$ a good category of commutative diagram.

Theorem 10. Let $C_i$ be abelian category with projective class $\xi_i$. Then every $[I, C_i]$ with $T_{ij}$ is imbedding in a good category $[I, C_i]$ with $T'_{ij}$.

Proof. We shall define new functors $T'_{ij}$.
Then it is clear that $T'_{ij}$ are cokernel preserving and $\psi'_{i,jk}=I_{C_k}$ and (***) is trivial. Furthermore, there exist unique natural transformations $\phi_{i,j}: T'_{ij} \rightarrow T_{ij}$ by Lemma 9. Put $C=[I, C_i]$ with $T_{ij}$ and $C'=[I, C_i]$ with $T'_{ij}$. We define a function $F$ of $C$ into $C'$ as follows: For $V=(V_i)$ with arrows $d_{ij}$ in $C$ we put $F(V)=\langle V_i \rangle$ with the following arrows $d'_{ij}$:

$$d'_{ii+1} = d_{ii+1},$$
$$d'_{ij} = d_{ij}\phi_{i,j}T'_{ij} \quad \text{for } i+1<j.$$ 

We have to show that $d'_{ij}$ satisfies (***) . We have a diagram for $i<j<k$ and $V_t \in C_t$

$$T'_{jk}(V_i) \xrightarrow{T'(\phi)} T'_{jk}T_{ij}(V_i) \xrightarrow{T'(d_{ij})} T'_{jk}(V_j) \quad \text{II}$$
$$\psi'_{i,jk} \quad I \quad T'_{jk}T_{ij}(V_i) \xrightarrow{\phi} T'_{jk}(V_j) \quad \text{III}$$
$$T_{ik}(V_i) \xrightarrow{d_{ik}} T'_{ik}(V_i) \rightarrow V_k.$$ 

I is commutative by Lemma 9, II is commutative by naturality of $\phi$ and so is III by (**). Hence, $d'_{ij}$ satisfies (***) . Define $F((f_i))=(f_i)$ for morphism $(f_i)$ in $C$. Then we can similarly show that $F$ is a functor. It is clear that $F$ is an imbedding functor. Since $\psi'_{i,jk}=I_{C_k}$, $K'(P_i)=0$ in (*) of [4], Lemma 3. 7. Hence, $C'$ is good by [4], Lemma 3. 7.

If every objects in $C$ are projective, $C$ is called a semi-simple category.

**Corollary.** Let $C_i$ be a semi-simple abelian category. Then $[I, C_i]$ is imbedding in an abelian hereditary category, (cf. [3], Theorem 5).

**Proof.** It is clear from Theorem 10 and [4], Theorem 3. 12.

Finally, we note that if $C_i$ have functor $\varphi_i$ satisfying I, II and III and $T_{ij}(\varphi_i(V_i)) \subseteq \varphi_jT_{ij}(V_i)$ on $V=(V_i)$ in $[I, C_i]$. Then

$$\varphi(V) = (\varphi_i(V_1), \varphi_i(V_2) \cup d_i(V_2), \ldots , \varphi_j(V_j) \cup \bigcup_{i<j} d_i(V_i), \ldots)$$

is a functor on $[I, C_i]$ satisfying I, II and III. If $\varphi^m=0$ for all $t$ then $\varphi^{nm}=0$.

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References


