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Let $C$ be an abelian category with exact direct limits, namely cocomplete $C_3$-category ([5], p. 83).

In this note we always assume that $C$ contains a generator $U$, and hence $C$ is locally small by [5], p. 71. In [2], Gabriel and Popesco have given a characterization of $U$ being projective and small by using the concept of localization in [1]. We shall give another proof without localization in the section 1.

In the section 2, we shall define a function $\phi$ of $C$ into itself, which is analogous to the radical of semi-primary ring.

We shall show that $C$ has such a function when the endomorphism ring $[U,U]$ is a semi-primary ring, and we shall give some criteria by means of $\phi$ that $U$ is small and projective.

In the section 3, we shall add some remarks in the previous author's work on category of tri-angular matrices, [4].

In this note we shall freely make use of concepts in categories from [5].

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1. Preliminary results

In this section we shall summarize all results which we need in the following sections.

Almost all results in this section have been proved in [2] and [6] by using the concepts of localization in [1]. However, we shall give here another approach to them by means of rather homological method.

Let $C$ be an abelian cocomplete $C_3$-category ([5], p. 81) and $U$ an object of $C$. Let $A=[U,U]$. By mod $A$ we mean the category of $A$-right modules. Let $T:C \to \text{mod } A$; $T(V)=[U,V]$ for any $V \in C$ be the functor of $C$ into mod $A$. In this case we can define a coadjoint $S$ of $T$ such that $S(M)=M \otimes_A U$ by [5], p. 143, namely $\eta: [M, T(V)] \approx [S(M), V]_C$. Furthermore, we have natural transformation $\gamma_V: ST(V) \to V$ and $\phi_M: M \to TS(M)$, (see [5], pp 118-119).

**Theorem 0** (Gabriel and Popesco [2]). Let $C$, $U$ and $A$ be as above. Then the following statements are equivalent:

1) $U$ is a generator.
2) \( T \) is a completely faithful (namely, full and faithful).
3) \( \psi_V \) is isomorphic for all \( V \in C \) and \( S \) is an exact functor.

Proof. 1)\( \Rightarrow \)2). See [2] or [5] in which we do not need the concept of localization.
3)\( \Rightarrow \)2). \([ST(V), V'] \approx [T(V), T(V')]\) and \([ST(V), V'] \approx [V, V']\) for \( V, V' \in C \).
2)\( \Rightarrow \)3). \([ST(V), V'] \approx [T(V), T(V')] \approx [V, V']\). Hence, \([ST(V), \_] \) and \([V, \_]\) give the equivalent functors. Therefore, \( \psi_V = \eta^{-1} \alpha^{-1} I_V \) is isomorphic.

Thus, it remains to show that \( S \) is exact. First, we show that if \( M \in \text{mod} A \) is contained in a free module \( F \), then \( 0 \rightarrow S(M) \rightarrow S(F) \) is exact. In order that, we assume first that \( M \) is finitely generated, say \( M = (m_1, m_2, \cdots, m_n) \) and hence we may assume that \( F \) is also finitely generated. Then we have a commutative diagram

\[
\begin{array}{ccc}
0 \& \rightarrow M \& \rightarrow \sum_{i=1}^{n} A v_{\beta_i} \& \rightarrow \kappa A = \sum_{k \in K} A w_{\nu_k} \\
& i & \downarrow \alpha & \\
& F = \sum_{i=1}^{m} A u_{a_i} & \rightarrow \sum_{i=1}^{m} A u_{a_i},
\end{array}
\]

where \( u_{a_i}, v_{\beta_i} \) and \( w_{\nu_k} \) are free bases and \( i \) is the inclusion map, \( f \) is a natural mapping such that \( f(v_{\beta_i}) = m_i \), \( \alpha = if \), and \( K = \ker f \).

Operating \( S \) on the above 1) we obtain commutative exact diagram:

\[
\begin{array}{ccc}
0 \& \rightarrow S(M) \& \rightarrow \sum_{i=1}^{n} U \& \rightarrow \sum_{i=1}^{n} V \\
& S(f) \downarrow \alpha & \\
& S(F) \rightarrow S(F)
\end{array}
\]

where \( V = \text{im} (\sum_{i=1}^{n} U) \) and \( K' = \ker S(\alpha) \).

It is clear that there exists the inclusion map \( i_i \) of \( V \) into \( K' \). Operating again \( T \) on 2) we have

\[
\begin{array}{ccc}
\sum_{i=1}^{n} A v_{\beta_i} \& \rightarrow T(V) \& \rightarrow T(\kappa U) \& \rightarrow \kappa A \\
& \sum_{i=1}^{n} A u_{a_i} \downarrow \alpha & \\
& T(i_i) \leftarrow T(\kappa U) \rightarrow \kappa A
\end{array}
\]
where the vertical line is exact and $T(i_1), T(i_2)$ are inclusions. Since $K$ is also $\ker \alpha$, there exists a unique isomorphism $\theta$ such that

\[
\begin{array}{c}
\sum \oplus A_{\beta_i} \\
\downarrow \theta \\
T(K') \\
\downarrow i_s \\
\end{array}
\]

is commutative. Let $a \in T(K')$ and put $k=\theta a$. Then $T(i_s)T(\beta)\varphi_{K_\Lambda}w_k=i_s a$

$=i_s \theta a=i_s k=i_s T(i_s)T(\beta)\varphi_{K_\Lambda}w_k$ by the naturality of $\varphi$. Put $b=T(\beta)\varphi_{K_\Lambda}w_k \in T(V)$,

$i_s a=i_s T(i_s)b$. Since $i_s$ is injective, $a=T(i_s)b$. Hence, $T(i_s)$ is isomorphic. Since $T$ is faithful, $i_s$ is isomorphic by [5], p. 56. Therefore, $0 \to S(M) \to S(F)$ is exact from (2). Next, let $M$ be any submodule of free $A$-module $F$: $0 \to M \to F$. Then $M$ is a direct limit of the family of finitely generated $A$-submodules $M_{ai}$; $M=\lim M_{ai}$. Since $S$ is colimit and exact preserving by [5], p. 85 and p. 55, $0 \to S(M)=\lim S(M_{ai}) \to S(F)$ is exact from the first argument. Hence, $\Tor^1(M, U)=0$ for all $M \in \text{mod} A$, ([5], p. 112, § 8), which implies that $S$ is exact.

From now on we fix a generator $t/U$ in $C$ and $A=\{U, F\}$. Then for any subobject $U'$ in $U$ it is clear that $[U, U']$ is identified to a right ideal in $A$, and we shall denote it by $r_{U'}$ or $r$. By $KU$ we mean the image of $f: \sum \limits_{k \in K} U_k \to U$ defined by $f(U_k)=kU$ for any subset $K$ in $A$. We note from the definitions that $r_{U'}U=S(T(U'))$. Then we have from [5], p. 71.

**Lemma 1.** For any subobject $U'$ in $U$ we have $U'=r_{U'}U$.

**Lemma 2.** Let $U$ be a generator in $C$ and $r_1, r_2$ right ideals in $A$. Then we have

1) $(r_1+r_2)U=r_1U \cup r_2U$.

2) $(r_1 \cap r_2)U=r_1U \cap r_2U$.

Proof. 1) is trivial from the definition.

2) We have the following row exact and commutative diagrams:

\[
\begin{array}{c}
0 \to r_1U \to r_1U \cup r_2U \to (r_1U \cup r_2U)/r_2U \to 0 \\
\uparrow \\
\end{array}
\]

4) $0 \to r_1U \cap r_2U \to r_1U \to r_1U/r_1U \cap r_2U \to 0$

$\uparrow \uparrow \approx$

$0 \to (r_1 \cap r_2)U \to r_1U \to r_1U/(r_1 \cap r_2)U$, and

$0 \to r_2 \to r_2 \cup r_1 \to (r_2 \cup r_1)/r_2 \to 0$

$\uparrow \uparrow \approx$

$0 \to r_1 \cap r_2 \to r_1 \cap r_1/r_1 \cap r_2 \to 0$

Since $S$ is an exact functor, we obtain $(r_1 \cap r_2)U=r_1U \cap r_2U$ from 4) by operating $S$ on 5).
The following proposition is an immediate consequence of [6], Prop. 1.1 and [5], p. 104. However, we shall prove it without localization.

**Proposition 3.** Let $C$, $U$ and $A$ be as above and $U$ a generator. Then the following statements are equivalent.

1) $S(\cdot) = \otimes U$ is an equivalent functor.
2) $T(\cdot) = [U, \cdot]$ and $S(\cdot)$ give a one-to-one correspondence between right ideals and subobjects in $U$.
3) For any maximal right ideal $r$ in $A$ $S(A/r) = 0$.
4) $U$ is projective and small in $C$.

**Proof.** 1)$\rightarrow$2)$\rightarrow$3) are trivial.
4)$\rightarrow$1) is proved in [5], p. 104.
3)$\rightarrow$4) It is clear from 3) that for any non-zero $A$-module $M$, $S(M) = M \otimes U \neq 0$, since $S$ is exact by Theorem 0. Let $V_1 \xrightarrow{\alpha} V_2 \rightarrow 0$ be exact in $C$ and $T(V_1) \rightarrow T(V_2) \rightarrow K \rightarrow 0$ be exact in mod $A$. Since $S$ is exact, $ST(V_1) = V_1 \xrightarrow{\alpha} ST(V_2) \rightarrow S(K) \rightarrow 0$ is exact. Hence, $S(K) = 0$, which means $K = 0$ from the above. Therefore, $T$ is exact and hence, $U$ is projective. Finally we shall show that $U$ is small. Let $f: U \rightarrow \sum \limits_{i \in I} V_i$ be a morphism in $C$, where $V_i$'s are any objects in $C$.

Put $U_j = f^{-1}(\sum \limits_{k \in J} V_k)$, where $J$ is a finite set of $I$. Since $C$ is $C_3$-category, $U = \bigcup \limits_{j \in J} U_j$ by [5] p. 83. Then $A = \bigcup \limits_{j \in J} r_j$ by Lemma 2 and 3), where $r_j = [U, U_j]$.

Put $1 = \sum \limits_{i \in I} f_i, f_i \in r_{j_i}$. Then $U = \bigcup \limits_{i \in I} U_{j_i}$, which implies $im f = \sum \limits_{i \in I} \sum \limits_{j \in J} V_i$.

An object $V$ in $C$ is called minimal if there exist no proper subobjects in $V$. If $V'$ is a directsum of minimal sub-objects, then $V'$ is called semi-simple. We note that some properties of semi-simple modules are valid in $C$.

**Lemma 4.** For any artinian and noetherian object $V$, $[V, V]$ is a semi-primary ring.

It is well known in mod $A$, and its proof is valid in $C$.

**2. Semi-primary category $C$**

Let $C$ be an abelian category mentioned in the section 1. We shall consider a function $\phi$ of object in $C$ into itself which is similar to the radical of a ring.

I. $\phi(C)$ is a subobject in $C$ for any $C$ in $C$ such that $C/\phi(C)$ is semi-simple.

II. $C = \phi(C)$ if and only if $C = 0$.

III. If $C/C'$ is semi-simple for some subobject $C'$ in $C$, then $C' \supset \phi(C)$.

Let $\phi, \phi_*$ be functions in $C$ satisfying I and II. We note in this case that every non-zero object contains a maximal subobject. If $\phi_*(C) \supset \phi_*(C)$ for all
Then we shall say \( \varphi_2 \) is smaller than \( \varphi_1 \). Furthermore, if \( \varphi_2 \) satisfies III and \( C \) is locally small, then \( \varphi_2 \) is a unique minimal function among those satisfying I and II, since \( \varphi_2(C) = \bigcap D \), where \( D \) runs all maximal subobjects in \( C \). In this case \( \varphi_2 \) is a functor which satisfies the following commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\varphi(C) & \xrightarrow{\varphi(f)} & \varphi(C') \\
\end{array}
\]

where \( f \in C \) and \( i, i' \) are inclusions and \( \varphi(f) \) is defined as follows: Let \( V \) be a maximal subobject in \( C' \) then \( f^{-1}(V) = C \) or \( C/f^{-1}(V) \cong C'/V \) ([5], pp. 22-24), and hence \( f(\varphi(C)) \subseteq V \), which implies \( \text{im}(f|\varphi(C)) \subseteq \varphi(C') \). Conversely, if \( \varphi \) satisfying I, II induces a functor in \( C \) satisfying 6), then \( \varphi \) satisfies III. In fact, let \( V \neq 0 \) in \( C \), then \( V \) contains a maximal subobject \( V_0 \). The commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\varphi(V)} & \varphi(V)/V_0 \\
\end{array}
\]

shows \( \varphi(V) \subseteq V_0 \).

We put \( \varphi'(U) = \varphi(U) \), \( \varphi^i(U) = \varphi(\varphi^{i-1}(U)) \).

**Lemma 5.** Let \( U \) be a generator of \( C \). If \( \varphi^i \) is defined in \( U \) such that \( \varphi^n(U) = 0 \) for some \( n \) and satisfies I, II (resp. I, II and III), then \( \varphi \) induces a function \( \Phi \) in \( C \) such that \( \Phi \) satisfies I, II (reps. I, II and III).

Proof. First, we define \( \Phi(\varphi^i(U)) = \varphi^{i+1}(U) \) for all \( i \). Let \( V \) be any object in \( C \) which is different from any \( \varphi^i(U) \), and \( g: \bigoplus_{i \in I} U_i \rightarrow V \) the canonical morphism defined by \( f: U_f \rightarrow V \). We assume that \( \text{im}(g|\bigoplus \varphi^i(U)) = V \) and \( \text{im}(g|\bigoplus \varphi^{i+1}(U)) \neq V \). Then define \( \Phi(V) = \text{im}(g|\bigoplus \varphi^{i+1}(U)) \). It is clear that \( V/\Phi(V) \) is semi-simple and that \( V/\Phi(V) \neq 0 \) if \( V \neq 0 \). Next, we assume \( \varphi \) satisfies III for \( U \). Let \( V_0 \) be a maximal subobject in \( V \), then \( f^{-1}(V_0) \supseteq \varphi(U) \). Therefore, \( \Phi(V) \subseteq V_0 \).

**Definition.** Let \( V \) be an object in \( C \). If \( [V, V] \) is a semi-primary ring, \( V \) is called a semi-primary object.

From Lemma 4, every artinian and noetherian object is semi-primary.

**Proposition 6.** Let \( U \) be a projective, small generator in an abelian \( C_\pi \)-category. Then \( U \) is semi-primary if and only if a function \( \varphi \) in \( U \) satisfying I, II and III is defined and \( U/\varphi(U) \) is a direct sum of finite many of simple objects and \( \varphi^n(U) = 0 \) for some \( n \).
Proof. It is clear from Theorem 0 and Proposition 3. We note here that \( \phi' \equiv S(n'U) \), where \( n \) is the radical of \([U, U]\).

The main purpose of this section is to study some structure of \( C(3) \)-category with semi-primary generator.

**Theorem 7.** Let \( C \) be an abelian \( C(3) \)-category with semi-primary generator \( U \). Then we can define a function \( \phi \) in \( C \) which satisfies I and II and \( U/\phi(U) \) is a finite direct sum of simple subobjects and \( \phi^n(U) = 0 \) for some \( n \).

Proof. Let \( A = [U, U] \) and \( n \) the radical of \( A \). Put \( U_i = n^i U \) for all \( i \). It is clear that \( U_i \supseteq U_{i+1} \). Put \( n_i = n^i \). Then \( n_i U_i = U_i \cap U_{i+1} = U_{i+1} \) by Lemma 2. Since \( n^i/n^{i+1} \) is semi-simple, say \( n^i/n^{i+1} = \sum \oplus n_a \), \( n_i \cdot n \supseteq n_a \rightarrow n_i U_{i+1} \), and \( n_a \) is simple. Put \( U_a = n_a U \).

If \( U_a = U_{i+1} \), \( U_a \cap n_i = 0 \), which is a contradiction. Hence, \( U_i \supseteq U_a \supseteq U_{i+1} \). We shall show that \( U_a = U_{i+1} \) is simple. Let \( V \) be a subobject such that \( U_a \supset V \supseteq U_{i+1} \). Then \( \tau V \supset n_a \), in fact if \( \tau V \supset n_a \), \( \tau V \cap n_a = \tau_{i+1} \), and hence, \( U_a = U_{i+1} \).

Since \( C \) is \( C(3) \)-category, \( U_i / U_{i+1} \) is semi-simple. We define \( \phi(U) = U_i \). Then \( U/\phi(U) \) is a finite direct sum of simple subobjects from the above, and \( \phi^n(U) = 0 \) if \( n^n = 0 \). Then we can define a function \( \phi \) in \( C \) from Lemma 5.

Let \( V_0 \) be a subobject in \( V \) such that \( V_0 + V' = V \) implies \( V = V' \) for any subobject \( V' \) in \( V \). \( V_0 \) is called negligible. By \([U: U_i]\) we mean the number of simple components in \( U/U_i \).

**Theorem 8.** Let \( C \) be an abelian \( C(3) \)-category with semi-primary generator \( U \). Then the following conditions are equivalent.

1) \( U \) is projective and small.
2) \([A: n] = [U: \phi(U)]\), where \( \phi(U) = n U, A = [U, U] \) and \( n \) is the radical of \( A \).
3) \( \phi(U) \) is negligible in \( U \).
4) \( \phi \) satisfies the condition III.
5) \( T: C \rightarrow \text{mod } A \) is preserving minimal objects.

Proof. If \( U \) is projective and small, then \( C \) is equivalent to \( \text{mod } A \) by Proposition 3. Hence, 2), 3), 4) and 5) are trivial. We assume 2). We put \( a = [U, n U] \). If we restrict the argument in the proof of Theorem 7 to the case of \( i = 1 \), we get \([A: a] = [U: n U]\), \( a = n \). Hence, \( a = n \). For every maximal right ideal \( r \), \( r/n = \sum_{i=1}^{n-1} \oplus r_a/n \), which implies \( U \supseteq \bigcup_{i=1}^{n-1} r_a U = r U \). Hence, we obtain 1) from Proposition 3.
Let $\alpha$ be as above. We assume $\alpha \phi n$. Then there exists a right ideal $b$ properly containing $n$ such that $b/n = (eA+n)/n$. Let $e$ be an idempotent element in $A$ such that $b/n = (eA+n)/n$. Since $b \supset n$, $bU \supset nU = aU$. Hence, $U = (a+b)U = bU$. Put $U_\circ = eU$. Then $U_\circ + nU = (eA+n)U = bU = U$. Therefore, $U_\circ = U$ by 3). Hence, $e=I_n$, which is a contradiction.

4) If $n \neq a$, we obtain the fact $U = U_\circ + nU$ and $U_\circ \neq U$. Since $U/U_\circ$ contains a maximal object from Theorem 7, there exists a maximal subobject $V \supset U_\circ$. Therefore, $V \supset nU$.

5) If $n \neq a$, then there exists a maximal subobject $V$ in $U$ such that $U = V + nU$. Since $0 \rightarrow [U, V] \rightarrow [U, U] \rightarrow [U, U/V]$ is exact and $[U, U/V]$ is minimal, $\tau_V = [U, V]$ is a maximal right ideal, and hence $\tau_V \supset n$, which is a contradiction.

It is clear that there are many examples in which semi-primary generators are not projective.

**Corollary 1.** Let $U$ be a semi-primary generator in $C$. If $A/n$ is a simple rings, $U$ is projective and small, where $A = [U, U]$ and $n$ is its radical.

**Proof.** Let $a = [U, nU]$. Since $U \neq nU$, and $a$ is a two-sided ideal, $a = n$.

**Corollary 2.** Let $B$ be a semi-primary ring and $U$ be a semi-primary generator in the category of $B$-right modules. Then $\eta_A U \supset U \eta_B$. $\eta_A U = U \eta_B$ if and only if $U$ is a finitely generated and projective, where, $A = [U, U]$ and $\eta_A$ (resp. $\eta_B$) is the radical of $A$ (resp. $B$).

**Proof.** Let $\phi(U) = U \eta_B$. Then $\phi$ is a functor in mod $B$ satisfying I, II and III. Hence, $\eta_A U \supset U \eta_B$ by Theorem 7. If $\eta_A U = U \eta_B$, a function $\phi'$ defined in the proof of Theorem 7 satisfies III. Hence, $U$ is projective and small. The converse is trivial.

**Example.** We shall show that there exists a generator $U$ such that $\phi'$ are defined in $U$ satisfying the following conditions: $U/\phi(U)$ is a finite directsum of simple object, $\phi'$ satisfies I, II and III for all $i$ and $\phi'(U) = 0$ for some $n$, however $U$ is not semi-primary.

Let $k$ be a field and $K = k(x)$. Let $A = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$ be a tri-angular matrix ring. Then $A$ is semi-primary with radical $n$. We define $\phi(U) = U n$ in mod $A$. Put $\tau = \begin{pmatrix} 0 & 0 \\ k[x] & 0 \end{pmatrix}$. Then $\tau$ is a right ideal in $A$. Then $[A/\tau, A/\tau] \cong \begin{pmatrix} k & 0 \\ k(x)/k[x] & k[x] \end{pmatrix}$ is not semi-primary. $U = A \oplus A/\tau$ is the desired generator.

### 3. Abelian category of commutative diagram

We recall the definition of abelian category of commutative diagram over abelian categories $C_i$ (see [4]).
Let $I = (1, 2, \ldots, n)$ be a finite linear ordered set and $\{C_i\}_{i \in I}$ a family of abelian categories. We assume that there are given cokernel preserving functors $T_{ij}: C_i \to C_j$ for $i < j$. Furthermore, we assume:

(*) There exist natural transformations

$$\psi_{ijk}: T_{jk}T_{ij} \to T_{ik} \quad \text{for all } i < j < k,$$

and

(**) For any $i < j < k < l$ and $V$ in $C_i$

$$T_{kh}T_{jk}T_{ij}(V) \xrightarrow{T_{kh}(\psi_{ijk})} T_{kh}T_{ik}(V) \xrightarrow{T_{ih}(\psi_{ijk})} T_{ih}(V)$$

is commutative.

We call a family of morphisms $d_{ij}: T_{ij}(V_i) \to V_j$ an arrow for $V_i \in C_i$, $V_j \in C_j$ and for all $i < j$, when the diagrams

$$T_{jk}T_{ij}(V_i) \xrightarrow{T_{jk}(d_{ij})} T_{jk}(V_j)$$

are commutative.

We define a commutative diagram $[I, C]$ as follows; Its objects consist of set $\{V_i\}_{i \in I}$ with arrows $\{d_{ij}\}$ and morphisms consist of set $\{(f_i)_{i \in I}; f_i: V_i \to V_i'\}$ in $C_i$ such that $d_{ij}T_{ij}(f_i) = f_jd_{ij}$.

**Lemma 9.** Let $T_{ij}$ be functors satisfying (**). Then the natural transformation of $T_{i_{mn}}T_{i_{m-1}n}T_{i_{m-2}n-1} \cdots T_{i_2i_1} \to T_{i_1i_n}$ does not depend on any choice of combination of $T_{i_{mn}}T_{i_{m-1}n} \cdots T_{i_2i_1}$.

**Proof.** We can prove the lemma by using induction on the number of functors and naturality of $\psi_{ijk}$. Namely, every natural transformation is equal to $T_{i_{mn}}(T_{i_{m-1}n}(T_{i_{m-2}n-1}(\cdots(T_{i_2i_1}) \to T_{i_1i_n}$.

We assume that all $C_i$ have projective class $\xi_i$. We define a functor $S_i: C_i \to [I, C]$ by setting $S_i(V_i) = (0, \ldots, 0, V_i, T_{ii+1}(V_i), \ldots, T_{in}(V_i))$. Then the projective objects in $[I, C]$ are of the form $\oplus S_i(P_i)$ and their retract, where $P_i$ is $\xi_i$-projective for all $i$, ([4], Prop. 1.2'). If the projective objects in $[I, C]$ are only of the former forms, we call $[I, C]$ a good category of commutative diagram.

**Theorem 10.** Let $C_i$ be abelian category with projective class $\xi_i$. Then every $[I, C_i]$ with $T_{ij}$ is embedding in a good category $[I, C_i]$ with $T'_{ij}$.

**Proof.** We shall define new functors $T'_{ij}$:
Then it is clear that $T'_{ij}$ are cokernel preserving and $\psi'_{ijk}=I_{C_k}$ and $($**$)$ is trivial. Furthermore, there exist unique natural transformations $\phi_{ij}: T'_{ij} \to T_{ij}$ by Lemma 9. Put $C=[I, C_i]$ with $T_{ij}$ and $C'=[I, C_i]$ with $T'_{ij}$. We define a function $F$ of $C$ into $C'$ as follows: For $V=(V_i)$ with arrows $d_{ij}$ in $C$ we put $F(V)=(V_i)$ with the following arrows $d'_{ij}$:

$$
\begin{align*}
T'_{ii+1} &= T_{ii+1} \\
T'_{ij} &= T_{j-1} T_{j-2j-1} \cdots T_{ii+1} \quad \text{for } i+1<j.
\end{align*}
$$

We have to show that $d'_{ij}$ satisfies (***) . We have a diagram for $i<j<k$ and $V_i \in C_t$

$$
\begin{array}{cccc}
T'_{jk}(V_i) & \xrightarrow{T'(\phi)} & T'_{ij}(V_i) & \xrightarrow{T'(d_{ij})} & T'_{jk}(V_j) \\
\downarrow {\psi'_{ijk}} & & \downarrow {\phi} & & \downarrow {\phi} \\
T_{jk}(V_i) & \xrightarrow{T(d_{ij})} & T_{ij}(V_i) & \xrightarrow{d_{jk}} & T_{jk}(V_j)
\end{array}
$$

I is commutative by Lemma 9, II is commutative by naturality of $\phi$ and so is III by (**). Hence, $d'_{ij}$ satisfies (***) . Define $F((f_i))=(f_i)$ for morphism $(f_i)$ in $C$. Then we can similarly show that $F$ is a functor. It is clear that $F$ is an imbedding functor. Since $\psi_{ijk}=I_{C_k}$, $K'(P_i)=0$ in (*) of [4], Lemma 3.7. Hence, $C'$ is good by [4], Lemma 3.7.

If every objects in $C$ are projective, $C$ is called a semi-simple category.

**Corollary.** Let $C_i$ be a semi-simple abelian category. Then $[I, C_i]$ is imbedding in an abelian hereditary category, (cf. [3], Theorem 5).

**Proof.** It is clear from Theorem 10 and [4], Theorem 3.12.

Finally, we note that if $C_i$ have functor $\varphi_i$ satisfying I, II and III and $T_{ij}(\varphi_i(V_i)) \subseteq \varphi_j T_{ij}(V_i)$ on $V=(V_i)$ in $[I, C_i]$. Then

$$
\varphi(V) = (\varphi_i(V_1), \varphi_i(V_2) \cup d_{i1}(V_2), \ldots, \varphi_j(V_j) \cup \bigcup_{i<j} d_{ij}(V_i), \ldots)
$$

is a functor on $[I, C_i]$ satisfying I, II and III. If $\varphi'^m=0$ for all $t$ then $\varphi'^m=0$. 

**Universidad de Buenos Aires and Osaka City University**
References


