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Osaka University
Let $C$ be an abelian category with exact direct limits, namely cocomplete $C_3$-category ([5], p. 83).

In this note we always assume that $C$ contains a generator $U$, and hence $C$ is locally small by [5], p. 71. In [2], Gabriel and Popesco have given a characterization of $U$ being projective and small by using the concept of localization in [1]. We shall give another proof without localization in the section 1.

In the section 2, we shall define a function $\varphi$ of $C$ into itself, which is analogous to the radical of semi-primary ring.

We shall show that $C$ has such a function when the endomorphism ring $[U, U]$ is a semi-primary ring, and we shall give some criteria by means of $\varphi$ that $U$ is small and projective.

In the section 3, we shall add some remarks in the previous author’s work on category of tri-angular matrices, [4].

In this note we shall freely make use of concepts in categories from [5].

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1. Preliminary results

In this section we shall summarize all results which we need in the following sections.

Almost all results in this section have been proved in [2] and [6] by using the concepts of localization in [1]. However, we shall give here another approach to them by means of rather homological method.

Let $C$ be an abelian cocomplete $C_3$-category ([5], p. 81) and $U$ an object of $C$. Let $A = [U, U]$. By mod $A$ we mean the category of $A$-right modules. Let $T : C \to \text{mod } A$; $T(V) = [U, V]$ for any $V \in C$ be the functor of $C$ into mod $A$. In this case we can define a coadjoint $S$ of $T$ such that $S(M) = M \otimes U$ by [5], p. 143, namely $\eta : [M, T(V)] \cong [S(M), V]_C$. Furthermore, we have natural transformation $\psi_V: ST(V) \to V$ and $\varphi_M: M \to TS(M)$, (see [5], pp 118-119).

**Theorem 0** (Gabriel and Popesco [2]). Let $C$, $U$ and $A$ be as above. Then the following statements are equivalent:

1) $U$ is a generator.
2) \( T \) is a completely faithful (namely, full and faithful).

3) \( \psi_V \) is isomorphic for all \( V \in C \) and \( S \) is an exact functor.

Proof. \( 1) \leftrightarrow 2) \). See [2] or [5] in which we do not need the concept of localization.

3) \( \rightarrow 2) \). \( [ST(V), V'] \approx [T(V), T(V')] \) and \( [ST(V), V'] \approx [V, V'] \) for \( V, V' \in C \).

2) \( \rightarrow 3) \). \( [ST(V), V'] \approx [T(V), T(V')] \approx [V, V'] \). Hence, \( [ST(V), ] \) and \( [V, ] \) give the equivalent functors. Therefore, \( \psi_V = \eta^{-1} \alpha^{-1} I_V \) is isomorphic.

Thus, it remains to show that \( S \) is exact. First, we show that if \( M \in \text{mod} \ A \) is contained in a free module \( F \), then \( 0 \rightarrow S(M) \rightarrow S(F) \) is exact. In order that, we assume first that \( M \) is finitely generated, say \( M=(m_1, m_2, \ldots, m_n) \) and hence we may assume that \( F \) is also finitely generated. Then we have a commutative diagram

\[
\begin{array}{c}
0 \leftarrow M \leftarrow \sum_{i=1}^{n} A \psi_{i} \leftarrow \kappa A = \sum_{i \in k} A \psi_{i} \\
\downarrow i \quad \quad \downarrow \alpha \\
F \leftarrow \sum_{i=1}^{m} A u_{ai} = \sum_{i=1}^{m} A u_{ai},
\end{array}
\]

where \( u_{ai} \), \( \psi_{i} \) and \( \psi_{i} \) are free bases and \( i \) is the inclusion map, \( f \) is a natural mapping such that \( f(\psi_{i}) = m_i, \alpha = \psi f, \) and \( K = \ker f \).

Operating \( S \) on the above \( 1) \) we obtain commutative exact diagram:

\[
\begin{array}{c}
0 \leftarrow S(M) \leftarrow \sum_{i=1}^{n} U \leftarrow V \\
\downarrow S(f) \quad \downarrow \beta \\
S(F) = S(F)
\end{array}
\]

where \( V = \text{im} (\kappa U \leftarrow \sum_{i} U) \) and \( K' = \ker S(\alpha) \).

It is clear that there exists the inclusion map \( i_i \) of \( V \) into \( K' \). Operating again \( T \) on \( 2) \) we have

\[
\begin{array}{c}
0 \leftarrow T(K') \leftarrow T(i_i) \\
\downarrow i_3 \\
\sum \psi_{i} \leftarrow T(V) \leftarrow T(\kappa U) \leftarrow \kappa A \\
\downarrow \alpha \\
\sum \psi_{i} \leftarrow T(i_2) \leftarrow T(V) \leftarrow T(\beta) \leftarrow \kappa A
\end{array}
\]
where the vertical line is exact and $T(i_1), T(i_2)$ are inclusions. Since $K$ is also \( \ker \alpha \), there exists a unique isomorphism $\theta$ such that

\[
\begin{array}{c}
T(K') \\
\downarrow i_3 \\
\sum \oplus A_{\beta_i}
\end{array} \xrightarrow{\theta} K
\]

is commutative. Let $a \in T(K')$ and put $k = \theta a$. Then $T(i_3)T(\beta)\varphi_{K\alpha}w_k = i_3a = i_3\theta a = i_3k = i_3T(i_2)T(\beta)\varphi_{K\alpha}w_k$ by the naturality of $\varphi$. Put $b = T(\beta)\varphi w_k \in T(V)$, $i_3a = i_3T(i_2)b$. Since $i_3$ is injective, $a = T(i_2)b$. Hence, $T(i_2)$ is isomorphic. Since $T$ is faithful, $i_2$ is isomorphic by [5], p. 56. Therefore, $0 \to S(M) \to S(F)$ is exact from (2). Next, let $M$ be any submodule of free $A$-module $F$: $0 \to M \to F$. Then $M$ is a direct limit of the family of finitely generated $A$-submodules $M_{a_i}$, $M = \lim_{\to} M_{a_i}$. Since $S$ is colimit and exact preserving by [5], p. 85 and p. 55, $0 \to S(M) = \lim_{\to} S(M_{a_i}) \to S(F)$ is exact from the first argument. Hence, $\text{Tor}^1(M, U) = 0$ for all $M \in \text{mod} \ A$, ([5], p. 112, § 8), which implies that $S$ is exact.

From now on we fix a generator $t$ in $C$ and $A = [U, U]$. Then for any subobject $U'$ in $U$ it is clear that $[U, U']$ is identified to a right ideal in $A$, and we shall denote it by $r_{U'}$ or $r$. By $KU$ we mean the image of $f: \sum_{k \in K} U_k \to U$ defined by $f(U_k) = kU$ for any subset $K$ in $A$. We note from the definitions that $r_{U'}U = ST(U')$. Then we have from [5], p. 71.

**Lemma 1.** For any subobject $U'$ in $U$ we have $U' = r_{U'}U$.

**Lemma 2.** Let $U$ be a generator in $C$ and $r_1$, $r_2$ right ideals in $A$. Then we have

1) \((r_1 + r_2)U = r_1U \cup r_2U\).
2) \((r_1 \cap r_2)U = r_1U \cap r_2U\).

**Proof.**

1) is trivial from the definition.

2) We have the following row exact and commutative diagrams:

\[
\begin{array}{cccc}
0 & \to & r_1U \to & r_1U \cup r_2U \to (r_1U \cup r_2U)/r_2U \to 0 \\
& \uparrow & \uparrow & \approx \\
0 & \to & r_1U \cap r_2U \to & r_1U \cap r_1U \cap r_2U \to 0 \\
& \uparrow & \uparrow & \approx \\
0 & \to & (r_1 \cap r_2)U \to & r_1U \cap (r_1 \cap r_2)U, \text{ and} \\
& \uparrow & \uparrow & \approx \\
0 & \to & r_1 \cap r_2 \cup r_1 \to (r_2 \cup r_1)/r_2 \to 0 \\
& \uparrow & \uparrow & \approx \\
0 & \to & r_1 \cap r_2 \to r_1 \to r_1/r_1 \cap r_2 \to 0
\end{array}
\]

Since $S$ is an exact functor, we obtain $(r_1 \cap r_2)U = r_1U \cap r_2U$ from 4) by operating $S$ on 5).
The following proposition is an immediate consequence of [6], Prop. 1.1 and [5], p. 104. However, we shall prove it without localization.

**Proposition 3.** Let $C$, $U$ and $A$ be as above and $U$ a generator. Then the following statements are equivalent.

1) $S(\cdot) \otimes U$ is an equivalent functor.

2) $T(\cdot) = [U, \cdot]$ and $S(\cdot)$ give a one-to-one correspondence between right ideals and subobjects in $U$.

3) For any maximal right ideal $r$ in $A$ $S(A/r) \neq 0$.

4) $U$ is projective and small in $C$.

**Proof.** 1) $\rightarrow$ 2) $\rightarrow$ 3) are trivial.

4) $\rightarrow$ 1) is proved in [5], p. 104.

3) $\rightarrow$ 4) It is clear from 3) that for any non-zero $A$-module $M$, $S(M) = M \otimes U \neq 0$, since $S$ is exact by Theorem 0. Let $V_1 \xrightarrow{\alpha} V_2 \rightarrow 0$ be exact in $C$ and $T(V_1) \rightarrow T(V_2) \rightarrow K \rightarrow 0$ be exact in mod $A$. Since $S$ is exact, $ST(V_1) = V_1 \xrightarrow{\alpha} ST(V_2) = V_2 \rightarrow S(K) \rightarrow 0$ is exact. Hence, $S(K) = 0$, which means $K = 0$ from the above. Therefore, $T$ is exact and hence, $U$ is projective. Finally we shall show that $U$ is small. Let $f: U \rightarrow \sum_{i \in I} V_i$ be a morphism in $C$, where $V_i$'s are any objects in $C$.

Put $U_j = f^{-1}(\sum_{i \in I_j} V_i)$, where $J$ is a finite set of $I$. Since $C$ is $C_A$-category, $U = \bigcup U_j$ by [5] p. 83. Then $A = \bigcup r_j$ by Lemma 2 and 3), where $r_j = [U, U_j]$.

Put $1 = \sum_{i \in I} f_i$, $f_i \in r_{j_i}$. Then $U = \bigcup U_{j_i}$, which implies $\text{im } f \subset \sum_{i \in I} \sum_{j \in I_i} V_i$.

An object $V$ in $C$ is called minimal if there exist no proper subobjects in $V$. If $V'$ is a direct sum of minimal sub-objects, then $V'$ is called semi-simple. We note that some properties of semi-simple modules are valid in $C$.

**Lemma 4.** For any artinian and noetherian object $V$, $[V, V]$ is a semi-primary ring.

It is well known in mod $A$, and its proof is valid in $C$.

2. **Semi-primary category $C$**

Let $C$ be an abelian category mentioned in the section 1. We shall consider a function $\varphi$ of object in $C$ into itself which is similar to the radical of a ring.

I. $\varphi(C)$ is a subobject in $C$ for any $C$ in $C$ such that $C/\varphi(C)$ is semi-simple.

II. $C = \varphi(C)$ if and only if $C = 0$.

III. If $C/C'$ is semi-simple for some subobject $C'$ in $C$, then $C' \supset \varphi(C)$.

Let $\varphi, \varphi_1$ be functions in $C$ satisfying I and II. We note in this case that every non-zero object contains a maximal subobject. If $\varphi_i(C) \supset \varphi_\alpha(C)$ for all
$C \subseteq C$, then we shall say $\varphi_2$ is smaller than $\varphi_1$. Furthermore, if $\varphi_2$ satisfies III and $C$ is locally small, then $\varphi_2$ is a unique minimal function among those satisfying I and II, since $\varphi_2(C) = \cap D$, where $D$ runs all maximal subobjects in $C$. In this case $\varphi_2$ is a functor which satisfies the following commutative diagram

$$
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\uparrow i & & \uparrow i' \\
\varphi(C) & \xrightarrow{\varphi(f)} & \varphi(C')
\end{array}
$$

where $f \in C$ and $i, i'$ are inclusions and $\varphi(f)$ is defined as follows: Let $V$ be a maximal subobject in $C'$ then $f^{-1}(V) = C$ or $C/f^{-1}(V) \cong C'/V$ ([5], pp. 22–24), and hence $f(\varphi(C)) \subseteq V$, which implies $\text{im} (f \varphi(C)) \subseteq \varphi(C')$. Conversely, if $\varphi$ satisfying I, II induces a functor in $C$ satisfying 6), then $\varphi$ satisfies III. In fact, let $V \neq 0$ in $C$, then $V$ contains a maximal subobject $V_\circ$. The commutative diagram

$$
\begin{array}{ccc}
V & \to & V/V_\circ \\
\uparrow \varphi & & \uparrow \varphi(V/V_\circ) \\
\varphi(V) & \to & \varphi(V/V_\circ) = 0
\end{array}
$$

shows $\varphi(V) \subseteq V_\circ$.

We put $\varphi^i(U) = \varphi(U)$, $\varphi^i(U) = \varphi(\varphi^{i-1}(U))$.

**Lemma 5.** Let $U$ be a generator of $C$. If $\varphi^i$ is defined in $U$ such that $\varphi^n(U) = 0$ for some $n$ and satisfies I, II (resp. I, II and III), then $\varphi$ induces a function $\Phi$ in $C$ such that $\Phi$ satisfies I, II (reps. I, II and III).

Proof. First, we define $\bar{\varphi}(\varphi^i(U)) = \varphi^{i+1}(U)$ for all $i$. Let $V$ be any object in $C$ which is different from any $\varphi(U)$, and $g: \sum_{i \in I} U_i \to V$ the canonical morphism defined by $f: U_i \to V$. We assume that $\text{im} (g \sum \varphi^i(U)) = V$ and $\text{im} (g \sum \varphi^{i+1}(U)) = V$. Then define $\Phi(V) = \text{im} (g \sum \varphi^{i+1}(U))$. It is clear that $V/\Phi(V)$ is semi-simple and that $V/\Phi(V) \neq 0$ if $V \neq 0$. Next, we assume $\Phi$ satisfies III for $U$. Let $V_\circ$ be a maximal subobject in $V$, then $f^{-1}(V_\circ) \supseteq \varphi(U)$. Therefore, $\Phi(V) \subseteq V_\circ$.

**Definition.** Let $V$ be an object in $C$. If $[V, V]$ is a semi-primary ring, $V$ is called a semi-primary object.

From Lemma 4, every artinian and noetherian object is semi-primary.

**Proposition 6.** Let $U$ be a projective, small generator in an abelian $C_\gamma$-category. Then $U$ is semi-primary if and only if a function $\varphi$ in $U$ satisfying I, II and III is defined and $U/\varphi(U)$ is a directsum of finite many of simple objects and $\varphi^n(U) = 0$ for some $n$. 

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Proof. It is clear from Theorem 0 and Proposition 3. We note here that 
\[ \varphi(t(U)) = S(t'U), \] where \( n \) is the radical of \([U, U]\).

The main purpose of this section is to study some structure of \( C \)-category
with semi-primary generator.

**Theorem 7.** Let \( C \) be an abelian \( C \)-category with semi-primary generator 
\( U \). Then we can define a function \( \varphi \) in \( C \) which satisfies I and II and \( U/\varphi(U) \) is
a finite direct sum of simple subobjects and \( \varphi^n(U) = 0 \) for some \( n \).

Proof. Let \( A = [U, U] \) and \( n \) the radical of \( A \). Put \( U_i = n^iU \) for all \( i \).
It is clear that \( U_i \supseteq U_{i+1} \). Then \( r_i = n^i \cap r_{i+1} \).
Then \( r_{i+1} U = U_i \cap U_{i+1} = U_{i+1} \) by Lemma 2. Since \( n^i/n^{i+1} \) is semi-simple, so is \( n^i/r_{i+1}^{i+1} \).
It is clear that \( C/r_i \supseteq C/r_{i+1} \).

Let \( V \) be a subobject such that \( U_{a_i} \supseteq V \supseteq U_{i+1} \). Then \( r_{a_i} \supseteq r_{i+1} \).
In fact if \( r_{a_i} \supseteq r_{i+1} \), then \( U_{a_i} = U_{i+1} \).
Hence, \( U_{a_i} = r_{a_i} U \).
Since \( U \) is \( C \)-category, \( U_{a_i}/U_{i+1} \) is semi-simple.
We define \( \varphi(U) = U_i \).
Then \( U/\varphi(U) \) is a finite direct sum of simple subobjects from the above, and \( \varphi^n(U) = 0 \) if \( n^n = 0 \). Then we can define a function \( \varphi \) in \( C \)
from Lemma 5.

Let \( V_0 \) be a subobject in \( V \) such that \( V_0 + V' = V \) implies \( V = V' \) for any
subobject \( V' \) in \( V \). \( V_0 \) is called negligible. By \([U:U_i] \) we mean the number of
simple components in \( U/U_i \).

**Theorem 8.** Let \( C \) be an abelian \( C \)-category with semi-primary generator 
\( U \), Then the following conditions are equivalent.
1) \( U \) is projective and small.
2) \([A: n] = [U: \varphi(U)]\), where \( \varphi(U) = nU, A = [U, U] \) and \( n \) is the radical of \( A \).
3) \( \varphi(U) \) is negligible in \( U \).
4) \( \varphi \) satisfies the condition III.
5) \( T: C \to \text{mod} \ A \) is preserving minimal objects.

Proof. If \( U \) is projective and small, then \( C \) is equivalent to \( \text{mod} \ A \) by
Proposition 3. Hence, 2), 3), 4) and 5) are trivial. We assume 2). We put \( a = [U, nU] \).
If we restrict the argument in the proof of Theorem 7 to the case of \( i = 1 \), we get \([A: a] = [U: nU], = n \). Hence, \( a = n \).
For every maximal right
ideal \( r, n = \sum a_i/n \), which implies \( U = \bigcup_i r_{a_i} U = r U \).
Hence, we obtain
1) from Proposition 3.
3) Let $\alpha$ be as above. We assume $\alpha \neq n$. Then there exists a right ideal $b$ properly containing $n$ such that $a/n \oplus b/n = A/n$. Let $e$ be an idempotent element in $A$ such that $b/n = (eA + n)/n$. Since $b \supseteq n$, $bU \supseteq nU = aU$. Hence, $U = (a + b)U = bU$. Put $U_0 = eU$. Then $U_0 + nU = (eA + n)U = bU = U$. Therefore, $U_0 = U$ by (3). Hence, $e = I_n$, which is a contradiction.

4) If $n \neq a$, we obtain the fact $U = U_0 + nU$ and $U_0 \neq U$. Since $U/U_0$ contains a maximal object from Theorem 7, there exists a maximal subobject $V \supset U_0$. Therefore, $V \supset U_0$.

5) If $n \neq a$, then there exists a maximal subobject $V$ in $U$ such that $U = V + nU$. Since $0 \rightarrow [U, V] \rightarrow [U, U] \rightarrow [U, U/V]$ is exact and $[U, U/V]$ is minimal, $\tau_V = [U, V]$ is a maximal right ideal, and hence $\tau_V \supset n$, which is a contradiction.

It is clear that there are many examples in which semi-primary generators are not projective.

**Corollary 1.** Let $U$ be a semi-primary generator in $C$. If $A/n$ is a simple ring, $U$ is projective and small, where $A = [U, U]$ and $n$ is its radical.

Proof. Let $a = [U, nU]$. Since $U \neq nU$, and $a$ is a two-sided ideal, $a = n$.

**Corollary 2.** Let $B$ be a semi-primary ring and $U$ be a semi-primary generator in the category of $B$-right modules. Then $\pi_A U \supset U \pi_B$, $\pi_A U = U \pi_B$ if and only if $U$ is a finitely generated and projective, where, $A = [U, U]$ and $\pi_A$ (resp. $\pi_B$) is the radical of $A$ (resp. $B$).

Proof. Let $\varphi(U) = U \pi_B$. Then $\varphi$ is a functor in mod $B$ satisfying I, II and III. Hence, $\pi_A U \supset U \pi_B$ by Theorem 7. If $\pi_A U = U \pi_B$, a function $\varphi'$ defined in the proof of Theorem 7 satisfies III. Hence, $U$ is projective and small. The converse is trivial.

**Example.** We shall show that there exists a generator $U$ such that $\varphi^i$ are defined in $U$ satisfying the following conditions: $U/\varphi(U)$ is a finite directsum of simple object, $\varphi^i$ satisfies I, II and III for all $i$ and $\varphi^i(U) = 0$ for some $n$, however $U$ is not semi-primary.

Let $k$ be a field and $K = k(x)$. Let $A = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$ be a tri-angular matrix ring. Then $A$ is semi-primary with radical $n$. We define $\varphi(U) = U \pi$ in mod $A$. Put $\tau = \begin{pmatrix} 0 & 0 \\ k[x] & 0 \end{pmatrix}$. Then $\tau$ is a right ideal in $A$. Then $[A/\tau, A/\tau] \approx \begin{pmatrix} k & 0 \\ k(x)/k[x] & k[x] \end{pmatrix}$ is not semi-primary. $U = A \oplus A/\tau$ is the desired generator.

3. **Abelian category of commutative diagram**

We recall the definition of abelian category of commutative diagram over abelian categories $C_i$ (see [4]).
Let \( I = (1, 2, \ldots, n) \) be a finite linear ordered set and \( \{C_i\}_{i \in I} \) a family of abelian categories. We assume that there are given cokernel preserving functors \( T_{ij} : C_i \rightarrow C_j \) for \( i < j \). Furthermore, we assume:

\((*)\) There exist natural transformations

\[ \psi_{ijk} : T_{jk} T_{ij} \rightarrow T_{ik} \]

for all \( i < j < k \), and

\((**)\) For any \( i < j < k < l \) and \( V \) in \( C_i \)

\[ T_{hi} T_{jk} T_{ij}(V) \xrightarrow{T_{hi}(\psi)} T_{hi} T_{ik}(V) \]

\[ T_{ji} T_{jk}(V) \xrightarrow{T_{ji}(\psi)} T_{ji}(V) \]

is commutative.

We call a family of morphism \( d_{ij} : T_{ij}(V_i) \rightarrow V_j \) an arrow for \( V_i \in C_i, V_j \in C_j \) and for all \( i < j \), when the diagrams

\[ T_{jk} T_{ij}(V_i) \xrightarrow{T_{jk}(d_{ij})} T_{jk}(V_j) \]

\[ T_{ik}(V_i) \xrightarrow{d_{ik}} V_k \]

are commutative.

We define a commutative diagram \([I, C_i]\) as follows; Its objects consist of set \( \{V_i\}_{i \in I} \) with arrows \( \{d_{ij}\} \) and morphisms consist of set \( \{(f_i)_{i \in I} ; f_i : V_i \rightarrow V_i'\} \) in \( C_i \) such that \( d'_i T_{ij}(f_i) = f'_j d_{ij} \).

**Lemma 9.** Let \( T_{ij} \) be functors satisfying \((**).\) Then the natural transformation of \( T_{im-n+i} T_{im-n+2i-1} \cdots T_{ij} \rightarrow T_{ij} \) does not depend on any choice of combination of \( T_{im-n+i}, \cdots, T_{ij} \).

Proof. We can prove the lemma by using induction on the number of functors and naturality of \( \psi_{ijk} \). Namely, every natural transformation is equal to \( T_{in-i'm}(T_{in-i'-1}(\cdots(T_{i+1}(T_{i}))(\cdots(T_{i+i-1}(T_{i})))) \rightarrow T_{in-i'} \).

We assume that all \( C_i \) have projective class \( \epsilon_i \). We define a functor \( S_i : C_i \rightarrow [I, C_i] \) by setting \( S_i(V_i) = (0, \ldots, 0, V_i, T_{ii+1}(V_i), \ldots, T_{in}(V_i)) \). Then the projective objects in \([I, C_i] \) are of the form \( \oplus S_i(P_i) \) and their retract, where \( P_i \) is \( \epsilon_i \)-projective for all \( i \) (\cite{4}, Prop. 1.2'). If the projective objects in \([I, C_i] \) are only of the former forms, we call \([I, C_i] \) a good category of commutative diagram.

**Theorem 10.** Let \( C_i \) be abelian category with projective class \( \epsilon_i \). Then every \([I, C_i]\) with \( T_{ij} \) is imbedding in a good category \([I, C_i]\) with \( T'_{ij} \).

Proof. We shall define new functors \( T'_{ij} : \)

\[ T_{ii+1} = T_{ii+1} \]
\[ T'_{ij} = T_{j-1} T_{j-2} \cdots T_{ii+1} \quad \text{for } i+1 < j . \]

Then it is clear that \( T'_{ij} \) are cokernel preserving and \( \psi'_{ijk} = I_{Ck} \) and (**) is trivial. Furthermore, there exist unique natural transformations \( \phi_{ij} : T'_{ij} \to T_{ij} \) by Lemma 9. Put \( C = [I, C_i] \) with \( T_{ij} \) and \( C' = [I, C_i] \) with \( T'_{ij} \). We define a function \( F \) of \( C \) into \( C' \) as follows. For \( V = (V_i) \) with arrows \( d_{ij} \) in \( C \) we put \( F(V) = (V_i) \) with the following arrows \( d'_{ij} \):

\[
\begin{align*}
\phi'_{ij} & \quad \text{for } i+1 < j . \\
d'_{ii+1} &= d_{ii+1} \\
d'_{ij} &= d_{ij} \phi_{ij} T'_{ij}
\end{align*}
\]

We have to show that \( d'_{ij} \) satisfies (***) . We have a diagram for \( i < j < k \) and \( V_t \in C_t \)

\[
\begin{array}{ccc}
T'_{jk} T'_{ij}(V_i) & \xrightarrow{T'(\phi)} & T'_{jk} T_{ij}(V_i) \\
\downarrow{\psi'_{ijk}} & & \downarrow{\phi} \\
T'_{jk}(V_i) & \xrightarrow{T'(d_{ij})} & T'_{jk}(V_j)
\end{array}
\]

I is commutative by Lemma 9, II is commutative by naturality of \( \phi \) and so is III by (**). Hence, \( d'_{ij} \) satisfies (***) . Define \( F((f_i)) = (f_i) \) for morphism \( (f_i) \) in \( C \). Then we can similarly show that \( F \) is a functor. It is clear that \( F \) is an imbedding functor. Since \( \psi'_{ijk} = I_{Ck} \), \( K^j(P_i) = 0 \) in (*) of [4], Lemma 3.7. Hence, \( C' \) is good by [4], Lemma 3.7.

If every objects in \( C \) are projective, \( C \) is called a semi-simple category.

**Corollary.** Let \( C_i \) be a semi-simple abelian category. Then \([I, C_i]\) is imbedding in an abelian hereditary category, (cf. [3], Theorem 5).

Proof. It is clear from Theorem 10 and [4], Theorem 3.12.

Finally, we note that if \( C_t \) have functor \( \varphi_t \) satisfying I, II and III and \( T_{ij}(\varphi_t(V_i)) \subseteq \varphi_j T_{ij}(V_i) \) on \( V = (V_i) \) in \([I, C_i]\). Then

\[ \varphi(V) = (\varphi(V_1), \varphi(V_2) \cup d_{it}(V_2), \cdots, \varphi_j(V_j) \cup d_{ij}(V_i), \cdots) \]

is a functor on \([I, C_i]\) satisfying I, II and III. If \( \varphi_t^m = 0 \) for all \( t \) then \( \varphi^m = 0 \).

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References