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ON SEMI-PRIMARY ABELIAN CATEGORIES

Dedicated to Professor Atuo Komatu for his 60th birthday

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Let $\boldsymbol{C}$ be an abelian category with exact direct limits, namely cocomplete C_3-category ([5], p. 83).

In this note we always assume that $\boldsymbol{C}$ contains a generator U, and hence $\boldsymbol{C}$ is locally small by [5], p. 71. In [2], Gabriel and Popesco have given a characterization of U being projective and small by using the concept of localization in [1]. We shall give another proof without localization in the section 1.

In the section 2, we shall define a function φ of $\boldsymbol{C}$ into itself, which is analogous to the radical of semi-primary ring.

We shall show that $\boldsymbol{C}$ has such a function when the endomorphism ring $[U,U]$ is a semi-primary ring, and we shall give some criteria by means of φ that U is small and projective.

In the section 3, we shall add some remarks in the previous author's work on category of tri-angular matrices, [4].

In this note we shall freely make use of concepts in categories from [5].

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1. Preliminary results

In this section we shall summarize all results which we need in the following sections.

Almost all results in this section have been proved in [2] and [6] by using the concepts of localization in [1]. However, we shall give here another approach to them by means of rather homological method.

Let $\boldsymbol{C}$ be an abelian cocomplete C_3-category ([5], p. 81) and U an object of $\boldsymbol{C}$. Let $A=[U,U]$. By mod A we mean the category of A-right modules. Let $T:\boldsymbol{C}\to \text{mod } A$; $T(V)=[U,V]$ for any $V\in\boldsymbol{C}$ be the functor of $\boldsymbol{C}$ into mod A. In this case we can define a coadjoint S of T such that $S(M)=M\otimes_A U$ by [5], p. 143, namely η: $[M, T(V)]\approx[S(M), V]_C$. Furthermore, we have natural transformation ψ_V: $ST(V)\to V$ and φ_M: $M\to TS(M)$, (see[5], pp 118–119).

Theorem 0 (Gabriel and Popesco [2]). *Let $\boldsymbol{C}$, U and A be as above. Then the following statements are equivalent:*

1) *U is a generator,*

2) *T is a completely faithful (namely, full and faithful).*
3) *ψ_V is isomorphic for all $V \in C$ and S is an exact functor.*

Proof. 1)$\leftrightarrow$2). See [2] or [5] in which we do not need the concept of localization.
3)$\rightarrow$2). $[ST(V),\ V'] \underset{\eta}{\approx} [T(V),\ T(V')]$ and $[ST(V),\ V'] \approx [V,\ V']$ for $V,\ V' \in \boldsymbol{C}$.
2)$\rightarrow$3). $[ST(V),\ V'] \underset{\eta}{\approx} [T(V),\ T(V')] \underset{\alpha}{\approx} [V, V']$. Hence, $[ST(V),\ \ ]$ and $[V,\ \ ]$ give the equivalent functors. Therefore, $\psi_V = \eta^{-1}\alpha^{-1}I_V$ is isomorphic.
Thus, it remains to show that S is exact. First, we show that if $M \in \operatorname{mod} A$ is contained in a free module F, then $0 \rightarrow S(M) \rightarrow S(F)$ is exact. In order that, we assume first that M is finitely generated, say $M = (m_1, m_2, \cdots, m_n)$ and hence we may assume that F is also finitely generated. Then we have a commutative diagram

1)
$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, "i"] & \sum_{i=1}^{n} \oplus Av_{\beta_i} \arrow[l, "f"'] \arrow[d, "\alpha"] & {}^{K}A = \sum_{k \in K} \oplus Aw_k \arrow[l] \\
& F = \sum_{i=1}^{m} \oplus Au_{\alpha_i} = & \sum_{i=1}^{m} \oplus Au_{\alpha_i}\,, &
\end{tikzcd}$$

where u_{α_i}, v_{β_i} and w_k are free bases and i is the inclusion map, f is a natural mapping such that $f(v_{\beta_i}) = m_i$, $\alpha = if$, and $K = \ker f$.

Operating S on the above 1) we obtain commutative exact diagram:

2)
$$\begin{tikzcd}
& & 0 \arrow[d] & \\
& & K' \arrow[d] & \\
0 & S(M) \arrow[l] \arrow[d] & \sum_{i=1}^{n} \oplus U \arrow[l, "S(f)"'] \arrow[d, "S(\alpha)"] & V \arrow[l, "i_2"] \arrow[ul, dashed, "i_1"'] \\
& S(F) \arrow[r, phantom, "="] & S(F) &
\end{tikzcd}$$

where $V = \operatorname{im}\left({}^{K}U \xrightarrow{\beta} \sum_1^n U\right)$ and $K' = \ker S(\alpha)$.
It is clear that there exists the inclusion map i_1 of V into K'. Operating again T on 2) we have

3)
$$\begin{tikzcd}
0 \arrow[d] & & & \\
T(K') \arrow[d, "i_3"] & & & \\
\sum \oplus Av_{\beta_i} \arrow[d, "\alpha"] & T(V) \arrow[l, "T(i_2)"] \arrow[ul, "T(i_1)"'] & T({}^{K}U) \arrow[l, "T(\beta)"] & {}^{K}A \arrow[l, "\varphi_{K_A}"] \\
\sum \oplus Au_{\alpha_i} & & &
\end{tikzcd}$$

where the vertical line is exact and $T(i_1)$, $T(i_2)$ are inclusions. Since K is also $\ker \alpha$, there exists a unique isomorphism θ such that

$$\begin{array}{ccc} T(K') & \xrightarrow{\theta} & K \\ {\scriptstyle i_3}\downarrow & \swarrow{\scriptstyle i_4} & \\ \sum \oplus A_{\beta_i} & & \end{array}$$

is commutative. Let $a \in T(K')$ and put $k=\theta a$. Then $T(i_2)T(\beta)\varphi_{KA}w_k=i_3a=i_4\theta a=i_4k=i_3T(i_1)T(\beta)\varphi_{KA}w_k$ by the naturality of φ. Put $b=T(\beta)\varphi w_k \in T(V)$, $i_3a=i_3T(i_1)b$. Since i_3 is injective, $a=T(i_1)b$. Hence, $T(i_1)$ is isomorphic. Since T is faithful, i_1 is isomorphic by [5], p. 56. Therefore, $0 \to S(M) \to S(F)$ is exact from 2). Next, let M be any submodule of free A-module F: $0 \to M \to F$. Then M is a direct limit of the family of finitely generated A-submodules M_{α_i}; $M=\varinjlim M_{\alpha_i}$. Since S is colimit and exact preserving by [5], p. 85 and p. 55, $0 \to S(M)=\varinjlim S(M_{\alpha_i}) \to S(F)$ is exact from the first argument. Hence, $\mathrm{Tor}^1(M, U)=0$ for all $M \in \mathrm{mod}\, A$, ([5], p. 112, § 8), which implies that S is exact.

From now on we fix a generator U in $\boldsymbol{C}$ and $A=[U, U]$. Then for any subobject U' in U it is clear that $[U, U']$ is identified to a right ideal in A, and we shall denote it by $\boldsymbol{r}_{U'}$ or $\boldsymbol{r}$. By $\boldsymbol{K}U$ we mean the image of f: $\sum_{k \in \boldsymbol{K}} U_k \to U$ defined by $f(U_k)=kU$ for any subset $\boldsymbol{K}$ in A. We note from the definitions that $\boldsymbol{r}_{U'}U=ST(U')$. Then we have from [5], p. 71.

Lemma 1. *For any subobject U' in U we have $U'=\boldsymbol{r}_{U'}U$.*

Lemma 2. *Let U be a generator in $\boldsymbol{C}$ and $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ right ideals in A. Then we have*

1) $(\boldsymbol{r}_1+\boldsymbol{r}_2)U=\boldsymbol{r}_1U \cup \boldsymbol{r}_2U$.
2) $(\boldsymbol{r}_1 \cap \boldsymbol{r}_2)U=\boldsymbol{r}_1U \cap \boldsymbol{r}_2U$.

Proof. 1) is trivial from the definition.

2) We have the following row exact and commutative diagrams:

4)
$$\begin{array}{ccccccccc} 0 & \to & \boldsymbol{r}_2U & \to & \boldsymbol{r}_1U \cup \boldsymbol{r}_2U & \to & (\boldsymbol{r}_1U \cup \boldsymbol{r}_2U)/\boldsymbol{r}_2U & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow \approx & & \\ 0 & \to & \boldsymbol{r}_1U \cap \boldsymbol{r}_2U & \to & \boldsymbol{r}_1U & \to & \boldsymbol{r}_1U/\boldsymbol{r}_1U \cap \boldsymbol{r}_2U & \to & 0 \\ & & \uparrow & & \uparrow \approx & & \uparrow & & \\ 0 & \to & (\boldsymbol{r}_1 \cap \boldsymbol{r}_2)U & \to & \boldsymbol{r}_1U & \to & \boldsymbol{r}_1U/(\boldsymbol{r}_1 \cap \boldsymbol{r}_2)U, & & \text{and} \end{array}$$

5)
$$\begin{array}{ccccccccc} 0 & \to & \boldsymbol{r}_2 & \to & \boldsymbol{r}_2 \cup \boldsymbol{r}_1 & \to & (\boldsymbol{r}_2 \cup \boldsymbol{r}_1)/\boldsymbol{r}_2 & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow \approx & & \\ 0 & \to & \boldsymbol{r}_1 \cap \boldsymbol{r}_2 & \to & \boldsymbol{r}_1 & \to & \boldsymbol{r}_1/\boldsymbol{r}_1 \cap \boldsymbol{r}_2 & \to & 0 \end{array}$$

Since S is an exact functor, we obtain $(\boldsymbol{r}_1 \cap \boldsymbol{r}_2)U=\boldsymbol{r}_1U \cap \boldsymbol{r}_2U$ from 4) by operating S on 5).

The following proposition is an immediate consequence of [6], Prop. 1.1 and [5], p. 104. However, we shall prove it without localization.

Proposition 3. *Let* $\boldsymbol{C}$, U *and* A *be as above and* U *a generator. Then the following statements are equivalent.*
1) $S(\ )=\ \otimes U$ *is an equivalent functor.*
2) $T(\ )=[U,\ ]$ *and* $S(\ )$ *give a one-to-one correspondence between right ideals and subobjects in* U.
3) *For any maximal right ideal* $\boldsymbol{r}$ *in* A $S(A/\boldsymbol{r})\neq 0$.
4) U *is projective and small in* $\boldsymbol{C}$.

Proof. 1)→2)→3) are trivial.
4)→1) is proved in [5], p. 104.
3)→4) It is clear from 3) that for any non-zero A-module M, $S(M)=M\otimes U \neq 0$, since S is exact by Theorem 0. Let $V_1 \xrightarrow{\alpha} V_2\to 0$ be exact in $\boldsymbol{C}$ and $T(V_1) \to T(V_2)\to K\to 0$ be exact in mod A. Since S is exact, $ST(V_1)=V_1 \xrightarrow{\alpha} ST(V_2)=V_2 \to S(K)\to 0$ is exact. Hence, $S(K)=0$, which means $K=0$ from the above. Therefore, T is exact and hence, U is projective. Finally we shall show that U is small. Let $f\colon U\to \sum_{i\in I} V_i$ be a morphism in $\boldsymbol{C}$, where V's are any objects in $\boldsymbol{C}$. Put $U_J=f^{-1}(\sum_{k\in J} V_k)$, where J is a finite set of I. Since $\boldsymbol{C}$ is C_3-category, $U=\cup U_J$ by [5] p. 83. Then $A=\bigcup_{J} \boldsymbol{r}_J$ by Lemma 2 and 3), where $\boldsymbol{r}_J=[U, U_J]$. Put $1=\sum_{i=1}^{t} f_i$, $f_i\in \boldsymbol{r}_{J_i}$. Then $U=\bigcup_{i=1}^{t} U_{J_i}$, which implies $\operatorname{im} f\subset \sum_{i=1}^{t}\sum_{l\in J_i} V_l$.

An object V in $\boldsymbol{C}$ is called minimal if there exist no proper subobjects in V. If V' is a directsum of minimal sub-objects, then V' is called semi-simple. We note that some properties of semi-simple modules are valid in $\boldsymbol{C}$.

Lemma 4. *For any artinian and noetherian object* V, $[V, V]$ *is a semi-primary ring.*

It is well known in mod A, and its proof is valid in $\boldsymbol{C}$.

2. Semi-primary category $\boldsymbol{C}$

Let $\boldsymbol{C}$ be an abelian category mentioned in the section 1. We shall consider a function φ of object in $\boldsymbol{C}$ into itself which is similar to the radical of a ring.

I. $\varphi(C)$ is a subobject in C for any C in $\boldsymbol{C}$ such that $C/\varphi(C)$ is semi-simple.

II. $C=\varphi(C)$ if and only if $C=0$.

III. If C/C' is semi-simple for some subobject C' in C, then $C'\supset\varphi(C)$.

Let φ, φ_1 be functions in $\boldsymbol{C}$ satisfying I and II. We note in this case that every non-zero object contains a maximal subobject. If $\varphi_1(C)\supseteqq\varphi_2(C)$ for all

$C \in \boldsymbol{C}$, then we shall say φ_2 is smaller than φ_1. Furthermore, if φ_2 satisfies III and $\boldsymbol{C}$ is locally small, then φ_2 is a unique minimal function among those satisfying I and II, since $\varphi_2(C) = \cap D$, where D runs all maximal subobjects in C. In this case φ_2 is a functor which satisfies the following commutative diagram

$$\text{6)} \qquad \begin{array}{ccc} C & \xrightarrow{f} & C' \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i'} \\ \varphi(C) & \xrightarrow{\varphi(f)} & \varphi(C'), \end{array}$$

where $f \in \boldsymbol{C}$ and i, i' are inclusions and $\varphi(f)$ is defined as follows: Let V be a maximal subobject in C' then $f^{-1}(V) = C$ or $C/f^{-1}(V) \approx C'/V$ ([5], pp. 22–24), and hence $f(\varphi(C)) \subset V$, which implies $\operatorname{im}(f \mid \varphi(C)) \subset \varphi(C')$. Conversely, if φ satisfying I, II induces a functor in $\boldsymbol{C}$ satisfying 6), then φ satisfies III. In fact, let $V \neq 0$ in $\boldsymbol{C}$, then V contains a maximal subobject V_0. The commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & V/V_0 \\ \uparrow & & \uparrow \\ \varphi(V) & \longrightarrow & \varphi(V/V_0) = 0 \end{array}$$

shows $\varphi(V) \subseteq V_0$.

We put $\varphi^1(U) = \varphi(U)$, $\varphi^i(U) = \varphi(\varphi^{i-1}(U))$.

Lemma 5. *Let U be a generator of $\boldsymbol{C}$. If φ^i is defined in U such that $\varphi^n(U) = 0$ for some n and satisfies I, II (resp. I, II and III), then φ induces a function $\tilde{\varphi}$ in $\boldsymbol{C}$ such that $\tilde{\varphi}$ satisfies I, II (reps. I, II and III).*

Proof. First, we define $\tilde{\varphi}(\varphi^i(U)) = \varphi^{i+1}(U)$ for all i. Let V be any object in $\boldsymbol{C}$ which is different from any $\varphi^i(U)$, and $g: \sum_{[U,V] \ni f} \oplus U_f \to V$ the canonical morphism defined by $f: U_f \to V$. We assume that $\operatorname{im}(g \mid \sum \oplus \varphi^i(U)) = V$ and $\operatorname{im}(g \mid \sum \oplus \varphi^{i+1}(U)) \neq V$. Then define $\tilde{\varphi}(V) = \operatorname{im}(g \mid \sum \oplus \varphi^{i+1}(U))$. It is clear that $V/\tilde{\varphi}(V)$ is semi-simple and that $V/\tilde{\varphi}(V) \neq 0$ if $V \neq 0$. Next, we assume φ satisfies III for U. Let V_0 be a maximal subobject in V, then $f^{-1}(V_0) \supset \varphi(U)$. Therefore, $\tilde{\varphi}(V) \subseteq V_0$.

DEFINITION. Let V be an object in $\boldsymbol{C}$. If $[V, V]$ is a semi-primary ring, V is called a *semi-primary* object.

From Lemma 4, every artinian and noetherian object is semi-primary.

Proposition 6. *Let U be a projective, small generator in an abelian C_3-category. Then U is semi-primary if and only if a function φ in U satisfying I, II and III is defined and $U/\varphi(U)$ is a directsum of finite many of simple objects and $\varphi^n(U) = 0$ for some n.*

Proof. It is clear from Theorem 0 and Proposition 3. We note here that $\varphi^i(U)=S(\mathfrak{n}^i U)$, where $\mathfrak{n}$ is the radical of $[U, U]$.

The main purpose of this section is to study some structure of C_3-category with semi-primary generator.

Theorem 7. *Let* $\boldsymbol{C}$ *be an abelian* C_3*-category with semi-primary generator* U*. Then we can define a function* φ *in* $\boldsymbol{C}$ *which satisfies I and II and* $U/\varphi(U)$ *is a finite directsum of simple subobjects and* $\varphi^n(U)=0$ *for some. n.*

Proof. Let $A=[U, U]$ and $\mathfrak{n}$ the radical of A. Put $U_i=\mathfrak{n}^i U$ for all i. It is clear that $U_i\supset U_{i+1}$. Put $\mathfrak{r}_i=[U, U_i]$. Then $\mathfrak{r}_t\supset\mathfrak{n}^t$. Put $\bar{\mathfrak{r}}_{i+1}=\mathfrak{n}^i\cap\mathfrak{r}_{i+1}$. Then $\bar{\mathfrak{r}}_{i+1}U=U_i\cap U_{i+1}=U_{i+1}$ by Lemma 2. Since $\mathfrak{n}^i/\mathfrak{n}^{i+1}$ is semi-simple, so is $\mathfrak{n}^i/\bar{\mathfrak{r}}_{i+1}$, say $\mathfrak{n}^i/\bar{\mathfrak{r}}_{i+1}=\sum_{i=I}\oplus\tilde{\mathfrak{r}}_{\alpha_i}$; $\mathfrak{n}^i\supset\mathfrak{r}_{\alpha_i}\supset\bar{\mathfrak{r}}_{i+1}$, and $\tilde{\mathfrak{r}}_{\alpha_i}$ is simple. Put $U_{\alpha_i}=\mathfrak{r}_{\alpha_i}U$. If $U_{\alpha_i}=U_{i+1}$, $\mathfrak{r}_{\alpha_i}\subset\mathfrak{n}^i\cap\mathfrak{r}_{i+1}$, which is a contradiction. Hence, $U_i\supset U_{\alpha_i}\supsetneqq U_{i+1}$. We shall show that U_{α_i}/U_{i+1} is simple. Let V be a subobject such that $U_{\alpha_i}\supset V\supsetneqq U_{i+1}$. Then $\mathfrak{r}_V\supset\mathfrak{r}_{\alpha_i}$, in fact if $\mathfrak{r}_V\not\supset\mathfrak{r}_{\alpha_i}$, $\mathfrak{r}_V\cap\mathfrak{r}_{\alpha_i}=\bar{\mathfrak{r}}_{i+1}$, and hence, $U_{i+1}=(\mathfrak{r}_V\cap\mathfrak{r}_{\alpha_i})U=V\cap U_{\alpha_i}=V$. Therefore, $V=\mathfrak{r}_V U\supset\mathfrak{r}_{\alpha_i}U=U_{\alpha_i}$. Since $\mathfrak{n}^i=\cup\,\mathfrak{r}_{\alpha_i}$, $U_i=\cup\,U_{\alpha_i}$. On the other hand, $\mathfrak{r}_{\alpha_i}\cap\bigcup_{i\neq j}\mathfrak{r}_{\alpha_j}=\bar{\mathfrak{r}}_{i+1}$. Hence, $U_{\alpha_i}\cap\cup\,U_{\alpha_j}=U_{i+1}$. Since $\boldsymbol{C}$ is C_3-category, $U_i/U_{i+1}\approx\sum\oplus U_{\alpha_i}/U_{i+1}$ is semi-simple. We define $\varphi^i(U)=U_i$. Then $U/\varphi(U)$ is a finite directsum of simple subobjects from the above, and $\varphi^n(U)=0$ if $\mathfrak{n}^n=0$. Then we can define a function $\tilde{\varphi}$ in $\boldsymbol{C}$ from Lemma 5.

Let V_0 be a subobject in V such that $V_0+V'=V$ implies $V=V'$ for any subobject V' in V. V_0 is called *negligible*. By $[U:U_1]$ we mean the number of simple components in U/U_1.

Theorem 8. *Let* $\boldsymbol{C}$ *be an abelian* C_3*-category with semi-primary generator* U*, Then the following conditions are equivalent.*

1) *U is projective and small.*
2) $[A:\mathfrak{n}]=[U:\varphi(U)]$, *where* $\varphi(U)=\mathfrak{n}U$, $A=[U, U]$ *and* $\mathfrak{n}$ *is the radical of* A.
3) $\varphi(U)$ *is negligible in* U.
4) φ *satisfies the condition III.*
5) T: $C\to$*mod* A *is preserving minimal objects.*

Proof. If U is projective and small, then $\boldsymbol{C}$ is equivalent to mod A by Proposition 3. Hence, 2) 3) 4) and 5) are trivial. We assume 2). We put $\mathfrak{a}=[U, \mathfrak{n}U]$. If we restrict the argument in the proof of Theorem 7 to the case of $i=1$, we get $[A:\mathfrak{a}]=[U:\mathfrak{n}U]$, $=n$. Hence, $\mathfrak{a}=\mathfrak{n}$. For every maximal right ideal $\mathfrak{r}$, $\mathfrak{r}/\mathfrak{n}=\sum_{i=1}^{n-1}\oplus\mathfrak{r}_{\alpha_i}/\mathfrak{n}$, which implies $U\neq\bigcup_{i=1}^{n-1}\mathfrak{r}_{\alpha_i}U=\mathfrak{r}U$. Hence, we obtain 1) from Proposition 3.

3) Let $\mathfrak{a}$ be as above. We assume $\mathfrak{a}\neq\mathfrak{n}$. Then there exists a right ideal $\mathfrak{b}$ properly containing $\mathfrak{n}$ such that $\mathfrak{a}/\mathfrak{n}\oplus\mathfrak{b}/\mathfrak{n}=A/\mathfrak{n}$. Let e be an idempotent element in A such that $\mathfrak{b}/\mathfrak{n}=(eA+\mathfrak{n})/\mathfrak{n}$. Since $\mathfrak{b}\supset\mathfrak{n}$, $\mathfrak{b}U\supset\mathfrak{n}U=\mathfrak{a}U$. Hence, $U=(\mathfrak{a}+\mathfrak{b})U=\mathfrak{b}U$. Put $U_0=eU$. Then $U_0+\mathfrak{n}U=(eA+\mathfrak{n})U=\mathfrak{b}U=U$. Therefore, $U_0=U$ by 3). Hence, $e=I_u$, which is a contradiction.

4) If $\mathfrak{n}\neq\mathfrak{a}$, we obtain the fact $U=U_0+\mathfrak{n}U$ and $U_0\neq U$. Since U/U_0 contains a maximal object from Theorem 7, there exists a maximal subobject $V\supset U_0$. Therefore, $V\not\supset\mathfrak{n}U$.

5) If $\mathfrak{n}\neq\mathfrak{a}$, then there exists a maximal subobject V in U such that $U=V+\mathfrak{n}U$. Since $0\to[U, V]\to[U, U]\to[U, U/V]$ is exact and $[U, U/V]$ is minimal, $\mathfrak{r}_V=[U, V]$ is a maximal right ideal, and hence $\mathfrak{r}_V\supset\mathfrak{n}$, which is a contradiction.

It is clear that there are many examples in which semi-primary generators are not projective.

Corollary 1. *Let U be a semi-primary generator in $\boldsymbol{C}$. If $A/\mathfrak{n}$ is a simple rings, U is projective and small, where $A=[U, U]$ and $\mathfrak{n}$ is its radical.*

Proof. Let $\mathfrak{a}=[U, \mathfrak{n}U]$. Since $U\neq\mathfrak{n}U$, and $\mathfrak{a}$ is a two-sided ideal, $\mathfrak{a}=\mathfrak{n}$.

Corollary 2. *Let B be a semi-primary ring and U be a semi-primary generator in the category of B-right modules. Then $\mathfrak{n}_A U\supset U\mathfrak{n}_B$. $\mathfrak{n}_A U=U\mathfrak{n}_B$ if and only if U is a finitely generated and projective, where, $A=[U, U]$ and $\mathfrak{n}_A$ (resp. $\mathfrak{n}_B$) is the radical of A (resp. B).*

Proof. Let $\varphi(U)=U\mathfrak{n}_B$. Then φ is a functor in mod B satisfying I, II and III. Hence, $\mathfrak{n}_A U\supset U\mathfrak{n}_B$ by Theorem 7. If $\mathfrak{n}_A U=U\mathfrak{n}_B$, a function φ' defined in the proof of Theorem 7 satisfies III. Hence, U is projective and small. The converse is trivial.

EXAMPLE. We shall show that there exists a generator U such that φ^i are defined in U satisfying the following conditions: $U/\varphi(U)$ is a finite directsum of simple object, φ^i satisfies I, II and III for all i and $\varphi^n(U)=0$ for some n, however U is not semi-primary.

Let k be a field and $K=k(x)$. Let $A=\begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$ be a tri-angular matrix ring. Then A is semi-primary with radical $\mathfrak{n}$. We define $\varphi(U)=U\mathfrak{n}$ in mod A. Put $\mathfrak{r}=\begin{pmatrix} 0 & 0 \\ k[x] & 0 \end{pmatrix}$. Then $\mathfrak{r}$ is a right ideal in A. Then $[A/\mathfrak{r}, A/\mathfrak{r}]\approx\begin{pmatrix} k & 0 \\ k(x)/k[x] & k[x] \end{pmatrix}$ is not semi-primary. $U=A\oplus A/\mathfrak{r}$ is the desired generator.

3. Abelian category of commutative diagram

We recall the definition of abelian category of commutative diagram over abelian categories $\boldsymbol{C}_i$ (see [4]).

Let $I=(1, 2, \ldots, n)$ be a finite linear ordered set and $\{\boldsymbol{C}_i\}_{i\in I}$ a family of abelian categories. We assume that there are given cokernel preserving functors $T_{ij}\colon \boldsymbol{C}_i\to\boldsymbol{C}_j$ for $i<j$. Furthermore, we assume:

(*) There exist natural transformations

$$\psi_{ijk}\colon T_{jk}T_{ij}\to T_{ik} \qquad \text{for all} \quad i<j<k, \quad \text{and}$$

(**) For any $i<j<k<l$ and V in $\boldsymbol{C}_i$

$$\begin{array}{ccc} T_{kl}T_{jk}T_{ij}(V) & \xrightarrow{T_{kl}(\psi)} & T_{kl}T_{ik}(V) \\ \downarrow{\scriptstyle \psi_{ikl}} & & \downarrow{\scriptstyle \psi_{ikl}} \\ T_{jl}T_{ij}(V) & \xrightarrow{\psi_{ijl}} & T_{il}(V) \end{array}$$

is commutative.

We call a family of morphism $d_{ij}\colon T_{ij}(V_i)\to V_j$ an arrow for $V_i\in\boldsymbol{C}_i$, $V_j\in\boldsymbol{C}_j$ and for all $i<j$, when the diagrams

$$(\text{***})\qquad \begin{array}{ccc} T_{jk}T_{ij}(V_i) & \xrightarrow{T_{jk}(d_{ij})} & T_{jk}(V_j) \\ \downarrow{\scriptstyle \psi_{ijk}} & & \downarrow \\ T_{ik}(V_i) & \xrightarrow{d_{ik}} & V_k \end{array}$$

are commutative.

We define a *commutative diagram* $[I, \boldsymbol{C}_i]$ as follows; Its objects consist of set $\{V_i\}_{i\in I}$ with arrows $\{d_{ij}\}$ and morphisms consist of set $\{(f_i)\}_{i\in I}; f_i\colon V_i\to V_i'$ in $\boldsymbol{C}_i$ such that $d'_{ij}T_{ij}(f_i)=f_jd_{ij}$.

Lemma 9. *Let T_{ij} be functors satisfying* (**). *Then the natural transformation of $T_{i_{n-1}i_n}T_{i_{n-2}i_{n-1}}\cdots T_{i_1i_2}\to T_{i_1i_n}$ does not depend on any choice of combination of $T_{i_ni_{n-1}}, \cdots, T_{i_1i_2}$.*

Proof. We can prove the lemma by using induction on the number of functors and naturality of ψ_{ijk}. Namely, every natural transformation is equal to $T_{i_{n-1}i_n}(T_{i_{n-2}i_{n-1}}(\cdots(T_{i_2i_3}T_{i_1i_2})\to T_{i_1i_n}$.

We assume that all $\boldsymbol{C}_i$ have projective class $\mathcal{E}_i$. We define a functor $S_i\colon\boldsymbol{C}_i\to[I, \boldsymbol{C}_i]$ by setting $S_i(V_i)=(0, \cdots, 0, V_i, T_{ii+1}(V_i), \cdots, T_{in}(V_i))$. Then the projective objects in $[I, \boldsymbol{C}_i]$ are of the form $\oplus S_i(P_i)$ and their retract, where P_i is $\mathcal{E}_i$-projective for all i, ([4], Prop. 1.2′). If the projective objects in $[I, \boldsymbol{C}_i]$ are only of the former forms, we call $[I, \boldsymbol{C}_i]$ a *good* category of commutative diagram.

Theorem 10. *Let $\boldsymbol{C}_i$ be abelian category with projective class $\mathcal{E}_i$. Then every $[I, \boldsymbol{C}_i]$ with T_{ij} is imbedding in a good category $[I, \boldsymbol{C}_i]$ with T'_{ij}.*

Proof. We shall define new functors T'_{ij}:

$$T'_{ii+1} = T_{ii+1}$$
$$T'_{ij} = T_{j-1j}T_{j-2j-1}\cdots T_{ii+1} \qquad \text{for} \quad i+1<j\,.$$

Then it is clear that T'_{ij} are cokernel preserving and $\psi'_{ijk}=I_{Ck}$ and (**) is trivial. Furthermore, there exist unique natural transformations $\phi_{ij}\colon T'_{ij}\to T_{ij}$ by Lemma 9. Put $\boldsymbol{C}=[I, \boldsymbol{C}_i]$ with T_{ij} and $\boldsymbol{C}'=[I, \boldsymbol{C}_i]$ with T'_{ij}. We define a function F of $\boldsymbol{C}$ into $\boldsymbol{C}'$ as follows: For $V=(V_i)$ with arrows d_{ij} in $\boldsymbol{C}$ we put $F(V)=(V_i)$ with the following arrows d'_{ij}:

$$d'_{ii+1} = d_{ii+1}$$
$$d'_{ij} = d_{ij}\phi_{ij}T'_{ij} \qquad \text{for} \quad i+1<j\,.$$

We have to show that d'_{ij} satisfies (***). We have a diagram for $i<j<k$ and $V_t\in\boldsymbol{C}_t$

$$\begin{tikzcd}
T'_{jk}T'_{ij}(V_i) \arrow[r, "T'(\phi)"] \arrow[d, equal, "\psi'_{ijk}"'] \arrow[dr, phantom, "\mathrm{I}"] & T'_{jk}T_{ij}(V_i) \arrow[r, "T'(d_{ij})"] \arrow[d, "\phi"] \arrow[dr, phantom, "\mathrm{II}"] & T'_{jk}(V_j) \arrow[d, "\phi"] \\
 & T_{jk}T_{ij}(V_i) \arrow[r, "T(d_{ij})"] \arrow[d] \arrow[dr, phantom, "\mathrm{III}"] & T_{jk}(V_j) \arrow[d, "d_{jk}"] \\
T'_{ik}(V_i) \arrow[r] & T_{ik}(V_i) \arrow[r, "d_{ik}"] & V_k\,.
\end{tikzcd}$$

I is commutative by Lemma 9, II is commutative by naturality of ϕ and so is III by (**). Hence, d'_{ij} satisfies (***). Define $F((f_i))=(f_i)$ for morphism (f_i) in $\boldsymbol{C}$. Then we can similarly show that F is a functor. It is clear that F is an imbedding functor. Since $\psi'_{ijk}=I_{Ck}$, $K^j(P_i)=0$ in (*) of [4], Lemma 3. 7. Hence, $\boldsymbol{C}'$ is good by [4], Lemma 3. 7.

If every objects in $\boldsymbol{C}$ are projective, $\boldsymbol{C}$ is called a *semi-simple category*.

Corollary. *Let $\boldsymbol{C}_i$ be a semi-simple abelian category. Then $[I, \boldsymbol{C}_i]$ is imbedding in an abelian hereditary category, (cf.* [3], *Theorem* 5).

Proof. It is clear from Theorem 10 and [4], Theorem 3. 12.

Finally, we note that if $\boldsymbol{C}_i$ have functor φ_i satisfying I, II and III and $T_{ij}(\varphi_i(V_i))\subseteq\varphi_jT_{ij}(V_i)$ on $V=(V_i)$ in $[I, \boldsymbol{C}_i]$. Then

$$\varphi(V) = (\varphi_1(V_1),\ \varphi_2(V_2)\cup d_{12}(V_2),\ \cdots,\ \varphi_j(V_j)\bigcup_{i<j} d_{ij}(V_i),\ \cdots)$$

is a functor on $[I, \boldsymbol{C}_i]$ satisfying I, II and III. If $\varphi_t{}^m=0$ for all t then $\varphi^{nm}=0$.

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