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A NOTE ON SCHWARZIAN DERIVATIVES AND SUGIYAMA-YASUDA LOCALLY EXACT DIFFERENTIALS

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Abstract

One main purpose of the present paper is to reorganize, in terms of the notion of a *Schwarz system*, a proof, by means of *Schwarzian derivatives*, of the existence of complex projective structures on compact hyperbolic Riemann surfaces and a proof, by means of *Sugiyama-Yasuda locally exact differentials*, of the existence of Frobenius-projective structures of level two on projective smooth curves in characteristic two. Moreover, we also construct quasi-Schwarz systems for certain Frobenius-affine and Frobenius-projective structures on projective smooth curves in positive characteristic.

Contents

Introduction

The author of the present paper pointed out a certain similarity between *Schwarzian derivatives* and locally exact differentials defined in [6] by Sugiyama and Yasuda, i.e., *Sugiyama-Yasuda locally exact differentials*, in the study of Frobenius-projective structures [cf. [2, Remark 7.1.1]]. One main purpose of the present paper is to reorganize

- the proof, by means of *Schwarzian derivatives*, of the existence of *complex projective structures* on compact hyperbolic Riemann surfaces given in [1, §9, (a), Corollary 2] and

- the proof, by means of *Sugiyama-Yasuda locally exact differentials*, of the existence of Frobenius-projective structures of level 2 on projective smooth curves in characteristic 2 given in [6, §3] [cf. also [2, §2] and [2, §3]]

in terms of the notion of a *Schwarz system*. Put another way, one main purpose of the present paper is to give one "*precise mathematical formulation*" of the similarity pointed out in [2,

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Remark 7.1.1].

Let X be a topological space. Then a *Schwarz system* [cf. Definition 1.2] on X is defined to be a suitable collection of data

$$(\mathcal{F}, \mathcal{A}, \mathfrak{U}, \{\sim_U\}_{U\in\mathfrak{U}}, \theta\colon \mathcal{F}\times\mathcal{F} \to \mathcal{A})$$

consisting of

- a sheaf $\mathcal{F}$ of sets on X,
- a sheaf $\mathcal{A}$ of abelian groups on X,
- an open basis $\mathfrak{U}$ of X,
- an equivalence relation $\sim_U$ on the set $\mathcal{F}(U)$ for each $U \in \mathfrak{U}$, and
- a morphism $\theta\colon \mathcal{F}\times\mathcal{F} \to \mathcal{A}$ of sheaves of sets on X.

Suppose that we are given a Schwarz system $\mathfrak{S} = (\mathcal{F}, \mathcal{A}, \mathfrak{U}, \{\sim_U\}_U, \theta)$ on X. Then we shall refer to

- a collection $(f_U)_{U\in\mathfrak{O}}$ of local sections $f_U \in \mathcal{F}(U)$ of the sheaf $\mathcal{F}$ — where $\mathfrak{O}$ is an open covering of X whose element is contained in $\mathfrak{U}$ — such that $f_U|_W \sim_W f_V|_W$ for each $U, V \in \mathfrak{O}$ and each $W \in \mathfrak{U}$ with $W \subseteq U \cap V$ as a *global object* associated to $\mathfrak{S}$ [cf. Definition 1.6] and

- an equivalence class, with respect to a certain equivalence relation defined by the $\sim_U$'s [cf. Definition 1.9, (i)], of global objects as a *global structure* associated to $\mathfrak{S}$ [cf. Definition 1.9, (ii)].

Moreover, one may associate, to $\mathfrak{S}$, a cohomology class in the first cohomology group $H^1(X, \mathcal{A})$. We shall refer to this class as the *obstruction class* of $\mathfrak{S}$ [cf. Definition 1.5].

The main result of the theory of Schwarz systems, i.e., established in §1 of the present paper, is as follows [cf. Theorem 1.7, (ii), and Theorem 1.12, (iv)].

Theorem A. *Let X be a topological space and $\mathfrak{S} = (\mathcal{F}, \mathcal{A}, \mathfrak{U}, \{\sim_U\}_{U\in\mathfrak{U}}, \theta)$ a Schwarz system on X. Then the following assertions hold*:

(i) *It holds that there exists a global structure associated to $\mathfrak{S}$ if and only if the obstruction class of $\mathfrak{S}$ vanishes.*

(ii) *Suppose that the obstruction class of the Schwarz system $\mathfrak{S}$ vanishes. Then the set of global structures associated to $\mathfrak{S}$ has a structure of $\Gamma(X, \mathcal{A})$-torsor.*

In §2 of the present paper, we reorganize the proof, by means of *Schwarzian derivatives*, of the existence of *complex projective structures* on compact hyperbolic Riemann surfaces given in [1, §9, (a), Corollary 2] from the point of view of Theorem A. More precisely, for a given Riemann surface X, we construct a Schwarz system $\mathfrak{S}_X^{\mathrm{P}}$ such that

(§2-a) the sheaf "$\mathcal{A}$" of the Schwarz system $\mathfrak{S}_X^{\mathrm{P}}$ is the invertible sheaf of *holomorphic quadratic differentials* on X,

(§2-b) the morphism "θ" of the Schwarz system $\mathfrak{S}_X^{\mathrm{P}}$ is given by the *Schwarzian derivatives* [cf., e.g., [1, p.164] and [1, p.167]], and

(§2-c) a *global structure associated to* $\mathfrak{S}_X^{\mathrm{P}}$ is essentially the same as a *complex projective* *structure* [cf., e.g., [1, p.167]] on X

[cf. Theorem 2.4 and Remark 2.4.1]. In particular, if the given Riemann surface X is *compact* and *hyperbolic*, then since [one verifies easily that] the first cohomology group of the invertible sheaf of holomorphic quadratic differentials on X — that contains the obstruction class of $\mathfrak{S}_X^{\mathrm{P}}$ [cf. (§2-a)] — *vanishes*, it follows from Theorem A that the set of *complex projective structures* on X [cf. (§2-c)] is *nonempty* and has a structure of torsor under the space of *global holomorphic quadratic differentials* [cf. (§2-a)] on X [cf. Corollary 2.5].

In §3 of the present paper, we reorganize the proof, by means of *Sugiyama-Yasuda locally exact differentials*, of the existence of Frobenius-projective structures of level 2 on projective smooth curves in characteristic 2 given in [6, §3] [cf. also [2, §2] and [2, §3]] from the point of view of Theorem A. More precisely, for a given projective smooth curve X over an algebraically closed field of characteristic 2, we construct a Schwarz system $\mathfrak{S}_X^{2,2,\mathrm{P}}$ whose obstruction class *vanishes* such that

(§3-a) the sheaf "$\mathcal{A}$" of the Schwarz system $\mathfrak{S}_X^{2,2,\mathrm{P}}$ is the invertible sheaf of *locally exact differentials* on X,

(§3-b) the morphism "θ" of the Schwarz system $\mathfrak{S}_X^{2,2,\mathrm{P}}$ is given by the *Sugiyama-Yasuda locally exact differentials* [cf. [6, Definition 2.8]], and

(§3-c) a *global structure associated to* $\mathfrak{S}_X^{2,2,\mathrm{P}}$ is essentially the same as a *Frobenius-projective structure of level* 2 [cf. [2, Definition 3.1]] on X

[cf. Theorem 3.5 and Remark 3.5.1, (ii)]. In particular, it follows from Theorem A that the set of *Frobenius-projective structures of level* 2 on X [cf. (§3-c)] is *nonempty* and has a structure of torsor under the space of *global locally exact differentials* [cf. (§3-a)] on X [cf. Corollary 3.6, (ii)].

Moreover, in the present paper, we also construct certain *quasi-Schwarz systems* [cf. Definition 1.2] on projective smooth curves in positive characteristic. More precisely, we construct,

- in §4 of the present paper, a quasi-Schwarz system on a projective smooth curve in characteristic 2 whose *global structure* is essentially the same as a *Frobenius-affine structure of level* 2 [cf. [3, Definition 3.1]] on the curve [cf. Theorem 4.9 and Remark 4.9.1, (ii)],

- in §5 of the present paper, a quasi-Schwarz system on a projective smooth curve in positive characteristic whose *global structure* is essentially the same as a *Frobenius-affine structure of level* 1 on the curve [cf. Theorem 5.5 and Remark 5.5.1, (ii)], and

- in §6 of the present paper, a quasi-Schwarz system on a projective smooth curve in positive characteristic whose *global structure* is essentially the same as a *Frobenius-projective structure of level* 1 on the curve [cf. Theorem 6.4 and Remark 6.4.1, (ii)].

1. Schwarz Systems

In the present §1, we introduce and discuss the notion of a *Schwarz system* [cf. Definition 1.2 below]. Let X be a topological space.

Definition 1.1. Let $\mathfrak{U}$ be an open basis of X. Then we shall say that an open covering $\mathfrak{O}$ of X is a $\mathfrak{U}$-*covering* of X if the inclusion $\mathfrak{O} \subseteq \mathfrak{U}$ holds.

Remark 1.1.1. Let $\mathfrak{U}$ be an open basis of X and $\mathfrak{O}_1$, $\mathfrak{O}_2$ two $\mathfrak{U}$-coverings of X. Then one verifies easily that there exists a *refinement* of both $\mathfrak{O}_1$ and $\mathfrak{O}_2$ that forms a $\mathfrak{U}$-*covering* of X.

Definition 1.2. We shall say that a collection of data

$$(\mathcal{F}, \mathcal{A}, \mathfrak{U}, \{\sim_U\}_{U\in\mathfrak{U}}, \theta\colon \mathcal{F}\times\mathcal{F} \to \mathcal{A})$$

consisting of

- a sheaf $\mathcal{F}$ of sets on X,
- a sheaf $\mathcal{A}$ of abelian groups on X,
- an open basis $\mathfrak{U}$ of X,
- an equivalence relation $\sim_U$ on the set $\mathcal{F}(U)$ for each $U \in \mathfrak{U}$, and
- a morphism $\theta\colon \mathcal{F}\times\mathcal{F} \to \mathcal{A}$ of sheaves of sets on X

is a *quasi-Schwarz system* on X if, for each $U \in \mathfrak{U}$, the following three conditions are satisfied:

(1) The set $\mathcal{F}(U)$ is nonempty, and the first cohomology group $H^1(U, \mathcal{A})$ vanishes.

(2) For each $f_1, f_2, f_3 \in \mathcal{F}(U)$, the cocycle condition

$$\theta(U)(f_1, f_3) = \theta(U)(f_1, f_2) + \theta(U)(f_2, f_3)$$

is satisfied.

(3) For each $f_0 \in \mathcal{F}(U)$, the map $\mathcal{F}(U) \to \mathcal{A}(U)$ of sets given by sending $f \in \mathcal{F}(U)$ to $\theta(U)(f, f_0) \in \mathcal{A}(U)$ factors through the natural quotient map $\mathcal{F}(U) \twoheadrightarrow \mathcal{F}(U)/\sim_U$, and, moreover, the resulting map $\mathcal{F}(U)/\sim_U \to \mathcal{A}(U)$ is injective:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[rr, "\theta(U)(-,f_0)"] \arrow[dr] & & \mathcal{A}(U) \\ & \mathcal{F}(U)/\sim_U \arrow[ur, hook, dashed] & \end{tikzcd}$$

Moreover, we shall say that a quasi-Schwarz system $(\mathcal{F}, \mathcal{A}, \mathfrak{U}, \{\sim_U\}_{U\in\mathfrak{U}}, \theta\colon \mathcal{F}\times\mathcal{F} \to \mathcal{A})$ is a *Schwarz system* if the following condition is satisfied:

(4) For each $U \in \mathfrak{U}$, $x \in U$, $f \in \mathcal{F}(U)$, and $a \in \mathcal{A}(U)$, there exist $V \in \mathfrak{U}$ and $g \in \mathcal{F}(V)$ such that $x \in V \subseteq U$, and, moreover, the equality $\theta(V)(g, f|_V) = a|_V$ holds.

Remark 1.2.1. Let $(\mathcal{F}, \mathcal{A}, \mathfrak{U}, \{\sim_U\}_{U\in\mathfrak{U}}, \theta)$ be a quasi-Schwarz system on X, U an element of $\mathfrak{U}$, and $f_1, f_2 \in \mathcal{F}(U)$ local sections of the sheaf $\mathcal{F}$. Then it follows from condition (2) of Definition 1.2 that

(i) the equalities $\theta(U)(f_1, f_1) = 0$, $\theta(U)(f_1, f_2) = -\theta(U)(f_2, f_1)$ hold.

In particular, it follows from condition (3) of Definition 1.2 that

(ii) it holds that $f_1 \sim_U f_2$ if and only if $\theta(U)(f_1, f_2) = 0$.

In the remainder of the present §1, let $\mathfrak{S} = (\mathcal{F}, \mathcal{A}, \mathfrak{U}, \{\sim_U\}_{U\in\mathfrak{U}}, \theta)$ be a quasi-Schwarz system on X.

Definition 1.3. Let $\mathfrak{O}$ be a $\mathfrak{U}$-covering of X and $(f_U)_{U\in\mathfrak{O}}$ a collection of local sections $f_U \in \mathcal{F}(U)$ of the sheaf $\mathcal{F}$. Then we shall refer to the image in the first cohomology group $H^1(X, \mathcal{A})$ of the element of the first Čech cohomology group $\check{H}^1(\mathfrak{O}, \mathcal{A})$ determined [cf. condition (2) of Definition 1.2] by the collection $(\theta(U \cap V)(f_U|_{U\cap V}, f_V|_{U\cap V}))_{U,V\in\mathfrak{O}}$ of local sections $\theta(U \cap V)(f_U|_{U\cap V}, f_V|_{U\cap V}) \in \mathcal{A}(U \cap V)$ of the sheaf $\mathcal{A}$ as the *cohomology class determined by* $(f_U)_U$.

Lemma 1.4. *Let $\mathfrak{O}_f$, $\mathfrak{O}_g$ be $\mathfrak{U}$-coverings of X and $(f_U)_{U\in\mathfrak{O}_f}$, $(g_V)_{V\in\mathfrak{O}_g}$ collections of local sections $f_U \in \mathcal{F}(U)$, $g_V \in \mathcal{F}(V)$ of the sheaf $\mathcal{F}$. Then the cohomology class determined by $(f_U)_U$ coincides with the cohomology class determined by $(g_V)_V$.*

Proof. Let us first observe that one verifies easily that, to verify Lemma 1.4, we may assume without loss of generality — by replacing $\mathfrak{O}_f$ and $\mathfrak{O}_g$ by a refinement of both $\mathfrak{O}_f$ and $\mathfrak{O}_g$ that forms a $\mathfrak{U}$-covering of X [cf. Remark 1.1.1] — that $\mathfrak{O}_f = \mathfrak{O}_g$. For each $U \in \mathfrak{O}_f$, write $a_U \stackrel{\text{def}}{=} \theta(U)(f_U, g_U) \in \mathcal{A}(U)$. Then it follows from condition (2) of Definition 1.2 and Remark 1.2.1, (i), that, for each U, $V \in \mathfrak{O}_f$, the equalities

$$\begin{aligned}&\theta(U \cap V)(f_U|_{U\cap V}, f_V|_{U\cap V})\\ &= \theta(U \cap V)(f_U|_{U\cap V}, g_U|_{U\cap V}) + \theta(U \cap V)(g_U|_{U\cap V}, g_V|_{U\cap V}) + \theta(U \cap V)(g_V|_{U\cap V}, f_V|_{U\cap V})\\ &= a_U|_{U\cap V} + \theta(U \cap V)(g_U|_{U\cap V}, g_V|_{U\cap V}) - a_V|_{U\cap V}\end{aligned}$$

hold. Thus, the cohomology class determined by $(f_U)_U$ *coincides* with the cohomology class determined by $(g_V)_V$, as desired. This completes the proof of Lemma 1.4. □

Definition 1.5. We shall refer to the cohomology class in $H^1(X, \mathcal{A})$ determined by some [cf. Lemma 1.4] collection of local sections of the sheaf $\mathcal{F}$ with respect to some $\mathfrak{U}$-covering of X [cf. condition (1) of Definition 1.2] as the *obstruction class* of $\mathfrak{S}$.

Definition 1.6. We shall say that a collection $(f_U)_{U\in\mathfrak{O}}$ of local sections $f_U \in \mathcal{F}(U)$ of the sheaf $\mathcal{F}$ — where $\mathfrak{O}$ is a $\mathfrak{U}$-covering of X — is a *global object* associated to $\mathfrak{S}$ if, for each U, $V \in \mathfrak{O}$ and each $W \in \mathfrak{U}$ with $W \subseteq U \cap V$, the local section $f_U|_W \in \mathcal{F}(W)$ is equivalent — i.e., with respect to the equivalence relation $\sim_W$ — to the local section $f_V|_W \in \mathcal{F}(W)$. We shall write

$$\Gamma(X, \mathfrak{S})$$

for the set of global objects associated to $\mathfrak{S}$.

The first main result of the theory of Schwarz systems is as follows.

Theorem 1.7. *Let X be a topological space and $\mathfrak{S} = (\mathcal{F}, \mathcal{A}, \mathfrak{U}, \{\sim_U\}_{U\in\mathfrak{U}}, \theta)$ a quasi-Schwarz system on X. Then the following assertions hold*:

(i) *If the set $\Gamma(X, \mathfrak{S})$ is nonempty, then the obstruction class of the quasi-Schwarz system*

$\mathfrak{S}$ *vanishes.*

(ii) *Suppose that the quasi-Schwarz system* $\mathfrak{S}$ *is a Schwarz system. Then it holds that the set* $\Gamma(X, \mathfrak{S})$ *is nonempty if and only if the obstruction class of the Schwarz system* $\mathfrak{S}$ *vanishes.*

Proof. First, we verify assertion (i). Let $(f_U)_{U\in\mathfrak{O}} \in \Gamma(X, \mathfrak{S})$ be a global object associated to $\mathfrak{S}$ — where $\mathfrak{O}$ is a $\mathfrak{U}$-covering of X. Then it is immediate that, to verify assertion (i), it suffices to show that the cohomology class determined by $(f_U)_U$ *vanishes*. On the other hand, this follows from Remark 1.2.1, (ii). This completes the proof of assertion (i).

Next, we verify assertion (ii). The necessity follows from assertion (i). To verify the sufficiency, suppose that the obstruction class of $\mathfrak{S}$ *vanishes*. Let $(f_U)_{U\in\mathfrak{U}}$ be a collection of local sections $f_U \in \mathcal{F}(U)$ of the sheaf $\mathcal{F}$ [cf. condition (1) of Definition 1.2]. Then since [we have assumed that] the obstruction class of $\mathfrak{S}$ *vanishes*, there exists a local section $a_U \in \mathcal{A}(U)$ of the sheaf $\mathcal{A}$ for each $U \in \mathfrak{U}$ such that, for each $U, V \in \mathfrak{U}$, the equality

$$\theta(U \cap V)(f_U|_{U\cap V}, f_V|_{U\cap V}) = a_U|_{U\cap V} - a_V|_{U\cap V}$$

holds [cf. condition (1) of Definition 1.2]. Now since [we have assumed that] $\mathfrak{S}$ is a *Schwarz system*, it follows from condition (4) of Definition 1.2 that there exist a $\mathfrak{U}$-covering $\mathfrak{O}$ of X and, for each $U \in \mathfrak{O}$, an element $\widetilde{U} \in \mathfrak{U}$ and a local section $g_U \in \mathcal{F}(U)$ of the sheaf $\mathcal{F}$ such that $U \subseteq \widetilde{U}$, and, moreover, $-a_{\widetilde{U}}|_U = \theta(U)(g_U, f_{\widetilde{U}}|_U)$. Then it follows from condition (2) of Definition 1.2 and Remark 1.2.1, (i), that, for each $U, V \in \mathfrak{O}$, the equalities

$$\begin{aligned}
&\theta(U \cap V)(g_U|_{U\cap V}, g_V|_{U\cap V}) \\
&= \theta(U \cap V)(g_U|_{U\cap V}, f_{\widetilde{U}}|_{U\cap V}) + \theta(U \cap V)(f_{\widetilde{U}}|_{U\cap V}, f_{\widetilde{V}}|_{U\cap V}) + \theta(U \cap V)(f_{\widetilde{V}}|_{U\cap V}, g_V|_{U\cap V}) \\
&= -a_{\widetilde{U}}|_{U\cap V} + (a_{\widetilde{U}}|_{\widetilde{U}\cap\widetilde{V}} - a_{\widetilde{V}}|_{\widetilde{U}\cap\widetilde{V}})|_{U\cap V} + a_{\widetilde{V}}|_{U\cap V} = 0
\end{aligned}$$

hold. Thus, it follows from Remark 1.2.1, (ii), that $g_U|_W \sim_W g_V|_W$ for each $W \in \mathfrak{U}$ with $W \subseteq U \cap V$, i.e., that the collection $(g_U)_{U\in\mathfrak{O}}$ forms a global object associated to $\mathfrak{S}$. This completes the proof of assertion (ii), hence also of Theorem 1.7. □

Corollary 1.8. *Suppose that the quasi-Schwarz system* $\mathfrak{S}$ *is a Schwarz system, and that the first cohomology group* $H^1(X, \mathcal{A})$ *vanishes. Then the set* $\Gamma(X, \mathfrak{S})$ *is nonempty.*

Proof. This is a formal consequence of Theorem 1.7, (ii). □

Definition 1.9.

(i) We shall define an equivalence relation $\sim_{\mathfrak{S}}$ on the set $\Gamma(X, \mathfrak{S})$ as follows: Let $(f_U)_{U\in\mathfrak{O}_f}$, $(g_V)_{V\in\mathfrak{O}_g} \in \Gamma(X, \mathfrak{S})$ be two global objects associated to $\mathfrak{S}$ — where $\mathfrak{O}_f$, $\mathfrak{O}_g$ are $\mathfrak{U}$-coverings of X. Then we shall write $(f_U)_U \sim_{\mathfrak{S}} (g_V)_V$ if $f_U|_W \sim_W g_V|_W$ for each $U \in \mathfrak{O}_f$, $V \in \mathfrak{O}_g$, $W \in \mathfrak{U}$ with $W \subseteq U \cap V$.

(ii) We shall refer to an equivalence class with respect to the equivalence relation $\sim_{\mathfrak{S}}$ as a *global structure* associated to $\mathfrak{S}$. We shall write

$$\Gamma_{/\sim}(X, \mathfrak{S}) \stackrel{\text{def}}{=} \Gamma(X, \mathfrak{S})/\sim_{\mathfrak{S}}$$

for the set of global structures associated to $\mathfrak{S}$.

Lemma 1.10. *Let $(f_U)_{U\in\mathfrak{D}_f}$, $(g_V)_{V\in\mathfrak{D}_g} \in \Gamma(X, \mathfrak{S})$ be two global objects associated to $\mathfrak{S}$ — where $\mathfrak{D}_f$, $\mathfrak{D}_g$ are $\mathfrak{U}$-coverings of X — and $\mathfrak{D}$ a refinement of both $\mathfrak{D}_f$ and $\mathfrak{D}_g$ that forms a $\mathfrak{U}$-covering of X. Moreover, let $\tau_f\colon \mathfrak{D} \to \mathfrak{D}_f$, $\tau_g\colon \mathfrak{D} \to \mathfrak{D}_g$ be maps of sets such that $U \subseteq \tau_f(U)$, $V \subseteq \tau_g(V)$ for each U, $V \in \mathfrak{D}$. Then the following assertions hold*:

(i) *Let U be an element of $\mathfrak{D}$. Then the element $\theta(U)(f_{\tau_f(U)}|_U, g_{\tau_g(U)}|_U) \in \mathcal{A}(U)$ does not depend on the choices of the maps τ_f, τ_g.*

(ii) *The collection*

$$(\theta(U)(f_{\tau_f(U)}|_U, g_{\tau_g(U)}|_U))_{U\in\mathfrak{D}}$$

of local sections $\theta(U)(f_{\tau_f(U)}|_U, g_{\tau_g(U)}|_U) \in \mathcal{A}(U)$ of the sheaf $\mathcal{A}$ arises from a global section of the sheaf $\mathcal{A}$.

(iii) *The global section of the sheaf $\mathcal{A}$ of* (ii) *does not depend on the choices of the refinement $\mathfrak{D}$ and the maps τ_f, τ_g.*

Proof. First, we verify assertion (i). Let $\tau'_f\colon \mathfrak{D} \to \mathfrak{D}_f$, $\tau'_g\colon \mathfrak{D} \to \mathfrak{D}_g$ be maps of sets such that $U \subseteq \tau'_f(U)$, $V \subseteq \tau'_g(V)$ for each U, $V \in \mathfrak{D}$. Then since $(f_U)_{U\in\mathfrak{D}_f}$, $(g_V)_{V\in\mathfrak{D}_g}$ are *global objects* associated to $\mathfrak{S}$, the equivalences $f_{\tau_f(U)}|_U \sim_U f_{\tau'_f(U)}|_U$, $g_{\tau_g(U)}|_U \sim_U g_{\tau'_g(U)}|_U$ hold for each $U \in \mathfrak{D}$. Thus, assertion (i) follows from condition (3) of Definition 1.2 and Remark 1.2.1, (i). This completes the proof of assertion (i).

Next, we verify assertion (ii). Let U, V be elements of $\mathfrak{D}$. Then it follows from condition (2) of Definition 1.2 that the equality

$$\begin{aligned}&\theta(U)(f_{\tau_f(U)}|_U, g_{\tau_g(U)}|_U)|_{U\cap V} - \theta(V)(f_{\tau_f(V)}|_V, g_{\tau_g(V)}|_V)|_{U\cap V}\\ &= \theta(U\cap V)(f_{\tau_f(U)}|_{U\cap V}, f_{\tau_f(V)}|_{U\cap V}) + \theta(U\cap V)(g_{\tau_g(V)}|_{U\cap V}, g_{\tau_g(U)}|_{U\cap V})\end{aligned}$$

holds. On the other hand, since $(f_U)_{U\in\mathfrak{D}_f}$, $(g_V)_{V\in\mathfrak{D}_g}$ are *global objects* associated to $\mathfrak{S}$, the equivalences $f_{\tau_f(U)}|_W \sim_W f_{\tau_f(V)}|_W$, $g_{\tau_g(U)}|_W \sim_W g_{\tau_g(V)}|_W$ hold for each $W \in \mathfrak{U}$ with $W \subseteq U\cap V$. Thus, it follows from Remark 1.2.1, (ii), that the equalities $\theta(U\cap V)(f_{\tau_f(U)}|_{U\cap V}, f_{\tau_f(V)}|_{U\cap V}) = \theta(U\cap V)(g_{\tau_g(V)}|_{U\cap V}, g_{\tau_g(U)}|_{U\cap V}) = 0$ hold. In particular, one may conclude that the collection of local sections of the sheaf $\mathcal{A}$ under consideration arises from a global section of the sheaf $\mathcal{A}$, as desired. This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertion (i), together with the various definitions involved. This completes the proof of Lemma 1.10. □

DEFINITION 1.11. Let $f, g \in \Gamma(X, \mathfrak{S})$ be two global objects associated to $\mathfrak{S}$. Then it follows from Lemma 1.10, (i), (ii), (iii) [cf. also Remark 1.1.1], that the pair (f, g) determines a global section of the sheaf $\mathcal{A}$. We shall write $\theta_{\mathfrak{S}}(f, g) \in \Gamma(X, \mathcal{A})$ for this global section of the sheaf $\mathcal{A}$.

The second main result of the theory of Schwarz systems is as follows.

Theorem 1.12. *Let X be a topological space and $\mathfrak{S} = (\mathcal{F}, \mathcal{A}, \mathfrak{U}, \{\sim_U\}_{U\in\mathfrak{U}}, \theta)$ a quasi-Schwarz system on X. Suppose that the set $\Gamma(X, \mathfrak{S})$ is nonempty* [*which thus implies that the obstruction class of the quasi-Schwarz system $\mathfrak{S}$ vanishes — cf.* Theorem 1.7, (i)]. *Let $f_0 \in \Gamma(X, \mathfrak{S})$ be a global object associated to $\mathfrak{S}$. Then the following assertions hold*:

(i) *The assignment "$(f, g) \mapsto \theta_{\mathfrak{S}}(f, g)$" determines a map*

$$\theta_{\mathfrak{S}} \colon \Gamma_{/\sim}(X, \mathfrak{S}) \times \Gamma_{/\sim}(X, \mathfrak{S}) \longrightarrow \Gamma(X, \mathcal{A}).$$

(ii) *The assignment "$f \mapsto \theta_{\mathfrak{S}}(f, f_0)$" determines an injective map*

$$\theta_{\mathfrak{S}}(-, f_0) \colon \Gamma_{/\sim}(X, \mathfrak{S}) \hookrightarrow \Gamma(X, \mathcal{A}).$$

If, moreover, the quasi-Schwarz system $\mathfrak{S}$ is a Schwarz system, then this injective map is bijective:

$$\theta_{\mathfrak{S}}(-, f_0) \colon \Gamma_{/\sim}(X, \mathfrak{S}) \xrightarrow{\sim} \Gamma(X, \mathcal{A}).$$

(iii) *Suppose that the quasi-Schwarz system $\mathfrak{S}$ is a Schwarz system. Then the map*

$$\Gamma_{/\sim}(X, \mathfrak{S}) \times \Gamma(X, \mathcal{A}) \longrightarrow \Gamma_{/\sim}(X, \mathfrak{S})$$

defined by the assignment "$(f, a) \mapsto \iota_{f_0}(\theta_{\mathfrak{S}}(f, f_0) + a)$" — where we write ι_{f_0} for the inverse of the bijective map of the second display of (ii) *— determines an action of the abelian group $\Gamma(X, \mathcal{A})$ on the set $\Gamma_{/\sim}(X, \mathfrak{S})$.*

(iv) *Suppose that the quasi-Schwarz system $\mathfrak{S}$ is a Schwarz system. Then the action of* (iii) *determines a structure of $\Gamma(X, \mathcal{A})$-torsor on the set $\Gamma_{/\sim}(X, \mathfrak{S})$.*

Proof. Assertion (i) follows immediately from condition (3) of Definition 1.2 and Remark 1.2.1, (i). Assertion (ii) follows immediately from conditions (3), (4) of Definition 1.2 [cf. also the fact that $\mathcal{A}$ is a *sheaf* on X]. Assertions (iii), (iv) follow immediately from the various definitions involved. □

2. Schwarz Systems by Schwarzian Derivatives

In the present §2, let us reorganize the proof of the existence of *complex projective structures on compact hyperbolic Riemann surfaces* given in [1, §9, (a), Corollary 2] in terms of the notion of a Schwarz system. Let X be a connected Riemann surface. Write $\mathbb{P}^1$ for the Riemann sphere and $\mathbb{C} \subseteq \mathbb{P}^1$ for the complex plane.

Definition 2.1. We shall write $\mathfrak{U}_X$ for the set of open subspaces of X biholomorphic to the complex upper half-plane. Note that it follows from the well-known classification of simply connected Riemann surfaces that the set $\mathfrak{U}_X$ forms an open basis of X.

Definition 2.2.

(i) We shall write C_X^{A} for the sheaf of sets on X that assigns, to an open subspace $U \subseteq X$, the set of locally biholomorphic maps $U \to \mathbb{C}$.

(ii) We shall write $\mathcal{D}_X^{\mathrm{A}}$ for the invertible sheaf of holomorphic differentials on X.

(iii) Let U be an element of $\mathfrak{U}_X$. Then we shall define an equivalence relation $\sim_U^{\mathrm{A}}$ on the set $C_X^{\mathrm{A}}(U)$ as follows: Let $f_1, f_2 \in C_X^{\mathrm{A}}(U)$ be local sections of the sheaf C_X^{A}. Then we shall write $f_1 \sim_U^{\mathrm{A}} f_2$ if there exists an automorphism ι of the complex plane $\mathbb{C}$ such that the diagram

$$\begin{tikzcd} & U \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ \mathbb{C} \arrow[rr, "\iota", "\sim"'] & & \mathbb{C} \end{tikzcd}$$

is commutative.

(iv) We shall write $\theta_X^{\mathrm{A}}\colon \mathcal{C}_X^{\mathrm{A}} \times \mathcal{C}_X^{\mathrm{A}} \to \mathcal{D}_X^{\mathrm{A}}$ for the morphism defined as follows: Let $U \subseteq X$ be an open subspace of X and $f_1, f_2 \in \mathcal{C}_X^{\mathrm{A}}(U)$ local sections of the sheaf $\mathcal{C}_X^{\mathrm{A}}$. Thus, there exists an open covering $\mathfrak{O}$ of U such that, for each $V \in \mathfrak{O}$, both $f_1|_V$ and $f_2|_V$ are injective, which thus implies that the restrictions $f_1|_V$, $f_2|_V$ determine biholomorphic maps $V \xrightarrow{\sim} f_1(V)$, $V \xrightarrow{\sim} f_2(V)$, respectively. For each $V \in \mathfrak{O}$ and each positive integer n, write $f_V^{(n)}\colon V \to \mathbb{C}$ for the composite of the biholomorphic map $f_2|_V\colon V \xrightarrow{\sim} f_2(V)$ and the n-th derivative of the holomorphic function $f_2(V) \overset{f_2|_V^{-1}}{\to} V \overset{f_1|_V}{\to} \mathbb{C}$ on $f_2(V)$ $(\subseteq \mathbb{C})$. Then $\theta_X^{\mathrm{A}}(U)(f_1, f_2)$ is defined to be the element of $\mathcal{D}_X^{\mathrm{A}}(U)$ determined by the collection of local sections

$$\frac{f_V^{(2)}}{f_V^{(1)}} d f_2|_V \in \mathcal{D}_X^{\mathrm{A}}(V)$$

[cf. the differential operator θ_1 of [1, p.164]].

Definition 2.3.

(i) We shall write $\mathcal{C}_X^{\mathrm{P}}$ for the sheaf of sets on X that assigns, to an open subspace $U \subseteq X$, the set of locally biholomorphic maps $U \to \mathbb{P}^1$.

(ii) We shall write $\mathcal{D}_X^{\mathrm{P}} \overset{\mathrm{def}}{=} \mathcal{D}_X^{\mathrm{A}} \otimes_{\mathcal{O}_X} \mathcal{D}_X^{\mathrm{A}}$ for the invertible sheaf of holomorphic quadratic differentials on X.

(iii) Let U be an element of $\mathfrak{U}_X$. Then we shall define an equivalence relation $\sim_U^{\mathrm{P}}$ on the set $\mathcal{C}_X^{\mathrm{P}}(U)$ as follows: Let $f_1, f_2 \in \mathcal{C}_X^{\mathrm{P}}(U)$ be local sections of the sheaf $\mathcal{C}_X^{\mathrm{P}}$. Then we shall write $f_1 \sim_U^{\mathrm{P}} f_2$ if there exists an automorphism ι of the Riemann sphere $\mathbb{P}^1$ such that the diagram

$$\begin{tikzcd} & U \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ \mathbb{P}^1 \arrow[rr, "\iota", "\sim"'] & & \mathbb{P}^1 \end{tikzcd}$$

is commutative.

(iv) We shall write $\theta_X^{\mathrm{P}}\colon \mathcal{C}_X^{\mathrm{P}} \times \mathcal{C}_X^{\mathrm{P}} \to \mathcal{D}_X^{\mathrm{P}}$ for the morphism defined as follows: Let $U \subseteq X$ be an open subspace of X and $f_1, f_2 \in \mathcal{C}_X^{\mathrm{P}}(U)$ local sections of the sheaf $\mathcal{C}_X^{\mathrm{P}}$. Thus, there exists an open covering $\mathfrak{O}$ of U such that, for each $V \in \mathfrak{O}$, both $f_1|_V$ and $f_2|_V$ are injective — which thus implies that the restrictions $f_1|_V$, $f_2|_V$ determine biholomorphic maps $V \xrightarrow{\sim} f_1(V)$, $V \xrightarrow{\sim} f_2(V)$, respectively — and, moreover, neither $\{0, \infty\} \subseteq f_1(V)$ nor $\{0, \infty\} \subseteq f_2(V)$ holds. For each $i \in \{1, 2\}$ and $V \in \mathfrak{O}$, write $F_{i,V}\colon V \to \mathbb{P}^1$ for the map obtained by forming $f_i|_V$ (respectively, $f_i^{-1}|_V$) if $\infty \notin f_1(V)$ (respectively, $\infty \in f_1(V)$) — which thus implies that the map $F_{i,V}$ gives a biholomorphic map from V onto an open subspace of the complex plane $\mathbb{C} \subseteq \mathbb{P}^1$. For each $V \in \mathfrak{O}$ and each positive integer n, write $F_V^{(n)}: V \to \mathbb{C}$ for the composite of the biholomorphic map $F_{2,V}: V \xrightarrow{\sim} F_{2,V}(V)$ and the n-th derivative of the holomorphic function $F_{2,V}(V) \xrightarrow{F_{2,V}^{-1}} V \xrightarrow{F_{1,V}} \mathbb{C}$ on $F_{2,V}(V)$ $(\subseteq \mathbb{C})$. Then $\theta_X^{\mathrm{P}}(U)(f_1, f_2)$ is defined to be the element of $\mathcal{D}_X^{\mathrm{P}}(U)$ determined [cf. the discussion of the paragraph that contains the displayed equality (4) of [1, p.166] and the discussion surrounding the displayed equality (6) of [1, p.169]] by the collection of local sections

$$\frac{2F_V^{(1)}F_V^{(3)} - 3(F_V^{(2)})^2}{2(F_V^{(1)})^2} dF_{2,V} \otimes dF_{2,V} \in \mathcal{D}_X^{\mathrm{P}}(V)$$

[cf. the *Schwarzian derivative* θ_2 of [1, p.164] and [1, p.167]].

Theorem 2.4. *Let X be a connected Riemann surface. Then, for each $\square \in \{\mathrm{A}, \mathrm{P}\}$, the collection of data*

$$\mathfrak{S}_X^{\square} \stackrel{\mathrm{def}}{=} (C_X^{\square}, \mathcal{D}_X^{\square}, \mathfrak{U}_X, \{\sim_U^{\square}\}_{U \in \mathfrak{U}_X}, \theta_X^{\square})$$

[*cf.* Definition 2.1*,* Definition 2.2*,* Definition 2.3] *forms a Schwarz system.*

Proof. Let us first observe that one verifies immediately that $\mathfrak{S}_X^{\square}$ satisfies condition (1) of Definition 1.2. Moreover, it follows from the discussion surrounding the displayed equality (6) of [1, p.169] that $\mathfrak{S}_X^{\square}$ satisfies condition (2) of Definition 1.2. In particular, it follows from the discussion of the paragraph that contains the displayed equality (4) of [1, p.166] that $\mathfrak{S}_X^{\square}$ satisfies condition (3) of Definition 1.2. Finally, it follows from the discussion of the first paragraph of the proof of [1, p.170, Theorem 19] that $\mathfrak{S}_X^{\square}$ satisfies condition (4) of Definition 1.2. This completes the proof of Theorem 2.4. $\square$

Remark 2.4.1. One verifies easily that a *global structure associated to* $\mathfrak{S}_X^{\mathrm{A}}$ (respectively, $\mathfrak{S}_X^{\mathrm{P}}$) [cf. Theorem 2.4] is essentially the same as a *complex affine structure* (respectively, *complex projective structure*) on the Riemann surface X [cf., e.g., [1, p.167]].

Corollary 2.5. *Suppose that the Riemann surface X is compact and hyperbolic. Then there exists a complex projective structure on X. Moreover, the set of complex projective structures on X has a structure of* $\Gamma(X, \mathcal{D}_X^{\mathrm{P}})$*-torsor.*

Proof. Since X is *compact* and *hyperbolic*, it follows that the invertible sheaf $\mathcal{D}_X^{\mathrm{A}}$ of holomorphic differentials on X is *of positive degree*, which implies that $\deg(\mathcal{D}_X^{\mathrm{P}}) = 2\deg(\mathcal{D}_X^{\mathrm{A}}) > \deg(\mathcal{D}_X^{\mathrm{A}})$. Thus, it follows from the *Serre duality theorem* that the first cohomology group $H^1(X, \mathcal{D}_X^{\mathrm{P}})$ *vanishes*. Thus, the conclusion of Corollary 2.5 is, in light of Theorem 2.4, a formal consequence of Corollary 1.8 and Theorem 1.12, (iv). This completes the proof of Corollary 2.5. $\square$

Remark 2.5.1. Note that it is well-known [cf., e.g., [1, §9, (e)]] that one may omit the assumption that X is *compact* and *hyperbolic* in the statement of Corollary 2.5. Put another way, for an arbitrary connected Riemann surface, the set of *complex projective structures* is *nonempty* and has a structure of torsor under the space of *global holomorphic quadratic* *differentials*.

3. Schwarz Systems by Sugiyama-Yasuda Locally Exact Differentials

In the present §3, let us reorganize the proof of the existence of *Frobenius-projective structures of level* 2 *on projective smooth curves in characteristic* 2 given in [6, §3] [cf. also [2, §2] and [2, §3]] in terms of the notion of a Schwarz system.

In the present §3, let p be a prime number, N a positive integer, k an algebraically closed field of characteristic p, and X a projective smooth curve over k. Write K_X for the function field of X, $\mathbb{P}^1_k$ for the projective line over k, and $\mathbb{A}^1_k \subseteq \mathbb{P}^1_k$ for the affine line over k. Let us fix a regular function t on $\mathbb{A}^1_k$ that determines an isomorphism $\mathbb{A}^1_k \xrightarrow{\sim} \mathrm{Spec}(k[t])$ of schemes over k. Thus, one verifies easily that, for each nonempty open subscheme $U \subseteq X$ of X, this fixed regular function t on $\mathbb{A}^1_k$ determines, by considering the image of t in K_X, a *bijective* map between

- the set of dominant morphisms $U \to \mathbb{P}^1_k$ over k and
- the complement $K_X \setminus k$ of k in K_X.

Let us identify these two sets by means of this bijective map. Now observe that one also verifies easily that this bijective map restricts to a *bijective* map between

- the subset of generically étale morphisms $U \to \mathbb{P}^1_k$ over k and
- the complement $K_X \setminus K_X^p$ of K_X^p in K_X.

Write $0 \in \mathbb{A}^1_k$ $(\subseteq \mathbb{P}^1_k)$ for the closed point of $\mathbb{A}^1_k$ [and of $\mathbb{P}^1_k$] that forms the *zero* of the fixed regular function t on $\mathbb{A}^1_k$, $\infty \in \mathbb{P}^1_k$ for the closed point of $\mathbb{P}^1_k$ that forms the complement of $\mathbb{A}^1_k$ in $\mathbb{P}^1_k$ [or, equivalently, forms the *pole* of the meromorphic function t on $\mathbb{P}^1_k$], Ω^1_X for the $\mathcal{O}_X$-module of *differentials* on X, $\Omega^1_{K_X}$ for the sheaf on X of *meromorphic differentials* on X, Φ for the *absolute Frobenius endomorphism* of X, and $\mathcal{B}_X \stackrel{\mathrm{def}}{=} \mathrm{Im}(\Phi_* d \colon \Phi_*\mathcal{O}_X \to \Phi_*\Omega^1_X)$ for the $\mathcal{O}_X$-module of *locally exact differentials* on X, i.e., the locally free coherent $\mathcal{O}_X$-module of rank $p-1$ obtained by forming the image of the homomorphism $\Phi_*\mathcal{O}_X \to \Phi_*\Omega^1_X$ of $\mathcal{O}_X$-modules determined by the exterior differentiation operator $d \colon \mathcal{O}_X \to \Omega^1_X$.

Definition 3.1. We shall write $\mathfrak{U}_X$ for the set of affine open subschemes $U \subseteq X$ of X such that there exists an étale morphism $U \to \mathbb{A}^1_k$ over k. Note that it follows from the smoothness of X that the set $\mathfrak{U}_X$ forms an open basis of X.

Definition 3.2.

(i) We shall write $\mathcal{P}^{\text{ét}}_X$ for the sheaf of sets on X that assigns, to an open subscheme $U \subseteq X$, the set of étale morphisms $U \to \mathbb{P}^1_k$ over k.

(ii) Let U be an element of $\mathfrak{U}_X$. Then we shall define an equivalence relation $\sim^{p,N,\mathrm{P}}_U$ on the set $\mathcal{P}^{\text{ét}}_X(U)$ as follows: Let $f_1, f_2 \in \mathcal{P}^{\text{ét}}_X(U)$ be local sections of the sheaf $\mathcal{P}^{\text{ét}}_X$. Write F_1, F_2 for the elements of $K_X \setminus K_X^p$ that correspond to the étale morphisms $f_1, f_2 \colon U \to \mathbb{P}^1_k$ over k, respectively. Then we shall write $f_1 \sim^{p,N,\mathrm{P}}_U f_2$ if there exist elements a_1, a_2, a_3, a_4 of K_X such that $a_1a_4 \neq a_2a_3$, and, moreover,

$$F_1 = \frac{a_1^{p^N} F_2 + a_2^{p^N}}{a_3^{p^N} F_2 + a_4^{p^N}}.$$

Remark 3.2.1. Let $U \subseteq X$ be an open subscheme of X and $f \in \mathcal{P}_X^{\text{ét}}(U)$ a local section of the sheaf $\mathcal{P}_X^{\text{ét}}$. Write F for the element of $K_X \setminus K_X^p$ that corresponds to the étale morphism $f\colon U \to \mathbb{P}_k^1$ over k. Then one verifies easily that $1, F, \ldots, F^{p^N-1} \in K_X$ forms a basis of the vector space K_X over $K_X^{p^N}$.

Definition 3.3. Let $U \subseteq X$ be an open subscheme of X and $f \in \mathcal{P}_X^{\text{ét}}(U)$ a local section of the sheaf $\mathcal{P}_X^{\text{ét}}$. Then we shall write $\partial_f\colon K_X \to K_X$ for the endomorphism of the vector space K_X over K_X^p given by "differentiating with respect to f", i.e., by the assignment

$$\sum_{i=0}^{p-1} a_i^p F^i \mapsto \sum_{i=1}^{p-1} i a_i^p F^{i-1}$$

— where we write F for the element of $K_X \setminus K_X^p$ that corresponds to the étale morphism $f\colon U \to \mathbb{P}_k^1$ over k, and $a_0, a_1, \ldots, a_{p-1}$ are elements of K_X [cf. Remark 3.2.1].

Next, let us recall the *Sugiyama-Yasuda locally exact differentials* defined in [6].

Definition 3.4. Suppose that $(p, N) = (2, 2)$. Let $U \subseteq X$ be an open subscheme of X and $f_1, f_2 \in \mathcal{P}_X^{\text{ét}}(U)$ local sections of the sheaf $\mathcal{P}_X^{\text{ét}}$. Write F_1, F_2 for the elements of $K_X \setminus K_X^p$ that correspond to the étale morphisms $f_1, f_2\colon U \to \mathbb{P}_k^1$ over k, respectively. Write a_0, a_1, a_2, a_3 for the elements of K_X such that $F_1 = a_0^4 + a_1^4 F_2 + a_2^4 F_2^2 + a_3^4 F_2^3 = A_0^2 + A_1^2 F_2$, where we write $A_0 \overset{\text{def}}{=} a_0^2 + a_2^2 F_2$ and $A_1 \overset{\text{def}}{=} a_1^2 + a_3^2 F_2$ [cf. Remark 3.2.1]. Now let us recall that it follows from [6, Theorem 2.10] that the meromorphic differential

$$a(f_1, f_2) \overset{\text{def}}{=} \frac{a_1^2 a_3^2 + a_2^4}{a_3^4 F_2^2 + a_1^4} dF_2 = (\partial_{f_2}(A_0)^2 + \partial_{f_2}(A_1)^2 F_2 + \partial_{f_2}(A_1) A_1) \partial_{f_2}(F_1)^{-1} dF_2 \in \Omega_{K_X}^1(U)$$

defined in [6, Definition 2.8] gives rise to an element of $\mathcal{B}_X(U)$. We shall write

$$\theta_X^{2,2,\mathrm{P}}(U)\colon \mathcal{P}_X^{\text{ét}}(U) \times \mathcal{P}_X^{\text{ét}}(U) \longrightarrow \mathcal{B}_X(U)$$

for the map given by sending (f_1, f_2) to this element of $\mathcal{B}_X(U)$ and

$$\theta_X^{2,2,\mathrm{P}}\colon \mathcal{P}_X^{\text{ét}} \times \mathcal{P}_X^{\text{ét}} \longrightarrow \mathcal{B}_X$$

for the morphism determined by the $\theta_X^{2,2,\mathrm{P}}(U)$'s.

Theorem 3.5. *Let k be an algebraically closed field of characteristic 2 and X a projective smooth curve over k. Then the collection of data*

$$\mathfrak{S}_X^{2,2,\mathrm{P}} \overset{\text{def}}{=} (\mathcal{P}_X^{\text{ét}}, \mathcal{B}_X, \mathfrak{U}_X, \{\sim_U^{2,2,\mathrm{P}}\}_{U \in \mathfrak{U}_X}, \theta_X^{2,2,\mathrm{P}})$$

[*cf.* Definition 3.1, Definition 3.2, Definition 3.4] *forms a Schwarz system whose obstruction class vanishes.*

Proof. Let us first observe that it is immediate that $\mathfrak{S}_X^{2,2,\mathrm{P}}$ satisfies condition (1) of Definition 1.2. Let us also observe that it follows from [6, Proposition 2.11] that $\mathfrak{S}_X^{2,2,\mathrm{P}}$ satisfies condition (2) of Definition 1.2.

Next, to verify conditions (3), (4) of Definition 1.2, let U be an element of $\mathfrak{U}_X$ and $f_0 \in \mathcal{P}_X^{\text{ét}}(U)$ a local section of the sheaf $\mathcal{P}_X^{\text{ét}}$. Then it follows from [6, Proposition 2.9, (2), (3)] that the map $\mathcal{P}_X^{\text{ét}}(U) \to \mathcal{B}_X(U)$ of sets given by sending $f \in \mathcal{P}_X^{\text{ét}}(U)$ to $\theta_X^{2,2,\mathrm{P}}(U)(f, f_0) \in \mathcal{B}_X(U)$ *factors* through the natural quotient map $\mathcal{P}_X^{\text{ét}}(U) \twoheadrightarrow \mathcal{P}_X^{\text{ét}}(U)/\sim_U^{2,2,\mathrm{P}}$. Moreover, it follows from [6, Proposition 2.9, (1)] and [6, Proposition 2.11] that the resulting map $\mathcal{P}_X^{\text{ét}}(U)/\sim_U^{2,2,\mathrm{P}} \to \mathcal{B}_X(U)$ is *injective*. In particular, it follows immediately, in light of [6, Theorem 2.10], from [6, Lemma 3.4] that the resulting map $\mathcal{P}_X^{\text{ét}}(U)/\sim_U^{2,2,\mathrm{P}} \to \mathcal{B}_X(U)$ is *bijective*. This completes the proof of the assertion that $\mathfrak{S}_X^{2,2,\mathrm{P}}$ satisfies conditions (3), (4) of Definition 1.2, hence also of the assertion that $\mathfrak{S}_X^{2,2,\mathrm{P}}$ is a *Schwarz system*.

Finally, it follows from [6, Theorem 3.6] that the obstruction class of the Schwarz system $\mathfrak{S}_X^{2,2,\mathrm{P}}$ *vanishes*, as desired. [Note that one verifies easily that the cohomology class $\beta(X)$ defined in [6, Proposition 3.3] *coincides* with the obstruction class of the Schwarz system $\mathfrak{S}_X^{2,2,\mathrm{P}}$ in the sense of Definition 1.5 of the present paper.] This completes the proof of Theorem 3.5. □

Remark 3.5.1. In the situation of Theorem 3.5:

(i) One verifies easily that the existence of a *global object associated to* $\mathfrak{S}_X^{2,2,\mathrm{P}}$ [cf. Theorem 3.5] is *equivalent* to the existence of a *pseudo-tame rational function* [cf. [6, Definition 2.2]] on the projective smooth curve X, or, equivalently [cf. [2, Remark 2.3.2]], a *pseudo-coordinate of level* 2 [cf. [2, Definition 2.3]] on X.

(ii) One also verifies easily from [2, Lemma 3.5, (i)] and [2, Proposition 3.7] that a *global structure associated to* $\mathfrak{S}_X^{2,2,\mathrm{P}}$ [cf. Theorem 3.5] is essentially the same as a *Frobenius-projective structure of level* 2 [cf. [2, Definition 3.1]] on the projective smooth curve X.

Corollary 3.6. *Suppose that* $p = 2$. *Then the following assertions hold*:

(i) *There exists a pseudo-tame rational function* [*cf.* [6, Definition 2.2]] *on X, or, equivalently* [*cf.* [2, Remark 2.3.2]], *a pseudo-coordinate of level* 2 [*cf.* [2, Definition 2.3]] *on* X.

(ii) *There exists a Frobenius-projective structure of level* 2 [*cf.* [2, Definition 3.1]] *on* X. *Moreover, the set of Frobenius-projective structures of level* 2 *on X has a structure of* $\Gamma(X, \mathcal{B}_X)$*-torsor.*

Proof. These assertions are formal consequences of Theorem 1.7, (ii), and Theorem 1.12, (iv), together with Theorem 3.5 and Remark 3.5.1. □

4. Quasi-Schwarz Systems for Frobenius-affine Structures: $(p, N) = (2, 2)$

In the present §4, we construct a *quasi-Schwarz system* whose global structure is essentially the same as a *Frobenius-affine structure*, studied in [3], of level 2 in characteristic 2. We maintain the notational conventions introduced at the beginning of the preceding §3.

Lemma 4.1. *Let $U \subseteq X$ be an open subscheme of X, $x \in U$ a closed point of U, $f \in \mathcal{P}_X^{\text{ét}}(U)$ a local section of the sheaf $\mathcal{P}_X^{\text{ét}}$ such that $f(x) = 0$, and $a_0, a_1, \ldots, a_{p^N-1}$ elements of K_X such that $(a_0, a_1, \ldots, a_{p^N-1}) \neq (0, \ldots, 0)$. Write F for the element of $K_X \setminus K_X^p$ that corresponds to the étale morphism $f\colon U \to \mathbb{P}_k^1$ over k and $v_x\colon K_X^\times \to \mathbb{Z}$ for the discrete valuation on K_X that corresponds to the closed point x and maps a uniformizer of $\mathcal{O}_{X,x}$ to $1 \in \mathbb{Z}$. Then the equality*

$$\begin{aligned} &v_x(a_0^{p^N} + a_1^{p^N} F + \cdots + a_{p^N-1}^{p^N} F^{p^N-1}) \\ &= \min\{p^N v_x(a_0), p^N v_x(a_1) + 1, \ldots, p^N v_x(a_{p^N-1}) + p^N - 1\} \end{aligned}$$

holds. In particular, if, moreover, $v_x(a_0^{p^N} + a_1^{p^N} F + \cdots + a_{p^N-1}^{p^N} F^{p^N-1}) = 1$, then

$$v_x(\partial_f(a_0^{p^N} + a_1^{p^N} F + \cdots + a_{p^N-1}^{p^N} F^{p^N-1})) = 0.$$

Proof. Since $f \in \mathcal{P}_X^{\text{ét}}(U)$, and $f(x) = 0$, it is immediate that $v_x(F) = 1$. Thus, Lemma 4.1 is immediate. □

Definition 4.2.

(i) We shall write $\mathcal{A}_X^{\text{ét}}$ for the sheaf of sets on X that assigns, to an open subscheme $U \subseteq X$, the set of étale morphisms $U \to \mathbb{A}_k^1$ over k. Let us regard $\mathcal{A}_X^{\text{ét}}$ as a subsheaf of $\mathcal{P}_X^{\text{ét}}$ by means of the injective map $\mathcal{A}_X^{\text{ét}} \hookrightarrow \mathcal{P}_X^{\text{ét}}$ induced by the natural open immersion $\mathbb{A}_k^1 \hookrightarrow \mathbb{P}_k^1$:

$$\mathcal{A}_X^{\text{ét}} \subseteq \mathcal{P}_X^{\text{ét}}.$$

(ii) Let U be an element of $\mathfrak{U}_X$. Then we shall define an equivalence relation $\sim_U^{p,N,\mathrm{A}}$ on the set $\mathcal{A}_X^{\text{ét}}(U)$ as follows: Let $f_1, f_2 \in \mathcal{A}_X^{\text{ét}}(U)$ be local sections of the sheaf $\mathcal{A}_X^{\text{ét}}$. Write F_1, F_2 for the elements of $K_X \setminus K_X^p$ that correspond to the étale morphisms $f_1, f_2\colon U \to \mathbb{P}_k^1$ over k, respectively. Then we shall write $f_1 \sim_U^{p,N,\mathrm{A}} f_2$ if there exist elements a_1, a_2 of K_X such that $a_1 \neq 0$, and, moreover,

$$F_1 = a_1^{p^N} F_2 + a_2^{p^N}.$$

Definition 4.3. Suppose that $(p, N) = (2, 2)$. Let $U \subseteq X$ be an open subscheme of X and $f_1, f_2 \in \mathcal{A}_X^{\text{ét}}(U)$ local sections of the sheaf $\mathcal{A}_X^{\text{ét}}$. Write F_1, F_2 for the elements of $K_X \setminus K_X^p$ that correspond to the étale morphisms $f_1, f_2\colon U \to \mathbb{P}_k^1$ over k, respectively. Write a_0, a_1, a_2, a_3 for the elements of K_X such that $F_1 = a_0^4 + a_1^4 F_2 + a_2^4 F_2^2 + a_3^4 F_2^3 = A_0^2 + A_1^2 F_2$, where we write $A_0 \overset{\text{def}}{=} a_0^2 + a_2^2 F_2$ and $A_1 \overset{\text{def}}{=} a_1^2 + a_3^2 F_2$ [cf. Remark 3.2.1]. Then we shall write

$$\begin{aligned} \delta(U)(f_1, f_2) &\overset{\text{def}}{=} \frac{a_3^2}{a_3^2 F_2 + a_1^2} dF_2 = \partial_{f_2}(A_1) A_1 \partial_{f_2}(F_1)^{-1} dF_2 \in \Omega_{K_X}^1(U), \\ \theta_X^{2,2,\mathrm{A}}(U)(f_1, f_2) &\overset{\text{def}}{=} \frac{a_3^4 F_2 + a_2^4}{a_3^4 F_2^2 + a_1^4} dF_2 = (\partial_{f_2}(A_0)^2 + \partial_{f_2}(A_1)^2 F_2) \partial_{f_2}(F_1)^{-1} dF_2 \\ &= \theta_X^{2,2,\mathrm{P}}(U)(f_1, f_2) + \delta(U)(f_1, f_2) \in \Omega_{K_X}^1(U). \end{aligned}$$

Lemma 4.4. *Suppose that we are in the situation of* Definition 4.3. *Write B_1, B_2 for the elements of K_X such that $F_2 = B_0^2 + B_1^2 F_1$* [*cf.* Remark 3.2.1]. *Then the following assertions hold*:

(i) *The equality $A_1 B_1 = 1$ holds.*

(ii) *The equalities $\partial_{f_2}(F_1) = A_1^2 = B_1^{-2} = \partial_{f_1}(F_2)^{-1}$ hold.*

(iii) *The equality $\delta(U)(f_1, f_2) = \partial_{f_2}(A_1)A_1^{-1}dF_2$ holds.*

Proof. These assertions follow immediately from straightforward computations. □

Lemma 4.5. *Suppose that we are in the situation of* Definition 4.3. *Let $f_3 \in \mathcal{A}_X^{\text{ét}}(U)$ be a local section of the sheaf $\mathcal{A}_X^{\text{ét}}$. Write F_3 for the element of $K_X \setminus K_X^p$ that corresponds to the étale morphism $f_3 \colon U \to \mathbb{P}_k^1$ over k. Write, moreover, C_1, C_2, D_1, D_2 for the elements of K_X such that $F_2 = C_0^2 + C_1^2 F_3$ and $F_1 = D_0^2 + D_1^2 F_3$* [*cf.* Remark 3.2.1]. *Then the following assertions hold*:

(i) *The equality $D_1 = A_1 C_1$ holds.*

(ii) *The equality $\delta(U)(f_1, f_3) = (\partial_{f_3}(A_1)A_1^{-1} + \partial_{f_3}(C_1)C_1^{-1})dF_3$ holds.*

Proof. These assertions follow immediately from straightforward computations, together with Lemma 4.4, (iii). □

Lemma 4.6. *Suppose that we are in the situation of* Definition 4.3. *Let $f_3 \in \mathcal{A}_X^{\text{ét}}(U)$ be a local section of the sheaf $\mathcal{A}_X^{\text{ét}}$. Then the following assertions hold*:

(i) *The equality $\delta(U)(f_1, f_2) = \delta(U)(f_2, f_1)$ holds.*

(ii) *The cocycle condition*

$$\delta(U)(f_1, f_3) = \delta(U)(f_1, f_2) + \delta(U)(f_2, f_3)$$

is satisfied.

(iii) *The cocycle condition*

$$\theta_X^{2,2,\mathrm{A}}(U)(f_1, f_3) = \theta_X^{2,2,\mathrm{A}}(U)(f_1, f_2) + \theta_X^{2,2,\mathrm{A}}(U)(f_2, f_3)$$

is satisfied.

Proof. First, we verify assertion (i). Suppose that we are in the situation of Lemma 4.4. Then it follows from Lemma 4.4 that

$$\begin{aligned}\delta(U)(f_2, f_1) &= \partial_{f_1}(B_1) \cdot B_1^{-1} \cdot dF_1 = \partial_{f_1}(F_2)\partial_{f_2}(A_1^{-1}) \cdot A_1 \cdot \partial_{f_2}(F_1)dF_2 \\ &= A_1^{-2}\partial_{f_2}(A_1)A_1^{-2} \cdot A_1 \cdot A_1^2 dF_2 = \partial_{f_2}(A_1)A_1^{-1}dF_2 = \delta(U)(f_1, f_2).\end{aligned}$$

This completes the proof of assertion (i).

Next, we verify assertion (ii). Suppose that we are in the situation of Lemma 4.5. Then it follows from Lemma 4.4 and Lemma 4.5 that

$$\begin{aligned}\delta(U)(f_1, f_2) + \delta(U)(f_2, f_3) &= \partial_{f_2}(A_1) \cdot A_1^{-1} \cdot dF_2 + \partial_{f_3}(C_1)C_1^{-1}dF_3 \\ &= \partial_{f_2}(F_3)\partial_{f_3}(A_1) \cdot A_1^{-1} \cdot \partial_{f_3}(F_2)dF_3 + \partial_{f_3}(C_1)C_1^{-1}dF_3\end{aligned}$$

$$= \partial_{f_3}(A_1)A_1^{-1}dF_3 + \partial_{f_3}(C_1)C_1^{-1}dF_3 = \delta(U)(f_1, f_3).$$

This completes the proof of assertion (ii).

Finally, since $\theta_X^{2,2,\mathrm{A}}(U) = \theta_X^{2,2,\mathrm{P}}(U) + \delta(U)$, assertion (iii) follows from assertion (ii), together with a similar cocycle condition for $\theta_X^{2,2,\mathrm{P}}$ [cf. [6, Proposition 2.11]]. This completes the proof of Lemma 4.6. □

Lemma 4.7. *Suppose that we are in the situation of* Definition 4.3. *Then the following two conditions are equivalent*:

(1) *The equality* $\theta_X^{2,2,\mathrm{A}}(U)(f_1, f_2) = 0$ *holds.*

(2) *The equivalence* $f_1 \sim_U^{2,2,\mathrm{A}} f_2$ *holds.*

Proof. Let us first observe that it is immediate that condition (1) is *equivalent* to the condition that $a_2 = a_3 = 0$. In particular, it follows from the definition of the a_i's that condition (1) is *equivalent* to the condition that there exist a_0, $a_1 \in K_X$ such that $F_1 = a_0^4 + a_1^4 F_2$, i.e., condition (2), as desired. This completes the proof of Lemma 4.7. □

Lemma 4.8. *Suppose that we are in the situation of* Definition 4.3. *Then the meromorphic differential* $\theta_X^{2,2,\mathrm{A}}(U)(f_1, f_2) \in \Omega^1_{K_X}(U)$ *is contained in the submodule* $\Omega^1_X(U) \subseteq \Omega^1_{K_X}(U)$. *In particular, the assignment "*$\theta_X^{2,2,\mathrm{A}}(U)$*" determines a morphism of sheaves*

$$\theta_X^{2,2,\mathrm{A}} \colon \mathcal{A}_X^{\text{ét}} \times \mathcal{A}_X^{\text{ét}} \longrightarrow \Omega^1_X.$$

Proof. Let $x \in U$ be a closed point of U. Let us observe that it follows immediately from Lemma 4.6, (iii), and Lemma 4.7 that, to verify the *regularity* of the meromorphic differential $\theta_X^{2,2,\mathrm{A}}(U)(f_1, f_2) \in \Omega^1_{K_X}(U)$ at x, we may assume without loss of generality — by replacing f_1, f_2 by suitable elements of $\mathcal{A}_X^{\text{ét}}(U)$ *equivalent*, i.e., with respect to $\sim_U^{2,2,\mathrm{A}}$, to f_1, f_2, respectively — that $f_1(x) = f_2(x) = 0$. Then it follows immediately from Lemma 4.1 that the meromorphic differential $\theta_X^{2,2,\mathrm{A}}(U)(f_1, f_2) \in \Omega^1_{K_X}(U)$ is *regular* at x, as desired. This completes the proof of Lemma 4.8. □

Theorem 4.9. *Let k be an algebraically closed field of characteristic* 2 *and X a projective smooth curve over k. Then the collection of data*

$$\mathfrak{S}_X^{2,2,\mathrm{A}} \stackrel{\text{def}}{=} (\mathcal{A}_X^{\text{ét}}, \Omega^1_X, \mathfrak{U}_X, \{\sim_U^{2,2,\mathrm{A}}\}_{U \in \mathfrak{U}_X}, \theta_X^{2,2,\mathrm{A}})$$

[*cf.* Definition 3.1, Definition 4.2, Definition 4.3, Lemma 4.8] *forms a quasi-Schwarz system.*

Proof. Let us first observe that it is immediate that $\mathfrak{S}_X^{2,2,\mathrm{A}}$ satisfies condition (1) of Definition 1.2. Let us also observe that it follows from Lemma 4.6, (iii), that $\mathfrak{S}_X^{2,2,\mathrm{A}}$ satisfies condition (2) of Definition 1.2. Moreover, it follows from Lemma 4.6, (iii), and Lemma 4.7 that $\mathfrak{S}_X^{2,2,\mathrm{A}}$ satisfies condition (3) of Definition 1.2. This completes the proof of Theorem 4.9. □

Remark 4.9.1. In the situation of Theorem 4.9:

(i) One verifies easily that the existence of a *global object associated to* $\mathfrak{S}_X^{2,2,\mathrm{A}}$ [cf. Theorem 4.9] is *equivalent* to the existence of a *Tango function of level* 2 [cf. [3, Definition 2.3]] on the projective smooth curve X.

(ii) One also verifies easily from [3, Lemma 3.5, (i)] and [3, Proposition 3.7] that a *global structure associated to* $\mathfrak{S}_X^{2,2,\mathrm{A}}$ [cf. Theorem 4.9] is essentially the same as a *Frobenius-affine structure of level* 2 [cf. [3, Definition 3.1]] on the projective smooth curve X.

(iii) It follows from (i), (ii) and [3, Theorem 2.9, (i)] that if X is *of even genus*, then there is *no global object associated to* $\mathfrak{S}_X^{2,2,\mathrm{A}}$, hence also *no global structure associated to* $\mathfrak{S}_X^{2,2,\mathrm{A}}$.

5. Quasi-Schwarz Systems for Frobenius-affine Structures: $N = 1$

In the present §5, we construct a *quasi-Schwarz system* whose global structure is essentially the same as a *Frobenius-affine structure*, studied in [3], of level 1. We maintain the notational conventions introduced at the beginning of §3.

Definition 5.1. Let $U \subseteq X$ be an open subscheme of X and $f_1, f_2 \in \mathcal{A}_X^{\text{ét}}(U)$ local sections of the sheaf $\mathcal{A}_X^{\text{ét}}$. Write F_1, F_2 for the elements of $K_X \setminus K_X^p$ that correspond to the étale morphisms $f_1, f_2 \colon U \to \mathbb{P}_k^1$ over k, respectively. Then we shall write

$$\theta_X^{p,1,\mathrm{A}}(U)(f_1, f_2) \overset{\text{def}}{=} \frac{\partial_{f_2}(\partial_{f_2}(F_1))}{\partial_{f_2}(F_1)} dF_2 \in \Omega^1_{K_X}(U)$$

[cf. also the morphism θ_X^{A} of Definition 2.2, (iv)].

Lemma 5.2. *Suppose that we are in the situation of* Definition 5.1. *Let* $f_3 \in \mathcal{A}_X^{\text{ét}}(U)$ *be a local section of the sheaf* $\mathcal{A}_X^{\text{ét}}$. *Then the cocycle condition*

$$\theta_X^{p,1,\mathrm{A}}(U)(f_1, f_3) = \theta_X^{p,1,\mathrm{A}}(U)(f_1, f_2) + \theta_X^{p,1,\mathrm{A}}(U)(f_2, f_3)$$

is satisfied.

Proof. Write F_3 for the element of $K_X \setminus K_X^p$ that corresponds to the étale morphism $f_3 \colon U \to \mathbb{P}_k^1$ over k. Then we obtain that

$$\begin{aligned}
\theta_X^{p,1,\mathrm{A}}(U)(f_1, f_3) &= \partial_{f_3}(\partial_{f_3}(F_1)) \cdot \partial_{f_3}(F_1)^{-1} \cdot dF_3 \\
&= \partial_{f_3}(\partial_{f_3}(F_2)\partial_{f_2}(F_1)) \cdot \partial_{f_3}(F_2)^{-1}\partial_{f_2}(F_1)^{-1} \cdot dF_3 \\
&= \Big(\partial_{f_3}(\partial_{f_3}(F_2))\partial_{f_2}(F_1) + \partial_{f_3}(F_2)\partial_{f_3}(\partial_{f_2}(F_1))\Big) \cdot \partial_{f_3}(F_2)^{-1}\partial_{f_2}(F_1)^{-1} \cdot dF_3 \\
&= \partial_{f_3}(\partial_{f_3}(F_2)) \cdot \partial_{f_3}(F_2)^{-1} \cdot dF_3 + \partial_{f_3}(\partial_{f_2}(F_1)) \cdot \partial_{f_2}(F_1)^{-1} \cdot dF_3 \\
&= \theta_X^{p,1,\mathrm{A}}(U)(f_2, f_3) + \partial_{f_3}(F_2)\partial_{f_2}(\partial_{f_2}(F_1)) \cdot \partial_{f_2}(F_1)^{-1} \cdot \partial_{f_2}(F_3) dF_2 \\
&= \theta_X^{p,1,\mathrm{A}}(U)(f_2, f_3) + \theta_X^{p,1,\mathrm{A}}(U)(f_1, f_2).
\end{aligned}$$

This completes the proof of Lemma 5.2. □

Lemma 5.3. *Suppose that we are in the situation of* Definition 5.1. *Then the following two conditions are equivalent*:

(1) *The equality* $\theta_X^{p,1,\mathrm{A}}(U)(f_1, f_2) = 0$ *holds.*

(2) *The equivalence* $f_1 \sim_U^{p,1,\mathrm{A}} f_2$ *holds.*

Proof. Write $a_0, a_1, \ldots, a_{p-1}$ for the elements of K_X such that $F_1 = a_0^p + a_1^p F_2 + \cdots + a_{p-1}^p F_2^{p-1}$ [cf. Remark 3.2.1]. Then let us observe that it is immediate that condition (1) is *equivalent* to the condition that $a_2 = a_3 = \cdots = a_{p-1} = 0$. In particular, it follows from the definition of the a_i's that condition (1) is *equivalent* to the condition that there exist a_0, $a_1 \in K_X$ such that $F_1 = a_0^p + a_1^p F_2$, i.e., condition (2), as desired. This completes the proof of Lemma 5.3. □

Lemma 5.4. *Suppose that we are in the situation of* Definition 5.1. *Then the meromorphic differential* $\theta_X^{p,1,\mathrm{A}}(U)(f_1, f_2) \in \Omega^1_{K_X}(U)$ *is contained in the submodule* $\Omega^1_X(U) \subseteq \Omega^1_{K_X}(U)$. *In particular, the assignment "*$\theta_X^{p,1,\mathrm{A}}(U)$*" determines a morphism of sheaves*

$$\theta_X^{p,1,\mathrm{A}} \colon \mathcal{A}_X^{\text{ét}} \times \mathcal{A}_X^{\text{ét}} \longrightarrow \Omega^1_X.$$

Proof. This assertion follows immediately from a similar argument to the argument applied in the proof of Lemma 4.8, together with Lemma 5.2 and Lemma 5.3. □

Theorem 5.5. *Let p be a prime number, k an algebraically closed field of characteristic p, and X a projective smooth curve over k. Then the collection of data*

$$\mathfrak{S}_X^{p,1,\mathrm{A}} \stackrel{\text{def}}{=} (\mathcal{A}_X^{\text{ét}}, \Omega^1_X, \mathfrak{U}_X, \{\sim_U^{p,1,\mathrm{A}}\}_{U \in \mathfrak{U}_X}, \theta_X^{p,1,\mathrm{A}})$$

[*cf.* Definition 3.1*,* Definition 4.2*,* Definition 5.1*,* Lemma 5.4] *forms a quasi-Schwarz system.*

Proof. Let us first observe that it is immediate that $\mathfrak{S}_X^{p,1,\mathrm{A}}$ satisfies condition (1) of Definition 1.2. Let us also observe that it follows from Lemma 5.2 that $\mathfrak{S}_X^{p,1,\mathrm{A}}$ satisfies condition (2) of Definition 1.2. Moreover, it follows from Lemma 5.2 and Lemma 5.3 that $\mathfrak{S}_X^{p,1,\mathrm{A}}$ satisfies condition (3) of Definition 1.2. This completes the proof of Theorem 5.5. □

Remark 5.5.1. In the situation of Theorem 5.5:

(i) One verifies easily that the existence of a *global object associated to* $\mathfrak{S}_X^{p,1,\mathrm{A}}$ [cf. Theorem 5.5] is *equivalent* to the existence of a *Tango function of level* 1 [cf. [3, Definition 2.3]] on the projective smooth curve X.

(ii) One also verifies easily from [3, Lemma 3.5, (i)] and [3, Proposition 3.7] that a *global structure associated to* $\mathfrak{S}_X^{p,1,\mathrm{A}}$ [cf. Theorem 5.5] is essentially the same as a *Frobenius-affine structure of level* 1 [cf. [3, Definition 3.1]] on the projective smooth curve X.

(iii) Let us recall from [3, Theorem 2.9, (ii)] that, for a generically étale morphism $X \to \mathbb{P}^1_k$ over k, it holds that f is a *Tango function of level* 1 if and only if the value $n(f)$ defined in [7, Definition 9] *coincides* with $(2g-2)/p$. In particular, it follows from (i), (ii) that if $2g-2$ is *not divisible* by p, then there is *no global object associated to* $\mathfrak{S}_X^{p,1,\mathrm{A}}$, hence also *no global structure associated to* $\mathfrak{S}_X^{p,1,\mathrm{A}}$.

Remark 5.5.2. Suppose that $p = 2$. Then one verifies easily that the morphism $\theta_X^{p,1,\mathrm{A}} \colon \mathcal{A}_X^{\text{ét}} \times \mathcal{A}_X^{\text{ét}} \to \Omega^1_X$ *factors* through the "zero subsheaf $0 \subseteq \Omega^1_X$" of Ω^1_X. Moreover, it follows from [3, Remark 2.7.1] that the collection of data

$$(\mathcal{A}_X^{\text{ét}}, 0, \mathfrak{U}_X, \{\sim_U^{p,1,\mathrm{A}}\}_{U\in\mathfrak{U}_X}, \theta_X^{p,1,\mathrm{A}})$$

forms a *Schwarz system* whose obstruction class *vanishes*.

Proposition 5.6. *Suppose that X is a Tango curve* [*cf.* [3, Definition 2.8, (ii)]]. *Then the obstruction class of the quasi-Schwarz system* $\mathfrak{S}_X^{p,1,\mathrm{A}}$ [*cf.* Theorem 5.5] *vanishes.*

Proof. This assertion follows, in light of [3, Corollary 2.11], from Theorem 1.7, (i), and Remark 5.5.1, (i). □

6. Quasi-Schwarz Systems for Frobenius-projective Structures: $N = 1$

In the present §6, we construct a *quasi-Schwarz system* whose global structure is essentially the same as a *Frobenius-projective structure*, studied in [2], of level 1. We maintain the notational conventions introduced at the beginning of §3.

Definition 6.1. We shall write $\mathcal{Q}_X \stackrel{\text{def}}{=} \Omega_X^1 \otimes_{\mathcal{O}_X} \Omega_X^1$ for the invertible sheaf on X of *quadratic differentials* on X and $\mathcal{Q}_{K_X}$ for the sheaf on X of *meromorphic quadratic differentials* on X.

Definition 6.2. Let $U \subseteq X$ be an open subscheme of X and $f_1, f_2 \in \mathcal{P}_X^{\text{ét}}(U)$ local sections of the sheaf $\mathcal{P}_X^{\text{ét}}$. Write F_1, F_2 for the elements of $K_X \setminus K_X^p$ that correspond to the étale morphisms $f_1, f_2\colon U \to \mathbb{P}_k^1$ over k, respectively. Then we shall write

$$\theta_X^{p,1,\mathrm{P}}(U)(f_1, f_2) \stackrel{\text{def}}{=} \frac{2\partial_{f_2}(F_1)\partial_{f_2}\Big(\partial_{f_2}(\partial_{f_2}(F_1))\Big) - 3\partial_{f_2}(\partial_{f_2}(F_1))^2}{2\partial_{f_2}(F_1)^2} dF_2 \otimes dF_2 \in \mathcal{Q}_{K_X}(U)$$

[cf. also the morphism θ_X^{P} of Definition 2.3, (iv)] if $p \neq 2$. We shall also write

$$\theta_X^{p,1,\mathrm{P}}(U)(f_1, f_2) \stackrel{\text{def}}{=} 0 \in \mathcal{Q}_{K_X}(U)$$

if $p = 2$.

Lemma 6.3. *Suppose that we are in the situation of* Definition 6.2. *Then the meromorphic quadratic differential $\theta_X^{p,1,\mathrm{P}}(U)(f_1, f_2) \in \mathcal{Q}_{K_X}(U)$ is contained in the submodule $\mathcal{Q}_X(U) \subseteq \mathcal{Q}_{K_X}(U)$. In particular, the assignment "$\theta_X^{p,1,\mathrm{P}}(U)$" determines a morphism of sheaves*

$$\theta_X^{p,1,\mathrm{P}}\colon \mathcal{P}_X^{\text{ét}} \times \mathcal{P}_X^{\text{ét}} \longrightarrow \mathcal{Q}_X.$$

Proof. This assertion follows immediately from a similar argument to the argument applied in the proof of Lemma 4.8, together with [4, Proposition 1] and [4, Proposition 2, (i)]. □

Theorem 6.4. *Let p be a prime number, k an algebraically closed field of characteristic p, and X a projective smooth curve over k. Then the collection of data*

$$\mathfrak{S}_X^{p,1,\mathrm{P}} \stackrel{\text{def}}{=} (\mathcal{P}_X^{\text{ét}}, \mathcal{Q}_X, \mathfrak{U}_X, \{\sim_U^{p,1,\mathrm{P}}\}_{U\in\mathfrak{U}_X}, \theta_X^{p,1,\mathrm{P}})$$

[*cf.* Definition 3.1*,* Definition 3.2*,* Definition 6.1*,* Definition 6.2*,* Lemma 6.3] *forms a quasi-Schwarz system.*

Proof. Let us first observe that it is immediate that $\mathfrak{S}_X^{p,1,\mathrm{P}}$ satisfies condition (1) of Definition 1.2. Let us also observe that it follows from [4, Proposition 1] that $\mathfrak{S}_X^{p,1,\mathrm{P}}$ satisfies condition (2) of Definition 1.2. Moreover, it follows from [4, Proposition 1] and [4, Proposition 2, (i)] that $\mathfrak{S}_X^{p,1,\mathrm{P}}$ satisfies condition (3) of Definition 1.2. This completes the proof of Theorem 6.4. □

Remark 6.4.1. In the situation of Theorem 6.4:

(i) One verifies easily that the existence of a *global object associated to* $\mathfrak{S}_X^{p,1,\mathrm{P}}$ [cf. Theorem 6.4] is *equivalent* to the existence of a *pseudo-coordinate of level* 1 [cf. [2, Definition 2.3]] on the projective smooth curve X.

(ii) If $p \neq 2$, then one also verifies easily from [2, Lemma 3.5, (i)] and [2, Proposition 3.7] that a *global structure associated to* $\mathfrak{S}_X^{p,1,\mathrm{P}}$ [cf. Theorem 6.4] is essentially the same as a *Frobenius-projective structure of level* 1 [cf. [2, Definition 3.1]] on the projective smooth curve X.

Remark 6.4.2. Suppose that $p \leq 3$. Then one verifies easily that the morphism $\theta_X^{p,1,\mathrm{P}} \colon \mathcal{P}_X^{\text{ét}} \times \mathcal{P}_X^{\text{ét}} \to \mathcal{Q}_X$ *factors* through the "zero subsheaf $0 \subseteq \mathcal{Q}_X$" of $\mathcal{Q}_X$. Moreover, it follows from [2, Proposition 2.8, (i)] that the collection of data

$$(\mathcal{P}_X^{\text{ét}}, 0, \mathfrak{U}_X, \{\sim_U^{p,1,\mathrm{P}}\}_{U \in \mathfrak{U}_X}, \theta_X^{p,1,\mathrm{P}})$$

forms a *Schwarz system* whose obstruction class *vanishes*.

Proposition 6.5. *Suppose that* $p \neq 2$*, and that* X *is of genus* ≥ 2*. Then the following assertions hold*:

(i) *There exists a global structure associated to the quasi-Schwarz system* $\mathfrak{S}_X^{p,1,\mathrm{P}}$ [*cf.* Theorem 6.4].

(ii) *The obstruction class of the quasi-Schwarz system* $\mathfrak{S}_X^{p,1,\mathrm{P}}$ *vanishes.*

(iii) *The quasi-Schwarz system* $\mathfrak{S}_X^{p,1,\mathrm{P}}$ *is not a Schwarz system.*

Proof. Assertion (i) follows from Remark 6.4.1, (ii), and [2, Corollary 5.9, (i)]. Assertion (ii) follows from assertion (i) and Theorem 1.7, (i). Next, we verify assertion (iii). Assume that the quasi-Schwarz system $\mathfrak{S}_X^{p,1,\mathrm{P}}$ is a *Schwarz system*. Then it follows, in light of the *Riemann-Roch theorem*, from assertion (i) and Theorem 1.12, (iv), that X has *infinitely many Frobenius-projective structures of level* 1. On the other hand, it follows from [2, Remark 4.4.1, (ii)] that this *infiniteness contradicts* the *finiteness* of the morphism $\overline{\mathcal{N}}_{g,r} \to \overline{\mathcal{M}}_{g,r}$ of the final display of [5, p.1030], i.e., derived from [5, Chapter II, Theorem 2.3]. This completes the proof of assertion (iii), hence also of Proposition 6.5. □

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References

[1] R.C. Gunning: Lectures on Riemann Surfaces, Princeton Mathematical Notes Princeton University Press, Princeton, N.J. 1966.
[2] Y. Hoshi: *Frobenius-projective structures on curves in positive characteristic*, Publ. Res. Inst. Math. Sci. **56** (2020), 401–430.
[3] Y. Hoshi: *Frobenius-affine structures and Tango curves*, Nagoya Math. J. **250** (2023), 385–409.
[4] Y. Ihara: *Schwarzian equations*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **21** (1974), 97–118.
[5] S. Mochizuki: *A theory of ordinary p-adic curves*, Publ. Res. Inst. Math. Sci. **32** (1996), 957–1152.
[6] Y. Sugiyama and S. Yasuda: *Belyi's theorem in characteristic two*, Compos. Math. **156** (2020), 325–339.
[7] H. Tango: *On the behavior of extensions of vector bundles under the Frobenius map*, Nagoya Math. J. **48** (1972), 73–89.

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