Hajiaghasi, S. and Azami, S.
Osaka J. Math.
61 (2024), 353–367

THE LOWER BOUNDS FOR THE FIRST EIGENVALUE OF THE BIHARMONIC AND p-BIHARMONIC OPERATORS ON FINSLER MANIFOLDS

Sakineh HAJIAGHASI and Shahroud AZAMI

(Received June 6, 2023)

Abstract
In this paper, we are going to estimate the lower bounds for the first eigenvalues of the buckling problem and clamped plate problem by considering a positive lower bound for the weighted Ricci curvature. Also, we extended the results for the p-biharmonic operator and we prove a Lichnerowicz-Obata-Cheng type estimate for the biharmonic operators.

1. Introduction

Let $(M, F, d\mu)$ be an n-dimensional compact connected Finsler manifold with smooth boundary ∂M. The clamped plate problem and buckling problem on this Finsler manifold introduced in [8] as follows:

$$\begin{cases} \Delta^{\nabla u}\Delta u = \Gamma u \ in \ M_u, \\ u|_{\partial\Omega} = g_{\overrightarrow{n}}(\overrightarrow{n}, \nabla u)|_{\partial M} = 0, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \Delta^{\nabla u}\Delta u = -\Lambda\Delta u \ in \ M_u, \\ u|_{\partial\Omega} = g_{\overrightarrow{n}}(\overrightarrow{n}, \nabla u)|_{\partial M} = 0, \end{cases} \tag{1.2}$$

here Δ and $\Delta^{\nabla u}$ are Laplacian and weighted Laplacian, $\overrightarrow{n}$ denotes the outer unit normal vector field of the boundary ∂M and $g_{\overrightarrow{n}}$ denotes the induced Riemannian structure on ∂M.
Finding the lower bound for the first eigenvalue of the biharmonic operator on Riemannian manifolds had been studied for a long time. For instance, Zhang and Zhao in [17] obtained the lower bounds of the first eigenvalues for the biharmonic operator (buckling and clamped plate problems) with considering the lower positive bound for the Ricci curvature. After a while, the Lichnerowicz-type estimate theorem investigated on both Riemannian and Kähler manifolds with the boundary condition for the integral Ricci curvature [3]. Recently, working on the first eigenvalue for different kinds of operator in Finsler geometry attracts much attention, since it has broader applications and plays an important role in Finsler geometry. For more study about the first eigenvalue of different operators we refer [8, 11, 12, 13, 14, 16].
For a domain Ω with compact closure and nonempty boundary $\partial\Omega$ of an n-dimensional Finsler manifold $(M, F, d\mu)$, $n \geq 2$, the first eigenvalue of the Dirichlet problem for the

2020 Mathematics Subject Classification. 53C20, 35P15, 53C60.

Laplace operator defines as follows (see [10]):

$$\lambda_1(\Omega) = \inf\left\{ \frac{\int_\Omega F(\nabla f)^2 d\mu}{\int_\Omega f^2 d\mu} : f \in L^2_{1,0}(\Omega) \setminus \{0\} \right\},$$

where F^* is the dual Finsler metric on T^*M and $L^2_{1,0}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the following norm:

$$\|\phi\|^2_\Omega = \int_\Omega \phi^2 d\mu + \int_\Omega (F^*(d\phi))^2 d\mu.$$

When M is a compact manifold without boundary, the first closed eigenvalue $\lambda_1(M)$ of M is defined as

$$\lambda_1(M) = \inf\left\{ \frac{\int_M F(\nabla f)^2 d\mu}{\int_M f^2 d\mu} : f \in L^2_{1,0}(M) \setminus \{0\}, \int_M f d\mu = 0 \right\}.$$

We recall the first eigenvalue of the both buckling and clamped plate problems from [8] as follows:

$$\Lambda_1(M) = \min_{u \in H_0^2(M), u \neq 0} \frac{\int_M (\Delta u)^2 d\mu}{\int_M |\nabla u|^2 d\mu},$$

and

$$\Gamma_1(M) = \min_{u \in H_0^2(M), u \neq 0} \frac{\int_M (\Delta u)^2 d\mu}{\int_M u^2 d\mu}.$$

Here $H^2(M)$ is defined as

$$H^2(M) = \{u : \ u, |\nabla u|, |\nabla^2 u|^2_{HS(\nabla u)} \in L^2(M)\},$$

where

$$\int_M |\nabla^2 u|^2_{HS(\nabla u)} d\mu := \int_{M_u} |\nabla^2 u|^2_{HS(\nabla u)} d\mu.$$

Here $M_u = \{x \in M| \quad \nabla u|_x \neq 0\}$ and $|.|^2_{HS}$ is the Hilbert-Schmidt norm such that for a bounded operator $A : H \to H$ that acts on a Hilbert space H with an orthonormal basis $\{e_i : i \in I\}$ define as

$$|A|^2_{HS} := \Sigma_{i \in I} |Ae_i|^2_H.$$

Also $H_0^2(M)$ which is the subset of $H^2(M)$ is

$$H_0^2(M) := \left\{ u \in H^2(M) : u|_{\partial M} = \left.\frac{\partial u}{\partial \vec{n}}\right|_{\partial M} = 0 \right\}.$$

Lately, Pan and Zhang in [8] considered $\mathrm{Ric}_N \geq (n-1)k$ where $N \in (n, \infty)$ and k is a positive constant, then they obtained the lower and upper bounds of the buckling and clamped plate problems on a Finsler manifold.

In this paper, we are going to estimate the lower bounds of the first eigenvalues for these two problems for an n-dimensional compact connected Finsler manifold with smooth boundary so that its weighted Ricci curvature Ric_∞ is bounded from below by a positive constant.

Also, we estimate the lower bound of the first eigenvalue for the following nonlinear eigenvalue problem:

$$\Delta_p^2 u = \lambda_p |u|^{p-2} u \qquad in\ M, \tag{1.3}$$

where $u \in W_0^{2,p}(M)$, and $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is an elliptic operator of fourth order, called p-biharmonic operator. Obviously, for $p = 2$, (1.3) describes clamped plate problem. All solutions of (1.3) in Riemannian manifolds are distributional in the sense that

$$\int_M |\Delta u|^{p-2} \Delta u \Delta \phi d\mu = \lambda_p \int_M |u|^{p-2} u \phi d\mu,$$

for all $\phi \in C_0^\infty(M)$ and if $\phi = u$, then we have

$$\lambda_p = \inf_u \frac{\int_M |\Delta u|^p d\mu}{\int_M |u|^p d\mu},$$

here the infimum will be taken over all $u \in W_0^{2,p}(M)$. If we consider a Finsler manifold with vanishing S-curvature, then the symmetry exchange holds for the Finsler Laplacian.
El Khalil et.al. in [4] proved that for any bounded domain Ω with $1 < p < +\infty$, Δ_p^2 satisfies the following:
(i) $\Delta_p^2 : W_0^{2,p}(\Omega) \to W^{-2,p'}(\Omega)$ is an hemicontinuous operator where $p' = \dfrac{p}{p-1}$.
(ii) It is a bounded monotonous, and coercive operator.
(iii) This operator is a bicontinuous operator.

They considered all weak solutions of the following problem on a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$

$$\Delta_p^2 u = \lambda \rho(x) |u|^{p-2} u \qquad in\, \Omega, \qquad u \in W_0^{2,p}(\Omega). \tag{1.4}$$

This means that for all $\phi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega |\Delta u|^{p-2} \Delta u \Delta \phi dx = \lambda \int_\Omega \rho(x) |u|^{p-2} u \phi dx,$$

here $\rho \in L^r(\Omega)$, $\rho \neq 0$. Also, they defined $A(u) = \dfrac{1}{p}||\Delta u||_p^p$ as a potential functional on $W_0^{2,p}(\Omega)$, then obtained their main result as follows:

Theorem 1.1. *Let* $\mathcal{M} = \left\{u \in W_0^{2,p}(\Omega); pB(u) = 1\right\}$*, where* $B(u) = \dfrac{1}{p}\int_\Omega \rho(x)|u|^p dx$*, and set*

$$\Gamma_k = \{K \subset \mathcal{M} : K \ is\ symmetric,\ compact\ and\ \gamma(K) \geq 1\},$$

here $\gamma(K) = k$ *is the smallest integer such that there exists an odd continuous map* $f : K \to \mathbb{R}^k - \{0\}$*. Then for any integer* $k \in \mathbb{N}^*$*, we get that*

$$\lambda_k := \inf_{k \in \Gamma_k} \max_{u \in K} \rho A(u),$$

is a critical value of A *restricted on* M*. Especially, there exists* $u_k \in K_k \in \Gamma_k$ *so that*

$$\lambda_k = \rho A(u_k) = \sup_{u \in K_k} \rho A(u),$$

and (λ_k, u_k) *is a solution of* (1.4) *corresponding to the positive eigenvalue* λ_k*, such that when* $k \to +\infty$*, then* $\lambda_k \to +\infty$*.*

Lately, Abolarinwa and his coauthor in [1] considered a nonlinear problem involving p-biharmonic operator and they studied monotonicity and differentiability of the first eigenvalue of this operator under consideration a Riemannian manifold endowed with Ricci flow

$$\begin{cases} \dfrac{\partial}{\partial t} g(t,x) = -2\mathrm{Ric}(t,x), \ (t,x) \in [0,T] \times M, \\ g(0,x) = g_0, \end{cases}$$

here Ric is the Ricci curvature tensor of g and $0 < T < T_{max}$ is taken to be the maximum time of existence for the flow, T_{max} is the first time when the flow blows up. Our purpose here is to peruse the first eigenvalue of Δ_p^2, the main assumption is that we take the lower bound $\mathrm{Ric}_\infty \geq C$ for some constant $C > 0$.
Actually, this paper contains three important sections. First of all, in section 2 we study the first eigenvalues for the biharmonic operator eigenvalue problems (buckling problem and clamped plate problem) on a compact Finsler manifold under some condition for weighted Ricci curvature Ric_∞. The second factor is the first eigenvalue of the Finsler p-biharmonic operator, which will be investigated in section 3. The most important reason for studying this operator is that it generalizes p-Laplacian, so that is interesting to ask if we could extend such results for the case of p-biharmonic operator. In this way considering different types of geometric Ricci flows on manifold endowed with this operator seems to have an essential result. The last but not least step in this paper is section 4 which is the generalization of Lichnerowicz and Obata's works (see [5], [6]). We take the same conditions as well as those works which leading ultimately to the geometric structure of such manifolds.

2. Finsler Geometry

Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle TM. Let (x, y) be a point of TM with $x \in M$, $y \in T_xM$, and let (x^i, y^i) be the local coordinate on TM with $y = y^i \dfrac{\partial}{\partial x^i}$. A Finsler metric on M is a function $F : TM \longrightarrow [0, \infty)$ satisfying the following properties:
(i) Regularity: F is C^∞ on the entire slit tangent bundle $TM \setminus \{0\}$,
(ii) positive homogeneity: $F(x, ay) = aF(x, y)$ for all $a > 0$,
(iii) strong convexity: the $n \times n$ Hessian matrix

$$(g_{ij}) := \Big([\frac{1}{2}F^2]_{y^i y^j}\Big),$$

is positive definite at every point of $TM \setminus \{0\}$.
Let $V = V^i \dfrac{\partial}{\partial x^i}$ be a non-vanishing vector field on an open subset $\mathcal{U} \subset M$. One can introduce a Riemannian metric $\tilde{g} = \mathbf{g}_V$ on the tangent bundle over $\mathcal{U}$ as follows:

$$\mathbf{g}_V(X, Y) := X^i Y^j g_{ij}(x, v), \quad \forall X = X^i \frac{\partial}{\partial x^i},\ Y = Y^i \frac{\partial}{\partial x^i}.$$

In particular, $\mathbf{g}_V(V, V) = F(V)^2$.
Let $\pi : TM \to M$ be the natural projection map, the pull-back bundle $\pi^* TM$ admits a

unique linear connection called Chern connection. The Chern connection determines by the following structure equations:

$$D_X^V Y - D_Y^V X = [X, Y],$$

this mentioned as torsion freeness and the almost g-compatibility is

$$X\mathbf{g}_V(Y, Z) = \mathbf{g}_V(D_X^V Y, Z) + \mathbf{g}_V(Y, D_X^V Z) + 2C_V(D_X^V V, Y, Z),$$

where $V \in T_xM - \{0\}$, $X, Y, Z \in TM$. Note that $D_X^V Y$ is the covariant derivative with respect to reference vector $V \in T_xM \setminus \{0\}$. The Cartan tensor defines as follows:

$$C_V(X, Y, Z) := C_{ijk}(V)X^iY^jZ^k = \frac{1}{4}\frac{\partial^3 F^2}{\partial V^i V^j V^k}(V)X^iY^jZ^k.$$

The coefficients of the Chern connection are

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\Big(\frac{\delta g_{kl}}{\delta x^j} + \frac{\delta g_{jl}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^l}\Big),$$

which is

$$D^V_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} := \Gamma^i_{jk}(x, V)\frac{\partial}{\partial x^k},$$

where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j},\ N_i^j = \frac{\partial G^j}{\partial y^i},\ G^i = \frac{1}{4}g^{il}\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\}.$$

Let ∇^V be the Chern connection, and then, the Chern curvature $\mathbf{R}^V(X, Y)Z$ for vector fields $X, Y, Z \in C(\pi^*TM)$ is defined by:

$$\mathbf{R}^V(X, Y)Z := \nabla_X^V\nabla_Y^V Z - \nabla_Y^V\nabla_X^V Z - \nabla^V_{[X,Y]}Z.$$

For a flag (V, W) consist of non-zero tangent vectors $V, W \in T_xM$ and a 2-plane $P \subset T_xM$ with $V \in P$ the flag curvature $K(V, W)$ is defined as follows:

$$K(V, W) := \frac{\mathbf{g}_V(R^V(V, W)W, V)}{\mathbf{g}_V(V, V)\mathbf{g}_V(W, W) - \mathbf{g}_V(V, W)^2},$$

here, W is a tangent vector such that V, W span the 2-plane P and $V \in T_xM$ is extended to a geodesic field, i.e., $\nabla_V^V V = 0$ near x. The Ricci curvature of V is defined as:

$$Ric(V) = \sum_{i=1}^{n-1} K(V, e_i),$$

here $e_1, \cdots, e_{n-1}, \dfrac{V}{F(V)}$ form an orthonormal basis of T_xM with respect to $\mathbf{g}_V$. Namely, one has $\mathrm{Ric}(aV) = a\mathrm{Ric}(V)$ for any $a > 0$.

The reversible function $\lambda : M \longrightarrow \mathbb{R}$ is defined by:

$$\lambda(x) = \max_{y \in T_xM\setminus\{0\}} \frac{F(y)}{F(-y)}.$$

It is clear that $1 \leq \lambda(x) < +\infty$ for any $x \in M$. Here $\lambda_F = \sup_{x\in M}\lambda(x)$ is called the

reversibility of (M, F), and (M, F) is called reversible if $\lambda_F = 1$.
The gradient vector field of a differentiable function f on M by the Legendre transformation $\mathcal{L} : T_xM \to T^*_xM$ is defined as

$$\nabla f := \mathcal{L}^{-1}(df).$$

Let $M_f = \{x \in M : \nabla f|_x \neq 0\}$. We define the Hessian of f on M_f as follows:

$$H(f)(X, Y) := XY(f) - \nabla^{\nabla f}_X Y(f), \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}).$$

For a given volume form $d\mu = \sigma(x)dx$ and vector $V \in T_xM \setminus \{0\}$, the distortion of M is defined by

$$\tau(V) := \ln \frac{\sqrt{\det(g_{ij}(V))}}{\sigma}.$$

Considering the rate of changes of the distortion along geodesics, leads to the so-called S-curvature as follows

$$S(V) := \frac{d}{dt}[\tau(\gamma(t), \dot{\gamma}(t)]_{t=0},$$

where $\gamma(t)$ is the geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = V$. Define

$$\dot{S}(V) := F^{-2}(V)\frac{d}{dt}[S(\gamma(t), \dot{\gamma}(t)]_{t=0}.$$

Then the weighted Ricci curvatures of M defined as follows

$$\begin{aligned}
Ric_n(V) &:= \begin{cases} Ric(V) + \dot{S}(V), & for\ S(V) = 0, \\ -\infty, & otherwise, \end{cases} \\
Ric_N(V) &:= Ric(V) + \dot{S}(V) - \frac{S(V)^2}{(N-n)F(V)^2}, \quad \forall N \in (n, \infty), \\
Ric_\infty(V) &:= Ric(V) + \dot{S}(V).
\end{aligned}$$

Fix a volume form $d\mu$, the divergence $\operatorname{div}(X)$ of X is defined as:

$$d(X \rfloor d\mu) = \operatorname{div}(X)d\mu.$$

For a given smooth function $f : M \longrightarrow \mathbb{R}$, the Laplacian Δf of f is defined by $\Delta f = \operatorname{div}(\nabla f) = \operatorname{div}(\mathcal{L}^{-1}(df))$.
Given a vector field V, the weighted gradient vector and the weighted Laplacian on the weighted Riemannian manifold $(M, \mathbf{g}_V)$ are defined by

$$\nabla^V u := \begin{cases} g^{ij}(V)\dfrac{\partial u}{\partial x_j}\dfrac{\partial}{\partial x_i}, & on\ M_u, \\ 0, & on\ M \setminus M_u, \end{cases} \qquad \Delta^V u := \operatorname{div}(\nabla^V u).$$

Here $M_u := \{x \in M | du(x) \neq 0\}$. We note that $\nabla u = \nabla^V u$, $\Delta u = \Delta^V u$.
Assume that $(M, F, d\mu)$ is a Finsler measure space with boundary ∂M, then we shall view ∂M as a hypersurface embedded in M. More importantly, ∂M is a Finsler manifold with a Finsler structure $F_{\partial M}$ induced by F. For any $x \in \partial M$, there exists exactly two unit normal vectors $\overrightarrow{n}$, which are characterized by

$$T_x(\partial M) = \{C \in T_xM | g_{\vec{n}}(\vec{n}, V) = 0, g_{\vec{n}}(\vec{n}, \vec{n}) = 1\}.$$

The normal vector $\vec{n}$ induces a volume form $d\mu_{\vec{n}}$ on ∂M from $d\mu$ by

$$V \rfloor d\mu = g_{\vec{n}}(\vec{n}, V) d\mu_{\vec{n}}, \quad \forall V \in TM.$$

The Stokes theorem holds as follows (see [10]):

$$\int_M \mathrm{div}(V) d\mu = \int_{\partial M} g_{\vec{n}}(\vec{n}, V) d\mu_{\vec{n}}.$$

Proving our main results, we may need the following formulas.

Lemma 2.1 ([7])**.** *Let $(M, F, d\mu)$ be a Finsler measure space, and $u : M \to \mathbb{R}$ a smooth function on M. Then*

$$\Delta^{\nabla u}\Big(\frac{F(\nabla u)^2}{2}\Big) - D(\Delta u)(\nabla u) = \mathrm{Ric}_\infty(\nabla u) + |\nabla^2 u|^2_{HS(\nabla u)}, \tag{2.1}$$

as well as

$$\Delta^{\nabla u}\Big(\frac{F(\nabla u)^2}{2}\Big) - D(\Delta u)(\nabla u) \geq \mathrm{Ric}_N(\nabla u) + \frac{(\Delta u)^2}{N}, \tag{2.2}$$

for $N \in (n, \infty)$, point-wise on M_u.

Also, we could easily obtain the p-Bochner formula

$$\begin{aligned} \frac{1}{p}\Delta^{\nabla u}(F(\nabla u)^p) &= (p-2)F(\nabla u)^{p-2}(\nabla^{\nabla u}(F(\nabla u)))^2 \\ &\quad + F(\nabla u)^{p-2}\Big[\|\nabla^2 u\|^2_{HS} + D(\Delta u)(\nabla u) + \|\nabla u\|^2 \mathrm{Ric}_\infty(\nabla u)\Big]. \end{aligned} \tag{2.3}$$

At first due to the definition of Laplacian in Finsler geometry we have

$$\frac{1}{p}\Delta^{\nabla u}(F(\nabla u)^p) = \frac{1}{p}\Delta^{\nabla u}(F(\nabla u)^2)^{p/2} = \frac{1}{p}\mathrm{div}\nabla^{\nabla u}(F(\nabla u)^2)^{p/2}.$$

It is easy to see that the direct computation due to the definitions of divergence and gradient in Finsler geometry concludes (2.3).

3. Eigenvalue estimation of biharmonic operator

In this section, we are going to improve estimate from [8] for the first eigenvalue of the clamped plate problem (1.1) and buckling problem (1.2).

Theorem 3.1. *Let (M, F) be an n-dimensional compact connected Finsler manifold with smooth boundary ∂M. Assume positive constant C as lower bound for the weighted Ricci curvature* Ric_∞*. Let $\Lambda_1(M)$ be the first eigenvalue of the buckling problem* (1.2)*, then for any vector field $X \in \Gamma(TM)$ such that $\|X\|_\infty = \sup_M F(X) < \infty$ and $\inf_M \mathrm{div}(X) > 0$, we have*

$$\Lambda_1(M) \geq \left(\frac{\inf_M \mathrm{div}X}{2\|X\|_\infty}\right)^2 + C.$$

Proof. We know that for any $f \in C_0^\infty(\Omega)$, the vector field $(F^*(df))^2X$ has compact support on M, so we compute

$$\begin{aligned}\operatorname{div}(XF(\nabla f)^2) &= 2F(\nabla f)(X(F(\nabla f))) + F(\nabla f)^2\operatorname{div}(X)\\ &\geq -2F(\nabla f)\|X\|_\infty \nabla F(\nabla f) + F(\nabla f)^2 \inf_M \operatorname{div}(X).\end{aligned} \tag{3.1}$$

Applying Young's inequality for any $\epsilon > 0$, we get

$$F(\nabla f)\nabla F(\nabla f) \leq \frac{(\nabla F(\nabla f))^2}{2\epsilon^2} + \frac{\epsilon^2 F(\nabla f)^2}{2}. \tag{3.2}$$

Now (3.1) changes as follows

$$\begin{aligned}\operatorname{div}(X(F^*(df))^2) \geq -2\|X\|_\infty&\Big(\frac{(\nabla F(\nabla f))^2}{2\epsilon^2} + \frac{\epsilon^2 F(\nabla f)^2}{2}\Big)\\ &+ (F^*(df))^2 \inf_M \operatorname{div}(X).\end{aligned} \tag{3.3}$$

Using divergence theorem, we obtain

$$\int_M \operatorname{div}(XF(\nabla f)^2)d\mu = \int_{\partial M} F(\nabla f)^2 g_{\vec{n}}(\vec{n}, X)d\mu_{\vec{n}} = 0.$$

From (3.2) and (3.3), we infer

$$\int_M \operatorname{div}(XF(\nabla f)^2)d\mu \geq \frac{\epsilon^2}{\|X\|_\infty}(\inf_M \operatorname{div}X - \epsilon^2\|X\|_\infty)\int_M F(\nabla f)^2 d\mu,$$

hence

$$\frac{\int_M \operatorname{div}(XF(\nabla f)^2)d\mu}{\int_M F(\nabla f)^2 d\mu} \geq \frac{\epsilon^2}{\|X\|_\infty}(\inf_M \operatorname{div}X - \epsilon^2\|X\|_\infty). \tag{3.4}$$

Consider $g(\epsilon) = \epsilon^2 \inf_M \operatorname{div}X - \epsilon^4\|X\|_\infty$, then a simple calculation get

$$g'(\epsilon) = 2\epsilon \inf_M \operatorname{div}X - 4\epsilon^3\|X\|_\infty, \qquad g''(\epsilon) = 2\inf_M \operatorname{div}X - 12\epsilon^2\|X\|_\infty.$$

Its clear that whenever $g'(\epsilon) = 0$, then there is $\epsilon_0 = \Big(\frac{\inf_M \operatorname{div}X}{2\|X\|_\infty}\Big)^{1/2}$ so that $g''(\epsilon_0) = -4\inf_M \operatorname{div}X$. Therefore, we infer

$$\max g(x) = \Big(\frac{\inf_M \operatorname{div}X}{2\|X\|_\infty^{1/2}}\Big)^2.$$

Consequently, it follows from (3.4), that

$$\frac{\int_M \operatorname{div}(XF(\nabla f)^2)d\mu}{\int_M F(\nabla f)^2 d\mu} \geq \Big(\frac{\inf_M \operatorname{div}X}{2\|X\|_\infty}\Big)^2. \tag{3.5}$$

Using the following formula

$$\int_M (\Delta f)^2 d\mu = \int_M \Big(\|\nabla^2 f\|_{HS}^2 + \operatorname{Ric}_\infty(\nabla f)\Big)d\mu,$$

since $\operatorname{Ric}_\infty \geq C$, and $(\nabla F(\nabla f))^2 < |\nabla^2 f|^2$, from (3.5) it follows that

$$\frac{\int_M (\Delta f)^2 d\mu}{\int_M F(\nabla f)^2 d\mu} \geq \Big(\frac{\inf_M \operatorname{div}X}{2\|X\|_\infty}\Big)^2 + C, \tag{3.6}$$

so as X was an arbitrary vector field, due to the definition of the first eigenvalue of (1.2), we conclude

$$\Lambda_1(M) \geq \left(\frac{\inf_M \mathrm{div}X}{2\|X\|_\infty}\right)^2 + C. \qquad \square$$

Theorem 3.2. *Consider the Finsler manifold (M, F) and the vector field $X \in \Gamma(TM)$ such that satisfy in the conditions that stated in the first theorem. Let $\Gamma_1(M)$ be the first eigenvalue of the clamped plate problem* (1.1) *and $\lambda_1(M)$ as the first eigenvalue of the Dirichlet eigenvalue problem for Laplace operator. Then we have*

$$\Gamma_1(M) \geq \left(\left(\frac{\inf_M \mathrm{div}X}{2\|X\|_\infty}\right)^2 + C\right)\lambda_1(M).$$

Proof. From (3.6) for all $f \in H_0^2(M)$, it follows that

$$\frac{\int_M (\Delta f)^2 d\mu}{\int_M f^2 d\mu} \geq \left(\left(\frac{\inf_M \mathrm{div}X}{2\|X\|_\infty}\right)^2 + C\right)\frac{\int_M F(\nabla f)^2 d\mu}{\int_M f^2 d\mu}. \tag{3.7}$$

Due to the fact that f is nonzero with $f|_{\partial M} = 0$, and from the definition of the first eigenvalue for Dirichlet problem, we have

$$\lambda_1(M) \leq \frac{\int_M F(\nabla f)^2 d\mu}{\int_M f^2 d\mu}. \tag{3.8}$$

Therefore, from the definition of $\Gamma_1(M)$, (3.7), and (3.8) together with the fact that X is arbitrary, we infer

$$\Gamma_1(M) \geq \left(\left(\frac{\inf_M \mathrm{div}X}{2\|X\|_\infty}\right)^2 + C\right)\lambda_1(M), \qquad \square$$

REMARK 3.3. Replacing $F(\nabla f)$ by f in (3.5) implies

$$\frac{\int_M F(\nabla f)^2 d\mu}{\int_M f^2 d\mu} \geq \left(\frac{\inf_M \mathrm{div}X}{2\|X\|_\infty}\right)^2,$$

which by (3.7) leads to

$$\Gamma_1(M) \geq \left(\frac{\inf_M \mathrm{div}X}{2\|X\|_\infty}\right)^2\left(\left(\frac{\inf_M \mathrm{div}X}{2\|X\|_\infty}\right)^2 + C\right).$$

We will need the following Hessian comparison theorem which was proved by Yin ea al.[16].

Theorem 3.4. *Let (M, F) be an n-dimensional Finsler manifold with Ricci curvature* Ric $\geq (n-1)k$*, and let $\gamma : [0, r(x)] \to M$ be a minimizing geodesic satisfying $\gamma(0) = p$, where $r(x)$ is the distance function $d_F(p, x)$ from any given point $p \in M$. Consider k as a flag curvature such that*

$$\mathbf{ct}_k(r) = \begin{cases} \sqrt{k}\cot(\sqrt{k}r), & k > 0, \\ \dfrac{1}{r} & k = 0, \\ \sqrt{-k}\coth(\sqrt{-k}r) & k < 0. \end{cases}$$

Then the Hessian trace of r satisfy

$$tr_{\nabla r}H(r) \leq (n-1)ct_k(r).$$

Here equality holds iff the radial flag curvature $K(\dot{\gamma}(t); .) \equiv k$ along the geodesic $\gamma(t)$. In this case, any Jacobi field $J(t)$ orthogonal to $\dot{\gamma}(t)$ can be written as $J(t) = s_k(t)E(t)$, where $E(t) \perp \dot{\gamma}(t)$ is a parallel vector field along γ and

$$\mathbf{s}_k = \begin{cases} \frac{1}{\sqrt{k}}\sin(\sqrt{k}t), & k > 0, \\ t, & k = 0, \\ \frac{1}{\sqrt{-k}}\sinh(\sqrt{-k}t), & k < 0. \end{cases}$$

As an application of this section's theorems, we can obtain the following results:

Corollary 3.5. *Let (M, F) be an n-dimensional complete Finsler manifold, and consider $B_M(p, R)$ as a geodesic ball with radius $R < inj(p)$. Let $k(p, R) = \sup\{K_M(x); x \in B_M(p, R)\}$. Here $K_M(x)$ is the flag curvature of M at x. If $k > 0$, $k(p, R) = k^2$, $R < \frac{\pi}{2k}$ and the weighted Ricci curvature $\mathrm{Ric}_\infty \geq C$, then we have*

$$\begin{aligned} \Lambda_1(B_M(p, R)) &\geq [(n-1)ct_k]/4r^2 + C, \\ \Gamma_1(B_M(p, R)) &\geq ([(n-1)ct_k]/4r^2 + C)\lambda_1(B_M(p, R)). \end{aligned}$$

Proof. Let $X = \nabla r$, then from Theorem 3.1, we get

$$\begin{aligned} \Lambda_1(B_M(p, R)) &\geq \left[\frac{\inf_{B_M(p,R)} \Delta r}{2\|\nabla r\|_\infty}\right]^2 + C \\ &\geq \left[\frac{(n-1)ct_k(r)}{2r}\right]^2 + C, \end{aligned}$$

and

$$\Gamma_1(B_M(p, R)) \geq \left(\left[\frac{(n-1)ct_k(r)}{2r}\right]^2 + C\right)\lambda_1(B_M(p, R)). \qquad \square$$

4. Eigenvalue estimation of *p*-biharmonic operator

In this section, we want to study the first eigenvalue of the so called p-biharmonic operator (1.3). Singular elliptic problems involving p-biharmonic operators have been studied by many authors (see for instance [2], [9]). It is well known that such problems like (1.3) with $M = \Omega \in \mathbb{R}^n$ and the condition $u = \Delta u = 0$ on $\partial\Omega$ has simple and isolated least positive eigenvalue λ_p^* in the sense that the set of all solutions with $\lambda = \lambda_p^*$ forms the one-dimensional linear space spanned by a positive eigenfunction u^* associated with λ_p^* so that f is strictly superharmonic. For more study, we refer to ([4]).
We shall use the same method as the last section for proving our main result.

Theorem 4.1. *Let $(M, F, d\mu)$ be an n-dimensional compact connected Finsler manifold with smooth boundary ∂M with $\mathrm{Ric}_\infty \geq C$, for $C > 0$. Then for any vector field $X \in \Gamma(TM)$ such that $\|X\|_\infty = \sup_M F(X) < \infty$ and $\inf_M \mathrm{div}(X) > 0$, we have*

$$(\Gamma_1)^{-2/p}(\lambda_1)^{2/p} \geq \Big(\frac{\epsilon^2}{\|X\|_\infty}(\inf_M \operatorname{div} X - \epsilon^2\|X\|_\infty)\Big) + C,$$

where Γ_1 *is the first eigenvalue of* (1.1) *and* λ_1 *is the first eigenvalue of the Dirichlet eigenvalue problem.*

Proof. Using Bochner formula (2.1) for $f \in C_0^\infty(M)$, we obtain

$$\begin{aligned} \int_M (\nabla F(\nabla f))^2 d\mu &\leq \int_M |\nabla^2 f|^2 d\mu \\ &\leq \Big(\int_M ((\Delta f)^2)^{p/2} d\mu\Big)^{2/p} (Vol(M))^{1-2/p} - C\int_M |\nabla f|^2 d\mu. \end{aligned} \tag{4.1}$$

Note that

$$\int_M |\nabla f|^p d\mu \leq \Big(\int_M (|\nabla f|^p)^{2/p} d\mu\Big)^{p/2} (Vol(M))^{1-p/2},$$

so

$$\int_M |\nabla f|^2 d\mu \geq \Big(\frac{1}{(Vol(M))^{1-p/2}}\int_M |\nabla f|^p d\mu\Big)^{2/p}. \tag{4.2}$$

Substituting (4.2) in (4.1) yields

$$\int_M (\nabla F(\nabla f))^2 d\mu \leq \Big(\int_M (\Delta f)^p\Big)^{2/p} (Vol(M))^{1-2/p} - C\Big(\frac{1}{(Vol(M))^{1-p/2}}\int_M |\nabla f|^p\Big)^{2/p}. \tag{4.3}$$

On the other hand, due to the (3.4) and substituting $\int_M |\nabla f|^2 d\mu$ from (4.2), and (4.3), we obtain

$$(\Gamma_1(M))^{-2/p}(\lambda_1(M))^{2/p} Vol(M)^\beta Vol(M)^\alpha - C \geq \frac{\epsilon^2}{\|X\|_\infty}(\inf_M \operatorname{div} X - \epsilon^2\|X\|_\infty).$$

Here $\beta = 1 - 2/p$ and $\alpha = (1 - p/2)(2/p)$, λ_1 is the first eigenvalue for the Laplacian. Thus we conclude

$$(\Gamma_1(M))^{-2/p}(\lambda_1(M))^{2/p} \geq \Big(\frac{\epsilon^2}{\|X\|_\infty}(\inf_M \operatorname{div} X - \epsilon^2\|X\|_\infty)\Big) + C. \qquad \square$$

REMARK 4.2. For a Finsler manifold $(M, F, d\mu)$ with compact closure and nonempty boundary, suppose that f is a first Dirichlet eigenfunction of Δ_p in M, and X be a vector field on M satisfying $\inf_M \operatorname{div}(X) > 0$. Then we have the following inequalities for the first eigenvalue of p-Laplacian from [13]:

(1) If there exist a point $x_0 \in M$ where $f(x_0) < 0$, then

$$\lambda_{1,p}(M) \geq \Big[\frac{\inf_M \operatorname{div}(X)}{p \sup_M F(X)}\Big]^p;$$

(2) If there exist a point $x_0 \in M$ where $f(x_0) > 0$, then

$$\lambda_{1,p}(M) \geq \Big[\frac{\inf_M \operatorname{div}(X)}{p \sup_M \overleftarrow{F}(X)}\Big]^p.$$

Corollary 4.3. *Let* $(m, F, d\mu)$ *be an n-dimensional compact connected Finsler manifold with compact closure and nonempty boundary* ∂M*, then for the first eigenvalue of* (1.1) *we*

have:
(1) *If there exist a point* $x_0 \in M$ *where* $f(x_0) < 0$*, then*

$$\Gamma_1(M) \geq \Big(\Big(\frac{\inf_M \operatorname{div} X}{2\|X\|_\infty}\Big)^2 + C\Big)\Big(\frac{\inf_M \operatorname{div}(X)}{p \sup_M F(X)}\Big)^p;$$

(2) *If there exist a point* $x_0 \in M$ *where* $f(x_0) > 0$*, then*

$$\Gamma_1(M) \geq \Big(\Big(\frac{\inf_M \operatorname{div} X}{2\|X\|_\infty}\Big)^2 + C\Big)\Big(\frac{\inf_M \operatorname{div}(X)}{p \sup_M \overleftarrow{F}(X)}\Big)^p.$$

5. Lichnerowicz-Obata-Cheng Type estimate

In [15], the authors proved that for a forward n-dimensional complete connected Finsler manifold $(M, F, d\mu)$ with $\mathrm{Ric}_N \geq (n-1)k$ and constant S-curvature, the first eigenvalue of Laplace operator satisfy

$$\lambda_1 \geq \frac{n-1}{N-1} Nk.$$

They also showed that if $\mathrm{Ric}_N = (n-1)k$, then $diam(M) = \sqrt{\frac{N-1}{n-1}}\frac{\pi}{\sqrt{k}}$.

In this section, considering same condition as [16], we study the first eigenvalue of biharmonic operator for both buckling and clamped plate problems. We are trying to obtain the same results as above under considering the lower bound for Ricci curvature. Here is our main results:

Theorem 5.1. *Let* $(M, F, d\mu)$ *be an n-dimensional complete Finsler manifold with* $\mathrm{Ric} \geq (n-1)k > 0$ *and constant S-curvature* $S = (n+1)cF$*. Then*
(i) *For the first eigenvalue of* (1.2)*, we have*

$$\Lambda_1(M) \geq \frac{(N-n)(n-1)Nk + (n+1)^2c^2N}{(N-n)(N-1)}.$$

(ii) *For the first eigenvalue of (1.1), we have*

$$\Gamma_1(M) \geq \Big(\frac{(N-n)(n-1)Nk + (n+1)^2c^2N}{(N-n)(N-1)}\Big)\lambda_1(M),$$

where $\lambda_1(M)$ *is the first eigenvalue of Laplacian and* $N \in (n, \infty)$*. Moreover, the diameter of M satisfies*

$$diam(M) \geq \pi\sqrt{\frac{(N-n)(n-1)N^2k + (n+1)^2c^2N^2}{(N-n)(N-1)\Gamma_1}}.$$

Proof. (i) From Bochner-Weitzenböck formula (2.2), we know

$$\Delta^{\nabla u}\Big(\frac{F(\nabla u)^2}{2}\Big) - D(\Delta u)(\nabla u) \geq F(\nabla u)^2 \mathrm{Ric}_N(\nabla u) + \frac{(\Delta u)^2}{N}. \tag{5.1}$$

Since S-curvature is constant, $\dot{S} = 0$ so

$$\mathrm{Ric}_N(\nabla u) = \mathrm{Ric}(\nabla u) + \frac{S(\nabla u)^2}{(N-n)F(\nabla u)^2}$$

$$\geq \quad (n-1)k + \frac{(n+1)^2c^2}{(N-n)}.$$

Integrating Equation (5.1) and using the divergence on M, we obtain

$$\int_M (\Delta u)^2 d\mu \geq \int_M \Big(((n-1)k + \frac{(n+1)^2c^2}{(N-n)})F(\nabla u)^2 + \frac{(\Delta u)^2}{N}\Big) d\mu.$$

Hence

$$\frac{\int_M (\Delta u)^2 d\mu}{\int_M F(\nabla u)^2 d\mu} \geq \frac{(N-n)(n-1)Nk + (n+1)^2c^2N}{(N-n)(N-1)}. \tag{5.2}$$

It follows from the definition of Λ_1 that

$$\Lambda_1(M) \geq \frac{(N-n)(n-1)Nk + (n+1)^2c^2N}{(N-n)(N-1)}.$$

(ii) Equation (5.2) gives

$$\frac{\int_M (\Delta u)^2 d\mu}{\int_M u^2 d\mu} \geq \Big(\frac{(N-n)(n-1)Nk + (n+1)^2c^2N}{(N-n)(N-1)}\Big)\frac{\int_M F(\nabla u)^2 d\mu}{\int_M u^2 d\mu}. \tag{5.3}$$

Let $f(x) := F(\nabla h)^2 + \frac{\lambda_1}{N}h^2$, then

$$\nabla u + \Sigma_{i=1}^n g_{\nabla h}(\nabla h, e_i)e_i = \Sigma_{i=1}^n h_i e_i,$$

where λ_1 is the first eigenvalue of Laplacian and h is the corresponding eigenfunction of λ_1. Thus

$$\begin{aligned} df(e_i) &= dg_{\nabla h}(\nabla h, \nabla h)(e_i) + 2\lambda_1 h dh(e_i) \\ &= 2g_{\nabla h}(\nabla^{\nabla h}_{e_i}\nabla h, \nabla h) + 2C_{\nabla h}(\nabla h, \nabla h, \nabla^{\nabla h}_{e_i}\nabla h) + 2\lambda_1 h h_i \\ &= 2H(u)(\nabla h, e_i) + 2\lambda_1 h h_i \\ &= 2h_i(h_{ii} + \lambda_1 h) = 0, \end{aligned}$$

which implies that f is constant on M. Now suppose that h takes its maximum and minimum at p and q, respectively. Then $h(p) = \frac{\lambda_1}{N}(h_{max})^2 = h(q) = \frac{\lambda_1}{N}(h_{min})^2$, i.e. $|h_{max}| = |h_{min}|$. Without loss of generality, we assume $h(p) = 1$ and $h(q) = -1$. Suppose $\gamma(s)$ is a minimum regular geodesic from p to q on (M, F) with tangent vector $\dot\gamma(s)$. Then we have

$$\frac{F(\nabla h)}{\sqrt{1-h^2}} = \sqrt{\frac{\lambda_1}{N}}.$$

Let $diam(M)$ be the diameter of M, so we get

$$\sqrt{\frac{\lambda_1}{N}}diam(M) \geq \int_\gamma F(\dot\gamma)\frac{\|\nabla u\|}{\sqrt{1-u^2}}ds \geq \pi,$$

therefore

$$diam(M) \geq \pi\sqrt{\frac{N}{\lambda_1}}.$$

Using (5.3) in the above gives the result. □

Remark 5.2. In [16], Yin and Zhang proved that with the same condition as above Theorem, the first eigenvalue of the Finsler-Laplacian satisfies

$$\lambda_1 \geq \frac{n(n-1)k + (n+1)^2 c^2 - (n+1)c}{n-1+(n+1)c},$$

and moreover the diameter of this manifold attains its maximum $\frac{\pi}{\sqrt{k}}$ and equality holds when the S-curvature vanishes.

As well in [5], Lichnerowicz proved if (M, g) be a complete connected Riemannian n-dimensional manifold with the same condition as Theorem 5.1, then the first closed eigenvalue of the Laplacian is not less than nk and Obata in [6] showed that if this eigenvalue attains its lower bound, thus the manifold is isometric to the Euclidean sphere $S^n(\frac{1}{\sqrt{k}})$.

References

[1] A. Abolarinwa, C. Yang and D. Zhang: *On the spectrum of the p-biharmonic operator under Ricci flow*, Results Math. **75** (2020), Paper No. 54, 16 pp.

[2] N. Ghoussoub and C. Yuan: *Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents*, Trans. Amer. Math. Soc. **352** (2000), 5703–5743.

[3] M.J. Habibi and S. Azami: *Lichnerowicz-type estimates for the first eigenvalue of biharmonic operator*, Complex Var. Elliptic Equ. **67** (2022), 2352–2359.

[4] A.E. Khalil, S. Kellati and A. Touzani: *On the spectrum of the p-biharmonic operator*; in Proceedings of the 2002 Fez Conference on Partial Differential Equations, Electron. J. Differ. Equ. Conf. **9**, Southwest Texas State University, San Marcos, TX, 2002, 161–170.

[5] A. Lichnerowicz: Geometrie des groupes de transformations, Travaux et Recherches Mathemtiques III, Dunod, Paris, 1958.

[6] M. Obata: *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Jpn. **14** (1962), 333–340.

[7] S. Ohta and K.T. Sturm: *Bochner-Weitzenböck formula and Li-Yau estimates on Finsler manifolds*, Adv. Math. **252** (2014), 429–448.

[8] S. Pan and L. Zhang: *The lower and upper bounds of the first eigenvalues for the bi-Laplace operator on Finsler manifolds*, Differential Geom. Appl. **47** (2016), 190–201.

[9] A. Razani and M.M. Chaharlang: *A Fourth order singular elliptic problem involving p-biharmonic operator*, Taiwanese J. Math. **23** (2019), 589–599.

[10] Z.M. Shen: Lectures on Finsler Geometry, World Scientific Publishing Co., Singapore, 2001.

[11] B-Y. Wu: *Comparison theorems in Finsler geometry with integral weighted Ricci curvature bounds and their applications*, Bull. Iranian Math. Soc. **47** (2021), 379–401.

[12] B-Y. Wu: *Comparison theorems in Riemann-Finsler geometry with line radial integral curvature bounds and related results*, J. Korean Math. Soc. **56** (2019), 421–437.

[13] S.T. Yin and Q. He: *Some eigenvalue comparison theorems of Finsler p-Laplacian*, Inernat. J. Math. **29** (2018), 1850044, 16 pp.

[14] S.T. Yin and Q. He: *The first eigenvalue of Finsler p-Laplacian*, Differential Geom. Appl. **35** (2014), 30–49.

[15] S.T. Yin, Q. He and Y.B. Shen: *On lower bounds of the first eigenvalue of Finsler Laplacian*, Publ. Math. Debrecen **83** (2013), 385–405.

[16] S.T. Yin and P. Zhang: *A Lichnerowicz-Obata-Cheng type theorems on Finsler manifolds*, Mathematics **6** (2018), 311.

[17] L. Zhang and Y. Zhao: *The lower bounds of the first eigenvalues for the biharmonic operator on manifolds*, J. Inequal. Appl. (2016), Paper No. 5, 9 pp.

Sakineh Hajiaghasi
Department of Pure Mathematics, Faculty of Science
Imam Khomeini International University
Qazvin
Iran
e-mail: s.hajiagasi@gmail.com

Shahroud Azami
Department of Pure Mathematics, Faculty of Science
Imam Khomeini International University
Qazvin
Iran
e-mail: azami@sci.ikiu.ac.ir