Title	THE BETTI NUMBERS OF REAL TORIC VARIETIES ASSOCIATED TO WEYL CHAMBERS OF TYPES E7 AND E8
Author(s)	Choi, Suyoung; Yoon, Younghan; Yu, Seonghyeon
Citation	Osaka Journal of Mathematics. 2024, 61(3), p. 409-417
Version Type	VoR
URL	https://doi.org/10.18910/97640
rights	
Note	

Choi, S., Yoon, Y. and Yu, S.
Osaka J. Math.
61 (2024), 409–417

THE BETTI NUMBERS OF REAL TORIC VARIETIES ASSOCIATED TO WEYL CHAMBERS OF TYPES E_7 AND E_8

Suyoung CHOI, Younghan YOON and Seonghyeon YU

(Received May 16, 2023, revised July 18, 2023)

Abstract

We compute the rational Betti numbers of the real toric varieties associated to Weyl chambers of types E_7 and E_8, completing the computations for all types of root systems.

1. Introduction

It is known that a root system of type R generates a non-singular complete fan Σ_R by its Weyl chambers and co-weight lattice [10], and that Σ_R corresponds to a smooth compact (complex) toric variety X_R by the fundamental theorem for toric geometry. In particular, the real locus of X_R is called *the real toric variety associated to the Weyl chambers*, denoted by $X^{\mathbb{R}}_R$.

It is natural to ask for the topological invariants of $X^{\mathbb{R}}_R$. By [6], the $\mathbb{Z}_2$-Betti numbers of $X^{\mathbb{R}}_R$ can be completely computed from the face numbers of Σ_R. In general, however, computing the rational Betti numbers of a real toric variety is much more difficult. In 2012, Henderson [8] computed the rational Betti numbers of $X^{\mathbb{R}}_{A_n}$. The computation of other classical and exceptional types has been carried out using the formulae for rational Betti numbers developed in [13] or [5]. At the time of writing this paper, results have been established for $X^{\mathbb{R}}_R$ of all types except E_7 and E_8.

For the classical types $R = A_n, B_n, C_n$, and D_n, the kth Betti numbers β_k of $X^{\mathbb{R}}_R$ are known to be as follows (see [8], [4], [3]):

$$\begin{aligned}
\beta_k(X^{\mathbb{R}}_{A_n};\mathbb{Q}) &= \binom{n+1}{2k} a_{2k},\\
\beta_k(X^{\mathbb{R}}_{B_n};\mathbb{Q}) &= \binom{n}{2k} b_{2k} + \binom{n}{2k-1} b_{2k-1},\\
\beta_k(X^{\mathbb{R}}_{C_n};\mathbb{Q}) &= \binom{n}{2k-2}\left(2^n - 2^{2k-2}\right) a_{2k-2} + \binom{n}{2k}(2b_{2k} - 2^{2k}a_{2k}), \text{ and}\\
\beta_k(X^{\mathbb{R}}_{D_n};\mathbb{Q}) &= \binom{n}{2k-4}\left(2^{2k-4} + (n-2k+2)2^{n-1}\right) a_{2k-4} + \binom{n}{2k}(2b_{2k} - 2^{2k}a_{2k}),
\end{aligned}$$

where a_r is the rth Euler zigzag number (A000111 in [11]) and b_r is the rth generalized Euler number (A001586 in [11]).

2020 Mathematics Subject Classification. 57S12, 14M25, 55U10, 57N65.

The authors were supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2019R1A2C2010989).

For the exceptional types $R = G_2, F_4$, and E_6, the Betti numbers of $X^{\mathbb{R}}_R$ are as in Table 1 (see [2, Proposition 3.3]).

Table 1. Nonzero Betti numbers of $X^{\mathbb{R}}_{G_2}$, $X^{\mathbb{R}}_{F_4}$, and $X^{\mathbb{R}}_{E_6}$

$\beta_k(X^{\mathbb{R}}_R)$	$R = G_2$	$R = F_4$	$R = E_6$
$k = 0$	1	1	1
$k = 1$	9	57	36
$k = 2$	0	264	1,323
$k = 3$	0	0	4,392

The purpose of this paper is to compute the Betti numbers for the remaining exceptional types E_7 and E_8. The reason these cases have remained unsolved to date is that, as remarked in [2], the corresponding fans are too large to be dealt with. We provide a technical method to decompose all facets of the Coxeter complex; using this method, we obtain explicit subcomplexes K_S that play an important role in our main computation. Furthermore, we obtain a smaller simplicial complex by removing vertices in K_S without changing its homology groups so that the Betti numbers can be computed.

Theorem 1.1. *The kth Betti numbers β_k of $X^{\mathbb{R}}_{E_7}$ and $X^{\mathbb{R}}_{E_8}$ are as follows.*

$$\beta_k(X^{\mathbb{R}}_{E_7};\mathbb{Q}) = \begin{cases} 1, & \text{if } k = 0 \\ 63, & \text{if } k = 1 \\ 8{,}127, & \text{if } k = 2 \\ 131{,}041, & \text{if } k = 3 \\ 122{,}976, & \text{if } k = 4 \\ 0, & \text{otherwise.} \end{cases}$$

$$\beta_k(X^{\mathbb{R}}_{E_8};\mathbb{Q}) = \begin{cases} 1, & \text{if } k = 0 \\ 120, & \text{if } k = 1 \\ 103{,}815, & \text{if } k = 2 \\ 6{,}925{,}200, & \text{if } k = 3 \\ 23{,}932{,}800, & \text{if } k = 4 \\ 0, & \text{otherwise.} \end{cases}$$

2. Real toric varieties associated to the Weyl chambers

We recall some known facts about the real toric varieties associated to the Weyl chambers, following the notation in [2] unless otherwise specified.

Let Φ_R be an irreducible root system of type R in a finite dimensional Euclidean space and W_R its Weyl group. The connected components of the complement of the reflection hyperplanes are called the *Weyl chambers*. We fix a particular Weyl chamber, called the *fundamental Weyl chamber* Ω, and *the fundamental co-weights* $\omega_1, \dots, \omega_n$ *form the set of* *its rays*. Then, $\mathbb{Z}(\{\omega_1, \dots, \omega_n\})$ has a lattice structure, called the co-weight lattice. Consider the set of Weyl chambers as a nonsingular complete fan Σ_R with the co-weight lattice. From the set $V = \{v_1, \dots, v_m\}$ of rays spanning Σ_R we obtain the simplicial complex K_R, called *the Coxeter complex* of type R on V, whose faces in K_R are obtained via the corresponding faces in Σ_R (see [1] for more details). The directions of rays on the co-weight lattice give a linear map $\lambda_R \colon V \to \mathbb{Z}^n$. In addition, the composition map $\Lambda_R \colon V \xrightarrow{\lambda_R} \mathbb{Z}^n \xrightarrow{\text{mod } 2} \mathbb{Z}_2^n$ can be expressed as an $n \times m$ (mod 2) matrix, called a (mod 2) *characteristic matrix*. Let S be an element of the row space $Row(\Lambda_R)$ of Λ_R, the vector space spanned by the row vectors of Λ_R. Since each column of Λ_R corresponds to a vertex $v \in V$, S can be regarded as a subset of V. Let us consider the induced subcomplex K_S of K_R with respect to S. It is known that the reduced Betti numbers of K_S are related to the Betti numbers of $X_R^{\mathbb{R}}$.

Theorem 2.1 ([2])**.** *For any root system Φ_R of type R, let W_R be the Weyl group of Φ_R. Then, there is a W_R-module isomorphism*

$$H_*(X_R^{\mathbb{R}}) \cong \bigoplus_{S \in Row(\Lambda_R)} \widetilde{H}_{*-1}(K_S),$$

where K_S is the induced subcomplex of K_R with respect to S.

The definition of the W_R-action on $Row(\Lambda_R)$ is explained in Lemma 3.1 in [2], and implies that

$$K_S \cong K_{gS} \text{ for } S \in Row(\Lambda_R) \text{ and } g \in W_R. \tag{2.1}$$

Combining Theorem 2.1 with (2.1), we need only investigate representatives K_S of the W_R-orbits in $Row(\Lambda_R)$.

Proposition 2.2 ([2])**.** *For type E_7, there are 127 nonzero elements in $Row(\Lambda_{E_7})$. In addition, there are exactly three orbits (whose representatives are denoted by S_1, S_2, and S_3), and the numbers of elements for each orbit are 63, 63, and 1, respectively.*

For type E_8, there are 255 nonzero elements in $Row(\Lambda_{E_8})$. There are only two orbits (whose representatives are denoted by S_4 and S_5), and the numbers of elements for each orbit are 120 and 135, respectively.

Thus, for our purpose, it is enough to compute the (reduced) Betti numbers of K_{S_i} for $1 \leq i \leq 5$. For practical reasons such as memory constraints and high time complexity, it is not easy to obtain K_S directly by computer programs. The remainder of this section is devoted to introducing an effective way to obtain K_S.

For a fixed fundamental co-weight ω, let H_ω be the isotropy subgroup of ω in W_R, and let K_ω be the subcomplex of K_R such that the set of facets of K_ω is $\{h \cdot \Omega \mid h \in H_\omega\}$, where Ω is the fundamental Weyl chamber.

Lemma 2.3. *The set of facets of K_R is decomposed as the union of the sets of all facets of $K^g = g \cdot K_\omega$ for all $g \in W_R/H_\omega$.*

Proof. For each facet $\sigma \in K_R$, there uniquely exists $g_\sigma \in W_R$ such that $g_\sigma \cdot \Omega = \sigma$ by Propositions 8.23 and 8.27 in [7]. Thus, there is exactly one $g_\sigma \cdot H_\omega \in W_R/H_\omega$ such that σ is a facet of K^{g_σ} as desired. $\square$

Obviously, the set of facets of K_S is then obtainable as the union of the sets of all facets of K_S^g for all $g \in W_R/H_\omega$.

In this paper, we fix the fundamental co-weight ω to correspond to α_1 for type E_7, and to correspond to α_8 for type E_8 in Figure 1.

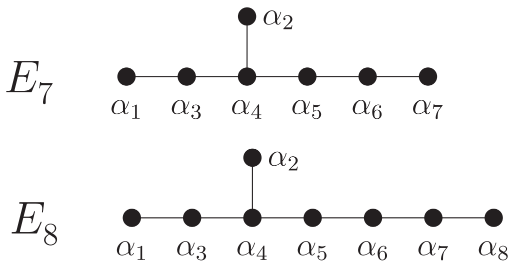

Fig. 1. The Dynkin diagrams for types E_7 and E_8

However, since K^g still has many facets, it is not easy to obtain K_S^g from K^g directly; see Table 2.

Table 2. Statistics for K_R when $R = E_7$ and E_8

	$R = E_7$	$R = E_8$
# vertices of K_R	17,642	881,760
# facets of K_R	2,903,040	696,729,600
$\lvert W_R/H_\omega \rvert$	126	240
# facets of K^g	23,040	2,903,040

Hence, we establish a lemma to improve the time complexity. Denote by V_S^g the set of vertices in K_S^g.

Lemma 2.4. *Let $g, h \in W_R/H_\omega$. If $g \cdot V_S^h = V_S^{gh}$, then $g \cdot K_S^h = K_S^{gh}$.*

Proof. For $g \in W_R/H_\omega$, we naturally consider g a simplicial isomorphism from K^h to K^{gh}. If $g \cdot V_S^h = V_S^{gh}$, then the restriction of g to K_S^h is well-defined. Thus, g is also regarded as a simplicial isomorphism between K_S^h and K_S^{gh}. □

By the above lemma, when $g \cdot V_S^h = V_S^{gh}$, K_S^{gh} is obtainable without any computation. Since checking the hypothesis of the lemma is much easier than forming K_S^g from K^g, a good deal of time can be saved. Using this method, one can obtain K_S within a reasonable time with standard computer hardware.

3. Simplicial complexes for types E_7 and E_8

Since each K_S for the types E_7 or E_8 is too large for direct computation, it is impossible to compute their Betti numbers directly using existing methods. In this section, we introduce the specific smaller simplicial complex $\widehat{K_S}$ whose homology group is isomorphic as a group to that of K_S.

Let K be a simplicial complex. The *link* $Lk_K(v)$ of v in K is a set of all faces $\sigma \in K$ such that $v \notin \sigma$ and $\{v\} \cup \sigma \in K$, while the (closed) *star* $St_K(v)$ of v in K is a set of all faces $\sigma \in K$ such that $\{v\} \cup \sigma \in K$. For a vertex v of K_S satisfying $Lk_K(v) \neq \emptyset$, we consider the following Mayer-Vietoris sequence:

$$\cdots \to \widetilde{H}_k(Lk_K(v)) \to \widetilde{H}_k(K-v) \oplus \widetilde{H}_k(St_K(v)) \to \widetilde{H}_k(K) \to \widetilde{H}_{k-1}(Lk_K(v)) \to \cdots,$$

where $K - v = \{\sigma \setminus \{v\} \mid \sigma \in K\}$ and k is a positive integer. We note that $\widetilde{H}_k(St_K(v)) = 0$ for $k \geq 0$ since $St_K(v)$ is a topological cone. Therefore, for $k \geq 0$, if $\widetilde{H}_k(Lk_K(v))$ is trivial, then $\widetilde{H}_k(K-v) \cong \widetilde{H}_k(K)$ as groups. In this case, we call v a *removable vertex* of K.

Let us consider the canonical action of the Weyl group W_R on the vertex set V_R of K_R. It is known that there are exactly n vertex orbits $V_1, \ldots, V_n$ of K_R, where n is the number of simple roots of W_R.

Theorem 3.1. *For a subcomplex L of K_R, the simplicial complex obtained by the algorithm below has the same homology group as L.*

Algorithm

```
1:  K ← L
2:  for i = 1, . . . , n do
3:      W ← ∅
4:      for each v ∈ V_i do
5:          if v is removable in K then
6:              W ← W ∪ {v}
7:          end if
8:      end for
9:      K ← K − W := {σ \ W | σ ∈ K}
10: end for
11: Return K
```

Proof. By Proposition 8.29 in [7], for each facet C of K_R, every vertex orbit of K_R contains exactly one vertex of C. That is, for any $v, w \in V_i$, v and w are not adjacent. Then, for any subcomplex K of K_R and $v, w \in V_i$, v is not contained in $Lk_K(w)$.

Note that, for removable vertices v and w of K, w is still removable in $K - v$ if w is not in the link of v in K, whereas there is no guarantee that w is removable in $K - v$ in general. Thus, we can remove all removable vertices of K in V_i from K at once without changing their homology groups. We do this procedure inductively for every vertex orbit to obtain K, and obviously, that $H_*(K) \cong H_*(L)$ as groups. □

If line 5 of the algorithm above is replaced with '**if** $Lk_K(v)$ forms a cone **then**', simplicial complex K returned in line 11 is unique up to isomorphism, regardless of any changes in the order of vertex orbits [9]. However, Theorem 3.1 is enough to compute the Betti numbers of K_{S_i} for $1 \leq i \leq 5$.

In this paper, we fix the order by size of orbit, with $|V_i| < |V_{i+1}|$. Let $\widehat{K_S}$ be the complex resulting from K_S as obtained by the algorithm in Theorem 3.1. Then, the sizes of $\widehat{K_S}$ obtained as in Table 3 are dramatically smaller than the sizes of K_S.

Table 3. Numbers of vertices of K_S and $\widehat{K}_S$

E_7	$S = S_1$	$S = S_2$	$S = S_3$
K_S	9,176	8,672	4,664
$\widehat{K}_S$	408	928	4,664

E_8	$S = S_4$	$S = S_5$
K_S	432,944	451,200
$\widehat{K}_S$	9,328	15,488

The following proposition establishes some properties of K_S and $\widehat{K}_S$.

Proposition 3.2.

(1) *K_{S_1} and K_{S_4} have two connected components; the other K_S are connected.*
(2) *For $S = S_1, S_4$, two components of K_S are isomorphic.*
(3) *All $\widehat{K}_S$ are pure simplicial complexes.*
(4) *Each component of $\widehat{K}_{S_1}$ is isomorphic to some induced subcomplex of K_{D_6}.*
(5) *Each component of $\widehat{K}_{S_4}$ is isomorphic to $\widehat{K}_{S_3}$.*

The above proposition was checked by a computer program. The Python codes used for validation are available at `https://github.com/Seonghyeon-Yu/E7-and-E8`. Note that to verify the correctness of these codes, we computed the Betti numbers for the types already known in Table 1 using the codes.

In conclusion, by Proposition 3.2, we only need to compute the Betti numbers of K_S for $S = S_2, S_3$, and S_5, since the Betti numbers of K_S of K_{D_6} are already computed in [3] for all $S \in Row(\Lambda_{D_6})$.

Remark 3.3.

(1) Each isomorphism in Proposition 3.2 (2) can be represented as one of simple roots; see Figure 1. For the type E_7, the simple root α_3 represents the isomorphism between the components of $\widehat{K}_{S_1}$; for the type E_8, the simple root α_2 represents the isomorphism between the components of $\widehat{K}_{S_4}$.
(2) Denote by $\bar{K}_S$ a connected component of $\widehat{K}_S$. The f-vectors $f(\bar{K}_S)$ of $\bar{K}_S$ as follows:

$f(\bar{K}_{S_1}) = (204, 1312, 1920)$
$f(\bar{K}_{S_2}) = (928, 6848, 15360, 11520)$
$f(\bar{K}_{S_3}) = (4664, 36288, 60480)$
$f(\bar{K}_{S_4}) = (4664, 36288, 60480)$
$f(\bar{K}_{S_5}) = (15488, 193536, 645120)$

As seen, the f-vectors of $\bar{K}_{S_3}$ and $\bar{K}_{S_4}$ are the same because of Proposition 3.2 (5). From the f-vectors, we can compute the Euler characteristic of K_S.

4. Computation of the Betti numbers

In this section, we shall use a computer program *SageMath 9.3* [12], to compute the Betti numbers of the given simplicial complexes. From Proposition 3.2, we already know the Betti numbers of $\widehat{K}_{S_1}$. For S_2 and S_3, we can compute the Betti numbers of $\widehat{K}_S$ within a reasonable time; see Table 4.

From Table 4, we can immediately conclude the following theorem.

Table 4. Nonzero reduced Betti numbers of K_S for S in $\mathrm{Row}(\Lambda_{E_7})$

$\widetilde{\beta}_k(K_S)$	$S = S_1$	$S = S_2$	$S = S_3$
$k = 0$	1	0	0
$k = 1$	0	129	0
$k = 2$	1,622	0	28,855
$k = 3$	0	1,952	0
# orbit	63	63	1

Theorem 4.1. *The kth Betti numbers β_k of $X^{\mathbb{R}}_{E_7}$ are as follows:*

$$\beta_k(X^{\mathbb{R}}_{E_7}) = \begin{cases} 1, & \text{if } k = 0 \\ 63, & \text{if } k = 1 \\ 8{,}127, & \text{if } k = 2 \\ 131{,}041, & \text{if } k = 3 \\ 122{,}976, & \text{if } k = 4 \\ 0, & \text{otherwise.} \end{cases}$$

By Proposition 3.2 and the above result, we now have the Betti numbers of $\widehat{K}_{S_4}$. For any vertex v of $\widehat{K}_{S_5}$, we can check $\widetilde{H}_0(Lk_{\widehat{K}_{S_5}}(v)) = \widetilde{H}_1(Lk_{\widehat{K}_{S_5}}(v)) = 0$ by the program. Hence, we have the Mayer-Vietoris sequence

$$0 = \widetilde{H}_1(Lk_{\widehat{K}_{S_5}}(v)) \to \widetilde{H}_1(\widehat{K}_{S_5} - v) \oplus \widetilde{H}_1(St_{\widehat{K}_{S_5}}(v)) \to \widetilde{H}_1(\widehat{K}_{S_5}) \to \widetilde{H}_0(Lk_{\widehat{K}_{S_5}}(v)) = 0.$$

Since $\widetilde{H}_1(St_{\widehat{K}_{S_5}}(v))$ is trivial, $\widetilde{H}_1(\widehat{K}_{S_5} - v)$ is isomorphic to $\widetilde{H}_1(\widehat{K}_{S_5})$. For the largest vertex orbit V of $\widehat{K}_{S_5}$, by the same proof argument as for Theorem 3.1, $\widetilde{H}_1(\widehat{K}_{S_5} - V)$ is isomorphic to $\widetilde{H}_1(\widehat{K}_{S_5})$. Note that the size of $\widehat{K}_{S_5} - V$ is much smaller than $\widehat{K}_{S_5}$. Thus, $\widetilde{\beta}_1(K_{S_5})$ can be computed within a reasonable time from $\widehat{K}_{S_5} - V$ instead of $\widehat{K}_{S_5}$. However, there is no vertex of $\widehat{K}_{S_5}$ such that $\widetilde{H}_2(Lk_{\widehat{K}_{S_5}}(v)) = 0$. Thus, for $k = 2, 3$ we must compute $\widetilde{\beta}_k(\widehat{K}_{S_5})$ directly, which takes a few days of run time. See Table 5 for the results.

Table 5. Nonzero reduced Betti numbers of K_S for S in $\mathrm{Row}(\Lambda_{E_8})$

$\widetilde{\beta}_k(K_S)$	$S = S_4$	$S = S_5$
$k = 0$	1	0
$k = 1$	0	769
$k = 2$	57,710	0
$k = 3$	0	177,280
# orbit	120	135

Table 5 implies the following theorem.

Theorem 4.2. *The kth Betti numbers β_k of $X^{\mathbb{R}}_{E_8}$ are as follows:*

$$\beta_k(X^{\mathbb{R}}_{E_8}) = \begin{cases} 1, & \text{if } k = 0 \\ 120, & \text{if } k = 1 \\ 103{,}815, & \text{if } k = 2 \\ 6{,}925{,}200, & \text{if } k = 3 \\ 23{,}932{,}800, & \text{if } k = 4 \\ 0, & \text{otherwise.} \end{cases}$$

The Euler characteristic number $\chi(X)$ of a topological space X is equal to the alternating sum of the Betti numbers $\beta_k(X)$ of X. We can use this fact as a confidence check for our results.

Remark 4.3. The $\mathbb{Z}_2$-cohomology ring of a real toric variety is completely determined by its fan [6], and then, it can be obtained that $\chi(X^{\mathbb{R}}_{E_7}) = 0$ and $\chi(X^{\mathbb{R}}_{E_8}) = 17{,}111{,}296$. Obviously, the alternating sums of the Betti numbers based on our results match $\chi(X^{\mathbb{R}}_{E_7})$ and $\chi(X^{\mathbb{R}}_{E_8})$, respectively.

References

[1] A. Björner: *Some combinatorial and algebraic properties of Coxeter complexes and Tits buildings*, Adv. in Math. **52** (1984), 173–212.

[2] S. Cho, S. Choi and S. Kaji: *Geometric representations of finite groups on real toric spaces*, J. Korean Math. Soc. **56** (2019), 1265–1283.

[3] S. Choi, S. Kaji and H. Park: *The cohomology groups of real toric varieties associated with Weyl chambers of types C and D*, Proc. Edinb. Math. Soc. (2) **62** (2019), 861–874.

[4] S. Choi, B. Park and H. Park: *The Betti numbers of real toric varieties associated to Weyl chambers of type B*, Chinese Ann. Math. Ser. B **38** (2017), 1213–1222.

[5] S. Choi and H. Park: *On the cohomology and their torsion of real toric objects*, Forum Math. **29** (2017), 543–553.

[6] M.W. Davis and T. Januszkiewicz: *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), 417–451.

[7] B. Hall: Lie groups, Lie Algebras, and Representations, Springer, Cham, 2015.

[8] A. Henderson: *Rational cohomology of the real Coxeter toric variety of type A*; in Configuration Spaces, Geometry, Combinatorics, and Topology, Pisa, 2012, 313–326.

[9] J. Matoušek: *LC reductions yield isomorphic simplicial complexes*, Contrib. Discrete Math. **3** (2008), 37–39.

[10] C. Procesi: *The toric variety associated to Weyl chambers*; in Mots, Lang. Raison. Calc., Hermès, Paris, 1990, 153–161.

[11] N.J.A. Sloane: *The on-line encyclopedia of integer sequences*, available at `http://oeis.org`.

[12] W.A. Stein et al: Sage Mathematics Software (Version 9.3), The Sage Development Team 2023, `http://www.sagemath.org`.

[13] A.I. Suciu and A. Trevisan: *Real toric varieties and abelian covers of generalized Davis-Januszkiewicz spaces*, preprint, 2012.

Suyoung Choi
Department of mathematics, Ajou University
206, World cup-ro, Yeongtong-gu
Suwon 16499
Republic of Korea
e-mail: schoi@ajou.ac.kr

Younghan Yoon
Department of mathematics, Ajou University
206, World cup-ro, Yeongtong-gu
Suwon 16499
Republic of Korea
e-mail: younghan300@ajou.ac.kr

Seonghyeon Yu
Department of mathematics, Ajou University
206, World cup-ro, Yeongtong-gu
Suwon 16499
Republic of Korea
e-mail: yoosh0319@ajou.ac.kr