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# THE BETTI NUMBERS OF REAL TORIC VARIETIES ASSOCIATED TO WEYL CHAMBERS OF TYPES $E_7$ AND $E_8$

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## Abstract

We compute the rational Betti numbers of the real toric varieties associated to Weyl chambers of types  $E_7$  and  $E_8$ , completing the computations for all types of root systems.

## 1. Introduction

It is known that a root system of type  $R$  generates a non-singular complete fan  $\Sigma_R$  by its Weyl chambers and co-weight lattice [10], and that  $\Sigma_R$  corresponds to a smooth compact (complex) toric variety  $X_R$  by the fundamental theorem for toric geometry. In particular, the real locus of  $X_R$  is called *the real toric variety associated to the Weyl chambers*, denoted by  $X_R^{\mathbb{R}}$ .

It is natural to ask for the topological invariants of  $X_R^{\mathbb{R}}$ . By [6], the  $\mathbb{Z}_2$ -Betti numbers of  $X_R^{\mathbb{R}}$  can be completely computed from the face numbers of  $\Sigma_R$ . In general, however, computing the rational Betti numbers of a real toric variety is much more difficult. In 2012, Henderson [8] computed the rational Betti numbers of  $X_{A_n}^{\mathbb{R}}$ . The computation of other classical and exceptional types has been carried out using the formulae for rational Betti numbers developed in [13] or [5]. At the time of writing this paper, results have been established for  $X_R^{\mathbb{R}}$  of all types except  $E_7$  and  $E_8$ .

For the classical types  $R = A_n, B_n, C_n$ , and  $D_n$ , the  $k$ th Betti numbers  $\beta_k$  of  $X_R^{\mathbb{R}}$  are known to be as follows (see [8], [4], [3]):

$$\begin{aligned}\beta_k(X_{A_n}^{\mathbb{R}}; \mathbb{Q}) &= \binom{n+1}{2k} a_{2k}, \\ \beta_k(X_{B_n}^{\mathbb{R}}; \mathbb{Q}) &= \binom{n}{2k} b_{2k} + \binom{n}{2k-1} b_{2k-1}, \\ \beta_k(X_{C_n}^{\mathbb{R}}; \mathbb{Q}) &= \binom{n}{2k-2} (2^n - 2^{2k-2}) a_{2k-2} + \binom{n}{2k} (2b_{2k} - 2^{2k} a_{2k}), \text{ and} \\ \beta_k(X_{D_n}^{\mathbb{R}}; \mathbb{Q}) &= \binom{n}{2k-4} (2^{2k-4} + (n-2k+2)2^{n-1}) a_{2k-4} + \binom{n}{2k} (2b_{2k} - 2^{2k} a_{2k}),\end{aligned}$$

where  $a_r$  is the  $r$ th Euler zigzag number (A000111 in [11]) and  $b_r$  is the  $r$ th generalized Euler number (A001586 in [11]).

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For the exceptional types  $R = G_2, F_4,$  and  $E_6,$  the Betti numbers of  $X_R^{\mathbb{R}}$  are as in Table 1 (see [2, Proposition 3.3]).

Table 1. Nonzero Betti numbers of  $X_{G_2}^{\mathbb{R}}, X_{F_4}^{\mathbb{R}},$  and  $X_{E_6}^{\mathbb{R}}$

$\beta_k(X_R^{\mathbb{R}})$	$R = G_2$	$R = F_4$	$R = E_6$
$k = 0$	1	1	1
$k = 1$	9	57	36
$k = 2$	0	264	1,323
$k = 3$	0	0	4,392

The purpose of this paper is to compute the Betti numbers for the remaining exceptional types  $E_7$  and  $E_8.$  The reason these cases have remained unsolved to date is that, as remarked in [2], the corresponding fans are too large to be dealt with. We provide a technical method to decompose all facets of the Coxeter complex; using this method, we obtain explicit sub-complexes  $K_S$  that play an important role in our main computation. Furthermore, we obtain a smaller simplicial complex by removing vertices in  $K_S$  without changing its homology groups so that the Betti numbers can be computed.

**Theorem 1.1.** *The  $k$ th Betti numbers  $\beta_k$  of  $X_{E_7}^{\mathbb{R}}$  and  $X_{E_8}^{\mathbb{R}}$  are as follows.*

$$\beta_k(X_{E_7}^{\mathbb{R}}; \mathbb{Q}) = \begin{cases} 1, & \text{if } k = 0 \\ 63, & \text{if } k = 1 \\ 8,127, & \text{if } k = 2 \\ 131,041, & \text{if } k = 3 \\ 122,976, & \text{if } k = 4 \\ 0, & \text{otherwise.} \end{cases}$$

$$\beta_k(X_{E_8}^{\mathbb{R}}; \mathbb{Q}) = \begin{cases} 1, & \text{if } k = 0 \\ 120, & \text{if } k = 1 \\ 103,815, & \text{if } k = 2 \\ 6,925,200, & \text{if } k = 3 \\ 23,932,800, & \text{if } k = 4 \\ 0, & \text{otherwise.} \end{cases}$$

**2. Real toric varieties associated to the Weyl chambers**

We recall some known facts about the real toric varieties associated to the Weyl chambers, following the notation in [2] unless otherwise specified.

Let  $\Phi_R$  be an irreducible root system of type  $R$  in a finite dimensional Euclidean space and  $W_R$  its Weyl group. The connected components of the complement of the reflection hyperplanes are called the *Weyl chambers*. We fix a particular Weyl chamber, called the *fundamental Weyl chamber*  $\Omega,$  and the *fundamental co-weights*  $\omega_1, \dots, \omega_n$  form the set of

its rays. Then,  $\mathbb{Z}(\{\omega_1, \dots, \omega_n\})$  has a lattice structure, called the co-weight lattice. Consider the set of Weyl chambers as a nonsingular complete fan  $\Sigma_R$  with the co-weight lattice. From the set  $V = \{v_1, \dots, v_m\}$  of rays spanning  $\Sigma_R$  we obtain the simplicial complex  $K_R$ , called the Coxeter complex of type  $R$  on  $V$ , whose faces in  $K_R$  are obtained via the corresponding faces in  $\Sigma_R$  (see [1] for more details). The directions of rays on the co-weight lattice give a linear map  $\lambda_R: V \rightarrow \mathbb{Z}^n$ . In addition, the composition map  $\Lambda_R: V \xrightarrow{\lambda_R} \mathbb{Z}^n \xrightarrow{\text{mod } 2} \mathbb{Z}_2^n$  can be expressed as an  $n \times m \pmod{2}$  matrix, called a  $(\text{mod } 2)$  characteristic matrix. Let  $S$  be an element of the row space  $\text{Row}(\Lambda_R)$  of  $\Lambda_R$ , the vector space spanned by the row vectors of  $\Lambda_R$ . Since each column of  $\Lambda_R$  corresponds to a vertex  $v \in V$ ,  $S$  can be regarded as a subset of  $V$ . Let us consider the induced subcomplex  $K_S$  of  $K_R$  with respect to  $S$ . It is known that the reduced Betti numbers of  $K_S$  are related to the Betti numbers of  $X_R^{\mathbb{R}}$ .

**Theorem 2.1** ([2]). *For any root system  $\Phi_R$  of type  $R$ , let  $W_R$  be the Weyl group of  $\Phi_R$ . Then, there is a  $W_R$ -module isomorphism*

$$H_*(X_R^{\mathbb{R}}) \cong \bigoplus_{S \in \text{Row}(\Lambda_R)} \tilde{H}_{*-1}(K_S),$$

where  $K_S$  is the induced subcomplex of  $K_R$  with respect to  $S$ .

The definition of the  $W_R$ -action on  $\text{Row}(\Lambda_R)$  is explained in Lemma 3.1 in [2], and implies that

$$(2.1) \quad K_S \cong K_{gS} \text{ for } S \in \text{Row}(\Lambda_R) \text{ and } g \in W_R.$$

Combining Theorem 2.1 with (2.1), we need only investigate representatives  $K_S$  of the  $W_R$ -orbits in  $\text{Row}(\Lambda_R)$ .

**Proposition 2.2** ([2]). *For type  $E_7$ , there are 127 nonzero elements in  $\text{Row}(\Lambda_{E_7})$ . In addition, there are exactly three orbits (whose representatives are denoted by  $S_1, S_2$ , and  $S_3$ ), and the numbers of elements for each orbit are 63, 63, and 1, respectively.*

*For type  $E_8$ , there are 255 nonzero elements in  $\text{Row}(\Lambda_{E_8})$ . There are only two orbits (whose representatives are denoted by  $S_4$  and  $S_5$ ), and the numbers of elements for each orbit are 120 and 135, respectively.*

Thus, for our purpose, it is enough to compute the (reduced) Betti numbers of  $K_{S_i}$  for  $1 \leq i \leq 5$ . For practical reasons such as memory constraints and high time complexity, it is not easy to obtain  $K_S$  directly by computer programs. The remainder of this section is devoted to introducing an effective way to obtain  $K_S$ .

For a fixed fundamental co-weight  $\omega$ , let  $H_\omega$  be the isotropy subgroup of  $\omega$  in  $W_R$ , and let  $K_\omega$  be the subcomplex of  $K_R$  such that the set of facets of  $K_\omega$  is  $\{h \cdot \Omega \mid h \in H_\omega\}$ , where  $\Omega$  is the fundamental Weyl chamber.

**Lemma 2.3.** *The set of facets of  $K_R$  is decomposed as the union of the sets of all facets of  $K^g = g \cdot K_\omega$  for all  $g \in W_R/H_\omega$ .*

*Proof.* For each facet  $\sigma \in K_R$ , there uniquely exists  $g_\sigma \in W_R$  such that  $g_\sigma \cdot \Omega = \sigma$  by Propositions 8.23 and 8.27 in [7]. Thus, there is exactly one  $g_\sigma \cdot H_\omega \in W_R/H_\omega$  such that  $\sigma$  is a facet of  $K^{g_\sigma}$  as desired.  $\square$

Obviously, the set of facets of  $K_S$  is then obtainable as the union of the sets of all facets of  $K_S^g$  for all  $g \in W_R/H_\omega$ .

In this paper, we fix the fundamental co-weight  $\omega$  to correspond to  $\alpha_1$  for type  $E_7$ , and to correspond to  $\alpha_8$  for type  $E_8$  in Figure 1.

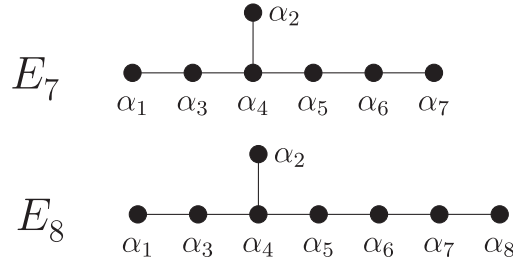


Fig. 1. The Dynkin diagrams for types  $E_7$  and  $E_8$

However, since  $K^g$  still has many facets, it is not easy to obtain  $K_S^g$  from  $K^g$  directly; see Table 2.

Table 2. Statistics for  $K_R$  when  $R = E_7$  and  $E_8$

	$R = E_7$	$R = E_8$
# vertices of $K_R$	17,642	881,760
# facets of $K_R$	2,903,040	696,729,600
$ W_R/H_\omega $	126	240
# facets of $K^g$	23,040	2,903,040

Hence, we establish a lemma to improve the time complexity. Denote by  $V_S^g$  the set of vertices in  $K_S^g$ .

**Lemma 2.4.** *Let  $g, h \in W_R/H_\omega$ . If  $g \cdot V_S^h = V_S^{gh}$ , then  $g \cdot K_S^h = K_S^{gh}$ .*

Proof. For  $g \in W_R/H_\omega$ , we naturally consider  $g$  a simplicial isomorphism from  $K^h$  to  $K^{gh}$ . If  $g \cdot V_S^h = V_S^{gh}$ , then the restriction of  $g$  to  $K_S^h$  is well-defined. Thus,  $g$  is also regarded as a simplicial isomorphism between  $K_S^h$  and  $K_S^{gh}$ .  $\square$

By the above lemma, when  $g \cdot V_S^h = V_S^{gh}$ ,  $K_S^{gh}$  is obtainable without any computation. Since checking the hypothesis of the lemma is much easier than forming  $K_S^g$  from  $K^g$ , a good deal of time can be saved. Using this method, one can obtain  $K_S$  within a reasonable time with standard computer hardware.

### 3. Simplicial complexes for types $E_7$ and $E_8$

Since each  $K_S$  for the types  $E_7$  or  $E_8$  is too large for direct computation, it is impossible to compute their Betti numbers directly using existing methods. In this section, we introduce the specific smaller simplicial complex  $\widehat{K}_S$  whose homology group is isomorphic as a group to that of  $K_S$ .

Let  $K$  be a simplicial complex. The *link*  $Lk_K(v)$  of  $v$  in  $K$  is a set of all faces  $\sigma \in K$  such that  $v \notin \sigma$  and  $\{v\} \cup \sigma \in K$ , while the (closed) *star*  $St_K(v)$  of  $v$  in  $K$  is a set of all faces  $\sigma \in K$  such that  $\{v\} \cup \sigma \in K$ . For a vertex  $v$  of  $K_S$  satisfying  $Lk_K(v) \neq \emptyset$ , we consider the following Mayer-Vietoris sequence:

$$\cdots \rightarrow \widetilde{H}_k(Lk_K(v)) \rightarrow \widetilde{H}_k(K - v) \oplus \widetilde{H}_k(St_K(v)) \rightarrow \widetilde{H}_k(K) \rightarrow \widetilde{H}_{k-1}(Lk_K(v)) \rightarrow \cdots,$$

where  $K - v = \{\sigma \setminus \{v\} \mid \sigma \in K\}$  and  $k$  is a positive integer. We note that  $\widetilde{H}_k(St_K(v)) = 0$  for  $k \geq 0$  since  $St_K(v)$  is a topological cone. Therefore, for  $k \geq 0$ , if  $\widetilde{H}_k(Lk_K(v))$  is trivial, then  $\widetilde{H}_k(K - v) \cong \widetilde{H}_k(K)$  as groups. In this case, we call  $v$  a *removable vertex* of  $K$ .

Let us consider the canonical action of the Weyl group  $W_R$  on the vertex set  $V_R$  of  $K_R$ . It is known that there are exactly  $n$  vertex orbits  $V_1, \dots, V_n$  of  $K_R$ , where  $n$  is the number of simple roots of  $W_R$ .

**Theorem 3.1.** *For a subcomplex  $L$  of  $K_R$ , the simplicial complex obtained by the algorithm below has the same homology group as  $L$ .*

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**Algorithm**

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1:  $K \leftarrow L$ 
2: for  $i = 1, \dots, n$  do
3:    $W \leftarrow \emptyset$ 
4:   for each  $v \in V_i$  do
5:     if  $v$  is removable in  $K$  then
6:        $W \leftarrow W \cup \{v\}$ 
7:     end if
8:   end for
9:    $K \leftarrow K - W := \{\sigma \setminus W \mid \sigma \in K\}$ 
10: end for
11: Return  $K$ 

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*Proof.* By Proposition 8.29 in [7], for each facet  $C$  of  $K_R$ , every vertex orbit of  $K_R$  contains exactly one vertex of  $C$ . That is, for any  $v, w \in V_i$ ,  $v$  and  $w$  are not adjacent. Then, for any subcomplex  $K$  of  $K_R$  and  $v, w \in V_i$ ,  $v$  is not contained in  $Lk_K(w)$ .

Note that, for removable vertices  $v$  and  $w$  of  $K$ ,  $w$  is still removable in  $K - v$  if  $w$  is not in the link of  $v$  in  $K$ , whereas there is no guarantee that  $w$  is removable in  $K - v$  in general. Thus, we can remove all removable vertices of  $K$  in  $V_i$  from  $K$  at once without changing their homology groups. We do this procedure inductively for every vertex orbit to obtain  $K$ , and obviously, that  $H_*(K) \cong H_*(L)$  as groups.  $\square$

If line 5 of the algorithm above is replaced with ‘**if**  $Lk_K(v)$  forms a cone **then**’, simplicial complex  $K$  returned in line 11 is unique up to isomorphism, regardless of any changes in the order of vertex orbits [9]. However, Theorem 3.1 is enough to compute the Betti numbers of  $K_{S_i}$  for  $1 \leq i \leq 5$ .

In this paper, we fix the order by size of orbit, with  $|V_i| < |V_{i+1}|$ . Let  $\widehat{K}_S$  be the complex resulting from  $K_S$  as obtained by the algorithm in Theorem 3.1. Then, the sizes of  $\widehat{K}_S$  obtained as in Table 3 are dramatically smaller than the sizes of  $K_S$ .

Table 3. Numbers of vertices of  $K_S$  and  $\widehat{K}_S$

$E_7$	$S = S_1$	$S = S_2$	$S = S_3$	$E_8$	$S = S_4$	$S = S_5$
$K_S$	9,176	8,672	4,664	$K_S$	432,944	451,200
$\widehat{K}_S$	408	928	4,664	$\widehat{K}_S$	9,328	15,488

The following proposition establishes some properties of  $K_S$  and  $\widehat{K}_S$ .

**Proposition 3.2.**

- (1)  $K_{S_1}$  and  $K_{S_4}$  have two connected components; the other  $K_S$  are connected.
- (2) For  $S = S_1, S_4$ , two components of  $K_S$  are isomorphic.
- (3) All  $\widehat{K}_S$  are pure simplicial complexes.
- (4) Each component of  $\widehat{K}_{S_1}$  is isomorphic to some induced subcomplex of  $K_{D_6}$ .
- (5) Each component of  $\widehat{K}_{S_4}$  is isomorphic to  $\widehat{K}_{S_3}$ .

The above proposition was checked by a computer program. The Python codes used for validation are available at <https://github.com/Seonghyeon-Yu/E7-and-E8>. Note that to verify the correctness of these codes, we computed the Betti numbers for the types already known in Table 1 using the codes.

In conclusion, by Proposition 3.2, we only need to compute the Betti numbers of  $K_S$  for  $S = S_2, S_3$ , and  $S_5$ , since the Betti numbers of  $K_S$  of  $K_{D_6}$  are already computed in [3] for all  $S \in Row(\Lambda_{D_6})$ .

**REMARK 3.3.**

- (1) Each isomorphism in Proposition 3.2 (2) can be represented as one of simple roots; see Figure 1. For the type  $E_7$ , the simple root  $\alpha_3$  represents the isomorphism between the components of  $\widehat{K}_{S_1}$ ; for the type  $E_8$ , the simple root  $\alpha_2$  represents the isomorphism between the components of  $\widehat{K}_{S_4}$ .
- (2) Denote by  $\bar{K}_S$  a connected component of  $\widehat{K}_S$ . The  $f$ -vectors  $f(\bar{K}_S)$  of  $\bar{K}_S$  as follows:  
 $f(\bar{K}_{S_1}) = (204, 1312, 1920)$                        $f(\bar{K}_{S_4}) = (4664, 36288, 60480)$   
 $f(\bar{K}_{S_2}) = (928, 6848, 15360, 11520)$          $f(\bar{K}_{S_5}) = (15488, 193536, 645120)$   
 $f(\bar{K}_{S_3}) = (4664, 36288, 60480)$

As seen, the  $f$ -vectors of  $\bar{K}_{S_3}$  and  $\bar{K}_{S_4}$  are the same because of Proposition 3.2 (5). From the  $f$ -vectors, we can compute the Euler characteristic of  $K_S$ .

**4. Computation of the Betti numbers**

In this section, we shall use a computer program *SageMath* 9.3 [12], to compute the Betti numbers of the given simplicial complexes. From Proposition 3.2, we already know the Betti numbers of  $\widehat{K}_{S_1}$ . For  $S_2$  and  $S_3$ , we can compute the Betti numbers of  $\widehat{K}_S$  within a reasonable time; see Table 4.

From Table 4, we can immediately conclude the following theorem.

Table 4. Nonzero reduced Betti numbers of  $K_S$  for  $S$  in  $\text{Row}(\Lambda_{E_7})$

$\widetilde{\beta}_k(K_S)$	$S = S_1$	$S = S_2$	$S = S_3$
$k = 0$	1	0	0
$k = 1$	0	129	0
$k = 2$	1,622	0	28,855
$k = 3$	0	1,952	0
# orbit	63	63	1

**Theorem 4.1.** *The  $k$ th Betti numbers  $\beta_k$  of  $X_{E_7}^{\mathbb{R}}$  are as follows:*

$$\beta_k(X_{E_7}^{\mathbb{R}}) = \begin{cases} 1, & \text{if } k = 0 \\ 63, & \text{if } k = 1 \\ 8,127, & \text{if } k = 2 \\ 131,041, & \text{if } k = 3 \\ 122,976, & \text{if } k = 4 \\ 0, & \text{otherwise.} \end{cases}$$

By Proposition 3.2 and the above result, we now have the Betti numbers of  $\widehat{K}_{S_4}$ . For any vertex  $v$  of  $\widehat{K}_{S_5}$ , we can check  $\widetilde{H}_0(Lk_{\widehat{K}_{S_5}}(v)) = \widetilde{H}_1(Lk_{\widehat{K}_{S_5}}(v)) = 0$  by the program. Hence, we have the Mayer-Vietoris sequence

$$0 = \widetilde{H}_1(Lk_{\widehat{K}_{S_5}}(v)) \rightarrow \widetilde{H}_1(\widehat{K}_{S_5} - v) \oplus \widetilde{H}_1(St_{\widehat{K}_{S_5}}(v)) \rightarrow \widetilde{H}_1(\widehat{K}_{S_5}) \rightarrow \widetilde{H}_0(Lk_{\widehat{K}_{S_5}}(v)) = 0.$$

Since  $\widetilde{H}_1(St_{\widehat{K}_{S_5}}(v))$  is trivial,  $\widetilde{H}_1(\widehat{K}_{S_5} - v)$  is isomorphic to  $\widetilde{H}_1(\widehat{K}_{S_5})$ . For the largest vertex orbit  $V$  of  $\widehat{K}_{S_5}$ , by the same proof argument as for Theorem 3.1,  $\widetilde{H}_1(\widehat{K}_{S_5} - V)$  is isomorphic to  $\widetilde{H}_1(\widehat{K}_{S_5})$ . Note that the size of  $\widehat{K}_{S_5} - V$  is much smaller than  $\widehat{K}_{S_5}$ . Thus,  $\widetilde{\beta}_1(K_{S_5})$  can be computed within a reasonable time from  $\widehat{K}_{S_5} - V$  instead of  $\widehat{K}_{S_5}$ . However, there is no vertex of  $\widehat{K}_{S_5}$  such that  $\widetilde{H}_2(Lk_{\widehat{K}_{S_5}}(v)) = 0$ . Thus, for  $k = 2, 3$  we must compute  $\widetilde{\beta}_k(\widehat{K}_{S_5})$  directly, which takes a few days of run time. See Table 5 for the results.

Table 5. Nonzero reduced Betti numbers of  $K_S$  for  $S$  in  $\text{Row}(\Lambda_{E_8})$

$\widetilde{\beta}_k(K_S)$	$S = S_4$	$S = S_5$
$k = 0$	1	0
$k = 1$	0	769
$k = 2$	57,710	0
$k = 3$	0	177,280
# orbit	120	135

Table 5 implies the following theorem.



**Theorem 4.2.** *The  $k$ th Betti numbers  $\beta_k$  of  $X_{E_8}^{\mathbb{R}}$  are as follows:*

$$\beta_k(X_{E_8}^{\mathbb{R}}) = \begin{cases} 1, & \text{if } k = 0 \\ 120, & \text{if } k = 1 \\ 103,815, & \text{if } k = 2 \\ 6,925,200, & \text{if } k = 3 \\ 23,932,800, & \text{if } k = 4 \\ 0, & \text{otherwise.} \end{cases}$$

The Euler characteristic number  $\chi(X)$  of a topological space  $X$  is equal to the alternating sum of the Betti numbers  $\beta_k(X)$  of  $X$ . We can use this fact as a confidence check for our results.

REMARK 4.3. The  $\mathbb{Z}_2$ -cohomology ring of a real toric variety is completely determined by its fan [6], and then, it can be obtained that  $\chi(X_{E_7}^{\mathbb{R}}) = 0$  and  $\chi(X_{E_8}^{\mathbb{R}}) = 17,111,296$ . Obviously, the alternating sums of the Betti numbers based on our results match  $\chi(X_{E_7}^{\mathbb{R}})$  and  $\chi(X_{E_8}^{\mathbb{R}})$ , respectively.

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