Let $R$ be a ring with identity element and $M$ be a unitary left $R$-module. $M$ is called quasi-injective if every element in $\text{Hom}_R(N, M)$ for any $R$-module $N$ in $M$ is extended to an element in $\text{Hom}_R(M, M)$. $M$ is an essential extension of $N$ if $M' \cap N \neq (0)$ for any non-zero $R$-submodule $M'$ of $M$ and we call in this case that $N$ is an essential submodule in $M$.

In Goldie [2] and Johnson, Wong [4] they have defined an $R$-submodule in $M$ for $R$-submodule $N$ as follows: $\text{cl}N = \{m \in M | (N ; m) \text{ is an essential left ideal in } R\}$. If $\text{cl}N = N$, then $N$ is called closed. We call $\text{cl}(0)$ the singular submodule of $M$.

Johnson and Wong studied structures of closed submodules of a quasi-injective $R$-module with zero singular submodule and Goldie has also considered rings with zero singular ideal in [3], [4] and [2], respectively.

In this short note we shall prove the following theorem:

Let $M$ be a quasi-injective $R$-module. Then $M$ is a direct-sum of $Z_2(M) = \text{clcl}(0)$ and any maximal submodule $M_0$ with zero singular submodule: $M = M_0 \oplus Z_2(M)$. Furthermore, every closed submodule in $M$ corresponds uniquely to a direct summand of $M_0$, which is closed in $M_0$.

From this result we know some results in [3], [4] are valid without assumption $\text{cl}(0) = (0)$.

In § 2 we shall study all types of quasi-injective modules in a case where either $R$ is a Dedekind domain or an algebra over a field with finite dimension.

We always assume that $R$ is a ring with identity and $M$ a unitary left $R$-module.

1. Closed submodules.

We shall denote the singular submodule $\text{cl}(0)$ of $M$ by $Z(M)$ and $\text{clcl}(0)$ by $Z_2(M)$ following to [2]. We also call $Z_2(M)$ the torsion submodule of $M$ and $M$ is torsion free if $Z_2(M) = (0)$. If $R$ is a commutative integral domain, then they coincide with the usual torsion submodules.
and torsion-free modules.

We note that if $M$ is an essential extension of $N$, then the left ideal $(N:_R m)$ is essential in $R$ for any element $m$ in $M$.

From [2], Lemma 2.2 we have

**Lemma 1.1.** $Z_2(M)$ is a closed submodule in $M$.

From the definition of closed module we have

**Lemma 1.2.** Every closed submodule of $M$ contains $Z_2(M)$.

**Lemma 1.3.** For submodules $N_1, N_2$ of $M$ we have

$$\text{cl}(N_1 \cap N_2) = \text{cl}N_1 \cap \text{cl}N_2.$$ 

Let $N$ be a submodule of $M$. If a submodule $B$ of $M$ is a maximal one with property $N \cap B = (0)$, then we call $B$ a complement of $N$ in $M$. We denote it by $N^c$.

**Lemma 1.4.** Let $N$ be a submodule of $M$ and $B$ a complement of $N$. Then $M$ is an essential extension of $B \oplus N$. Hence, $\text{cl}(B \oplus N) = M$.

**Proposition 1.5.** Let $M$ be a quasi-injective $R$-module. Then every closed submodule $N$ is a direct summand of $M$, namely $M = N \oplus N^c$ (cf. [4], Proposition 1.5).

Proof. Let $N$ be a closed submodule and $B$ a complement of $N$ in $M$. Put $M_0 = B \oplus N$. Let $p$ be a projection of $M_0$ to $N$. Then there exists an element $g \in \text{Hom}_R(M, M)$ such that $g | M_0 = p$. Since $g^{-1}(0) \supseteq B$ and $g^{-1}(0) \cap N = (0)$, $g^{-1}(0) = B$. Furthermore, since $\text{cl}M_0 = M$ by Lemma 1.4, there exists an essential left ideal $L$ for any element $m$ in $M$ such that $Lm \subseteq M_0$. Therefore, $Lg(m) = g(Lm) \subseteq N$. Since $\text{cl}N = N$, $g(m) \in N$. Hence, $g(M) = N$. Therefore, $M = g^{-1}(0) + g(M) = B \oplus N$.

**Corollary.** Let $M$ be a quasi-injective. If $N$ is closed, then $N$ is quasi-injective (cf. [3], Theorem 1.6).

Proof. Since it is clear that a direct summand of a quasi-injective module is quasi-injective, we have the corollary from Proposition 1.5. If we consider $R$ as a left $R$-module, we have from the definition

**Lemma 1.6.** Let $M \supseteq N$ be $R$-modules. Then 1) $Z(R)M \subseteq Z(M)$, 2) $Z_2(R)M \subseteq Z_2(M)$, 3) $Z(N) = N \cap Z(M)$ and 4) $Z_2(N) = N \cap Z_2(M)$.

**Theorem 1.7.** Let $M$ be a quasi-injective $R$-module and $M_0$ a submodule which is a maximal one with $Z(M_0) = (0)$. Then $M = M_0 \oplus Z_2(M)$. A submodule $N$ of $M$ is closed if and only if $N$ contains $Z_2(M)$ and $M_0 \cap N$ is a direct summand of $M_0$. 


Proof. From Lemma 1.6 we obtain that $M_0 \cap Z(M) = (0)$ and $M_0$ is a complement of $Z(M)$. Hence, $M = M_0 \oplus Z(M)$ by Proposition 1.5, since $Z(M)$ is closed. If $N$ is a closed submodule of $M$, then $N \supseteq Z(M)$ by Lemma 1.1 and $N = M_0 \cap N \oplus Z(M)$. Since $N \cap M_0$ is closed in $M_0$, $N \cap M_0$ is a direct summand of $M_0$ by Proposition 1.5. Conversely, we assume that $N \supseteq Z(M)$ and $N \cap M_0$ is a direct summand of $M_0$; $M_0 = N \cap M_o \oplus N_1$. Considering in $M_0$, $M_0 = \text{cl}(N \cap M_0)$ by Lemma 1.2, $N$ is closed in $M_0$. Let $x \in \text{cl}(N)$, $x = m_o + y$, where $m_o \in M_0$, $y \in Z(M)$. Since $Lx \subseteq N$ for an essential left ideal $L$, $Lm_o \subseteq N \cap M_0$. Hence, $m_o \in N \cap M_0$. Therefore, $x \in N$.

**Corollary.** Let $M$ be a quasi-injective. If $N_1$, $N_2$ are closed in $M$, then $N_1 + N_2$ is closed. Hence, $\text{cl}(N_1 + N_2) = \text{cl}(N_1) + \text{cl}(N_2)$ for any submodules $S_1$ and $S_2$ (cf. [3], Theorem 1.4 and [4], Theorem 1.2).

Proof. Since $N_1$ is closed, $N_1$ contains $Z(M)$. Hence, it is sufficient to show that $N_1 \cap M_0 + N_2 \cap M_0$ is a direct summand of $M_0$ by Theorem 1.7, where $M = M_0 \oplus Z(M)$. Thus, we may assume $Z(M) = (0)$. $N_1 \cap N_2$ is closed by Lemma 1.2. Hence, $M = N_1 \oplus (N_1 \cap N_2) \oplus M'$. Since $N_1 \cap (N_1 \cap M') = (0)$, there exists a submodule $N_1'$ such that $N_1' \supseteq N_1 \cap M'$ and $M = N_1 \oplus N_1'$. Furthermore, $N_2' = (N_1 \cap M') \oplus N_2'$. Therefore, $M = N_1 \oplus (N_1 \cap M') \oplus N_1'$. On the other hand $N_1 = (N_1 \cap N_2) \oplus N_1 \cap M'$. Hence, $N_1 + N_2 = N_2 + (N_1 \cap M')$. Therefore, $M = (N_1 + N_2) \oplus N_1'$. The second half is clear from the first.

**Proposition 1.8.** Let $M$ be quasi-injective. Then the set of closed submodules coincides with the set of complement submodules containing $Z(M)$. Especially, if we assume $Z(M) = (0)$, then every complement of a submodule $N$ is isomorphic to each other and $N^c$ containing $N$ coincides with $\text{cl}(N)$.

Proof. Let $N = N_1 \supseteq Z(M)$. For any element $n \in N_1 \cap \text{cl}(N)$ we have $Ln \subseteq N_1 \cap N = (0)$, where $L$ is an essential left ideal. Hence, $n \in Z(M) \cap N_1 = (0)$. Therefore, $\text{cl}(N) = N$. The converse is clear from Proposition 1.5. We assume $Z(M) = (0)$. In this case we note that $\text{cl}(N) = \text{cl}(N)$. By Lemmas 1.3 and 1.4 and Corollary to Theorem 1.7 we have $M = \text{cl}(N \oplus N^c) = \text{cl}(N \oplus N^c)$ for any submodule $N$. Hence, $N^c \approx M/\text{cl}(N)$. Furthermore, we obtain $M = N^c \oplus N^c \approx \text{cl}(N \oplus N^c)$ by Proposition 1.5. If $N^c \supseteq N$, then $N^c \supseteq \text{cl}(N)$. Hence, $N^c = \text{cl}(N)$.

2. Special cases.

First we consider some relations between a quasi-injective module $M$ and its injective envelope $E(M)$.
Proposition 2.1. Let $M$ be an $R$-module. Then $E(M) = E(Z_2(M)) \oplus E(B)$ and $Z_2(E(M)) = E(Z_2(M))$, where $B$ is a maximal torsion-free submodule in $M$.

Proof. We assume $Z_2(M) = (0)$ and $E=E(M)$. Then $E=E_0 \oplus Z_2(E)$ by Theorem 1.7. Let $p$ be a projection of $E$ to $E_0$. If $p(m)=0$ for $m \in M$, then $m \in M \cap Z_2(E) = Z_2(M) = (0)$ by Lemma 1.6. Hence, $M$ is monomorphic to $E_0$. Therefore, $Z_2(E) = (0)$. If $Z_2(M) \neq M$, then $M \subseteq E(M) \subseteq Z_2(E)$. Hence, $Z_2(E) = E$. Since $M$ is an essential extension of $B \oplus Z_2(M)$, $E=E(B) \oplus E(Z_2(M))$.

Lemma 2.2. Let $M$ be an $R$-module and $K=\text{Hom}_R(E(M), E(M))$. $M$ is quasi-injective if and only if $M$ is a $K$-module. (See [3], Theorem 1.1.)

Proposition 2.3. Let $M$ be quasi-injective. If $E(M) = N_1 \oplus N_2$, then $M = M \cap N_1 \oplus M \cap N_2$, and $N_i = E(M \cap N_i)$.

Proof. Let $p$ be a projection of $E(M)$ to $N$. Since $p \in K$, $p(M) \subseteq M$ by Lemma 2.2. Hence, $M = M \cap N_1 \oplus M \cap N_2$.

Corollary. Let $R$ be a commutative integral domain. Then every injective module is a direct sum of the torsion submodule and a maximal torsion-free submodule. An injective envelope of torsion (resp. torsion-free) module is torsion (resp. torsion-free).

Proposition 2.4. Let $M_1, M_2$ be quasi-injective such that $E(M_1) \cong E(M_2)$. Then $M_1 \oplus M_2$ is quasi-injective if and only if $M_1 \cong M_2$.

Proof. $E(M_1 \oplus M_2) = E(M_1) \oplus E(M_2)$. If $M_1 \approx M_2$, $M = M_1 \oplus M_2$ is a $\text{Hom}_R(E(M), E(M))$-module, and hence $M$ is quasi-injective by Lemma 2.2. Conversely, we assume that $M$ is quasi-injective. Let $f$ be an element in $K=\text{Hom}_R(E(M), E(M))$ such that $f|E(M_2) \equiv 0$, $f|E(M_1)$ induces the isomorphism $\varphi$. Then $f(M) = f(M_1) \subseteq M \cap E(M_2) = M_2$ by Proposition 2.3. If we consider the same argument on $M_2$ for $\varphi^{-1}$, we can find $g \in K$ such that $gf|E(M_1) \equiv \text{identity}|E(M_1)$. Hence, $M_1 = gf(M_1) \subseteq E(M_2) \subseteq M_1$. Therefore, $M_1 \approx M_2$.

Proposition 2.5. Let $R$ be a left noetherian ring and $M$ a quasi-injective $R$-module. Then $M$ is a direct-sum of indecomposable quasi-injective $R$-modules. Furthermore, this decomposition is unique up to isomorphism, ([10], Theorem 4.5).

Proof. $E(M) = \Sigma \oplus M_\alpha$ by [6], Theorem 2.5, where $M_\alpha$ is an indecomposable injective $R$-module. Put $M_\alpha = M \cap \overline{M}_\alpha$. Then $M = \Sigma \oplus M_\alpha$ by Lemma 2.2. Since $M_\alpha$ is a direct summand of $M$, $M_\alpha$ is quasi-injective. It is clear that $M_\alpha$ is indecomposable. We assume $M = \Sigma \oplus M_\alpha'$.
is a second decomposition. Since $R$ is left noetherian, $E(M) = \sum \oplus E(M_\alpha)$ and $E(M_\alpha)$ is indecomposable by Proposition 2.3. Then there exists an automorphism $\varphi$ in $K$ such that $\varphi : \tilde{M}_\alpha \approx E(M_{\rho(\alpha)}/\alpha)$ for all $\alpha$ by [5], Proposition 2.7, where $\rho$ is a permutation of indices $\alpha$. Since $\varphi(M) \subseteq M$, $\varphi(M_\alpha) = \varphi(M \cap \tilde{M}_\alpha) \subseteq M \cap E(M_{\rho(\alpha)}) = M_{\rho(\alpha)}$. Taking $\varphi^{-1}$, we obtain $M_\alpha \approx M_{\rho(\alpha)}$.

Now we assume that $R$ is a commutative noetherian ring. Then we know by [6], Proposition 3.1 that every indecomposable injective $R$-module is isomorphic to $E(R/P)$, where $P$ is a prime ideal in $R$.

**Proposition 2.6.** Let $M$ be an indecomposable quasi-injective $R$-module. If $M$ is torsion-free, then $M$ is injective.

Proof. Since $E(M) = E(R/P)$ is torsion-free by Proposition 2.1, $Z(R/P) = (0)$. Hence, $P$ is not essential in $R$. There exists a non-zero ideal $Q$ such that $P \cap Q = (0)$. Therefore, $P$ is minimal prime and $(0)_P = PR_P$. From [6], Theorem 3.6 $E(R/P)$ is $R_P$-injective. Since $R_P$ is the quotient field $K$ of $R/P$, $E(M) = Kn$. Furthermore, $K \cong Hom_R(E(R/P), E(R/P))$. Hence, $M = KM = E(M)$.

**Corollary.** Let $R$ be a Dedekind domain and $M$ a quasi-injective $R$-module. Then $M$ is either injective or a direct-sum of $R$-modules $E(R/P)$ and $R/S_j$. Conversely, such a module is quasi-injective, where $\{P_i, S_j\}$ is a set of non-zero distinct primes in $R$.

Proof. $M = M_0 \oplus Z(M)$. If $M_0 \cong 0$, then $M_0 \cong \sum \oplus Q$ by Propositions 2.5 and 2.6, where $Q$ is the quotient field of $R$. Since $Z(M)$ is torsion, $Z(M) = \sum \oplus (E(R/P_i))^{*} \oplus \sum (R/(S_j))^{y_j}$ by Proposition 2.5 and [5], Theorem 10. However if $M_0 \neq 0$ then there exist natural epimorphisms of $M_0$ to $E(R/S_j)$. Hence, $f = \phi$ by Lemma 2.2, which means $M$ is injective. If $M_0 = 0$, then $M = \sum \oplus E(R/P_i)^{*} \oplus \sum (R/(S_j))^{y_j}$ and $\{P_i, S_j\}$ is a set of non-zero distinct primes by Proposition 2.4. The converse is clear.

Next, we consider a case of algebra $A$ over a field $K$ with finite dimension. Then we know from [7] that every indecomposable $A$-injective module $M$ is isomorphic to $(eA)^* = Hom_K(eA, K)$, where $e$ is a primitive idempotent in $A$. Hence, there exists a non-degenerated bilinear mapping $(,)$ of $eA \otimes M$ to $K$ with respect to $A$. It is clear that the adjoint elements of $Hom_A(M, M) = B$ is equal to $eAe$. Hence for an $A$-submodule $N$ of $M$, ann $N = \{x \in eA, (x, N) = 0\}$ is an $eAe$-module if and only if $N$ is a $B$-module. Thus from Lemma 2.2 we have

1) $M^*$ means a directsum of $\alpha$-copies of $M$. 

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Proposition 2.7. Let $A$ be an algebra over a field $K$ such that $[A:K]<\infty$. Then every quasi-injective $A$-module is a direct-sum of modules $(e_iA/e_iR_i)^*$, where $e_i$ is a primitive idempotents and $R_i$ is a right ideal in $A$ such that $e_iAe_iR_i\subseteq R_i$.

Corollary. Let $A$ be as above. If $A$ is a generalized uniserial ring, then every sub-module of indecomposable injective module is quasi-injective.

Remark. The converse of Proposition 2.7 is, in general, not true. Finally, we consider the singular ideal of quasi-Frobenius ring.

Proposition 2.8. Let $R$ be quasi-Frobenius. Then $Z(R)$ is equal to the radical $N$ of $R$ and $R$ is a direct sum of semi-simple subring and a quasi-Frobenius ring $R_1$ such that $Z_2(R)=R_1$.

Proof. Let $S$ be a left socle of $R$, namely the sum of minimal left ideals in $R$. Then $S$ is a unique minimal essential left ideal of $R$. Hence, $Z(R)$ is the right annihilator $S_r$ of $S$ in $R$. Since $N_r=Z_r$ by [8] and $S=N_r$, $Z(R)=S_r=N_r=N_r$ by [8]. Furthermore, $R$ is left $R$-injective by [1]. Hence, $R=L\oplus Z(R)$ as a left $R$-module by Theorem 1.7. Since $Z_2(R)$ is a direct-sum of non-nilpotent minimal left ideals. Since $SN=(0)$, $S'=L'$. It is clear that $Z_2(R)=(S')$. Therefore, $L'Z_2(R)=(0)$, which means that $L$ is a two-sided ideal. Since $L$ is completely reducible, $L$ is semi-simple.

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