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<th>Note on quasi-injective modules</th>
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Let $R$ be a ring with identity element and $M$ be a unitary left $R$-module. $M$ is called quasi-injective if every element in $\text{Hom}_R(N, M)$ for any $R$-module $N$ in $M$ is extended to an element in $\text{Hom}_R(M, M)$. $M$ is an essential extension of $N$ if $M' \cap N \neq (0)$ for any non-zero $R$-submodule $M'$ of $M$ and we call in this case that $N$ is an essential submodule in $M$.

In Goldie [2] and Johnson, Wong [4] they have defined an $R$-submodule in $M$ for $R$-submodule $N$ as follows: $\text{cl}N = \{ m \in M \mid (N; m) \text{ is an essential left ideal in } R \}$. If $\text{cl}N = N$, then $N$ is called closed. We call $\text{cl}(0)$ the singular submodule of $M$.

Johnson and Wong studied structures of closed submodules of a quasi-injective $R$-module with zero singular submodule and Goldie has also considered rings with zero singular ideal in [3], [4] and [2], respectively.

In this short note we shall prove the following theorem:

Let $M$ be a quasi-injective $R$-module. Then $M$ is a direct-sum of $Z_2(M) = \text{clcl}(0)$ and any maximal submodule $M_0$ with zero singular submodule: $M = M_0 \oplus Z_2(M)$. Furthermore, every closed submodule in $M$ corresponds uniquely to a direct summand of $M_0$, which is closed in $M_0$.

From this result we know some results in [3], [4] are valid without assumption $\text{cl}(0) = (0)$.

In § 2 we shall study all types of quasi-injective modules in a case where either $R$ is a Dedekind domain or an algebra over a field with finite dimension.

We always assume that $R$ is a ring with identity and $M$ a unitary left $R$-module.

1. Closed submodules.

We shall denote the singular submodule $\text{cl}(0)$ of $M$ by $Z(M)$ and $\text{clcl}(0)$ by $Z_2(M)$ following to [2]. We also call $Z_2(M)$ the torsion submodule of $M$ and $M$ is torsion free if $Z_2(M) = (0)$. If $R$ is a commutative integral domain, then they coincide with the usual torsion submodules.
and torsion-free modules.

We note that if $M$ is an essential extension of $N$, then the left ideal $(N:m)$ is essential in $R$ for any element $m$ in $M$.

From [2], Lemma 2.2 we have

**Lemma 1.1.** $Z_2(M)$ is a closed submodule in $M$.

From the definition of closed module we have

**Lemma 1.2.** Every closed submodule of $M$ contains $Z_2(M)$.

**Lemma 1.3.** For submodules $N_1$, $N_2$ of $M$ we have

$$\text{cl}(N_1 \cap N_2) = \text{cl}N_1 \cap \text{cl}N_2.$$  

Let $N$ be a submodule of $M$. If a submodule $B$ of $M$ is a maximal one with property $N \cap B = (0)$, then we call $B$ a complement of $N$ in $M$. We denote it by $N^\perp$.

**Lemma 1.4.** Let $N$ be a submodule of $M$ and $B$ a complement of $N$. Then $M$ is an essential extension of $B\oplus N$. Hence, $\text{cl}(B\oplus N) = M$.

**Proposition 1.5.** Let $M$ be a quasi-injective $R$-module. Then every closed submodule $N$ is a direct summand of $M$, namely $M = N \oplus N^\perp$ (cf. [4], Proposition 1.5).

Proof. Let $N$ be a closed submodule and $B$ a complement of $N$ in $M$. Put $M_0 = B \oplus N$. Let $p$ be a projection of $M_0$ to $N$. Then there exists an element $g \in \text{Hom}_R(M, M)$ such that $g|_{M_0} = p$. Since $g^{-1}(0) \subseteq B$ and $g^{-1}(0) \cap N = (0)$, $g^{-1}(0) = B$. Furthermore, since $\text{cl}M_0 = M$ by Lemma 1.4, there exists an essential left ideal $L$ for any element $m$ in $M$ such that $Lm \subseteq M_0$. Therefore, $Lg(m) = g(Lm) \subseteq N$. Since $\text{cl}N = N$, $g(m) \in N$. Hence, $g(M) = N$. Therefore, $M = g^{-1}(0) + g(M) = B \oplus N$.

**Corollary.** Let $M$ be a quasi-injective. If $N$ is closed, then $N$ is quasi-injective (cf. [3], Theorem 1.6).

Proof. Since it is clear that a direct summand of a quasi-injective module is quasi-injective, we have the corollary from Proposition 1.5.

If we consider $R$ as a left $R$-module, we have from the definition

**Lemma 1.6.** Let $M \supseteq N$ be $R$-modules. Then 1) $Z(R)M \subseteq Z(M)$, 2) $Z_2(R)M \subseteq Z_2(M)$, 3) $Z(N) = N \cap Z(M)$ and 4) $Z_2(N) = N \cap Z_2(M)$.

**Theorem 1.7.** Let $M$ be a quasi-injective $R$-module and $M_0$ a submodule which is a maximal one with $Z(M_0) = (0)$. Then $M = M_0 \oplus Z_2(M)$. A submodule $N$ of $M$ is closed if and only if $N$ contains $Z_2(M)$ and $M_0 \cap N$ is a direct summand of $M_0$. 
Proof. From Lemma 1.6 we obtain that \( M_0 \cap Z_2(M) = (0) \) and \( M_0 \) is a complement of \( Z_2(M) \). Hence, \( M = M_0 \oplus Z_2(M) \) by Proposition 1.5, since \( Z_2(M) \) is closed. If \( N \) is a closed submodule of \( M \), then \( N \supseteq Z_2(M) \) by Lemma 1.1 and \( N = M_0 \cap N \oplus Z_2(M) \). Since \( N \cap M_0 \) is closed in \( M_0 \), \( N \cap M_0 \) is a direct summand of \( M_0 \) by Proposition 1.5. Conversely, we assume that \( N \supseteq Z_2(M) \) and \( N \cap M_0 \) is a direct summand of \( M_0 \); \( M_0 = N \cap M_0 \oplus N_1 \). Considering in \( M_0 \), \( M_0 = \text{cl}M_0 = \text{cl}(N \cap M_0) + \text{cl}N_1 \). Since \( \text{cl}(N \cap M_0) \cap \text{cl}N_1 = \text{cl}(N \cap M_0 \cap N_1) = \text{cl}(0) = (0) \) by Lemma 1.2, \( N \) is closed in \( M_0 \). Let \( x \in \text{cl}N : x = m_0 + y \), where \( m_0 \in M_0, y \in Z_2(M) \). Since \( Lx \subseteq N \) for an essential left ideal \( L \), \( Lm_0 \subseteq N \cap M_0 \). Hence, \( m_0 \in N \cap M_0 \). Therefore, \( x \in N \).

**Corollary.** Let \( M \) be a quasi-injective. If \( N_1, N_2 \) are closed in \( M \), then \( N_1 + N_2 \) is closed. Hence, \( \text{cl}(S_1 + S_2) = \text{cl}(S_1) + \text{cl}(S_2) \) for any submodules \( S_1 \) and \( S_2 \) (cf. [3], Theorem 1.4 and [4], Theorem 1.2).

Proof. Since \( N_i \), it is sufficient to show that \( N_i \cap M_0 \) is a direct summand of \( M_0 \) by Theorem 1.7, where \( M = M_0 \oplus Z_2(M) \). Thus, we may assume \( Z(M) = (0) \). \( N_i \) is closed by Lemma 1.2. Hence, \( M = N_i \oplus N'_i = (N_i \cap M_0) \oplus M' \). Since \( N_i \cap (N_i \cap M') = (0) \), there exists a submodule \( N'_i \) such that \( N'_i \supseteq N_i \cap M' \) and \( M = N_i \oplus N'_i \). Furthermore, \( N'_2 = (N_i \cap M') \oplus N'_i \). Therefore, \( M = N_2 \oplus (N_i \cap M') \oplus N'_i \). On the other hand \( N_i = (N_i \cap N_2) \oplus N_i \cap M' \). Hence, \( N_1 + N_2 = N_i + (N_i \cap M') \). Therefore, \( M = (N_1 + N_2) \oplus N'_i \). The second half is clear from the first.

**Proposition 1.8.** Let \( M \) be quasi-injective. Then the set of closed submodules coincides with the set of complement submodules containing \( Z(M) \). Especially, if we assume \( Z(M) = (0) \), then every complement of a submodule \( N \) is isomorphic to each other and \( N^c \) containing \( N \) coincides with \( \text{cl}N \).

Proof. Let \( N = N_i \supseteq Z(M) \). For any element \( n \in N_i \cap \text{cl}N \) we have \( L_n \cap N = (0) \), where \( L \) is an essential left ideal. Hence, \( n \in Z(M) \cap N_i = (0) \). Therefore, \( \text{cl}N = N \). The converse is clear from Proposition 1.5. We assume \( Z(M) = (0) \). In this case we note that \( \text{cl}N = \text{cl}N \). By Lemmas 1.3 and 1.4 and Corollary to Theorem 1.7 we have \( M = \text{cl}(N \oplus N^c) = \text{cl}N \oplus N^c \) for any submodule \( N \). Hence, \( N^c \approx M \text{cl}N \). Furthermore, we obtain \( M = N^c \oplus N^c = \text{cl}(N \oplus N^c) \) by Proposition 1.5. If \( N^c \supseteq N \), then \( N^c \supseteq \text{cl}N \). Hence, \( N^c = \text{cl}N \).

2. Special cases.

First we consider some relations between a quasi-injective module \( M \) and its injective envelope \( E(M) \).
Proposition 2.1. Let $M$ be an $R$-module. Then $E(M) = E(Z_2(M)) \oplus E(B)$ and $Z_2(E(M)) = E(Z_2(M))$, where $B$ is a maximal torsion-free submodule in $M$.

Proof. We assume $Z_2(M) = (0)$ and $E = E(M)$. Then $E = E_0 \oplus Z_2(E)$ by Theorem 1.7. Let $p$ be a projection of $E$ to $E_0$. If $p(m) = 0$ for $m \in M$, then $m \in M \cap Z_2(E) = Z_2(M) = (0)$ by Lemma 1.6. Hence, $M$ is monomorphic to $E_0$. Therefore, $Z_2(E) = (0)$. If $Z_2(M) \neq M$, then $M \subset E(M) \subseteq Z_2(E)$. Hence, $Z_2(E) = E$. Since $M$ is an essential extension of $B \oplus Z_2(M)$, $E = E(B) \oplus E(Z_2(M))$.

Lemma 2.2. Let $M$ be an $R$-module and $K = \text{Hom}_R(E(M), E(M))$. $M$ is quasi-injective if and only if $M$ is a $K$-module. (See [3], Theorem 1.1.)

Proposition 2.3. Let $M$ be quasi-injective. If $E(M) = N_1 \oplus N_2$, then $M = M \cap N_1 \oplus M \cap N_2$, and $N_i = E(M \cap N_i)$.

Proof. Let $p$ be a projection of $E(M)$ to $N$. Since $p \in K$, $p(M) \subseteq M$ by Lemma 2.2. Hence, $M = M \cap N_1 \oplus M \cap N_2$.

Corollary. Let $R$ be a commutative integral domain. Then every injective module is a direct sum of the torsion submodule and a maximal torsion-free submodule. An injective envelope of torsion (resp. torsion-free) module is torsion (resp. torsion-free).

Proposition 2.4. Let $M_1, M_2$ be quasi-injective such that $E(M_1) \cong E(M_2)$. Then $M_1 \oplus M_2$ is quasi-injective if and only if $M_1 \cong M_2$.

Proof. $E(M_1 \oplus M_2) = E(M_1) \oplus E(M_2)$. If $M_1 \cong M_2$, $M = M_1 \oplus M_2$ is a $\text{Hom}_R(E(M), E(M))$-module, and hence $M$ is quasi-injective by Lemma 2.2. Conversely, we assume that $M$ is quasi-injective. Let $f$ be an element in $K = \text{Hom}_R(E(M), E(M))$ such that $f|E(M_2) \equiv 0$, $f|E(M_1)$ induces the isomorphism $\varphi$. Then $f(M) = f(M_1) \subseteq M \cap E(M_2) = M_2$ by Proposition 2.3. If we consider the same argument on $M_2$ for $\varphi^{-1}$, we can find $g \in K$ such that $gf|E(M_1) \equiv \text{identity}|E(M_1)$. Hence, $M_1 = gf(M_1) \subseteq M_2 \subseteq M_1$. Therefore, $M_1 \cong M_2$.

Proposition 2.5. Let $R$ be a left noetherian ring and $M$ a quasi-injective $R$-module. Then $M$ is a direct-sum of indecomposable quasi-injective $R$-modules. Furthermore, this decomposition is unique up to isomorphism, ([10], Theorem 4.5).

Proof. $E(M) = \sum \oplus M_\alpha$ by [6], Theorem 2.5, where $M_\alpha$ is an indecomposable injective $R$-module. Put $M_\alpha = M \cap M_\alpha$. Then $M = \sum \oplus M_\alpha$ by Lemma 2.2. Since $M_\alpha$ is a direct summand of $M$, $M_\alpha$ is quasi-injective. It is clear that $M_\alpha$ is indecomposable. We assume $M = \sum \oplus M_\alpha'$.
is a second decomposition. Since $R$ is left noetherian, $E(M) = \bigoplus E(M_{\alpha})$ and $E(M_{\alpha})$ is indecomposable by Proposition 2.3. Then there exists an automorphism $\varphi$ in $K$ such that $\varphi: \tilde{M}_{\alpha} \approx E(M_{\rho_{\alpha}})$ for all $\alpha$ by [5], Proposition 2.7, where $\rho$ is a permutation of indices $\alpha$. Since $\varphi(M) \subseteq M$, $\varphi(M) \cap \tilde{M}_{\alpha} \subseteq M \cap E(M_{\rho_{\alpha}}) = M_{\rho_{\alpha}}$. Taking $\varphi^{-1}$, we obtain $M_{\alpha} \approx M_{\rho_{\alpha}}$.

Now we assume that $R$ is a commutative noetherian ring. Then we know by [6], Proposition 3.1 that every indecomposable injective $R$-module is isomorphic to $E(R/P)$, where $P$ is a prime ideal in $R$.

**Proposition 2.6.** Let $M$ be an indecomposable quasi-injective $R$-module. If $M$ is torsion-free, then $M$ is injective.

Proof. Since $E(M) = E(R/P)$ is torsion-free by Proposition 2.1, $Z(R/P) = (0)$. Hence, $P$ is not essential in $R$. There exists a non-zero ideal $Q$ such that $P \cap Q = (0)$. Therefore, $P$ is minimal prime and $(0)_{P} = PR_{P}$. From [6], Theorem 3.6 $E(R/P)$ is $R_{P}$-injective. Since $R_{P}$ is the quotient field of $R/P$, $E(M) = Km$. Furthermore, $K \subseteq \text{Hom}_{R}(E(R/P), E(R/P))$. Hence, $M = KM = E(M)$.

**Corollary.** Let $R$ be a Dedekind domain and $M$ a quasi-injective $R$-module. Then $M$ is either injective or a direct-sum of $R$-modules $E(R/P_{i})$ and $R/S_{j}$. Conversely, such a module is quasi-injective, where $\{P_{i}, S_{j}\}$ is a set of non-zero distinct primes in $R$.

Proof. $M = M_{0} \oplus Z_{A}(M)$. If $M_{0} \neq (0)$, then $M_{0} \approx \bigoplus Q$ by Propositions 2.5 and 2.6, where $Q$ is the quotient field of $R$. Since $Z_{A}(M)$ is torsion, $Z_{A}(M) = \bigoplus \oplus (E(R/P_{i}))^{*} \oplus \bigoplus (R/(S_{j}))^{*}$. Theorem 10. However if $M_{0} = (0)$ then there exist natural epimorphisms of $M_{0}$ to $E(R/S_{j})$. Hence, $J = \phi$ by Lemma 2.2, which means $M$ is injective. If $M_{0} = (0)$, then $M = \bigoplus \oplus E(R/P_{i})^{*} \oplus \bigoplus (R/(S_{j}))^{*}$. The set of non-zero distinct primes by Proposition 2.4. The converse is clear.

Next, we consider a case of algebra $A$ over a field $K$ with finite dimension. Then we know from [7] that every indecomposable $A$-injective module $M$ is isomorphic to $(eA)^{*} = \text{Hom}_{K}(eA, K)$, where $e$ is a primitive idempotent in $A$. Hence, there exists a non-degenerated bilinear mapping $(,)$ of $eA \otimes M$ to $K$ with respect to $A$. It is clear that the adjoint elements of $\text{Hom}_{A}(M, M) = B$ is equal to $eAe$. Hence for an $A$-submodule $N$ of $M$, ann $N = \{x | (x, N) = 0\}$ is an $eAe$-module if and only if $N$ is a $B$-module. Thus from Lemma 2.2 we have

1) $M^{*}$ means a direcsum of $\alpha$-copies of $M$. 


Proposition 2.7. Let $A$ be an algebra over a field $K$ such that $[A : K] < \infty$. Then every quasi-injective $A$-module is a direct-sum of modules $(e_iA/e_iR_i)^*$, where $e_i$ is a primitive idempotents and $R_i$ is a right ideal in $A$ such that $e_iAe_i \subseteq R_i$.

Corollary. Let $A$ be as above. If $A$ is a generalized uniserial ring, then every sub-module of indecomposable injective module is quasi-injective.

Remark. The converse of Proposition 2.7 is, in general, not true. Finally, we consider the singular ideal of quasi-Frobenius ring.

Proposition 2.8. Let $R$ be quasi-Frobenius. Then $Z(R)$ is equal to the radical $N$ of $R$ and $R$ is a direct sum of semi-simple subring and a quasi-Frobenius ring $R_1$ such that $Z_2(R_1) = R_1$.

Proof. Let $S$ be a left socle of $R$, namely the sum of minimal left ideals in $R$. Then $S$ is a unique minimal essential left ideal of $R$. Hence, $Z(R) = S$ of $S$ in $R$. Since $N_1 = N_2$ by [8] and $S = N_1 = N_2 = N_1 = N_2$ by [8]. Furthermore, $R$ is left $R$-injective by [1]. Hence, $R = L \oplus Z_2(R)$ as a left $R$-module by Theorem 1.7. Since $Z_2(R) \supseteq N$, $NL \subseteq N_1 \subseteq L = (0)$. Hence, $L \subseteq S$. $S = S \cap N \oplus L'$, where $L' \supseteq L$ is a direct-sum of non-nilpotent minimal left ideals. Since $SN = (0)$, $S' = L'$. It is clear that $Z_2(R) = (S')_r$. Therefore, $L'Z_2(R) = (0)$, which means that $L$ is a two-sided ideal. Since $L$ is completely reducible, $L$ is semi-simple.

References