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NON-SOLVABLE GROUPS, WHOSE CHARACTER DEGREES ARE PRODUCTS OF AT MOST TWO PRIME NUMBERS*)

BERTRAM HUPPERT AND OLAF MANZ

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1. Introduction

If $n \in N$ has the prime-number-decomposition $n = \prod_{i=1}^k p_i^{a_i}$, we define $\omega(n) = \sum_{i=1}^k a_i$. If $\text{Irr}(G)$ is furthermore the set of irreducible complex characters of the finite group G , we define $\omega(G) = \max_{\chi \in \text{Irr}(G)} \omega(\chi(1))$.

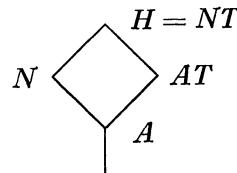
Suppose first that $\omega(G) = 1$, which means that all non-linear characters have prime-number-degrees. By a theorem of M. Isaacs and D. Passman (cf. Isaacs [6], 14.4), G must be solvable. But this conclusion does not hold, if $\omega(G) = 2$; for example $cd(A_5) = \{1, 3, 4, 5\}$ and $cd(A_7) = \{1, 6, 10, 14, 15, 21, 35\}$ (cf. McKay [8]; cd = character degrees).

There seem to be many solvable groups G with $\omega(G) = 2$. In a later paper we shall consider these; in particular we shall show that they have derived length at most 4.**)

The class of non-solvable groups G with $\omega(G) = 2$ is quite small. It is completely described by the following theorem.

Theorem. *Suppose that G is non-solvable. Then $\omega(G) = 2$ if and only if G is a direct product of an abelian group with a group H of the following type:*

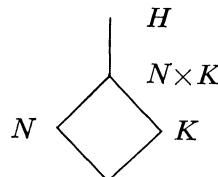
- (1) $H \cong A_7$.
- (2) $H \cong A_5$.
- (3) $H = NT$, where N is a normal abelian 2-subgroup of H , $T \cong A_5$, $N = N_0 \times A$, where A is the natural module for $SL(2, 4) \cong A_5$ and $[N, T] \leq A$.



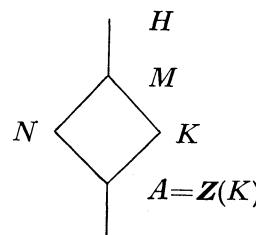
*) This paper is a contribution to the research project "Darstellungstheorie" of the DFG.

**) Arch. Math. 46 (1986), 387-392.

(4) H has a normal subgroup $M=N \times K$ of index 2, where $K \cong A_5$ and N is an abelian 2-group. Further $H/N \cong S_5$ and H/K is abelian.



(5) H is a central product of $SL(2, 5)$ with an abelian 2-group.
 (6) H has a normal subgroup M of index 2. M is a central product of $K \cong SL(2, 5)$ with an abelian 2-group N , where $N \leq Z(G)$ and H/K is abelian.



2. Proof of the Theorem

We start considering a simple group G , here we use the classification of all finite simple groups.

Lemma 1. *If G is simple, non-abelian with $\omega(G)=2$, then*

$$G \cong A_5 \quad \text{or} \quad G \cong A_7.$$

Proof. (1) By [10], G isn't sporadic.

(2) Let $G \cong A_n$ ($n \geq 5$). We consider the Young-tableau, corresponding to the partition $(n-3, 1, 1, 1)$.

n	$n-4$	\dots	2	1
3				
2				
1				

As the hook-product $H_{(n-3,1,1,1)}$ is $n \cdot (n-4)! \cdot 3 \cdot 2$, the character $\chi_{(n-3,1,1,1)} \in \text{Irr}(S_n)$ has degree

$$(n!)/H_{(n-3,1,1,1)} = (n-1)(n-2)(n-3)/6 \quad (\text{cf. Müller [9] 6.36}).$$

But for $n \neq 7$ the partition $(n-3, 1, 1, 1)$ isn't self-associated and we obtain $(n-3)(n-2)(n-1)/6 \in \text{cd}(A_n)$. The hypothesis $\omega(G)=2$ yields that one of $n-1$, $n-2$ and $n-3$ divides 6 and consequently $n \leq 9$. As $8 \in \text{cd}(A_6)$, $20 \in \text{cd}(A_8)$ and $8 \in \text{cd}(A_9)$ (cf. McKay [8]), we conclude $G \cong A_5$ or $G \cong A_7$.

(3) Suppose now that G is a Chevalley-group or a twisted type in characteristic p , say. Then G has the Steinberg-character σ of degree $\sigma(1)=p^a$, where p^a is the exact p -part of $|G|$. An inspection of those Chevalley-type groups, whose orders have p -parts at most p^2 , yields $G \cong \text{PSL}(2, p)$ or $G \cong \text{PSL}(2, p^2)$. We set $q=p$ or $q=p^2$. Now $\text{PSL}(2, q)$ has character degrees $q-1$ and $q+1$ and as $\omega(G)=2$ and $q > 3$, there exists a prime r such that $q-1=2r$. Therefore $q+1=2(r+1)$ and $r+1$ must also be a prime number, hence $r=2$. This yields $q=5$ and $G \cong \text{PSL}(2, 5) \cong A_5$.

The composition structure of a group G , satisfying $\omega(G)=2$, is not too complicated, namely

Lemma 2. *If G is a non-solvable group with $\omega(G)=2$, then G has a solvable normal subgroup N such that*

$$G/N \cong A_7 \text{ or } A_5 \text{ or } S_5.$$

Proof. Let M be the solvable residue of G and M/L a chief-factor of G . By lemma 1, $M/L \cong A_5$ or A_7 . We define $N/L = C_{G/L}(M/L)$. As $\text{Aut}(A_5) \cong S_5$ and $\text{Aut}(A_7) \cong S_7$, $G/N \cong A_5$, S_5 , A_7 or S_7 . But S_7 can be ruled out, because $20 \in \text{cd}(S_7)$ (cf. Kerber-James [7] page 350). Suppose that N is non-solvable, then we obtain a chief-factor $S/R \cong A_5$ or A_7 with $S \leq N$. If $C/R = C_{G/R}(S/R)$, we have $C/R \times S/R \trianglelefteq G/R$, where C/R involves a composition factor A_5 or A_7 . This, however, yields $\omega(C/R \times S/R) > 2$, a contradiction.

As lemma 2 indicates, there are two cases to consider, namely the case, where A_7 is involved and where A_5 is involved. We start with the first situation, which turns out to be the simplest one. We remind the reader that by Huppert-Manz [5] the group A_7 has the following subgroups $U < A_7$ with $\omega(|A_7: U|) \leq 2$:

type	index
$\text{PSL}(2, 7)$	$3 \cdot 5$
A_6	7
S_5	$3 \cdot 7$
$(A_4 \times Z_3) \cdot Z_2$	$5 \cdot 7$

Notice that A_7 has no subgroup of index 3^2 (cf. Huppert-Blackburn [4], XII

10.12).

Lemma 3. *Let $\omega(G)=2$, $N \trianglelefteq G$, N solvable and $G/N \cong A_7$. Then $G=N \times E$, where $N'=1$ and $E \cong A_7$.*

Proof. By a trivial induction argument, we may suppose that N is an irreducible G -module.

(1) We may assume that N is faithful, because the Schur-extensions of A_7 by Z_2 and Z_3 have character degrees 20 and 24, respectively (cf. Humphreys [2]).

(2) Let $1 \neq \lambda \in \text{Irr}(N)$. Then $\mathbf{T}(\lambda)/N$ must be one of those subgroups, listed above (Huppert [3] 17.11).

(3) Suppose that $\mathbf{T}(\lambda)/N \cong \text{PSL}(2, 7)$, S_5 or $(A_4 \times Z_3) \cdot Z_2$. We consider $\lambda^{T(\lambda)} = \sum_i e_i \chi_i$, $\chi_i \in \text{Irr}(\mathbf{T}(\lambda))$. Then $\chi_i^G \in \text{Irr}(G)$, $\chi_i(1) = e_i$ and $\chi_i^G(1) = e_i \cdot 3 \cdot 5$ or $e_i \cdot 3 \cdot 7$ or $e_i \cdot 5 \cdot 7$. As $\omega(G)=2$, we conclude that $e_i=1$ and χ_i is an extension of λ . By a theorem of Gallagher (cf. Isaacs [6] 6.17), we have $\{\chi_i\} = \{\chi_i \varphi \mid \varphi \in \text{Irr}(\mathbf{T}(\lambda)/N)\}$. But this contradicts the fact that in each case $\mathbf{T}(\lambda)/N$ has non-linear characters.

(4) It remains to investigate the situation, where all $\mathbf{T}(\lambda)/N \cong A_6$. As the subgroups of type A_6 are conjugate under the action of A_7 , we can define $p^s = |\mathbf{C}_{\text{Irr}(N)}(U)|$, where $U \cong A_6$ and $|N|=p^n$. A double counting yields

$$7 \cdot (p^s - 1) = |\{(\lambda, U) \mid 1 \neq \lambda \in \text{Irr}(N), U \cong A_6, U = \mathbf{T}(\lambda)/N\}| = p^n - 1$$

and consequently $7 = 1 + p^s + \dots + p^{(n/s-1)s}$. This, however, yields a faithful A_7 -module of type (2,2,2), a contradiction.

It remains to deal with the case that A_5 is involved. For this purpose we list the subgroups $U < A_5$ with $\omega(|A_5: U|) \leq 2$:

type	index
A_4	5
D_{10}	2·3
D_6	2·5
$Z_2 \times Z_2$	3·5

Lemma 4. *Let $\omega(G)=2$, $M \trianglelefteq G$ and M an irreducible non-trivial module for G/M of type (p, \dots, p) . Furthermore let $G/M \cong A_5$ or $G/M \cong \text{SL}(2, 5)$, which means there is a central subgroup L/M of G/M of order at most 2. Then we have $p=2$ and $n=4$.*

Proof. (1) Of course $\omega(|G: \mathbf{T}(\lambda)|) \leq 2$ for all $1 \neq \lambda \in \text{Irr}(M)$. In particular $2 \mid |\mathbf{T}(\lambda)/M|$, if $G/M \cong \text{SL}(2, 5)$. As $\text{SL}(2, 5)$ has only one involution, we have $L \leq \mathbf{T}(\lambda)$.

(2) $\mathbf{T}(\lambda)/L \cong D_{10}$ and $\cong D_6$:

If not, λ would be extendible to $\hat{\lambda} \in \text{Irr}(\mathbf{T}(\lambda))$, because the Sylow-subgroups of $\mathbf{T}(\lambda)/M$ are cyclic (Isaacs [6] 11.31). This, however, means $\lambda^{T(\lambda)} = \hat{\lambda}\varphi + \dots$ with $\varphi \in \text{Irr}(\mathbf{T}(\lambda)/M)$, $\varphi(1) = 2$ and $(\hat{\lambda}\varphi)^c(1) = |G : \mathbf{T}(\lambda)| \cdot 2$, in both cases a contradiction.

(3) By (2), $\mathbf{T}(\lambda)/L$ contains just one subgroup of type $Z_2 \times Z_2$, hence $\mathbf{T}(\lambda)/M$ is one Sylow-subgroup of G/M . Therefore

$$\begin{aligned} |\mathrm{Syl}_2(G/M)| \cdot (|\mathcal{C}_{\mathrm{Irr}(M)}(Q)| - 1) &= \\ |\{(\lambda, Q) \mid 1 \neq \lambda \in \mathrm{Irr}(M), Q \in \mathrm{Syl}_2(G/M), Q \leq \mathcal{T}(\lambda)/M\}| &= \\ |\mathrm{Irr}(M)| - 1 &= p^n - 1. \end{aligned}$$

If we put $p^s = |\mathcal{C}_{\text{Irr}(M)}(Q)|$, we obtain $5 \cdot (p^s - 1) = p^n - 1$. This yields $s \mid n$, $5 = 1 + p^s + \dots + p^{(n/s-1)s}$ and consequently $p = 2$, $s = 2$ and $n = 4$.

The lemma above handles the case that A_5 acts on an irreducible module. We suppose now that A_5 acts on an arbitrary solvable group.

Lemma 5. *Let $\omega(G)=2$, $G=G'$ and $N \trianglelefteq G$ with $G/N \cong A_5$. Then there is an abelian 2-group A , such that $A \leq N$, $A \trianglelefteq G$ and either $N=A$ or $G/A \cong SL(2, 5)$.*

Proof. a) We first show that N is a 2-group. Put $L = O^2(N)$ and suppose $L \neq 1$. We choose a chief factor L/M ; then L/M is of type (p, \dots, p) for an odd prime p . We can assume further on that $M = 1$.

$$\begin{array}{c} G \\ \hline N \\ \hline L \end{array} \left. \begin{array}{c} \\ \\ \end{array} \right\} \cong A_5 \quad 2 \cdots \left. \begin{array}{c} \\ \\ \end{array} \right\} (p, \underset{n}{\cdots} p), p \neq 2.$$

(1) As $\omega(N) \leq 2$, we have $\text{cd}(N) \subseteq \{1, 2, 4\}$.

(2) $\text{cd}(N) \subseteq \{1, 2\}$: Suppose there is $\tau \in \text{Irr}(N)$ with $\tau(1)=4$. Then τ is fixed under the action of G and consequently $\tau^G = \sum_i e_i \chi_i$, where $\chi_i \in \text{Irr}(G)$ and $(\chi_i)_N = e_i \tau$. Now $\omega(G)=2$ forces $e_i=1$ and τ is extendible to G . By Gallagher's theorem, $\tau(1) \cdot d \in \text{cd}(G)$ for all $d \in \text{cd}(A_5)$, a contradiction.

(3) $\mathrm{cd}(N)=\{1, 2\}$: Suppose $N'=1$, which means $N=S \times L$ with $S \in \mathrm{Syl}_2(N)$. If we consider G/S , lemma 4 implies the trivial action of $G/N \cong A_5$ on L , a contradiction to $G=G'$.

(4) By Isaacs [6] 12.11, we have one of the following assertions:

(i) N has a characteristic abelian subgroup U of index 2. As $G=G'$, we have $G/U \cong \mathrm{SL}(2, 5)$ and we obtain the same contradiction as in (3), using lemma 4.

(ii) $|N/\mathbf{Z}(N)|=2^2$ or 2^3 . But as A_5 has no irreducible GF(2)-module of dimension 2 or 3, $N/\mathbf{Z}(N)$ is central in $G/\mathbf{Z}(N)$, a contradiction to $G=G'$, because the Schur-multiplier of A_5 has order 2.

Altogether we have shown that N is a 2-group.

b) It remains to show that N has a characteristic abelian subgroup A of index at most 2.

(1) $\text{cd}(N) \subseteq \{1, 2\}$: Use the arguments of a) (1) and (2).

(2) By Isaacs [6] 12.11, we have to rule out $|N/\mathbf{Z}(N)|=2^2$ or 2^3 . But this is done as in a) (4) (ii).

For our later arguments we need the knowledge of those extensions G of A_5 by an irreducible GF(2)-module which have $\omega(G)=2$.

EXAMPLE 6. The group A_5 has three irreducible modules over GF(2), namely the trivial module M_0 , the augmented permutation module M_1 and the module M_2 , belonging to the representation $A_5 \cong \text{SL}(2, 4)$. Let $M \trianglelefteq G$, $G/M \cong A_5$ and M an irreducible GF(2)- A_5 -module.

a) $M \cong M_0$: If the extension is non-splitting, we have $G \cong \text{SL}(2, 5)$. As $\text{cd}(\text{SL}(2, 5)) = \{1, 2, 3, 4, 5, 6\}$ (cf. Dornhoff [1], page 228), $\omega(G)=2$ holds.

b) $M \cong M_1$: In this case we have $M \cong \left\{ \sum_{i=1}^5 k_i v_i \mid k_i \in \text{GF}(2), \sum_{i=1}^5 k_i = 0 \right\}$, where $\bigoplus_{i=1}^5 \text{GF}(2)v_i$ is the permutation module for A_5 . Obviously, the stabilizer of $v_1 + v_2$ in A_5 is isomorphic to S_3 . As $M \cong M_1$ is self-dual, we have $M \cong \text{Irr}(M)$ and therefore there is $\lambda \in \text{Irr}(M)$, such that $\mathbf{T}(\lambda)/M \cong S_3$. As in lemma 4 (2), λ is extendible to $\mathbf{T}(\lambda)$ and we obtain $2 \cdot |G: \mathbf{T}(\lambda)| = 2^2 \cdot 5 \in \text{cd}(G)$. This shows $\omega(G) > 2$.

c) $M \cong M_2$: Now $\text{SL}(2, 4)$ acts transitively on $M \setminus 1$. Therefore $|G: \mathbf{T}(\lambda)| = 15$ for all $1 \neq \lambda \in \text{Irr}(M)$, because $M \cong M_2$ is self-dual. Remark that $\mathbf{T}(\lambda)/M$ is a Kleinian four-group. By Prince [11], G must split over M ; hence the linear characters λ are extendible to $\widehat{\lambda} \in \text{Irr}(\mathbf{T}(\lambda))$. So we have

$$\lambda^{T(\lambda)} = \sum_{\alpha \in \text{Irr}(\mathbf{T}(\lambda)/M)} \alpha(1) \alpha \widehat{\lambda}, \text{ where } (\alpha \widehat{\lambda})^G \in \text{Irr}(G) \text{ has degree } (\alpha \widehat{\lambda})(1) \cdot |G: \mathbf{T}(\lambda)| = 15.$$

From this we conclude $\text{cd}(G) = \{1, 3, 4, 5, 15\}$, hence $\omega(G)=2$.

The notation M_0, M_1, M_2 for the irreducible modules of A_5 over GF(2) will be used from now on.

Lemma 7. *Suppose $\omega(G)=2$, $G=G'$ and $G/A \cong \text{SL}(2, 5)$. Then $A=1$.*

Proof. By lemma 5, A is an abelian 2-group. Suppose $A \neq 1$. Let A/B be a chief-factor of G . We can assume $B=1$. As $\text{SL}(2, 5)$ has trivial Schur-multiplier, $A \neq M_0$. Define $N/A = \mathbf{Z}(G/A)$, then N centralizes A , hence N is abelian. Moreover A , considered as an G/N -module, is one of the modules M_1 and M_2 .

(1) $A \cong M_1$. A Sylow-5-subgroup S of A_5 has fixed points on N/A , but not on $A \cong M_1$. Hence N is elementary abelian. As M_1 is of defect 0, $N \cong A \oplus C$, where $C \cong M_0$ as A_5 -modules. But by example 6 b), we have the contradiction $\omega(G/C) > 2$.

(2) Now we assume $A \cong M_2$. We proceed as in example 6 c). Again, A_5 operates transitively on $A \setminus 1$ and also on $\text{Irr}(A) \setminus 1$. If $1 \neq \lambda \in \text{Irr}(A)$, then $|G : \mathbf{T}(\lambda)| = 15$. As before $\lambda^{T(\lambda)} = \sum_i e_i \chi_i$, $\chi_i(1) = e_i$, $\chi_i^G \in \text{Irr}(G)$ and $\chi_i^G(1) = 15 \cdot e_i$. As $\omega(G) = 2$, all $e_i = 1$. Hence χ_1 is an extension of λ to $\mathbf{T}(\lambda)$ and $\lambda^{T(\lambda)} = \chi_1 \sum_{\varphi \in \text{Irr}(\mathbf{T}(\lambda)/A)} \varphi(1) \varphi$. But as $\mathbf{T}(\lambda)/A$ now is a quaternion group of order 8, it has an irreducible character φ of degree 2. Then $(\chi_1 \varphi)^G$ is irreducible and has degree 30.

Lemma 8. *Let $\omega(G) = 2$, $G = G'$ and $G/N \cong A_5$. If $N \neq 1$, then either $G \cong \text{SL}(2,5)$ or G is the splitting extension of A_5 with M_2 .*

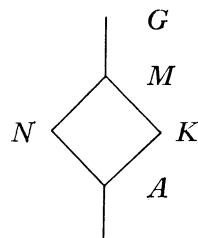
Proof. If G has a factor group isomorphic $\text{SL}(2,5)$, then by lemma 7 $G \cong \text{SL}(2,5)$. Hence by lemma 5, we have $G/A \cong A_5$, A an abelian 2-group, and every chief factor A/B is not isomorphic to M_0 . By example 6 b), also $A/B \not\cong M_1$, so $A/B \cong M_2$. To prove $B = 1$ we can assume that B is an irreducible A_5 -module.

We claim $\mathbf{T}(\lambda) < G$ for all $1 \neq \lambda \in \text{Irr}(A)$: This follows from the fact that $\text{Irr}(A)$ has no submodule isomorphic to M_0 , because A has no factor module isomorphic to M_0 .

Let $1 \neq \lambda \in \text{Irr}(A)$. As $\omega(|G : \mathbf{T}(\lambda)|) \leq 2$, the arguments of lemma 4 show that $\mathbf{T}(\lambda)/A$ contains exactly one Sylow-2-subgroup of G/A . Again, a double counting yields

$$\begin{aligned} & 5 \cdot (|\mathbf{C}_{\text{Irr}(A)}(Q)| - 1) = \\ & |\{(\lambda, Q) \mid 1 \neq \lambda \in \text{Irr}(A), Q \in \text{Syl}_2(G/A), Q \leq \mathbf{T}(\lambda)/A\}| = \\ & |\text{Irr}(A)| - 1. \text{ We put } 2^s = |\mathbf{C}_{\text{Irr}(A)}(Q)|. \text{ Then} \\ & 5 \cdot (2^s - 1) = \begin{cases} 2^8 - 1 & \text{if } B \not\cong M_0 \\ 2^5 - 1 & \text{if } B \cong M_0 \end{cases}, \text{ a contradiction.} \end{aligned}$$

Lemma 9. *Let $\omega(G) = 2$ and suppose that A_5 is involved. Then G has the following normal series:*



where

- (i) $K/A \cong A_5$, $|G/M| \leq 2$ and $G/N \cong S_5$ in case of $|G/M| = 2$.
- (ii) $(N/A)' = 1$.
- (iii) $A = 1$; or $A \cong M_0$ and $K \cong SL(2,5)$; or $A \cong M_2$ and K splits over A .

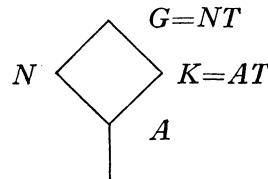
Proof. Let K be the solvable residue of G . If K/A is a chief-factor, then $K/A \cong A_5$ (by lemma 2). We put $N/A = C_{G/A}(K/A)$. Then $M/A := N/A \times K/A$ is a normal subgroup of G/A of index at most 2, where in case of $|G/M| = 2$ we have $G/N \cong S_5$. As $\omega(M/A) = 2$, we conclude that N/A is abelian. The application of lemma 8 to K finally yields (iii).

We shall use the notation of lemma 9.

Lemma 10. *The case $|G/M| = 2$ and $A \cong M_2$ does not occur.*

Proof. As $K/A \cong A_5$ acts transitively on M_2 , we have $|G: \mathbf{T}(\lambda)| = 15$ for all $1 \neq \lambda \in \text{Irr}(A)$; in particular $\mathbf{T}(\lambda)/A$ contains a Sylow-2-subgroup of G/A . As the Sylow-2-subgroups of S_5 are isomorphic to D_8 , $\mathbf{T}(\lambda)/A$ is non-abelian. Let $\lambda^{T(\lambda)} = \sum_i e_i \chi_i$ with $\chi_i \in \text{Irr}(\mathbf{T}(\lambda))$. Then $\chi_i^G \in \text{Irr}(G)$, $\chi_i^G(1) = 15 \cdot \chi_i(1)$; hence $\omega(G) = 2$ does imply $\chi_i(1) = 1$. Therefore χ_i extends λ and $\{\chi_i\} = \{\chi_i \varphi \mid \varphi \in \text{Irr } \mathbf{T}(\lambda)/A\}$. As $\mathbf{T}(\lambda)/A$ is non-abelian, we obtain a contradiction to $\chi_i(1) = 1$.

Lemma 11. *Now we assume that $\omega(G) = 2$ and that the solvable residue K of G is the splitting extension of $A \cong M_2$ by A_5 . We also assume that $G = M$.*



Then N is abelian and $N = N_1 \times A$. Neglecting abelian direct factors of G , we can assume that N is a 2-group.

Proof. a) Let T be a complement of A in K . Then obviously T also is a complement of N in G . Certainly, $\text{Hom}_T(A, A) \cong \text{GF}(4)$. Hence $N/C_N(A)$ has order 1 or 3. We assume at first that $|N/C_N(A)| = 3$. Suppose $S/A \in \text{Syl}_3(N/A)$ and $|S/A| = 3^m$. We consider the normal subgroup $R = SK = ST$ of G . Obviously $\omega(R) = 2$. As T operates transitively on the characters ($\neq 1$) of A , we have $|R: \mathbf{T}_R(\lambda)| = 15$ for every $1 \neq \lambda \in \text{Irr}(A)$. $\mathbf{T}_R(\lambda)$ splits over A , for the Sylow-2-subgroup of $\mathbf{T}_R(\lambda)$ does so (Gaschütz's theorem). Hence there exists an extension $\hat{\lambda}$ of λ to $\mathbf{T}_R(\lambda)$, and we obtain $\lambda^R = (\sum_j \hat{\lambda} \psi_j(1) \psi_j)^R$ ($\psi_j \in \text{Irr}(\mathbf{T}_R(\lambda)/A)$). Also $(\hat{\lambda} \psi_j)^R \in \text{Irr}(R)$. As $\omega(R) = 2$ and $(\hat{\lambda} \psi_j)^R(1) = |R: \mathbf{T}_R(\lambda)| \psi_j(1)$

$=15\psi_j(1)$, this forces $\psi_j(1)=1$. Hence $\overline{\mathbf{T}_R(\lambda)/A}$ is abelian. We write $\bar{H}=HA/A$ for any $H\leq R$. Then $|\overline{\mathbf{T}_R(\lambda)}\cap\bar{K}|=|\overline{\mathbf{T}_K(\lambda)}|=4$ and thus $|\overline{\mathbf{T}_R(\lambda)}\cdot\bar{K}|=|\overline{\mathbf{T}_R(\lambda)}|\cdot|\bar{K}|/|\overline{\mathbf{T}_R(\lambda)}\cap\bar{K}|=3^m\cdot60=|\bar{R}|$. This shows $\overline{\mathbf{T}_R(\lambda)}\not\cong\overline{\mathbf{C}_R(A)}\cdot\bar{K}<\bar{R}$. Hence there exists an element $\bar{s}\cdot\bar{k}\in\overline{\mathbf{T}_R(\lambda)}$, where $\bar{s}\in\bar{S}\setminus\overline{\mathbf{C}_R(A)}$, $\bar{k}\in\bar{K}$ and the order of $\bar{s}\cdot\bar{k}$ a power of 3. As $S/\mathbf{C}_S(A)$ operates fixedpointfreely on A (namely by multiplication with an element $\neq 1$ of $\text{GF}(4)^\times$), \bar{s} does not stabilize any character $\neq 1$ of A . Hence $\bar{k}\neq 1$, so $\bar{k}^3=1$ and \bar{k} does not centralize any Sylow-2-subgroup of \bar{K} . As $\overline{\mathbf{T}_R(\lambda)}$ contains $\bar{s}\cdot\bar{k}$ and a Sylow-2-subgroup of \bar{K} , $\overline{\mathbf{T}_R(\lambda)}$ is not abelian, a contradiction. This shows finally $\mathbf{C}_N(A)=N$.

b) We can assume that N is an abelian 2-group and $N=N_1\times A$ for some subgroup N_1 of N :

As $N'\leq A\leq Z(N)$ (a) and lemma 9), N is nilpotent. Neglecting abelian direct factors of G , we hence can assume that N is a 2-group. As $N\trianglelefteq G$, we have $N'\trianglelefteq G$ and $N'\leq A$. But $N'=A$ is impossible, for then a 5-element of T would operate trivially on N/N' , but non-trivially on N' . By the same argument, $A\not\cong\Phi(N)$, hence $A\cap\Phi(N)=1$. This implies $N=N_1\times A$ for some N_1 .

Now we show that all the groups described in lemma 11 have indeed $\omega(G)=2$.

Lemma 12. *Suppose G has the structure described in lemma 11, namely*

$$\begin{array}{ccc} & G=NT & \\ N & \diagdown \quad \diagup & K=AT \\ & A & \end{array}$$

with an abelian 2-group N . Then G has the character degrees 1, 3, 4, 5, 15, so $\omega(G)=2$.

Proof. a) Let S be a Sylow-2-subgroup of T . We can assume that S operates on $A\cong\text{GF}(4)^{(2)}$ by matrices of the form

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \quad (k\in\text{GF}(4)).$$

This shows that for any $1\neq s\in S$

$$\mathbf{C}_A(S)=\mathbf{C}_A(s)=\langle a^{s^{-1}} \mid a\in A \rangle.$$

If $f\in Z^1(T, A)$, then $f(s)\cdot f(s)^s=f(s^2)=f(1)=1$.

Hence $f(s)\in\mathbf{C}_A(s)=\mathbf{C}_A(S)$.

b) Let $\lambda\in\text{Irr}(N)$. As $N=N_1\times A$, we can write $\lambda=\gamma\alpha$, where $\gamma\in\text{Irr}(N_1)$ and $\alpha\in\text{Irr}(A)$. As $\mathbf{T}(\lambda)$ splits over N , there exists an extension $\hat{\lambda}$ of λ to $\mathbf{T}(\lambda)$ and

$$\lambda^G = \left(\sum_j \hat{\lambda} \psi_j(1) \psi_j \right)^G \quad (\psi_j \in \text{Irr}(\mathbf{T}(\lambda)/N)).$$

If $\alpha=1$, then $\mathbf{T}(\lambda)=G$ and we obtain only irreducible characters $\hat{\lambda} \psi_j$ of G of degrees 1, 3, 4, 5.

Suppose $\alpha \neq 1$. Then $|\mathbf{T}(\alpha)/N|=4$. We show that for any $s \in \mathbf{T}(\alpha)$ also $\lambda^s = \lambda$: Obviously, for $a \in A$

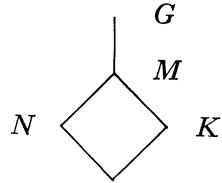
$$\lambda^s(a) = \lambda(a^s) = \alpha^s(a) = \alpha(a) = \lambda(a).$$

If $n \in N_1$ and $n^s = n \cdot f(s)$ with $f(s) \in A$, then $\lambda^s(n) = \lambda(n^s) = \gamma(n) \alpha(f(s))$. But $\alpha^s = \alpha$ implies by a) that $f(s) \in C_A(S) = \langle a^{s-1} \mid a \in A \rangle = \text{Ker } \alpha$. So $\lambda^s(n) = \gamma(n) = \lambda(n)$. This shows $|\mathbf{T}(\lambda)/N|=4$. Then $(\hat{\lambda} \psi_j)^G(1) = |G : \mathbf{T}(\lambda)| = 15$.

The lemmas 10–12 complete our proof for the case $A \cong M_2$. There remain by lemma 9 the cases $A=1$ or $A \cong M_0$ (=trivial module).

Lemma 13. *We suppose the conditions of lemma 9 with $A=1$ or $A \cong M_0$. Then G/K is abelian and $N \leq Z(G)$.*

Proof. a) We show that G/K is abelian: As N/A is abelian (lemma 9), we may assume that $A=1$ and $|G/M|=2$.



Let $1 \neq \lambda \in \text{Irr}(N)$ and $\chi \in \text{Irr}(K)$, such that $\chi(1)=4$. If $\mathbf{T}(\lambda)=M$, then $\mathbf{T}(\lambda\chi)=M$, so $(\lambda\chi)^G \in \text{Irr}(G)$ and $(\lambda\chi)^G(1)=8$, a contradiction. Hence G fixes all characters of N , so also the elements of N . Therefore $N \leq Z(G)$ and so G/K is abelian. From now on we may also assume $A \cong M_0$, hence $K \cong \text{SL}(2,5)$.

b) We show at first that $[N, K]=1$: If $x \in K$ and $n \in N$, then $n^x \cdot n^{-1} \in A$. Hence the automorphism induced by x on N has at most order 2. As $K \cong \text{SL}(2,5)$ has no non-trivial 2-factor-group, K centralizes N . Hence $M=NK$ is an epimorphic image of $N \times K$.

c) Suppose now that N is non-abelian, hence $N'=A$. As $K \cong \text{SL}(2,5)$, K has an irreducible character χ with $\chi(1)=4$ and $\chi(a)=-4$ for $1 \neq a \in A$. On the other hand, N has an irreducible character φ with $\varphi(1)>1$ and $\varphi(a)=-\varphi(1)$, because $A=N' \leq Z(N)$. Now $\varphi\chi \in \text{Irr}(N \times K)$ and $\varphi\chi((a, a))=\varphi\chi(1)$. Hence the kernel $\langle(a, a)\rangle$ of the epimorphism of $N \times K$ onto NK lies in the kernel of $\varphi\chi$. So $\varphi\chi \in \text{Irr}(NK)$ and $\varphi\chi(1)=4 \cdot \varphi(1)$. This contradicts $\omega(NK) \leq \omega(G)=2$.

d) Suppose $N \not\leq Z(G)$, hence $|G/M|=2$. Let $G=M\langle t \rangle$. There exists an $\lambda \in \text{Irr}(N)$ such that $\lambda^t \neq \lambda$ and then $A \not\leq \text{Ker } \lambda$. Also, $K \cong \text{SL}(2,5)$ has an irreducible character χ with $\chi(1)=4$, $A \not\leq \text{Ker } \chi$. Hence $\lambda\chi$ is a character of

$NK=M$. As t permutes at most the two classes of elements of K of order 5 resp. 10 and χ takes on these classes the values 1, 1 resp. $-1, -1$, so $\chi^t=\chi$ (cf. Dornhoff [1] page 228). Also $(\lambda\chi)^t=\lambda^t\chi\neq\lambda\chi$. This shows that $\mathbf{T}(\lambda\chi)=M$. Hence $(\lambda\chi)^G\in\text{Irr}(G)$ and $(\lambda\chi)^G(1)=8$, contradicting $\omega(G)=2$.

Lemma 14. *We again suppose the conditions of lemma 9 and $A=1$ or $A\cong M_0$. Neglecting abelian direct factors, we have one of the following cases:*

- (1) $A=1, |G/M|=1$: Then $G\cong A_5$.
- (2) $A=1, |G/M|=2$: Then $G/N\cong S_5$ and G/K is an abelian 2-group.
- (3) $A\cong M_0, |G/M|=1$: Now G is a central product of $K\cong SL(2,5)$ with the abelian 2-group N .
- (4) $A\cong M_0, |G/M|=2$: Now M is a central product of $K\cong SL(2,5)$ with an abelian 2-group N and G/K is abelian. Also $N\leq Z(G)$.

Proof. (1) If $A=1$ and $|G/M|=1$, then $G=N\times K$, where N is abelian and $K\cong A_5$.

(2) Let $A=1$ and $|G/M|=2$. By lemma 13, N is central in G . Hence the 2-complement of N is a direct summand of G . As $N=C_G(K)$, G/N is a group of automorphisms of K , hence $G/N\cong S_5$.

(3) Suppose $A\cong M_0$ and $|G/M|=1$. Then G is a central product of $K\cong SL(2,5)$ (lemma 9) with the abelian group N (lemma 13). Obviously, we can assume that N is a 2-group.

(4) Finally suppose $A\cong M_0$ and $|G/M|=2$. Then $M=NK$ has the structure described in (3), and by lemma 13, G/K is abelian and $N\leq Z(G)$.

Lemma 15. *All the groups G described in lemma 14 have $\omega(G)=2$.*

Proof. (1) Clearly, $\omega(A_5)=2$.

(2) Now N is central in G (lemma 13). Let $\lambda\chi\in\text{Irr}(N\times K)$, where $\lambda\in\text{Irr}(N)$ and $\chi\in\text{Irr}(K)$. The behaviour of the irreducible characters of $K\cong A_5$ under the automorphism induced by $G/N\cong S_5$ shows that $\mathbf{T}(\chi)=G$, if $\chi(1)=4$. In this case $\chi\lambda$ has an extension to G , as G/M is cyclic. In the other cases $\omega(\chi(1))\leq 1$, hence the irreducible components ψ of $(\lambda\chi)^G$ have $\omega(\psi(1))\leq 2$.

(3) As N is abelian and $K\cong SL(2,5)$, we have $\omega(N\times K)=\omega(K)=2$. (By Dornhoff [1] page 228, the character degrees of $SL(2,5)$ are 1, 2, 3, 4, 5, 6, without multiplicities.) As $G=NK$ is an epimorphic image of $N\times K$, so also $\omega(G)=2$.

(4) As $M=NK$ is a central product, any character of M is of the form $\lambda\chi$, where $\lambda\in\text{Irr}(N)$ and $\chi\in\text{Irr}(K)$. As $N\leq Z(G)$, so $\mathbf{T}(\lambda)=G$. If $\chi(1)=4$ or 6, then inspection of the character table of $SL(2,5)$ shows that χ is stable under any automorphism of $SL(2,5)$. Hence $\mathbf{T}(\lambda\chi)=G$ in this case and $\lambda\chi$ can be extended to G . Otherwise, $\chi(1)\in\{1, 2, 3, 5\}$ and then all irreducible components ψ of $(\lambda\chi)^G$ have $\omega(\psi(1))\leq 2$.

References

- [1] L. Dornhoff: *Group representation theory, part A*, Marcel Dekker, New York, 1971.
- [2] J. Humphreys: *Projective character tables for the finite simple groups of order less than one million*. Comm. Algebra **11** (1983), 725–751.
- [3] B. Huppert: *Endliche Gruppen I*, Springer, Berlin, 1979.
- [4] B. Huppert and N. Blackburn: *Finite groups II, III*, Springer, Berlin, 1982.
- [5] B. Huppert and O. Manz: *Squarefree character degrees*, Arch. Math. **45** (1985), 125–132.
- [6] M. Isaacs: *Character theory of finite groups*. Academic Press, New York, 1976.
- [7] G. James and A. Kerber: *The representation theory of the symmetric group*. Addison-Wesley, London, 1981.
- [8] J. McKay: *The non-abelian simple groups G , $|G| < 10^6$, character tables*, Comm. Algebra **7** (1979), 1407–1445.
- [9] W. Müller: *Darstellungstheorie von endlichen Gruppen*. Teubner, Stuttgart, 1980.
- [10] J. Neubüser: *Character tables of sporadic simple groups*. Aachen, 1983.
- [11] A. Prince: *On 2-groups admitting A_5 or A_6 with an element of order 5 acting fixed point freely*. J. Algebra **49** (1977), 374–386.

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