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On a Two-Dimensional Space of Projective Connection Associated with a Surface in $\boldsymbol{R}_3$

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Denote by $\boldsymbol{R}_n$ an n-dimensional space of projective connection. First, in this paper, we treat the development of a curve in $\boldsymbol{R}_2$ by a method analogous to the theory on an ordinary projective plane curve. Next, we associate $\boldsymbol{R}_2$ with a surface S in $\boldsymbol{R}_3$ by a method of projection and investigate some properties of $\boldsymbol{R}_2$ and othe1 relations between $\boldsymbol{R}_2$ and $\boldsymbol{R}_3$.

1. Let $\boldsymbol{R}_n$ be an n-dimensional space of projective connection, in which a moving point is determined by a system of coordinates (u^i). If a natural frame[1)] of reference $[A_0 A_1 \cdots A_n]$ is associated with the moving point A_0 in the tangential space of n dimensions at A_0 of $\boldsymbol{R}_n$, the infinitesimal displacement of the frame is given by

$$dA_\alpha = \omega_\alpha^\beta A_\beta\,, \quad \omega_\alpha^\beta = \Pi_{\alpha k}^\beta\, du^k\,, \tag{1}$$

and

$$\begin{cases} \omega_0^0 = 0\,, & \omega_0^i = du^i\,, \\ \Pi_{ik}^i = 0\,, & \Pi_{\beta 0}^\alpha = \Pi_{0\beta}^\alpha = \delta_\beta^\alpha\,, \end{cases} \tag{2}$$

where we denote by Greek letters α, β, etc. the indices which take the values $0, 1, \cdots, n$, and by Latin letters i, j, etc. those which take $1, 2, \cdots, n$.

Consider a curve C passing through A_0 of $\boldsymbol{R}_n$, where u^i are functions of a parameter t. Then we have along C

$$\frac{dA_\alpha}{dt} = p_\alpha^\beta A_\beta\,, \quad \omega_\alpha^\beta = p_\alpha^\beta dt\,, \tag{3}$$

and

$$\frac{d^2 A_0}{dt^2} = p_0^i p_i^0 A_0 + \left(\frac{dp_0^i}{dt} + p_0^h p_h^i\right) A_i\,,$$

$$\frac{d^3 A_0}{dt^3} = \left\{\frac{d}{dt}\,(p_0^i p_i^0) + p_k^0\left(\frac{dp_0^k}{dt} + p_0^h p_h^k\right)\right\} A_0$$
$$+ \left\{\frac{d}{dt}\left(\frac{dp_0^i}{dt} + p_0^h p_h^i\right) + p_0^i p_0^h p_h^0 + p_k^i\left(\frac{dp_0^k}{dt} + p_0^h p_h^k\right)\right\} A_i\,,$$

$$\begin{aligned}\frac{d^4A_0}{dt^4} = \Big\{\frac{d}{dt}\Big[\frac{d}{dt}\Big(\frac{dp_0^i}{dt}+p_0^hp_h^i\Big)+p_0^ip_0^hp_h^0+\Big(\frac{dp_0^k}{dt}+p_0^hp_h^k\Big)p_k^i\Big]\\ +\Big[\frac{d}{dt}(p_0^hp_h^0)+\Big(\frac{dp_0^k}{dt}+p_0^hp_h^k\Big)p_k^0\Big]p_0^i\\ +\Big[\frac{d}{dt}\Big(\frac{dp_0^k}{dt}+p_0^hp_h^k\Big)+p_0^kp_0^hp_h^0+\Big(\frac{dp_0^l}{dt}+p_0^hp_h^l\Big)p_l^k\Big]p_k^i\Big\}A_i\\ +(\quad)A_0.\end{aligned}$$

Consequently the point on the image Γ of C corresponding to $t+dt$ is given by

$$A_0+\frac{dA_0}{dt}dt+\frac{1}{2!}\frac{d^2A_0}{dt^2}(dt)^2+\cdots=\rho(A_0+x^iA_i),$$

where

$$\begin{aligned}(4)\qquad x^i &= p_0^idt+\frac{1}{2}\Big(\frac{dp_0^i}{dt}+p_0^jp_j^i\Big)(dt)^2\\ &+\frac{1}{6}\Big\{\frac{d}{dt}\Big(\frac{dp_0^i}{dt}+p_0^hp_h^i\Big)-2p_0^ip_0^hp_h^0+\Big(\frac{dp_0^k}{dt}+p_0^hp_h^k\Big)p_k^i\Big\}(dt)^3\\ &+\frac{1}{24}\Big\{\frac{d}{dt}\Big[\frac{d}{dt}\Big(\frac{dp_0^i}{dt}+p_0^hp_h^i\Big)+p_0^ip_0^hp_h^0+\Big(\frac{dp_0^k}{dt}+p_0^hp_h^k\Big)p_k^i\Big]\\ &\quad+\Big[\frac{d}{dt}\Big(\frac{dp_0^k}{dt}+p_0^hp_h^k\Big)+p_0^hp_h^0p_0^k+\Big(\frac{dp_0^l}{dt}+p_0^hp_h^l\Big)p_l^k\Big]p_k^i\\ &\quad-3\Big[\frac{d}{dt}(p_0^hp_h^0)+\Big(\frac{dp_0^k}{dt}+p_0^hp_h^k\Big)p_k^0\Big]p_0^i\\ &\quad-6\,p_0^hp_h^0\Big(\frac{dp_0^i}{dt}+p_0^kp_k^i\Big)\Big\}(dt)^4+\cdots.\end{aligned}$$

2. In the case of $n=2$, by means of (4), the image Γ can be expressed by the equation, referred to the frame $[A_0A_1A_2]$ of reference,

$$(5)\qquad x^2=\sum_{m=1}^{\infty}\frac{a_m}{m!\,(p_0^1)^m}(x^1)^m,$$

where

$$\begin{aligned}a_1&=p_0^2,\\ a_2&=\frac{dp_0^2}{dt}+p_0^hp_h^2-\frac{p_0^2}{p_0^1}\Big(\frac{dp_0^1}{dt}+p_0^hp_h^1\Big),\\ a_3&=\frac{d}{dt}\Big(\frac{dp_0^2}{dt}+p_0^hp_h^2\Big)+p_0^2p_0^hp_h^0+\Big(\frac{dp_0^k}{dt}+p_0^hp_h^k\Big)p_k^2\\ &\quad-\frac{p_0^2}{p_0^1}\Big[\frac{d}{dt}\Big(\frac{dp_0^1}{dt}+p_0^hp_h^1\Big)+p_0^1p_0^hp_h^0+\Big(\frac{dp_0^k}{dt}+p_0^hp_h^k\Big)p_k^1\Big]\\ &\quad-3\frac{1}{p_0^1}\Big(\frac{dp_0^1}{dt}+p_0^hp_h^1\Big)a_2\\ &=\frac{da_2}{dt}+\frac{1}{p_0^1}\Big\{\Big(\frac{dp_0^1}{dt}+p_0^hp_h^1\Big)\Big(-2a_2-p_0^ip_i^2+\frac{p_0^2}{p_0^1}p_0^ip_i^1\Big)\\ &\qquad+\frac{dp_0^1}{dt}p_0^hp_h^2-\frac{dp_0^2}{dt}p_0^hp_h^1\Big\},\end{aligned}$$

$$
\begin{aligned}
a_4 = \frac{d}{dt}\Bigg[& \frac{d}{dt}\left(\frac{dp_0^2}{dt}+p_0^h p_h^2\right)+p_0^2 p_0^h p_h^0+\left(\frac{dp_0^k}{dt}+p_0^h p_h^k\right)p_k^2\Bigg] \\
& +\left[\frac{d}{dt}\left(\frac{dp_0^l}{dt}+p_0^h p_h^l\right)+p_0^l p_0^h p_h^0+\left(\frac{dp_0^k}{dt}+p_0^h p_h^k\right)p_k^l\right]p_l^2 \\
& -\frac{p_0^2}{p_0^1}\Bigg\{\frac{d}{dt}\left[\frac{d}{dt}\left(\frac{dp_0^1}{dt}+p_0^h p_h^1\right)+p_0^1 p_0^h p_h^0+\left(\frac{dp_0^k}{dt}+p_0^h p_h^k\right)p_k^1\right] \\
& \qquad +\left[\frac{d}{dt}\left(\frac{dp_0^l}{dt}+p_0^h p_h^l\right)+p_0^l p_0^h p_h^0+\left(\frac{dp_0^k}{dt}+p_0^h p_h^k\right)p_k^l\right]p_l^1\Bigg\} \\
& -\frac{6}{p_0^1}\left(\frac{dp_0^1}{dt}+p_0^h p_h^1\right)a_3 \\
& -\frac{a_2}{(p_0^1)^2}\Bigg\{3\left(\frac{dp_0^1}{dt}+p_0^h p_h^1\right)^2 \\
& +4p_0^1\left[\frac{d}{dt}\left(\frac{dp_0^1}{dt}+p_0^h p_h^1\right)-\frac{1}{2}\, p_0^1 p_0^h p_h^0+\left(\frac{dp_0^k}{dt}+p_0^h p_h^k\right)p_k^1\right]\Bigg\} .
\end{aligned}
$$

If we denote by K_2 the osculating conic at A_0 of Γ on the plane $A_0A_1A_2$, K_2 is expressed by the equation

$$p_0^1 x^2 - p_0^2 x^1 = \sum_{i,j=1}^{2} C_{ij} x^i x^j \quad (C_{ij} = C_{ji}),$$

where

$$
\begin{aligned}
C_{11} &= \frac{1}{18 p_0^1 (a_2)^3}\left\{3(a_2)^2\left[3(a_2)^2-2a_1a_3\right]+(a_1)^2\left[3a_2a_4-4(a_3)^2\right]\right\}, \\
C_{12} &= \frac{1}{18(a_2)^3}\left\{a_3\left[3(a_2)^2-a_1a_3\right]-a_1\left[3a_2a_4-5(a_3)^2\right]\right\}, \\
C_{22} &= \frac{p_0^1}{18(a_2)^3}\left\{3\,a_2a_4-4\,(a_3)^2\right\}.
\end{aligned}
$$

We put

$$
\begin{aligned}
& B_0 = A_0, \quad B_1 = p_0^i A_i, \\
& B_2 = \frac{3a_2a_4-5(a_3)^2}{18(a_2)^2} A_0 - \frac{p_0^1 a_3}{3a_2} A_1 + \frac{3(a_2)^2-a_1a_3}{3a_2} A_2 .
\end{aligned}
$$

The points B_1 and B_2 lie on the tangent and the osculating conic K_2 at A_0 of Γ respectively and the line B_0B_2 is the polar of B_1 with respect to K_2.

If we associate the frame constituted by the points B_0, B_1, B_2 with the point $A_0(=B_0)$ of the development Γ of C, we get by means of (3)

$$
(6) \qquad \left\{
\begin{aligned}
\frac{dB_0}{dt} &= B_1, \\
\frac{dB_1}{dt} &= kB_0 + hB_1 + B_2, \\
\frac{dB_2}{dt} &= \Theta B_0 + kB_1 + 2hB_2,
\end{aligned}
\right.
$$

where

$$(7)\quad \begin{cases} h = \dfrac{a_3}{3a_2} + \dfrac{1}{p_0^1}\left(\dfrac{dp_0^1}{dt} + p_0^h p_h^1\right), \\ k = p_0^h p_h^0 - \dfrac{3a_2a_4 - 5(a_3)^2}{18(a_2)^2}, \\ \Theta = \dfrac{d}{dt}\left(\dfrac{3a_2a_4 - 5(a_3)^2}{18(a_2)^2}\right) - \dfrac{3a_2a_4 - 5(a_3)^2}{9(a_2)^2} h \\ \qquad - \dfrac{a_3}{3a_2} p_0^h p_h^0 + p_2^0 a_2 . \end{cases}$$

Thus. we get the equation for Γ

$$(8)\quad \begin{aligned} z^2 &= \frac{1}{2}(z^1)^2 + \frac{\Theta}{20}(z^1)^5 + \frac{1}{120}\left(\frac{d\Theta}{dt} - 3h\Theta\right)(z^1)^6 \\ &\quad + \frac{1}{840}\left\{3\Theta K + \frac{7}{6\Theta}\left(\frac{d\Theta}{dt} - 3h\Theta\right)^2\right\}(z^1)^7 + \cdots, \\ K &= k - \frac{dh}{dt} + \frac{1}{2}(h)^2 + \frac{1}{3}\left\{\frac{1}{\Theta}\frac{d^2\Theta}{dt^2} - \frac{7}{6}\left(\frac{1}{\Theta}\frac{d\Theta}{dt}\right)^2\right\}, \end{aligned}$$

z^1, z^2 being the nonhomogeneous coordinates of a point referred to the frame $[B_0B_1B_2]$.

3. When we make the transformation of coordinates[2)]

$$(9)\qquad \bar{u}^i = \bar{u}^i(u^1, \cdots, u^n), \quad (\nu)^{n+1} = \frac{\partial(\bar{u}^1, \cdots, \bar{u}^n)}{\partial(u^1, \cdots, u^n)} \neq 0,$$

we have the following relations for the vertices of the natural frame and the parameters of connection:

$$(10)\quad \begin{cases} \bar{A}_\alpha = \nu Q_\alpha^\beta A_\beta, \quad \nu A_\alpha = P_\alpha^\beta \bar{A}^\beta, \\ \bar{\Pi}_{\alpha i}^\beta = P_\lambda^\beta\left(Q_\alpha^\sigma Q_i^\tau \Pi_{\sigma\tau}^\lambda + \dfrac{\partial Q_\alpha^\lambda}{\partial \bar{u}^i}\right), \\ \bar{\Pi}_{0\alpha}^\beta = \bar{\Pi}_{\alpha 0}^\beta = P_\lambda^\beta Q_\alpha^\sigma Q_0^\tau \Pi_{\sigma\tau}^\lambda = \delta_\alpha^\beta, \end{cases}$$

where we put

$$P_0^0 = 1, \quad P_i^0 = -\frac{\partial \log \nu}{\partial u^i}, \quad P_0^i = 0, \quad P_j^i = \frac{\partial \bar{u}^i}{\partial u^j},$$

$$Q_0^0 = 1, \quad Q_i^0 = \frac{\partial \log \nu}{\partial \bar{u}^i}, \quad Q_0^i = 0, \quad Q_j^i = \frac{\partial u^i}{\partial \bar{u}^j}.$$

Hence we have for the vertices of the frame $[B_0B_1B_2]$ and the quantities h, k, Θ, K

$$(11)\quad \begin{cases} \bar{B}_0 = \nu B_0, \\ \bar{B}_1 = \nu\left(\dfrac{d\log\nu}{dt} B_0 + B_1\right), \\ \bar{B}_2 = \nu\left\{\dfrac{1}{2}\left(\dfrac{d\log\nu}{dt}\right)^2 B_0 + \dfrac{d\log\nu}{dt} B_1 + B_2\right\}, \end{cases}$$

$$\left\{\begin{aligned} \bar{k} &= k-\frac{d\log\nu}{dt}h-\frac{1}{2}\left(\frac{d\log\nu}{dt}\right)^2+\frac{d^2\log\nu}{dt^2}, \\ \bar{h} &= h+\frac{d\log\nu}{dt}, \\ \bar{\Theta} &= \Theta, \quad \bar{K}=K. \end{aligned}\right.$$

If we make the transformation $\bar{t}=f(t)$, we get

$$\text{(12)} \qquad \left\{\begin{aligned} &\bar{B}_0=B_0, \quad \bar{B}_1=\frac{1}{f'}B_1, \quad \bar{B}_2=\frac{1}{(f')^2}B_2, \\ &\bar{h}=\frac{1}{f'}\left(h-\frac{f''}{f'}\right), \quad \bar{k}=\frac{1}{(f')^2}k, \\ &\bar{\Theta}=\frac{1}{(f')^3}\Theta, \quad \bar{K}=\frac{1}{(f')^2}K. \end{aligned}\right.$$

Therefore (11) and (12) show that $\Theta(dt)^3$ and $K(dt)^2$ are invariant for the transformation of coordinates (9) and the change of parameter $\bar{t}=f(t)$.

By means of (8), the osculating conic K_2 is represented by

$$z^2=\frac{1}{2}(z^1)^2.$$

The projective normal[3)] at B_0 of Γ is the line joining B_0 with the point

$$\left(\frac{d\Theta}{dt}-3h\Theta\right)B_1+3\Theta B_2.$$

The cubic K_3 which has a contact of the sixth order with Γ at B_0 and meets the projective normal at B_0 of Γ at the conjugate points with respect to K_2 is represented by the equation

$$\left\{z^2-\frac{1}{2}(z^1)^2\right\}(1+az^1+bz^2)=\frac{\Theta}{5}z^1(z^2)^2+\left\{\frac{1}{15}\left(\frac{d\Theta}{dt}-3h\Theta\right)+\frac{2}{5}\Theta a\right\}(z^2)^3,$$

a, b satisfying the relation

$$\frac{1}{6\Theta}\left(\frac{d\Theta}{dt}-3h\Theta\right)^2+a\left(\frac{d\Theta}{dt}-3h\Theta\right)+3\Theta b=0,$$

from which we get

$$z^2=\frac{1}{2}(z^1)^2+\frac{\Theta}{20}(z^1)^5+\frac{1}{120}\left(\frac{d\Theta}{dt}-3h\Theta\right)(z^1)^6 +\frac{1}{720\Theta}\left(\frac{d\Theta}{dt}-3h\Theta\right)^2(z^1)^7+\cdots.$$

Hence we can say as follows.

Let B be a point which does not lie on the tangent B_0B_1 of Γ, and P, P_1, P_2, P_3 be the points of intersection of a line passing through B

with B_0B_1, Γ, K_2, K_3 *respectively in the neighbourhood of* B_0. *Then the principal parts of the anharmonic ratios* $[BPP_1P_2]$, $[BPP_2P_3]$ *are*

$$\frac{\Theta}{10}(dt)^3, \quad \frac{K}{14}(dt)^2$$

respectively.[3), 4)]

By means of (11) and (12), we can choose the system of coordinates (u^1, u^2) and the parameter t in such a way that we have $h = k = 0$ for C. Then we have from (7)

$$(13) \qquad \frac{d}{dt}\left(\frac{dp_0^i}{dt}+p_0^h p_h^i\right)+p_0^i p_0^h p_h^0+\left(\frac{dp_0^k}{dt}+p_0^h p_h^k\right)p_k^i = 0, \quad (i = 1, 2)$$

4. Consider another two-dimensional space $\boldsymbol{R}_2'$ of projective connection, where the infinitesimal displacement of the natural frame is given by

$$dA_\alpha' = \omega_\beta^{\alpha\prime} A_\beta',$$

and the coordinates of a moving point are (u^i). Suppose that the corresponding points of $\boldsymbol{R}_2$ and $\boldsymbol{R}_2'$ have the same value of u^i, the corresponding curve C, C' in $\boldsymbol{R}_2$, $\boldsymbol{R}_2'$ are defined by $u^i = u^i(t)$, and the homologous points A_0 and A_0' correspond to $(u^i)_0 = u^i(0) = 0$. Then we have

$$u^i(t) = p_0^i t + \frac{1}{2}\frac{dp_0^i}{dt}(t)^2 + \cdots.$$

We develop $\boldsymbol{R}_2$, $\boldsymbol{R}_2'$ along C, C', such as A_0, A_0' have a common image P and the frames $[A_0A_1A_2]$, $[A_0'A_1'A_2']$ take a common initial position, and take, in the neighbourhood of P, the image Q, Q' of the homologous points on C, C' respectively. By means of (4), the écart $[QQ']$ is given by

$$\frac{1}{2}\sum_{i=1}^{2}\left|(\Pi_{jk}^i - \Pi_{jk}'^i)\frac{du^j}{dt}\frac{du^k}{dt}(t)^2\right|,$$

excepting the terms of higher orders.

If we have

$$(14) \qquad \Pi_{jk}^i + \Pi_{kj}^i = \Pi_{jk}'^i + \Pi_{kj}'^i,$$

$[QQ']$ is an infinitesimal of the third order at least with respect to the écart $[PQ]$. In this case, it is said that $\boldsymbol{R}_2$ and $\boldsymbol{R}_2'$ *are projectively deformable.*[5)]

In the case that (14) is not satisfied, $[QQ']$ *is an infinitesimal of the third order with respect to* $[PQ]$ *along the two curves defined by*

$$(15) \qquad (\Pi_{jk}^i - \Pi_{jk}'^i)\frac{du^j}{dt}\frac{du^k}{dt} = 0,$$

if we have

$$(\Pi^1_{jk}+\Pi^1_{kj})-(\Pi'^1_{jk}+\Pi'^1_{kj})=\rho\left\{(\Pi^2_{jk}+\Pi^2_{kj})-(\Pi'^2_{jk}+\Pi'^2_{kj})\right\}.$$

5. Consider a surface S passing through A_0 in $\boldsymbol{R}_3$. Suppose that S is defined by $u^3=0$, this being possible, for, if S is expressed by an equation $f(u^1, u^2, u^3)=0$, we can choose a new system of coordinates $\bar{u}^i$ such as $\bar{u}^3=f(u^1, u^2, u^3)$. Along a curve C on S, we have

$$\text{(16)}\qquad \begin{cases} dA_0=du^iA_i\,, \\ dA_i=\Pi^0_{ik}\,du^kA_0+\Pi^j_{ik}\,du^kA_j+\Pi^3_{ik}\,du^kA_3\,, \\ dA_3=\Pi^0_{3k}\,du^kA_0+\Pi^j_{3k}\,du^kA_j+\Pi^3_{3k}\,du^kA_3\,, \\ \qquad (i, j, k=1, 2\,;\ du^3=0)\,. \end{cases}$$

Take a point

$$\bar{A}_3=\xi^0A_0+\xi^iA_i+A_3\,,$$

in the tangential projective space $\boldsymbol{E}_3$ at A_0 of $\boldsymbol{R}_3$. Then (16) becomes

$$\text{(17)}\qquad \begin{cases} dA_0=du^iA_i\,, \\ dA_i=\bar{\Pi}^0_{ik}\,du^kA_0+\bar{\Pi}^j_{ik}\,du^kA_j+\Pi^3_{ik}\,du^k\bar{A}_3\,, \\ d\bar{A}_3=\cdots, \\ \bar{\Pi}^\alpha_{ik}=\Pi^\alpha_{ik}-\xi^\alpha\Pi^3_{ik}\quad (\alpha=0, 1, 2\,;\ i, k=1, 2)\,. \end{cases}$$

The images of the tangents of curves passing through A_0 on S lie on the plane $A_0A_1A_2$. Now we consider the two-dimensional space $\boldsymbol{R}_2$ of projective connection defined by the connections $\bar{\Pi}^\alpha_{ik}\,du^k$ relating to S. It may be supposed that the tangential projective plane $\boldsymbol{E}_2$ at A_0 of $\boldsymbol{R}_2$ coincides with the plane $A_0A_1A_2$, the frame of reference associated with $\boldsymbol{R}_2$ has the common initial position with $[A_0A_1A_2]$, and the infinitesimal displacement of the frame is given by the projections of the variations of $[A_0A_1A_2\bar{A}_3]$ on the plane $A_0A_1A_2$ from $\bar{A}_3$. Namely we get for $\boldsymbol{R}_2$ from (17)

$$\text{(18)}\qquad \begin{cases} dA_0=du^iA_i\,, \\ dA_i=\bar{\Pi}^\alpha_{ik}\,du^kA_\alpha\,. \end{cases}$$

If we choose ξ^i in such a way that

$$\text{(19)}\qquad \xi^i\,\Pi^3_{ik}=-\Pi^3_{3k}\quad (i, k=1, 2)\,,$$

the frame $[A_0A_1A_2]$ is natural, for, since the frame $[A_0A_1A_2A_3]$ is natural, we have the condition

$$\sum_{i=1}^{2}\bar{\Pi}^i_{ik}=\sum_{i=1}^{3}\Pi^i_{ik}=0\,.$$

The point $\bar{A}_3$ in this case lies on the line

$$\sum_{i=1}^{3}z^i\,\Pi^3_{ik}=0\quad (k=1, 2)$$

in E_3, when the rank of the matrix

$$\begin{pmatrix} \Pi^3_{11} & \Pi^3_{21} & \Pi^3_{31} \\ \Pi^3_{12} & \Pi^3_{22} & \Pi^3_{32} \end{pmatrix}$$

is two, z^i being the coordinates of a point referred to the frame $[A_0A_1A_2A_3]$.

6. Project the development Γ of a curve $C[u^i = u^i(t),\ u^3 = 0]$ on S on the plane $A_0A_1A_2$ from A_3, and we have by (4)

$$x^i = p^i_0 dt + \frac{1}{2}\left(\frac{dp^i_0}{dt} + \Pi^i_{jk}\, p^j_0 p^k_0\right)(dt)^2 + \cdots$$
$$(p^3_0 = 0\,,\quad i = 1, 2)\,,$$

while the image $\bar{\Gamma}$ of the curve $\bar{C}[u^i = u^i(t)]$ of R_2 mentioned in the preceeding paragraph is expressed by

$$\bar{x}^i = p^i_0 dt + \frac{1}{2}\left(\frac{dp^i_0}{dt} + \bar{\Pi}^i_{jk}\, p^j_0 p^k_0\right)(dt)^2 + \cdots \quad (i = 1, 2)\,.$$

Consider a point-correspondence between S and R_2, the homologous points having the same values of u^i. Let Q and $\bar{Q}$ be the homologous points in the neighbourhood of A_0 on Γ and $\bar{\Gamma}$ respectively. Then, similarly as n°4, the écart $[Q\bar{Q}]$ is an infinitesimal of the third order with respect to $[A_0Q]$, when the equation equivalent to (15) is satisfied. Then we have by means of (17)

(20) $$\xi^i\, \Pi^3_{jk}\, du^j du^k = 0 \quad (i = 1, 2)\,.$$

On the other hand, $\Pi^3_{jk}\, du^j du^k = 0$ defines the asymptotic curves[6] of S. If $\xi^i = 0$, (20) is an identity. Hence we can say as follows:

Let S be a surface in R_3, C be a curve passing through a point A_0 on S, $[A_0A_1A_2A_3]$ be a natural frame in the tangential projective space E_3 at A_0 of R_3, and the plane $A_0A_1A_2$ be the image of the tangent plane at A_0 of S. Denote by Γ the projection of the development of C on the plane$A_0A_1A_2$ from A_3. Associate with S the two-dimensional space R_2 of projective connection in which the infinitesimal dispacements of the frame $[A_0A_1A_2]$ are defined by the projections of the variations of the frame $[A_0A_1A_2\bar{A}_3]$ on the plane $A_0A_1A_2$ from a point $\bar{A}_3$ which does not lie on the plane $A_0A_1A_2$ in E_3. Consider a point-correspondence between S and R_2 in such a way that the homologous points on them correspond to the same values in the system of coordinates determining points of R_3, and let $\bar{C}, \bar{\Gamma}$ be the figures with respect to R_2 homologous to C, Γ. Take the homologous points $Q, \bar{Q}$ in the neighbourhood of A_0 on $C, \bar{C}$. If the écart $[Q\bar{Q}]$ for the images is an infinitesimal of the third order with respect to $[A_0Q]$, C is an asymptotic curve of S. If $\bar{A}_3$ lies on the line

A_0A_3 , $\boldsymbol{R}_2$ *is projectively deformable to the space similar to* $\boldsymbol{R}_2$ *with* A_3 *as the centre of projection.*

If the relations (20) is identically satisfied for any values of ξ^i and any curve, we have

$$\Pi^3_{jk}+\Pi^3_{kj}=0 \quad (j, k=1, 2),$$

which is the condition that S is totally geodesic.[7] Hence *it is necessary and sufficient that S is totally geodesic, in order that the spaces* $\boldsymbol{R}_2$ *corresponding to the different centres* $\bar{A}_3$ *of projection are projectively deformable to each other.*

7. The displacement associated with an infinitesimal closed cycle on S of $\boldsymbol{R}_3$ is given by $R^\beta_{\alpha hk}[du^h du^k]$ with $du^3=0$, where

$$R^\beta_{\alpha hk}=\frac{\partial \Pi^\beta_{\alpha h}}{\partial u^k}-\frac{\partial \Pi^\beta_{\alpha k}}{\partial u^h}+\Pi^\lambda_{\alpha k}\Pi^\beta_{\lambda k}-\Pi^\lambda_{\alpha k}\Pi^\beta_{\lambda h} \tag{21}$$

$$(\alpha, \beta, \lambda=0, 1, 2, 3\,;\ h, k=1, 2),$$

and $[du^h du^k]$ represents the exterior product. On the other hand, $\boldsymbol{R}_2$ (n°5) associated with S, under the condition (19), has the tensor of curvature and torsion

$$\bar{R}^\beta_{\alpha hk}=\frac{\partial \bar{\Pi}^\beta_{\alpha h}}{\partial u^k}-\frac{\partial \bar{\Pi}^\beta_{\alpha k}}{\partial u^h}+\bar{\Pi}^\lambda_{\alpha h}\bar{\Pi}^\beta_{\lambda k}-\bar{\Pi}^\lambda_{\alpha k}\bar{\Pi}^\beta_{\lambda h}$$

$$(\alpha, \beta, \lambda=0, 1, 2\,;\ h, k=1, 2).$$

Reducing this by means of (17), we get

$$\begin{aligned}\bar{R}^\beta_{\alpha hk}&=R^\beta_{\alpha hk}-\xi^\beta R^3_{\alpha hk}+\Pi^3_{\alpha h}\frac{\partial \xi^\beta}{\partial u^k}-\Pi^3_{\alpha k}\frac{\partial \xi^\beta}{\partial u^h}\\&\quad+(\Pi^3_{\alpha k}\Pi^\beta_{\lambda h}-\Pi^3_{\alpha h}\Pi^\beta_{\lambda k})\xi^\lambda\\&\quad+\Pi^3_{\alpha k}\Pi^\beta_{3h}-\Pi^3_{\alpha h}\Pi^\beta_{3k}\end{aligned} \tag{22}$$

$$(\alpha, \beta, \lambda=0, 1, 2\,;\ h, k=1, 2),$$

so that

$$\bar{R}^\beta_{0hk}=R^\beta_{0hk}-\xi^\beta R^3_{0hk}\,. \tag{23}$$

Hence *if* $\boldsymbol{R}_3$ *is the space of zero torsion, the space* $\boldsymbol{R}_2$ *associated with the surface S in* $\boldsymbol{R}_3$ *by projection (n°5) is so, too.*

If S is totally geodesic, we have

$$\Pi^3_{jk}+\Pi^3_{kj}=0 \quad (j, k=1, 2),$$

so that from (22) we have

$$\begin{aligned}\bar{R}^\beta_{i12}&=R^\beta_{i12}-\xi^\beta R^3_{i12}\\&\quad-\delta^1_i\Pi^3_{12}\left(\frac{\partial \xi^\beta}{\partial u^1}-\Pi^\beta_{\lambda 1}\xi^\lambda-\Pi^\beta_{31}\right)\\&\quad+\delta^2_i\Pi^3_{21}\left(\frac{\partial \xi^\beta}{\partial u^2}-\Pi^\beta_{\lambda 2}\xi^\lambda-\Pi^\beta_{32}\right).\end{aligned}$$

If the tensor of torsion for $\boldsymbol{R}_3$ is zero, moreover, we have

$$\Pi^3_{jk} = 0,$$

and accordingly by (21)

$$R^3_{ihk} = 0 \quad (i, h, k = 1, 2).$$

Thus *if* $\boldsymbol{R}_3$ *is a space of zero torsion and* S *is a totally geodesic surface in* $\boldsymbol{R}_3$, *we have for* $\boldsymbol{R}_2$ *associated with* S

$$\bar{R}^\beta_{\alpha hk} = R^\beta_{\alpha hk} \quad (\alpha, \beta = 0, 1, 2; \; h, k = 1, 2).$$

Also, the relation (23) shows that, *the tensor of torsion for* $\boldsymbol{R}_2$ *is equal to the components of the tensor of torsion associated with an infinitesimal cycle on* S *of* $\boldsymbol{R}_3$, *when*

$$R^3_{\zeta hk} = 0 \quad (h, k = 1, 2),$$

which is the necessary and snfficient condition in order that the conjugate tangents at A_0 *of* S *are in involution.*[8)]

8. Now we consider as an example a surface S in a projective space $\boldsymbol{E}_3$ of three dimensions. The displacement of the Darboux frame $[A_0A_1A_2A_3]$ associated with a moving point A_0 of S is given by

$$\begin{cases} dA_0 = \omega_0^i A_i, \\ dA_i = \omega_i^0 A_0 + \omega_i^l A_l + \omega_i^3 A_3, \\ dA_3 = \omega_3^0 A_0 + \omega_3^l A_l, \end{cases}$$

where

$$\begin{aligned} &\omega_0^i = du^i, \quad \omega_i^0 = M_{ij} du^j \; (M_{ij} = M_{ji}), \\ &\omega_i^l = (K^l_{ij} + \Gamma^l_{ij}) du^j \quad (K^l_{ij} = K^l_{ji}, \; \Gamma^l_{ij} = \Gamma^l_{ji}, \; K^i_{ij} = 0), \\ &\omega_i^3 = H_{ij} du^j \qquad\qquad (H_{ij} = H_{ji}), \end{aligned}$$

and the indices i, j, l, etc. take the values 1, 2.

By projecting the variations of A_α on the plane $A_0A_1A_2$ from the point $\xi^\alpha A_\alpha + A_3$ $(\alpha = 0, 1, 2)$, we get the two-dimensional space $\boldsymbol{R}_2$ of projective connection associated with S, in which the displacement is defined by

$$\begin{cases} dA_0 = \omega_0^i A_i, \\ dA_i = (\omega_i^\alpha - \xi^\alpha \omega_i^3) A_\alpha. \end{cases}$$

The frame $[A_0A_1A_2]$ is natural, if $\xi^i (i = 1, 2)$ satisfy $\omega_i^i - \xi^i \omega_i^3 = 0$, which becomes $\xi^i H_{ij} = \Gamma^i_{ij}$, or $\xi^l = H^{ij}\Gamma^l_{ij}$.

Since the parameters of connection of $\boldsymbol{R}_2$ are

$$\begin{aligned} &\Pi^l_{0i} = \Pi^l_{i0} = \delta^l_i, \\ &\Pi^0_{ij} = M_{ij} - \xi^0 H_{ij}, \\ &\Pi^l_{ij} = K^l_{ij} + \Gamma^l_{ij} - \xi^l H_{ij}, \end{aligned}$$

these quantities are symmetric with respect to the lower indices. Hence $\boldsymbol{R}_2$ *is a space of torsion zero.* This follows from the result of the preceeding paragraph, for $\boldsymbol{E}_3$ is the space in which the tensor of curvature and torsion is zero.

Since the tensor of torsion of $\boldsymbol{R}_2$ is zero, $\boldsymbol{R}_2$ is applicable on the tangent plane $A_0A_1A_2$ of S, excepting an infinitesimal of the fourth order, by the equation

$$(24) \qquad x^i = u^i + \frac{1}{2} \Pi^i_{jk} u^j u^k + \frac{1}{6}\left(\frac{\partial \Pi^i_{jk}}{\partial u^l} + \Pi^\lambda_{jk} \Pi^i_{\lambda l}\right) u^j u^k u^l - \frac{1}{2} \Pi^0_{jk} u^i u^j u^k ,$$

which defines the point-correspondence between the points (x^i) on the plane $A_0A_1A_2$ and (u^i) on $\boldsymbol{R}_2$. If we make $h = k = 0$ for a curve $C\,[u^i = u^i(t)]$ in $\boldsymbol{R}_2$, the relations (13) are satisfied. By expanding $u^i(t)$ into a power series of dt by making use of (13), and substituting the expansion in place of u^i of (24), we obtain the equation of the curve C' on the plane $A_0A_1A_2$ corresponding to C. On the other hand, the development Γ of C on $A_0A_1A_2$ is given by (4).

If the development Γ has a contact of the fourth order with the curve C' corresponding to C with respect to the correspondence (24), we have

$$R^i_{hkl} \frac{dp^k}{dt} p^h p^l = 0 .$$

If this relation is satisfied, whatever the curve C may be, the applicability of $\mathbf{R}_2$ on $A_0A_1A_2$ is of the fourth order. Then we have

$$R^i_{\alpha hl} = 0 .$$

Hence *the space* $\boldsymbol{R}_2$ *is normal,*[9] *if* $\boldsymbol{R}_2$ *admits an applicability of the fourth order on* $A_0A_1A_2$.

The tensor of curvature and torsion of $\boldsymbol{R}_2$ is in general

$$\begin{aligned} R^i_{hkl} = {} & \frac{\partial}{\partial u^l}(K^i_{hk} + \Gamma^i_{hk} - \xi^i H_{hk}) - \frac{\partial}{\partial u^k}(K^i_{hl} + \Gamma^i_{hl} - \xi^i H_{hl}) \\ & + (M_{hk} - \xi^0 H_{hk})\,\delta^i_l - (M_{hl} - \xi^0 H_{hl})\,\delta^i_k \\ & + (K^j_{hk} + \Gamma^j_{hk} - \xi^j H_{hk})(K^i_{jl} + \Gamma^i_{jl} - \xi^i H_{jl}) \\ & - (K^j_{hl} + \Gamma^j_{hl} - \xi^j H_{hl})(K^i_{jk} + \Gamma^i_{jk} - \xi^i H_{jk}) , \end{aligned}$$

and consequently we have for $\boldsymbol{R}_2$

$$R^i_{ikl} = 0 .$$

By means of Bianchi's identity in the case of torsion zero

$$R^i_{lkh} + R^i_{khl} + R^i_{hlk} = 0 ,$$

and

$$R^i_{khi} = -R^i_{kih},$$

we get

$$R^i_{ikh} = R^i_{hki} - R^i_{khi},$$

which reduces to

$$R^i_{hki} = R^i_{khi}.$$

Therefore *the tensor* R_{hk} *is symmetric for the space* $\boldsymbol{R}_2$, *putting*

$$R_{hk} = R^i_{hki}.$$

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