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## ON DIFFERENTIABILITY AND ANALYTICITY OF SOLUTIONS OF WEIGHTED ELLIPTIC BOUNDARY VALUE PROBLEMS

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The present paper is concerned with the differentiability and analyticity of solutions of weighted elliptic boundary value problems (see [2] for the definition of weighted ellipticity)

$$A(x, t, D_x, D_t)u(x, t) = f(x, t), \quad x \in \Omega, \quad (0.1)$$

$$B_j(x, t, D_x, D_t)u(x, t) = 0, \quad x \in \partial\Omega, \quad j=1, \dots, m, \quad (0.2)$$

in some cylindrical domain with  $\Omega$  as its base, where we denote the order type of  $A$  by  $(2m, l)$ . We first investigate such regularity properties of the solution  $u$  considered as a function of  $t$  with values in  $L^2(\Omega)$  or  $H_{2m}(\Omega)$  and then the same properties of  $u$  as a numerical function of all independent variables  $(x, t)$ . In [2] S. Agmon and L. Nirenberg proved the differentiability and analyticity in  $t$  of the solutions of (0.1)-(0.2) in  $L^p(\Omega)$ ,  $1 < p < \infty$ , under the corresponding hypothesis on  $f$  in case in which all the coefficients of  $A$  and  $\{B_j\}_{j=1}^m$  do not depend on  $t$  with the aid of their general results on abstract differential equations

$$\frac{1}{i} \frac{du}{dt} - Au = f(t) \quad (0.3)$$

in a Banach space. Recently in [4] A. Friedman obtained such kind of regularity theorems for the solutions of abstract differential equations

$$\frac{1}{i} \frac{du}{dt} - A(t)u = f(t) \quad (0.4)$$

in a Hilbert space using Fourier transform in  $t$ . In his results  $A(t)$  may depend on  $t$  but is assumed to have a constant domain. In [11] the author showed that A. Friedman's method can be applied to the problem with time-dependent boundary conditions

$$\partial u(x, t) / \partial t + A(x, t, \partial / \partial x)u(x, t) = f(x, t), \quad x \in \Omega, \quad (0.5)$$

$$B_j(x, t, \partial / \partial x)u(x, t) = 0, \quad x \in \partial\Omega, \quad j=1, \dots, m, \quad (0.6)$$

where  $A(x, t, \partial/\partial x)$  is an elliptic operator of order  $2m$ , provided that the positive and negative imaginary axes are of minimal growth in the sense of S. Agmon [1] with respect to  $(A, \{B_j\}, \Omega)$ . In the present paper we shall generalize this result to the higher order problem (0.1)–(0.2), the whole contents being based on a slight extension of the inequality (14.6) in [2]. As in [4] essential use is made of Plancherel theorem, therefore we are obliged to take  $L^2(\Omega)$  as the basic Banach space. Roughly speaking our first main result is stated as follows: putting  $d=2m/l$  if  $A(x, t, D_x, \pm D_y^d)$  is elliptic in  $(x, y) \in \bar{\Omega} \times (-\infty, \infty)$  and the Complementing Condition ([3]) is satisfied by  $(A(x, t, D_x, \pm D_y^d), \{B_j(x, t, D_x, \pm D_y^d)\})$  in  $\bar{\Omega} \times (-\infty, \infty)$  for each fixed  $t$ , then the solution  $u$  of (0.1)–(0.2) is a smooth function of  $t$  with values in  $L^2(\Omega)$  provided that the coefficients of  $A$  and  $\{B_j\}$  as well as  $f$  are sufficiently smooth and  $u$  is analytic in  $t$  provided that all the coefficients together with some of their  $x$ -derivatives and  $f$  are analytic in  $t$ . With the aid of this result we shall finally prove that the solutions of (0.1)–(0.2) are analytic in all variables if the coefficients of  $A$ ,  $\{B_j\}$  and  $f$  are all analytic functions of  $(x, t) \in \bar{\Omega} \times (-\infty, \infty)$  and the boundary of  $\Omega$  is an analytic manifold. The proof of this last statement is so lengthy in the general case (0.1)–(0.2) that we shall confine ourselves to the special case (0.5)–(0.6).

We shall investigate solutions of (0.1) satisfying the homogenous boundary conditions (0.2). However, unless the boundary system is equivalent to another one whose coefficients are independent of  $t$ , the boundary conditions satisfied by  $D_t^k u$  are not necessarily homogenous if  $u$  is a solution of (0.1)–(0.2). For the the same reason the tangential derivatives of  $u$  in the space directions may not satisfy the homogenous boundary conditions (0.2). Therefore in sections 2 and 3 we shall obtain some estimate for the solutions of the inhomogenous boundary value problems. Based on this estimate we shall prove the differentiability in  $t$  of the solutions by means of difference quotient method and their analyticity in  $t$  following L. Hörmander's proof of the interior analyticity of the solutions of elliptic differential equations with analytic coefficients ([5], pp. 178–180). In the proof of the analyticity in all independent variables we follow the method of C. B. Morrey and L. Nirenberg [10] and show that the Cauchy data of  $u$  are analytic on the boundary so that we may apply Holmgren's theorem to obtain the desired result.

Finally we note that in [13] the analyticity in the abstract sense was proved for the solutions of (0.5)–(0.6) in  $L^p(\Omega)$ ,  $1 < p < \infty$ , when  $\{B_j\}$  is normal and all the rays  $\{re^{i\theta} : 0 < r < \infty\}$  with  $\pi/2 \leq \theta \leq 3\pi/2$  are of minimal growth with respect to  $(A, \{B_j\}, \Omega)$  as an application of a result on the existence problem of abstract differential equations in a

Banach space ([6]). We mention also [12], [9], [7], [8] and [15] for related topics.

**1. Notations and assumptions**

We denote by  $\Omega$  a domain in the  $n$ -dimensional Euclidean space  $E_n$  and by  $\partial\Omega$  its boundary. Let  $(x, t) = (x_1, \dots, x_n, t)$  be the generic point in  $E_{n+1}$ . We put  $D_x = \left( \frac{1}{(-1)^{1/2}} \frac{\partial}{\partial x_1}, \dots, \frac{1}{(-1)^{1/2}} \frac{\partial}{\partial x_n} \right)$ ,  $D_t = \frac{1}{(-1)^{1/2}} \frac{\partial}{\partial t}$  and denote by  $D_x^\alpha$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$ , the  $x$ -derivative  $D_1^{\alpha_1} \dots D_n^{\alpha_n} = \left( \frac{1}{(-1)^{1/2}} \frac{\partial}{\partial x_1} \right)^{\alpha_1} \dots \left( \frac{1}{(-1)^{1/2}} \frac{\partial}{\partial x_n} \right)^{\alpha_n}$ .  $|\alpha|$  denotes the length of the multi-index  $\alpha$ :  $|\alpha| = \alpha_1 + \dots + \alpha_n$ .

For any integer  $k$  we denote by  $H_k(\Omega)$  the class of all complex valued measurable functions whose distribution derivatives of order up to  $k$  are square integrable in  $\Omega$ , the norm of  $H_k(\Omega)$  being denoted by  $\| \cdot \|_{k, \Omega}$ :

$$\|u\|_{k, \Omega}^2 = \sum_{|\alpha| \leq k} \int_{\Omega} |D_x^\alpha u(x)|^2 dx.$$

Especially  $H_0(\Omega) = L^2(\Omega)$ .

$H_{k-1/2}(\partial\Omega)$  is to be the class of functions  $\phi$  which are the boundary values of functions  $v$  belonging to  $H_k(\Omega)$ . In this class we introduce the norm

$$\langle \phi \rangle_{k, \partial\Omega} = \inf \|v\|_{k, \Omega},$$

where inf is taken over all functions  $v$  in  $H_k(\Omega)$  which equal  $\phi$  on the boundary.

Let  $m$  and  $l$  be positive intergers and let  $d = 2m/l$ . We assume that  $d$  is also an integer.  $A(x, t, D_x, D_t)$  is a linear differential operator of the form

$$A(x, t, D_x, D_t) = \sum_{k=0}^l A_{l-k}(x, t, D_x) D_t^k \tag{1.1}$$

where

$$A_{l-k}(x, t, D_x) = \sum_{|\alpha| \leq 2m - kd} a_{l-k, \alpha}(x, t) D_x^\alpha, \quad 0 \leq k < l, \tag{1.2}$$

are differential operators in  $x$  of order  $2m - kd$  at the most with coefficients defined in  $\bar{\Omega} \times \{t : -\infty < t < \infty\}$  and  $A_0(x, t, D_x) \equiv 1$ . Let  $m_j$  be non negative integers smaller than  $2m$  and let  $l_j = [m_j/d]$  = the integral part of  $m_j/d$ .  $B_j(x, t, D_x, D_t)$ ,  $j = 1, \dots, m$ , are differential operators of the form

$$B_j(x, t, D_x, D_t) = \sum_{k=0}^{l_j} B_{j, l_j - k}(x, t, D_x) D_t^k \tag{1.3}$$

where

$$B_{j,l_j-k}(x, t, D_x) = \sum_{|\beta| \leq m_j - kd} b_{j,l_j-k,\beta}(x, t) D_x^\beta \tag{1.4}$$

are differential operators in  $x$  of order  $m_j - kd$  at the most with coefficients defined on  $\partial\Omega \times \{t : -\infty < t < \infty\}$ . In what follows we shall assume without restriction that the coefficients of  $B(x, t, D_x, D_t)$ ,  $j=1, \dots, m$ , are defined not only on  $\partial\Omega \times \{t : -\infty < t < \infty\}$  but also in  $\bar{\Omega} \times \{t : -\infty < t < \infty\}$ .

Denote by  $A_{l-k}^*(x, t, D_x)$ ,  $k=0, \dots, l$ , the sum of terms in  $A_{l-k}(x, t, D_x)$  which are of precise order  $2m - kd$ , letting  $A_{l-k}^* = 0$  if there are no such terms. Following [2] the weighted principal part  $A^*$  of  $A$  is defined as

$$A^*(x, t, D_x, D_t) = \sum_{k=0}^l A_{l-k}^*(x, t, D_x) D_t^k. \tag{1.5}$$

Similarly we denote by  $B_{j,l_j-k}^*(x, t, D_x)$ ,  $k=0, \dots, l_j$ ,  $j=1, \dots, m$ , the sum of terms in  $B_{j,l_j-k}$  which are of precise order  $m_j - kd$ , letting  $B_{j,l_j-k}^* = 0$  if there are no such terms. The weighted principal part  $B_j^*$  of  $B_j$  is

$$B_j^*(x, t, D_x, D_t) = \sum_{k=0}^{l_j} B_{j,l_j-k}^*(x, t, D_x) D_t^k. \tag{1.6}$$

Let  $y$  be an auxiliary real variable and we denote by  $\Gamma$  the infinite cylinder :

$$\Gamma = \{(x, y) : x \in \Omega, -\infty < y < \infty\}.$$

For each fixed  $t$ ,  $A(x, t, D_x, \pm D_y^a)$  is a linear differential operator in  $(x, y)$  of order  $2m$  with coefficients defined in  $\bar{\Gamma}$ . Similarly for each  $i=1, \dots, m$  and  $t \in (-\infty, \infty)$ ,  $B_j(x, t, D_x, \pm D_y^a)$  is a linear differential operator in  $(x, y)$  of order  $m_j$  at most with coefficients defined in  $\bar{\Gamma}$ . Clearly the principal part  $A'(x, t, D_x, \pm D_y^a)$  of  $A(x, t, D_x, \pm D_y^a)$  is

$$A'(x, t, D_x, \pm D_y^a) = A^*(x, t, D_x, \pm D_y^a). \tag{1.7}$$

Similarly if the order of  $B_j(x, t, D_x, \pm D_y^a)$  is equal to  $m_j$ , its principal part is

$$B_j'(x, t, D_x, \pm D_y^a) = B_j^*(x, t, D_x, \pm D_y^a). \tag{1.8}$$

ASSUMPTIONS. (I) For each  $t$ ,  $A(x, t, D_x, \pm D_y^a)$  is a uniformly elliptic operator of order  $2m$  in  $\Gamma$ ; hence  $A(x, t, D_x, D_t)$  is a uniformly weighted elliptic operator of order type  $(2m, l)$  in the terminology of [2]. For every  $(x, t) \in \partial\Omega \times (-\infty, \infty)$  and for every set of real vectors  $\xi \in E_n, \nu \in E_n, \tau \in E_1$  such that  $(\xi, \tau) \neq 0$  and  $\nu \neq 0$  the polynomial  $A^*(x, t, \xi + s\nu, \tau)$  in  $s$  has exactly  $m$  roots with a positive imaginary part.

(II) For each  $j=1, \dots, m$  and  $t \in (-\infty, \infty)$ , the order of  $B_j(x, t, D_x, \pm D_y^a)$  is equal to  $m_j$ .

(III) Let  $(x, t)$  be any point on  $\partial\Omega \times (-\infty, \infty)$ . Let  $\nu$  be the normal to  $\partial\Omega$  and  $\xi \in E_n$  be a real vector parallel to  $\partial\Omega$  at  $x$ . Whenever  $\tau$  is a real number such that  $(\xi, \tau) \neq 0$ , the polynomials in  $s: B^*(x, t, \xi + s\nu, \tau)$ ,  $j=1, \dots, m$ , are linearly independent modulo the polynomial  $\prod_{k=1}^m (s - s_k^+(\xi, \tau))$  where  $s_k^+(\xi, \tau)$ ,  $k=1, \dots, m$ , are the roots of  $A^*(x, t, \xi + s\nu, \tau)$  with positive imaginary part. In other words the Complementing Condition is satisfied by  $(A(x, t, D_x, D_y^a), \{B_j(x, t, D_x, D_y^a)\}_{j=1}^m, \Gamma)$  and  $(A(x, t, D_x, -D_y^a), \{B_j(x, t, D_x, -D_y^a)\}_{j=1}^m, \Gamma)$ .

(IV) For  $|\alpha| \leq 2m - kd$ ,  $k=0, \dots, l$ ,  $a_{l-k, \alpha}(x, t)$  (recall (1.2)) are continuous in  $\bar{\Omega} \times \{t: -\infty < t < \infty\}$ . For  $|\beta| \leq m_j - kd$ ,  $k=0, \dots, l_j$ ,  $|\kappa| \leq 2m - m_j$ ,  $j=1, \dots, m$  and  $i=0, \dots, l+1$ ,  $D_x^i b_{j, l_j - k, \beta}(x, t)$  and  $D_t^i b_{j, l_j - k, \beta}(x, t)$  (recall (1.4)) are continuous in  $\bar{\Omega} \times \{t: -\infty < t < \infty\}$ .

(V)  $\Omega$  is a bounded domain of class  $C^{2m}$ .

We shall first prove some results on differentiability in  $t$  of solutions of (0.1)-(0.2) assuming further differentiability of the coefficients. The problem is local, therefore without loss of generality we shall assume

(IV') All the functions in (IV) are uniformly continuous and bounded in  $\bar{\Omega} \times \{t: -\infty < t < \infty\}$ .

Throughout this paper it is understood that any solution of (0.1)-(0.2) is a function  $u$  such that  $D_t^k u(t)$  is a strongly continuous function of  $t$  with values in  $H_{2m-kd}(\Omega)$  for  $k=0, 1, \dots, 2m$  and such that (0.1)-(0.2) hold for  $u$ . In what follows we denote by  $C_1, C_2, \dots$  constants dependent only on the assumptions (I)~(V) and (IV') unless otherwise stated.

**2. Estimate in case of time-independent coefficients**

As a preparation we obtain some estimates in the special case in which all the coefficients of  $A$  and  $\{B_j\}_{j=1}^m$  are independent of  $t$ :

$$A(x, t, D_x, D_t) = A(x, D_x, D_t),$$

$$B_j(x, t, D_x, D_t) = B_j(x, D_x, D_t), \quad j = 1, \dots, m.$$

From now on we shall usually write the abbreviated forms  $\|u\|_k, \langle \phi \rangle_k$  omitting  $\Omega$  and  $\partial\Omega$  respectively.

**Lemma 2.1.** *Under the assumptions of section 1 for any function  $u \in H_{2m}(\Omega)$  and real number  $\lambda$  we have*

$$\sum_{k=0}^{2m} |\lambda|^{(2m-k)/d} \|u\|_k \leq C_1 \{ \|A(x, D_x, \lambda)u\|_0 + \sum_{j=1}^m |\lambda|^{(2m-m_j)/d} \|w_j(\lambda)\|_0 + \sum_{j=1}^m \|w_j(\lambda)\|_{2m-m_j} + \|u\|_0 \}, \tag{2.1}$$

where  $w_j$  ( $j=1, \dots, m$ ) is an arbitrary function in  $H_{2m-m_j}(\Omega)$  which satisfies the boundary condition

$$B_j(x, D_x, \lambda)u(x) = w_j(x, \lambda), \quad x \in \partial\Omega, \quad -\infty < \lambda < \infty, \quad j=1, \dots, m. \tag{2.2}$$

Proof. We follow the proof of Theorem 5.2 of [2]. Let  $\zeta(y)$  be an infinitely differentiable function on the real line such that  $\zeta=1$  for  $|y| \leq 1$ ,  $\zeta=0$  for  $|y| \geq 2$ . We introduce the function

$$v(x, y) = \zeta(y) e^{(-1)^{1/2} \mu y} u(x)$$

where  $\mu$  is a real number and  $u \in H_{2m}(\Omega)$ . By the assumptions and the remark in the proof of Theorem 5.2 of [2], we have

$$\|v\|_{2m, \Gamma} \leq C_2 \{ \|A(x, D_x, D_y^\alpha)v\|_{0, \Gamma} + \sum_{j=1}^m \langle B_j(x, D_x, D_y^\alpha)v \rangle_{2m-m_j, \partial\Gamma} + \|v\|_{0, \Gamma} \}.$$

From the obvious relation

$$A(x, D_x, D_y^\alpha)v(x, y) = \zeta(y) e^{(-1)^{1/2} \mu y} A(x, D_x, \mu^d)u(x) + \sum_{k=1}^l A_{l-k}(x, D_x)u(x) \sum_{p=0}^{kd-1} \binom{kd}{p} D_y^{kd-p} \zeta(y) \mu^p e^{(-1)^{1/2} \mu y},$$

it follows that

$$\|A(x, D_x, D_y^\alpha)v\|_{0, \Gamma} \leq C_3 \{ \|A(x, D_x, \mu^d)u\|_0 + \sum_{k=1}^l (1 + |\mu|^{kd-1}) \|u\|_{2m-kd} \}. \tag{2.3}$$

Recalling the definition of the boundary norm  $\langle \rangle_k$  and noting that  $B_j(x, D_x, \mu^d)u(x) = w_j(x, \mu^d)$  on  $\partial\Omega$ , we get

$$\begin{aligned} \langle B_j(x, D_x, D_y^\alpha)v \rangle_{2m-m_j, \partial\Gamma} &\leq \| \zeta e^{(-1)^{1/2} \mu y} w_j(\mu^d) \|_{2m-m_j, \Gamma} \\ &+ \| \sum_{k=1}^{l_j} B_{l_j-k}(x, D_x)u \sum_{p=0}^{kd-1} \binom{kd}{p} D_y^{kd-p} \zeta \cdot \mu^p e^{(-1)^{1/2} \mu y} \|_{2m-m_j, \Gamma} \\ &\leq C_4 \left\{ \sum_{q=0}^{2m-m_j} (1 + |\mu|)^{2m-m_j-q} \|w_j(\mu^d)\|_q \right. \\ &\left. + \sum_{k=1}^{l_j} \sum_{q=0}^{2m-m_j} (1 + |\mu|)^{kd-1+q} \|u\|_{2m-q-kd} \right\}. \end{aligned} \tag{2.4}$$

Noting that  $v(x, y) = e^{(-1)^{1/2} \mu y} u(x)$  for  $|y| \leq 1$ , we obtain

$$\|v\|_{2m, \Gamma}^2 \geq \|u e^{(-1)^{1/2} \mu y}\|_{2m, \Gamma}^2 \geq \sum_{k=0}^{2m} \|u\|_k^2 |\mu|^{2m-k}. \tag{2.5}$$

Thus if  $|\mu|$  is sufficiently large we find

$$\sum_{k=0}^{2m} |\mu|^{2m-k} \|u\|_k \leq C_5 \{ \|A(x, D_x, \mu^d)u\|_0 + \sum_{j=1}^m \sum_{k=0}^{2m-m_j} |\mu|^{2m-m_j-k} \|w_j(\mu^d)\|_k + |\mu|^{-1} \sum_{k=0}^{2m-1} |\mu|^{2m-k} \|u\|_k \} . \quad (2.6)$$

Replacing  $A(x, D_x, D_y^d)$  and  $\{B_j(x, D_x, D_y^d)\}_{j=1}^m$  by  $A(x, D_x, -D_y^d)$  and  $\{B_j(x, D_x, -D_y^d)\}_{j=1}^m$  respectively, we obtain an estimate similar to (2.6). Putting  $\lambda = \mu^d$  or  $\lambda = -\mu^d$ , we can show that if  $\lambda$  is a real number with a sufficiently large absolute value

$$\sum_{k=0}^{2m} |\lambda|^{(2m-k)/d} \|u\|_k \leq C_6 \{ \|A(x, D_x, \lambda)u\|_0 + \sum_{j=1}^m \sum_{k=0}^{2m-m_j} |\lambda|^{(2m-m_j-k)/d} \|w_j(\lambda)\|_k \} .$$

The proof of the lemma will be easily completed noting that for  $0 < k < 2m - m_j$

$$\begin{aligned} |\lambda|^{(2m-m_j-k)/d} \|w_j\|_k &\leq c_0 \|w_j\|_{2m-m_j}^{k/(2m-m_j)} \|w_j\|_0^{(2m-m_j-k)/(2m-m_j)} |\lambda|^{(2m-m_j-k)/d} \\ &\leq c_0 (\|w_j\|_{2m-m_j} + |\lambda|^{(2m-m_j)/d} \|w_j\|_0) . \end{aligned}$$

If  $u(x, t)$  is a function defined for  $x \in \Omega, -\infty < t < \infty$ , we denote by  $\hat{u}(x, \lambda)$  the Fourier transform of  $u$  with respect to  $t$ :

$$\hat{u}(x, \lambda) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-(-1)^{1/2} \lambda t} u(x, t) dt . \quad (2.7)$$

**Lemma 2.2.** *Let  $u$  be a function of  $t$  with values in  $H_{2m}(\Omega)$  satisfying*

$$A(x, D_x, D_t)u(x, t) = f(x, t), \quad x \in \Omega, \quad -\infty < t < \infty, \quad (2.8)$$

$$B_j(x, D_x, D_t)u(x, t) = g_j(x, t), \quad x \in \partial\Omega, \quad -\infty < t < \infty, \quad (2.9)$$

$$j = 1, \dots, m,$$

where  $f$  and  $g_j, j = 1, \dots, m$ , are functions of  $t$  with values in  $L^2(\Omega)$  and  $H_{2m-m_j}(\Omega)$  respectively. Then we have

$$\begin{aligned} &\sum_{k=0}^{2m} \int_{-\infty}^{\infty} (|\lambda|^{(2m-k)/d} \|\hat{u}(\lambda)\|_k)^2 d\lambda \leq C_7 \left\{ \int_{-\infty}^{\infty} \|f(t)\|_0^2 dt \right. \\ &+ \sum_{j=1}^m \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} \|\hat{g}_j(\lambda)\|_0)^2 d\lambda + \\ &\left. + \sum_{j=1}^m \int_{-\infty}^{\infty} \|g_j(t)\|_{2m-m_j}^2 dt + \int_{-\infty}^{\infty} \|u(t)\|_0^2 dt \right\} \end{aligned} \quad (2.10)$$

if all the terms of the above inequality are finite.

Note that (2.9) is assumed to hold only on the boundary although the functions on both sides are defined also in the interior.

Proof.  $\hat{u}(x, \lambda)$  satisfies



$$\begin{aligned} A(x, D_x, \lambda) \hat{u}(x, \lambda) &= \hat{f}(x, \lambda), & x \in \Omega, \quad -\infty < \lambda < \infty, \\ B_j(x, D_x, \lambda) \hat{u}(x, \lambda) &= \hat{g}_j(x, \lambda), & x \in \partial\Omega, \quad -\infty < \lambda < \infty. \end{aligned}$$

If we apply Lemma 2.1 to  $\hat{u}(x, \lambda)$  taking  $\hat{g}_j(x, \lambda)$  as  $w_j(x, \lambda)$ , we get

$$\begin{aligned} \sum_{k=0}^{2m} (|\lambda|^{(2m-k)/d} \|\hat{u}(\lambda)\|_k)^2 &\leq C_\delta \{ \|\hat{f}(\lambda)\|_0^2 + \sum_{j=1}^m (|\lambda|^{(2m-m_j)/d} \|\hat{g}_j(\lambda)\|_0)^2 \\ &\quad + \sum_{j=1}^m \|\hat{g}_j(\lambda)\|_{2m-m_j}^2 + \|\hat{u}(\lambda)\|_0^2 \}. \end{aligned}$$

Integrating the above relation over  $-\infty < \lambda < \infty$  and applying Plancherel theorem we complete the proof.

### 3. Estimates in general case

In this section we obtain some estimates in the general case of time-dependent coefficients.

**Lemma 3.1.** *Let  $v$  be a solution of*

$$A(x, t, D_x, D_t)v(x, t) = f(x, t), \quad x \in \Omega, \quad -\infty < t < \infty, \quad (3.1)$$

$$\begin{aligned} B_j(x, t, D_x, D_t)v(x, t) &= g_j(x, t), & x \in \partial\Omega, \quad -\infty < t < \infty, & (3.2) \\ & & j=1, \dots, m. & \end{aligned}$$

*If the support of  $v$  as a function of  $t$  is sufficiently small, then the same estimate as (2.10) holds for  $v$  replacing  $C_7$  by another constant if necessary.*

*Proof.* Let  $s$  be an arbitrary real number and suppose

$$v(x, t) = 0 \quad \text{if} \quad |t-s| > \delta \quad (3.3)$$

where  $\delta$  is some small positive number. Let  $\varphi(t)$  be a smooth real valued function such that  $\varphi(t)=1$  for  $|t| \leq 1$  and  $\varphi(t)=0$  for  $|t| > 2$ . If we write  $\psi(t) = \varphi((t-s)/\delta)$ , then  $\psi(t)=1$  on the support of  $v$ . Clearly  $v$  satisfies

$$A(x, s, D_x, D_t)v(x, t) = F(x, t), \quad x \in \Omega, \quad -\infty < t < \infty, \quad (3.4)$$

$$\begin{aligned} B_j(x, s, D_x, D_t)v(x, t) &= G_j(x, t), & x \in \partial\Omega, \quad -\infty < t < \infty, & (3.5) \\ & & j=1, \dots, m, & \end{aligned}$$

where

$$F = f + \sum_{k=0}^l \sum_{|\alpha| \leq 2m-kd} \psi(t)(a_{l-k, \alpha}(x, s) - a_{l-k, \alpha}(x, t)) D_x^\alpha D_t^k v, \quad (3.6)$$

$$G_j = g_j + \sum_{k=0}^{l_j} \sum_{|\beta| \leq m_j - kd} \gamma_{j, l_j - k, \beta}(x, s, t) D_x^\beta D_t^k v, \quad (3.7)$$

with the notation

$$\gamma_{j,l_j-k,\beta}(x, s, t) = \psi(t)(b_{j,l_j-k,\beta}(x, s) - b_{j,l_j-k,\beta}(x, t)). \tag{3.8}$$

As is easily seen

$$|\gamma_{j,l_j-k,\beta}(x, s, t)| \leq C_{10}\delta, \tag{3.9}$$

$$|D_t^{l+1}\gamma_{j,l_j-k,\beta}(x, s, t)| \leq C_{11}\delta^{-l}, \tag{3.10}$$

and hence

$$|\hat{\gamma}_{j,l_j-k,\beta}(x, s, \lambda)| \leq C_{12}\delta^2, \tag{3.11}$$

$$|\lambda^{l+1}\hat{\gamma}_{j,l_j-k,\beta}(x, s, \lambda)| \leq C_{13}\delta^{1-l}. \tag{3.12}$$

It follows from the above two inequalities that

$$\int_{-\infty}^{\infty} |\hat{\gamma}_{j,l_j-k,\beta}(x, s, \lambda)| d\lambda \leq C_{14}\delta, \tag{3.13}$$

$$\int_{-\infty}^{\infty} |\lambda|^{(2m-m_j)/d} |\hat{\gamma}_{j,l_j-k,\beta}(x, s, \lambda)| d\lambda \leq C_{15}\delta^{1-(2m-m_j)/d} \quad \text{if } m_j > 0. \tag{3.14}$$

The Fourier transform of  $G_j(x, t)$  is

$$\hat{G}_j(x, \lambda) = \hat{g}_j(x, \lambda) + \sum_{k=0}^{l_j} \sum_{|\beta| \leq m_j - kd} \hat{\gamma}_{j,l_j-k,\beta}(x, s, \cdot) * (D_x^\beta D_t^k v)^\wedge(x, \cdot). \tag{3.15}$$

If we notice the following lemma :

**Lemma 3.2.** *If  $f$  and  $g$  are complex valued functions on the real line  $-\infty < \lambda < \infty$ , we have for  $\gamma > 0$*

$$\begin{aligned} \left( \int_{-\infty}^{\infty} (|\lambda|^\gamma |(f * g)(\lambda)|)^2 d\lambda \right)^{1/2} &\leq 2^\gamma \int_{-\infty}^{\infty} |f(\lambda)| d\lambda \left( \int_{-\infty}^{\infty} (|\lambda|^\gamma |g(\lambda)|)^2 d\lambda \right)^{1/2} \\ &+ 2^\gamma \int_{-\infty}^{\infty} |\lambda|^\gamma |f(\lambda)| d\lambda \left( \int_{-\infty}^{\infty} |g(\lambda)|^2 d\lambda \right)^{1/2} \end{aligned} \tag{3.16}$$

*provided that all the terms of the inequality are finite;*

we get from (3.11), (3.9), (3.10)

$$\begin{aligned} \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} |\hat{G}_j(x, \lambda)|)^2 d\lambda &\leq C_{16} \left\{ \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} |\hat{g}_j(x, \lambda)|)^2 d\lambda \right. \\ &+ \delta^2 \sum_{k=0}^{l_j} \sum_{|\beta| \leq m_j - kd} \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} |\lambda^k D_x^\beta \hat{\psi}(x, \lambda)|)^2 d\lambda \\ &+ \sum_{k=0}^{l_j} \sum_{|\beta| \leq m_j - kd} \delta^{2(1-(2m-m_j)/d)} \int_{-\infty}^{\infty} |\lambda^k D_x^\beta \hat{\psi}(x, \lambda)|^2 d\lambda. \end{aligned}$$

Integrating over  $x \in \Omega$  we get

$$\int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} \|\hat{G}_j(\lambda)\|_0)^2 d\lambda \leq C_{17} \left\{ \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} \|\hat{g}_j(\lambda)\|_0)^2 d\lambda \right.$$

$$\begin{aligned}
& + \delta^2 \sum_{k=0}^{l_j} \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d+k} \|\hat{\psi}(\lambda)\|_{m_j-kd})^2 d\lambda \\
& + \left. \sum_{k=0}^{l_j} \delta^{2(1-(2m-m_j)/d)} \int_{-\infty}^{\infty} (|\lambda|^k \|\hat{\psi}(\lambda)\|_{m_j-kd})^2 d\lambda \right\}. \tag{3.17}
\end{aligned}$$

Since

$$\begin{aligned}
|\lambda|^k & = (\delta^{(2m-m_j)/d} |\lambda|^{(2m-m_j+kd)/d})^{kd/(2m-m_j+kd)} \delta^{-(2m-m_j)k/(2m-m_j+kd)} \\
& \leq \delta^{(2m-m_j)/d} |\lambda|^{(2m-m_j+kd)/d} + \delta^{-k},
\end{aligned}$$

we have

$$\begin{aligned}
& \delta^{2(1-(2m-m_j)/d)} \int_{-\infty}^{\infty} (|\lambda|^k \|\hat{\psi}(\lambda)\|_{m_j-kd})^2 d\lambda \\
& \leq 2\delta^2 \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d+k} \|\hat{\psi}(\lambda)\|_{m_j-kd})^2 d\lambda \\
& + 2\delta^{2(1-(2m-m_j)/d-k)} \int_{-\infty}^{\infty} \|\hat{\psi}(\lambda)\|_{m_j-kd}^2 d\lambda. \tag{3.18}
\end{aligned}$$

Noting

$$\begin{aligned}
\|\hat{\psi}(\lambda)\|_{m_j-kd} & \leq c_0 (\delta^{(2m-m_j)/d+k} \|\hat{\psi}(\lambda)\|_{2m})^{(m_j-kd)/2m} (\delta^{-(m_j-kd)/d} \|\hat{\psi}(\lambda)\|_0)^{(2m-m_j+kd)/2m} \\
& \leq c_0 (\delta^{(2m-m_j)/d+k} \|\hat{\psi}(\lambda)\|_{2m} + \delta^{-(m_j-kd)/d} \|\hat{\psi}(\lambda)\|_0),
\end{aligned}$$

we have also

$$\begin{aligned}
& \delta^{2(1-(2m-m_j)/d-k)} \int_{-\infty}^{\infty} \|\hat{\psi}(\lambda)\|_{m_j-kd}^2 d\lambda \\
& \leq 2c_0 \delta^2 \int_{-\infty}^{\infty} \|\hat{\psi}(\lambda)\|_{2m}^2 d\lambda + 2c_0^2 \delta^{2(1-l)} \int_{-\infty}^{\infty} \|\hat{\psi}(\lambda)\|_0^2 d\lambda. \tag{3.19}
\end{aligned}$$

It should be noted here that (3.19) is true for  $k=0$  and also for  $k=l_j$  if  $l_j=m_j/d$ . Summing up we get

$$\begin{aligned}
& \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} \|\hat{G}_j(\lambda)\|_0)^2 d\lambda \leq C_{18} \left\{ \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} \|\hat{g}_j(\lambda)\|_0)^2 d\lambda \right. \\
& + \delta^2 \sum_{k=0}^{l_j} \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j+kd)/d} \|\hat{\psi}(\lambda)\|_{m_j-kd})^2 d\lambda \\
& \left. + \delta^2 \int_{-\infty}^{\infty} \|\hat{\psi}(\lambda)\|_{2m}^2 d\lambda + \delta^{2(1-l)} \int_{-\infty}^{\infty} \|\hat{\psi}(\lambda)\|_0^2 d\lambda \right\}. \tag{3.20}
\end{aligned}$$

The following two inequalities are easily proved

$$\begin{aligned}
& \int_{-\infty}^{\infty} \|G_j(t)\|_{2m-m_j}^2 dt \leq C_{19} \left\{ \int_{-\infty}^{\infty} \|g_j(t)\|_{2m-m_j}^2 dt \right. \\
& \left. + \varepsilon(\delta) \sum_{k=0}^{l_j} \int_{-\infty}^{\infty} (|\lambda|^k \|\hat{\psi}(\lambda)\|_{2m-kd})^2 d\lambda \right\}, \tag{3.21}
\end{aligned}$$

$$\int_{-\infty}^{\infty} \|F(t)\|_0^2 dt \leq C_{20} \left\{ \int_{-\infty}^{\infty} \|f(t)\|_0^2 dt + \varepsilon(\delta) \sum_{k=0}^l \int_{-\infty}^{\infty} \|D_k^* v(t)\|_{2m-kd}^2 dt \right\}, \tag{3.22}$$

where  $\varepsilon(\delta)$  is a function such that  $\varepsilon(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ . Making use of (3.20), (3.21), (3.22) in the application of Lemma 2.2 to  $v$ , we get

$$\begin{aligned} & \sum_{k=0}^{2m} \int_{-\infty}^{\infty} (|\lambda|^{(2m-k)/d} \|\hat{\phi}(\lambda)\|_k)^2 d\lambda \leq C_{21} \left\{ \int_{-\infty}^{\infty} \|f(t)\|_0^2 dt \right. \\ & + \sum_{j=1}^m \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} \|\hat{g}_j(\lambda)\|_0)^2 d\lambda + \sum_{j=1}^m \int_{-\infty}^{\infty} \|g_j(t)\|_{2m-m_j}^2 dt \\ & \left. + (\delta^{2(1-l)} + 1) \int_{-\infty}^{\infty} \|v(t)\|_0^2 dt + \varepsilon(\delta) \sum_{k=0}^{2m} \int_{-\infty}^{\infty} (|\lambda|^{(2m-k)/d} \|\hat{\phi}(\lambda)\|_k)^2 d\lambda \right\}; \end{aligned} \tag{3.23}$$

the desired inequality for  $v$  follows from (3.23) when  $0 < \delta \leq \delta_0$ ,  $\delta_0$  being some small positive number independent of  $s$  and  $u$ .

Let  $a$  and  $b$  be any real numbers satisfying  $0 < b - a \leq 2\delta_0$ , where  $\delta_0$  is a positive number such that (3.3) holds whenever the length of the support of  $v$  is not larger than  $2\delta_0$  (cf. the end of the proof of Lemma 3.1). Let  $\delta$  and  $\delta'$  be any pair of positive numbers satisfying  $\delta' < \delta < \delta_0$  and  $\varphi(t)$  be a smooth real valued function such that  $\varphi(t) = 0$  if  $t < a + \delta'$  or  $t > b + \delta'$  and  $\varphi(t) = 1$  if  $a + \delta \leq t \leq b - \delta$  and

$$|D_t^l \varphi(t)| \leq K(\delta - \delta')^{-k}, \quad k = 1, \dots, l, \tag{3.24}$$

with some constant  $K$  independent of  $\delta$  and  $\delta'$ .  $\eta$  is to be a smooth function having a compact support in the real line and satisfying  $\eta(t) = 1$  in some open set containing the closed interval  $[a, b]$ .

**Lemma 3.3.** *Using the above notations for any solution  $u$  of*

$$A(x, t, D_x, D_t)u(x, t) = f(x, t), \quad x \in \Omega, \quad a < t < b, \tag{3.25}$$

$$\begin{aligned} B_j(x, t, D_x, D_t)u(x, t) &= g_j(x, t), \quad x \in \partial\Omega, \quad a < t < b, \\ & j = 1, \dots, m, \end{aligned} \tag{3.26}$$

we have

$$\begin{aligned} & \sum_{k=0}^l \int_{a+\delta}^{b-\delta} \|D_t^{l-k} u(t)\|_{ka}^2 dt \leq C_{22} \left\{ \int_{a+\delta'}^{b-\delta'} \|f(t)\|_0^2 dt \right. \\ & + \sum_{j=1}^m \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} \|\hat{\varphi} g_j(\lambda)\|_0)^2 d\lambda + \sum_{j=1}^m \int_{a+\delta'}^{b-\delta'} \|g_j(t)\|_{2m-m_j}^2 dt \\ & \left. + (M_0 K)^2 \sum_{\substack{i+k \leq l \\ i \geq 1}} \frac{1}{(\delta - \delta')^{2i}} \int_{a+\delta'}^{b-\delta'} \|D_t^{l-k-i} u(t)\|_{ka}^2 dt + \int_{a+\delta'}^{b-\delta'} \|u(t)\|_0^2 dt \right\}, \end{aligned} \tag{3.27}$$

where  $M_0$  is a constant such that

$$|a_{l-k, \alpha}(x, t)| \leq M_0, \quad x \in \Omega, \quad -\infty < t < \infty, \quad k = 0, \dots, l, \quad |\alpha| \leq 2m,$$

$$\begin{aligned} |D_x^\kappa b_{j, l_j - k, \beta}(x, t)| &\leq M_0, \quad \int_{-\infty}^{\infty} |\hat{\eta} b_{j, l_j - k, \beta}(x, \lambda)| d\lambda \leq M_0, \quad x \in \Omega, \quad -\infty < t < \infty, \\ & k = 0, \dots, l_j, \quad |\beta| \leq m_j - kd, \quad |\kappa| \leq 2m - m_j, \quad j = 1, \dots, m. \end{aligned}$$

Proof. If we write

$$v(x, t) = \varphi(t)u(x, t),$$

then the length of the support of  $v$  is not larger than  $2\delta_0$  and  $v$  satisfies

$$\begin{aligned} A(x, t, D_x, D_t)v(x, t) &= F(x, t), & x \in \Omega, \quad -\infty < t < \infty, \\ B_j(x, t, D_x, D_t)v(x, t) &= G_j(x, t), & x \in \partial\Omega, \quad -\infty < t < \infty, \\ & & j = 1, \dots, m, \end{aligned}$$

where

$$\begin{aligned} F &= \varphi f + \sum_{k=1}^l A_{l-k}(x, t, D_x) \sum_{p=0}^{k-1} \binom{k}{p} D_t^{k-p} \varphi \cdot D_t^p u, \\ G_j &= \varphi g_j + \sum_{k=1}^{l_j} B_{j, l_j-k}(x, t, D_x) \sum_{p=0}^{k-1} \binom{k}{p} D_t^{k-p} \varphi \cdot D_t^p u \\ &= \varphi g_j + \sum_{k=1}^{l_j} \sum_{p=0}^{k-1} \binom{k}{p} \sum_{|\beta| \leq m_j - kd} \eta b_{j, l_j-k, \beta} D_x^\beta D_t^{k-p} \varphi \cdot D_t^p u. \end{aligned}$$

As in the proof of Lemma 3.1 we get

$$\begin{aligned} \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} \|\hat{G}_j(\lambda)\|_0)^2 d\lambda &\leq C_{23} \left\{ \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} \|\hat{\varphi} g_j(\lambda)\|_0)^2 d\lambda \right. \\ &+ M_0^2 \sum_{k=1}^{l_j} \sum_{p=0}^{k-1} \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} \|(D_t^{k-p} \varphi \cdot D_t^p u)^\wedge(\lambda)\|_{m_j - kd})^2 d\lambda \\ &\left. + M_0^2 \sum_{k=1}^{l_j} \sum_{p=0}^{k-1} \int_{-\infty}^{\infty} \|(D_t^{k-p} \varphi \cdot D_t^p u)^\wedge(\lambda)\|_{m_j - kd}^2 d\lambda \right\}. \end{aligned}$$

If  $l_j = m_j/d$ ,  $(2m - m_j)/d$  is an integer and we can immediately apply Parseval theorem to all the summands in the second sum on the right. If  $m_j/d$  is not an integer, we shall replace these summands by the sums of other integrals in which only integers occur as the exponents of  $|\lambda|$ . Hence suppose  $m_j/d$  is not an integer. We notice

$$\frac{2m - m_j}{d} = (l - l_j - 1) \frac{m_j - kd}{(l_j + 1 - k)d} + (l - k) \frac{(l_j + 1)d - m_j}{(l_j + 1 - k)d}, \quad k = 1, \dots, l_j. \quad (3.29)$$

By (3.29) we get for any function  $w \in H_{m_j - kd}(\Omega)$

$$\begin{aligned} &|\lambda|^{(2m-m_j)/d} \|w\|_{m_j - kd} \\ &\leq c_0 (|\lambda|^{l-l_j-1} \|w\|_{(l_j+1-k)d})^{(m_j - kd)/(l_j+1-k)d} (|\lambda|^{l-k} \|w\|_0)^{((l_j+1)d - m_j)/(l_j+1-k)d} \\ &\leq c_0 (|\lambda|^{l-l_j-1} \|w\|_{(l_j+1-k)d} + |\lambda|^{l-k} \|w\|_0). \end{aligned} \quad (3.30)$$

By (3.30) with  $(D_t^{k-p} \varphi \cdot D_t^p u)^\wedge(\lambda)$  in place of  $w$  we get

$$\begin{aligned}
 & \sum_{k=1}^{l_j} \sum_{\beta=0}^{k-1} \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} (D_t^{k-p} \varphi \cdot D_t^p u)^\wedge(\lambda) \|_{m_j-kd})^2 d\lambda \\
 & \leq c_0^2 \sum_{k=1}^{l_j} \sum_{\beta=0}^{k-1} \int_{-\infty}^{\infty} (|\lambda|^{l-l_j-1} \| (D_t^{k-p} \varphi \cdot D_t^p u)^\wedge(\lambda) \|_{(l_j+1-k)d})^2 d\lambda \\
 & + c_0^2 \sum_{k=1}^{l_j} \sum_{\beta=0}^{k-1} \int_{-\infty}^{\infty} (|\lambda|^{l-k} \| (D_t^{k-p} \varphi \cdot D_t^p u)^\wedge(\lambda) \|_0)^2 d\lambda \\
 & \leq c_0^2 \sum_{k=1}^{l_j} \sum_{\beta=0}^{k-1} \int_{-\infty}^{\infty} \| D_t^{l-l_j-1} (D_t^{k-p} \varphi(t) \cdot D_t^p u(t)) \|_{(l_j+1-k)d}^2 dt \\
 & + c_0^2 \sum_{k=1}^{l_j} \sum_{\beta=0}^{k-1} \int_{-\infty}^{\infty} \| D_t^{l-k} (D_t^{k-p} \varphi(t) \cdot D_t^p u(t)) \|_0^2 dt \\
 & \leq c_0^2 \sum_{k=1}^{l_j} \sum_{\beta=0}^{k-1} \sum_{i=0}^{l-l_j-1} \binom{l-l_j-1}{i}^2 \left\{ \frac{K}{(\delta-\delta')^{l-l_j-1-i+k-p}} \right\}^2 \int_{a+\delta'}^{b-\delta'} \| D_t^{p+i} u(t) \|_{(l_j+1-k)d}^2 dt \\
 & + c_0^2 \sum_{k=1}^{l_j} \sum_{\beta=0}^{k-1} \sum_{i=0}^{l-k} \binom{l-k}{i}^2 \left\{ \frac{K}{(\delta-\delta')^{l-k-i+k+p}} \right\}^2 \int_{a+\delta'}^{b-\delta'} \| D_t^{p+i} u(t) \|_0^2 dt \\
 & \leq (c_0 C_{24} K)^2 \sum_{\substack{i+k \leq l \\ 1 \leq i}} \frac{1}{(\delta-\delta')^{2i}} \int_{a+\delta'}^{b-\delta'} \| D_t^{l-k-i} u(t) \|_{kd}^2 dt .
 \end{aligned}$$

Clearly

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \| (D_t^{k-p} \varphi \cdot D_t^p u)^\wedge(\lambda) \|_{m_j-kd}^2 d\lambda \leq \int_{-\infty}^{\infty} \| D_t^{k-p} \varphi(t) \cdot D_t^p u(t) \|_{(l-k)d}^2 dt \\
 & \leq \frac{K^2}{(\delta-\delta')^{2(k-p)}} \int_{a+\delta'}^{b-\delta'} \| D_t^p u(t) \|_{(l-k)d}^2 dt .
 \end{aligned}$$

Summing up we obtain

$$\begin{aligned}
 & \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} \| \hat{G}_j(\lambda) \|_0)^2 d\lambda \leq C_{25} \left\{ \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} \| \hat{\varphi} g_j(\lambda) \|_0)^2 d\lambda \right. \\
 & \left. + (M_0 c_0 K)^2 \sum_{\substack{i+k \leq l \\ 1 \leq i}} \frac{1}{(\delta-\delta')^{2i}} \int_{a+\delta'}^{b-\delta'} \| D_t^{l-k-i} u(t) \|_{kd}^2 dt \right. .
 \end{aligned}$$

As is easily seen

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \| F(t) \|_0^2 dt \leq C_{26} \left\{ \int_{a+\delta'}^{b-\delta'} \| f(t) \|_0^2 dt \right. \\
 & \left. + (M_0 K)^2 \sum_{\substack{i+k \leq l \\ 1 \leq i}} \frac{1}{(\delta-\delta')^{2i}} \int_{a+\delta'}^{b-\delta'} \| D_t^{l-k-i} u(t) \|_{kd}^2 dt \right\} , \\
 & \int_{-\infty}^{\infty} \| G_j(t) \|_{2m-m_j}^2 dt \leq C_{27} \left\{ \int_{a+\delta'}^{b-\delta'} \| g_j(t) \|_{2m-m_j}^2 dt \right. \\
 & \left. + (M_0 K)^2 \sum_{\substack{i+k \leq l \\ 1 \leq i}} \frac{1}{(\delta-\delta')^{2i}} \int_{a+\delta'}^{b-\delta'} \| D_t^{l-k-i} u(t) \|_{kd}^2 dt \right\} .
 \end{aligned}$$

Hence with the aid of Lemma 3.1 we get (3.27).

**4. Differentiability in  $t$**

From now on we shall write  $f^{(p)}(x, t) = D_t^p f(x, t)$  for any function  $f$  of  $(x, t)$ . By  $q$  we denote a natural number. In addition to the assumptions of section 1 here we make the following

ASSUMPTIONS. (VI<sub>q</sub>)  $a_{i-k, \alpha}(x, t)$ ,  $|\alpha| \leq 2m - kd$ ,  $k = 0, \dots, l - 1$ , are  $q$  times continuously differentiable in  $t$  in  $\bar{\Omega} \times (-\infty, \infty)$ ;

(VII<sub>q</sub>)  $D_x^k b_{j, l_j - k, \beta}(x, t)$  and  $b_{j, l_j - k, \beta}(x, t)$ ,  $|\beta| \leq m_j - kd$ ,  $k = 0, \dots, l_j$ ,  $|\kappa| \leq 2m - m_j$ ,  $j = 1, \dots, m$ , are  $q$  times and  $q + l + 1$  times continuously differentiable in  $t$  respectively in  $\bar{\Omega} \times (-\infty, \infty)$ .

Let  $M_p$ ,  $p = 0, \dots, q$ , be positive constants such that for  $p = 0, \dots, q$

$$\sup_{\substack{x \in \Omega \\ -\infty < t < \infty}} |a_{i-k, \alpha}^{(p)}(x, t)| \leq M_p, \quad |\alpha| \leq 2m - kd, \quad k = 0, \dots, l - 1,$$

$$\sup_{\substack{x \in \Omega \\ -\infty < t < \infty}} |D_x^k b_{j, l_j - k, \beta}^{(p)}(x, t)| \leq M_p,$$

$$\sup_{x \in \Omega} \int_{-\infty}^{\infty} |(\eta b_{j, l_j - k, \beta}^{(p)})^\wedge(x, \lambda)| d\lambda \leq M_p,$$

$$\sup_{x \in \Omega} \int_{-\infty}^{\infty} |\lambda|^{(2m - m_j)/d} |(\eta b_{j, l_j - k, \beta}^{(p)})^\wedge(x, \lambda)| d\lambda \leq M_p \quad \text{if } m_j > 0,$$

$$|\beta| \leq m_j - kd, \quad k = 0, \dots, l_j, \quad |\kappa| \leq 2m - m_j, \quad j = 1, \dots, m,$$

where  $\eta$  is an arbitrary smooth real valued function with a compact support of some fixed length such that

$$\eta(t) = 1 \text{ in some interval of length } 2\delta_0 + 1, \quad |D_t^k \eta(t)| \leq K_1 \text{ for } k = 0, \dots, l + 1, \quad -\infty < t < \infty, \text{ with some fixed constant } K_1.$$

**Lemma 4.1.** *Suppose for some  $q$  the assumption (VI<sub>q</sub>) and (VII<sub>q</sub>) are satisfied in addition to the assumptions in section 1. Suppose  $f(x, t)$  is  $q$  times continuously differentiable in  $t$  when it is considered as a function with values in  $L^2(\Omega)$ . If  $u$  is a solution of*

$$A(x, t, D_x, D_t)u(x, t) = f(x, t), \quad x \in \Omega, \quad -\infty < t < \infty, \quad (4.1)$$

$$B_j(x, t, D_x, D_t)u(x, t) = 0, \quad x \in \partial\Omega, \quad -\infty < t < \infty, \quad (4.2)$$

$$j = 1, \dots, m,$$

such that  $\|D_t^{i+p} u(t)\|_{(I-\delta)Q}$ ,  $i = 0, \dots, l$ ,  $p = 0, \dots, q$ , are locally square integrable, then for any  $a, b$  such that  $0 < b - a \leq 2\delta_0$  and  $\delta, \delta'$  such that  $0 < \delta' < \delta < \delta_0$ , we have

$$\begin{aligned}
 & \sum_{k=0}^l \left( \int_{a+\delta}^{b-\delta} \|D_t^{l-k} u^{(q)}(t)\|_{ka}^2 dt \right)^{1/2} \leq C_{28} \left\{ \left( \int_{a+\delta'}^{b-\delta'} \|f^{(q)}(t)\|_0^2 dt \right)^{1/2} \right. \\
 & + K \sum_{r=0}^{q-1} \binom{q}{r} M_{q-r} \sum_{k=0}^l \left( \int_{a+\delta'}^{b-\delta'} \|D_t^{l-k} u^{(r)}(t)\|_{ka}^2 dt \right)^{1/2} \\
 & + K \sum_{r=0}^q \binom{q}{r} M_{q-r} \sum_{k=0}^{l-1} \sum_{i=1}^{l-k} \frac{1}{(\delta-\delta')^i} \left( \int_{a+\delta'}^{b-\delta'} \|D_t^{l-k-i} u^{(r)}(t)\|_{ka}^2 dt \right)^{1/2} \quad (4.3) \\
 & \left. + \left( \int_{a+\delta'}^{b-\delta'} \|u^{(q)}(t)\|_0^2 dt \right)^{1/2} \right\}.
 \end{aligned}$$

Proof.  $u^{(q)}(x, t)$  is a solution of

$$\begin{aligned}
 A(x, t, D_x, D_t)u^{(q)}(x, t) &= F(x, t), & x \in \Omega, & -\infty < t < \infty, \\
 B_j(x, t, D_x, D_t)u^{(q)}(x, t) &= G_j(x, t), & x \in \partial\Omega, & -\infty < t < \infty,
 \end{aligned}$$

where letting  $A_{l-k}^{(q-r)}$  and  $B_{j,l_j-k}^{(q-r)}$  stand for the operators obtained by differentiating in  $t$  ( $q-r$ ) times the corresponding coefficients of  $A_{l-k}$  and  $B_{j,l_j-k}$  respectively

$$\begin{aligned}
 F &= f^{(q)} - \sum_{r=0}^{q-1} \sum_{k=0}^l \binom{q}{r} A_{l-k}^{(q-r)}(x, t, D_x) D_t^k u^{(r)}, \\
 G_j &= - \sum_{r=0}^{q-1} \sum_{k=0}^{l_j} \binom{q}{r} B_{j,l_j-k}^{(q-r)}(x, t, D_x) D_t^k u^{(r)}, \quad j=1, \dots, m.
 \end{aligned}$$

By the assumptions  $F(t) \in L^2(\Omega)$  and  $G_j(t) \in H_{2m-m_j}(\Omega)$ ,  $j=1, \dots, m$ , for each  $t$ , hence we can apply Lemma 3.3 to  $u^{(q)}$ . The estimation of  $\int_{a+\delta'}^{b-\delta'} \|F(t)\|_0^2 dt$  and  $\int_{a+\delta'}^{b-\delta'} \|G_j(t)\|_{2m-m_j}^2 dt$ ,  $j=1, \dots, m$ , is straightforward. As in the proof of Lemma 3.3 we get

$$\begin{aligned}
 & \left( \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} \|\widehat{\varphi} G_j(\lambda)\|_0)^2 d\lambda \right)^{1/2} \\
 & \leq C_{29} K \sum_{r=0}^{q-1} \binom{q}{r} M_{q-r} \left\{ \sum_{k=0}^l \left( \int_{a+\delta'}^{b-\delta'} \|D_t^{l-k} u^{(r)}(t)\|_{ka}^2 dt \right)^{1/2} \right. \\
 & \left. + \sum_{k=0}^{l-1} \sum_{i=1}^{l-k} \frac{1}{(\delta-\delta')^i} \left( \int_{a+\delta'}^{b-\delta'} \|D_t^{l-1-i} u^{(r)}(t)\|_{ka}^2 dt \right)^{1/2} \right\}.
 \end{aligned}$$

Thus the proof of the present lemma is completed.

With the aid of Lemma 4.1 we can proceed by difference quotient argument to prove

**Theorem 4.1.** *Suppose the assumptions (I)~(V), (VI<sub>q</sub>) and (VII<sub>q</sub>) are satisfied for some  $q > 0$  and  $f$  is a  $q$  times continuously differentiable function of  $t$  with values in  $L^2(\Omega)$ . Let  $u$  be a solution of (4.1)–(4.2). If  $\sum_{k=0}^l \|D_t^{l-k} u(t)\|_{kd}$  is locally square integrable in  $t$ , then so is  $\sum_{k=0}^l \|D_t^{l-k+q} u(t)\|_{kd}$ ;*



especially  $u$  is a  $l+q-1$  times continuously differentiable function of  $t$  with values in  $L^2(\Omega)$ .

**5. Analyticity in  $t$**

In this section we prove the abstract analyticity in  $t$  of solutions of (4.1)-(4.2) assuming that the coefficients of  $A$ ,  $\{B_j\}$  as well as  $f$  are all analytic functions of  $t$ .

**Theorem 5.1.** *Suppose in addition to the assumptions (I)~(V)*

$$\begin{aligned} a_{l-k,\alpha}(x, t), \quad |\alpha| \leq 2m - kd, \quad k=0, \dots, l-1, \\ D_x^\kappa b_{j,l_j-k,\beta}(x, t), \quad |\beta| \leq m_j - kd, \quad k=0, \dots, l_j, \quad |\kappa| \leq 2m - m_j, \\ j=1, \dots, m, \end{aligned}$$

are analytic functions of  $t$  in  $\bar{\Omega} \times (-\infty, \infty)$  with  $t$ -derivatives of all orders continuous in  $\bar{\Omega} \times (-\infty, \infty)$ . If  $f$  is analytic in  $t$  when considered as a function with values in  $L^2(\Omega)$ , then so is the solution  $u$  of (4.1)-(4.2).

Proof. By Theorem 4.2  $\sum_{k=0}^l \|D_t^{l-k} u^{(q)}(t)\|_{kd}$  is locally square integrable. Write

$$N_\delta(v) = \sum_{k=0}^l \left( \int_{a+\delta}^{b-\delta} \|D_t^{l-k} v(t)\|_{kd}^2 dt \right)^{1/2}$$

whenever the right side is finite. Let  $0 < b-a \leq \min(2\delta_0, 1)$ . By the assumptions it is easy to see that there exist constants  $M$  and  $L_0$  such that

$$M_p \leq M_0 M^p p!, \tag{5.1}$$

$$\left( \int_a^b \|f^{(p)}(t)\|_0^2 dt \right)^{1/2} \leq L_0 M^p p! \tag{5.2}$$

for any non-negative integer  $p$ . Now we apply Lemma 4.1 with  $\delta = (q+1)\varepsilon$  and  $\delta' = q\varepsilon$ , where  $\varepsilon$  is some small positive number. Noting

$$\begin{aligned} N_{q\varepsilon}(u^{(r)}) &\leq N_{(r+1)\varepsilon}(u^{(r)}) && \text{if } 0 \leq r \leq q-1, \\ N_{q\varepsilon}(u^{(r-i)}) &\leq N_{(r-i+1)\varepsilon}(u^{(r-i)}) && \text{if } l \leq r \leq q, 1 \leq i \leq l, \\ \left( \int_{a+q\varepsilon}^{b-q\varepsilon} \|u^{(q)}(t)\|_0^2 dt \right)^{1/2} &\leq N_{(q-l+1)\varepsilon}(u^{(q-l)}) && \text{if } q > l, \end{aligned} \tag{5.3}$$

we get when  $q > l$

$$\begin{aligned} \varepsilon^{q+l} N_{(q+1)\varepsilon}(u^{(q)}) &\leq C_{28} \{ \varepsilon^{q+l} L_0 M^q q! \\ + \sum_{r=0}^{q-1} \frac{q!}{r!} M_0 M^{q-r} \varepsilon^{q+l} N_{(r+1)\varepsilon}(u^{(r)}) &+ \sum_{r=l}^q \frac{q!}{r!} M_0 M^{q-r} \sum_{i=1}^l \varepsilon^{q+l-i} N_{(r-i+1)\varepsilon}(u^{(r-i)}) \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{r=0}^{l-1} \frac{q!}{r!} M_0 M^{q-r} \sum_{i=1}^l \varepsilon^{q+l-i} \sum_{k=0}^{l-i} \left( \int_{a+q\varepsilon}^{b-q\varepsilon} \|D_t^{l-k-i} u^{(r)}(t)\|_{k\alpha}^2 dt \right)^{1/2} \\
 &+ \varepsilon^{q+l} N_{(q-l+1)\varepsilon}(u^{(q-l)}) \}. \tag{5.3}
 \end{aligned}$$

We prove that there exists a positive constant  $L$  such that for any positive integer  $p$

$$\varepsilon^{p+l} N_{(p+1)\varepsilon}(u^{(p)}) \leq L^{l+p+1}. \tag{5.4}$$

It is clear that (5.4) is true for  $p=0, \dots, l$  if  $L$  is sufficiently large. Suppose (5.4) is known to hold for  $r=0, \dots, q-1$  with some  $q>l$ . If  $q\varepsilon>1$  the left member of (5.3) vanishes. If  $q\varepsilon \leq 1$

$$\begin{aligned}
 &\sum_{r=0}^{q-1} q!(r!)^{-1} M_0 M^{q-r} \varepsilon^{q+l} N_{(r+1)\varepsilon}(u^{(r)}) \\
 &\leq M_0 M^q \sum_{r=0}^{q-1} q!(r!)^{-1} \varepsilon^{q-r} M^{-r} \varepsilon^{r+l} N_{(r+1)\varepsilon}(u^{(r)}) \\
 &\leq M_0 M^q L^{l+1} \sum_{r=0}^{q-1} (LM^{-1})^r \leq 2M_0 ML^{l+q}
 \end{aligned}$$

if  $L \geq 2M$ . Estimating similarly the other terms we get if  $L \geq \max(2M, 1)$

$$\begin{aligned}
 \varepsilon^{q+l} N_{(q+1)\varepsilon}(u^{(q)}) &\leq C_{28} \{L_0 M^q q! + 2M_0 ML^{l+q} + 4M_0 L^{q+l} \\
 &+ M_0 M^q \sum_{r=0}^{l-1} M^{-r} \sum_{i=1}^l \sum_{k=0}^{l-i} \left( \int_a^b \|D_t^{l-k-i} u^{(r)}(t)\|_{k\alpha}^2 dt \right)^{1/2} + L^{q+1} \}.
 \end{aligned}$$

Therefore if  $L$  is sufficiently large it follows that (5.4) holds also for  $p=q$ . If  $0 < \delta < \delta_0$  and  $(q+1)\varepsilon = \delta$ , then from (5.4) it follows

$$\sum_{k=0}^l \left( \int_{a+\delta}^{b-\delta} \|D_t^{l-k} u^{(q)}(t)\|_{k\alpha}^2 dt \right)^{1/2} \leq L^{l+q+1} (q+1)^{q+l} \delta^{-q-l}$$

which implies the desired analyticity of  $u(t)$ .

**6. Analyticity in all variables**

We conclude this paper by showing that the solutions of (0.1)–(0.2) are analytic in all variables in  $\bar{\Omega} \times (-\infty, \infty)$  if the coefficients, the known function and the boundary of  $\Omega$  are all analytic. In order to avoid an inessential complication we confine ourselves to the case  $l=1$ , hence the problem (0.1)–(0.2) is reduced to

$$D_t u(x, t) + A(x, t, D_x) u(x, t) = f(x, t), \quad x \in \Omega, \tag{6.1}$$

$$B_j(x, t, D_x) u(x, t) = 0, \quad x \in \partial\Omega, \quad j=1, \dots, m. \tag{6.2}$$

Here  $A(x, t, D_x) = \sum_{|\alpha| \leq 2m} a_\alpha(x, t) D_x^\alpha$  and  $B_j(x, t, D_x) = \sum_{|\beta| \leq m_j} b_{j,\beta}(x, t) D_x^\beta$  ( $j=1, \dots, m$ ) are differential operators in  $x$  of order  $2m$  and  $m_j$  respectively all of which do not contain  $D_t$ .

Our assumptions are restated as follows :

ASSUMPTIONS. (I') for each  $t \pm D_y^{2m} + A(x, t, D_x)$  is an elliptic operator in  $(x, y) \in \Gamma$  of order  $2m$ , and the Complementing Condition is satisfied by  $(\pm D_y^{2m} + A(x, t, D_x), \{B_j(x, t, D_x)\}_{j=1}^m, \Gamma)$ .

(II') The boundary  $\partial\Omega$  of  $\Omega$  is an analytic manifold.

(III') All the functions  $a_\alpha(x, t)$ ,  $|\alpha| \leq 2m$ ,  $b_{j,\beta}(x, t)$ ,  $|\beta| \leq m_j$ ,  $j=1, \dots, m$ ,  $f(x, t)$  are analytic in  $(x, t) \in \bar{\Omega} \times (-\infty, \infty)$ .

Before proving the main result we note that if the coefficients of  $A$ ,  $\{B_j\}$ , the function  $f$  and the boundary of  $\Omega$  are all infinitely differentiable, then the solution is infinitely differentiable up to the boundary. This statement can be proved by differentiating (6.1)-(6.2) in  $t$  successively or by starting from

$$\begin{aligned} \|v\|_{2m+k, \Gamma} &\leq C^{(k)} \{ \|(\pm D_y^{2m} + A(x, D_x))v\|_{k, \Gamma} \\ &\quad + \sum_{j=1}^m \langle B_j(x, D_x)v \rangle_{2m-m_j+k, \partial\Gamma} + \|v\|_{0, \Gamma} \} \end{aligned} \quad (6.3)$$

instead of the one with  $k=0$  in the proof of Lemma 2.1. Thus it suffices to verify that the Cauchy data of  $u$  on the boundary are analytic, since once this has been proved we can apply Holmgren's theorem to show the analyticity of  $u$  near the boundary and the interior analyticity of  $u$  is easier to be proved. Furthermore we want to notice here that the analyticity of  $u$  as a function of  $t$  with values in  $H_{2m+k}(\Omega)$  follows for any  $k>0$  under the present assumptions for the same reason that implied the infinite differentiability of  $u$  above.

By means of an analytic transformation we may suppose that the origin is located on a part of  $\partial\Omega$  which is contained in the hyperplane  $x_n=0$ , and we shall prove that the Cauchy data of  $u$  are analytic near the origin  $x=0, t=0$ . In what follows we denote by  $C_{29}, \dots, C_{42}$  constants dependent only on the assumption (I') as well as certain smoothness properties of the coefficients of  $A$ ,  $\{B_j\}$  and the boundary  $\partial\Omega$ .

We shall employ the following semi-norms and norms :

$$\begin{aligned} |v|_i^2 &= |v|_{i, \Omega}^2 = \sum_{|\kappa|=i} \int_{\Omega} |D_x^\kappa v(x)|^2 dx, \\ |v|_{i, r}^2 &= \sum_{|\kappa|=i} \int_{\substack{|x|<r \\ x_n>0}} |D_x^\kappa v(x)|^2 dx, \\ \|v\|_{k, r}^2 &= \sum_{j=0}^k |v|_{j, r}^2. \end{aligned}$$

We may take positive numbers  $c_0, c_1$  in such a manner that we have

$$|v|_i \leq c_0 |v|_j^{i/j} |v|_0^{(j-i)/j} + c_1 |v|_0, \tag{6.4}$$

$$|v|_{i,r} \leq c_0 |v|_{j,r}^{i/j} |v|_{0,r}^{(j-i)/j} + c_1 r^{-i} |v|_{0,r}, \tag{6.5}$$

if  $0 < i < j \leq 2m$  and  $0 < r$ . Furthermore following M. Schechter [11] we use the boundary semi-norms for a function defined on  $\partial\Omega$ :

$$[\varphi]_{k,\partial\Omega} = \inf |v|_{k,\Omega},$$

where the infimum is taken over all functions  $v$  coinciding  $\phi$  on  $\partial\Omega$ .

First we consider the case in which all the operators  $A, B_j, j=1, \dots, m$ , have only highest order derivatives with constant coefficients:

$$A(x, t, D_x) = A(D_x) = \sum_{|\alpha|=2m} a_\alpha D_x^\alpha,$$

$$B_j(x, t, D_x) = B_j(D_x) = \sum_{|\beta|=m_j} b_{j,\beta} D_x^\beta.$$

We begin with some estimation of a function  $u$  which satisfies

$$D_t u(x, t) + A(D_x)u(x, t) = f(x, t), \quad x \in \Omega, \tag{6.6}$$

$$B_j(D_x)u(x, t) = g_j(x, t), \quad x \in \partial\Omega, \quad j=1, \dots, m. \tag{6.7}$$

By Theorem 3.1 of [11] we have

$$|v|_{2m,\Gamma} \leq C_{29} (|(\pm D_y^{2m} + A(D_x))v|_{0,\Gamma} + \sum_{j=1}^m [B_j(D_x)u]_{2m-m_j,\partial\Gamma} + |v|_{0,\Gamma})$$

for any function  $v$  which is defined and smooth in  $\Gamma$  and vanishes for  $|y| > 1$ . Hence the same argument as in section 2 yields

**Lemma 6.1.** *If  $u$  satisfies (6.6)–(6.7) and has a compact support, then we have*

$$\begin{aligned} & \sum_{k=0}^{2m} \int_{-\infty}^{\infty} (|\lambda|^{(2m-k)/2m} |\hat{v}(\lambda)|_k)^2 d\lambda \leq C_{30} \left\{ \int_{-\infty}^{\infty} |f(t)|_0^2 dt \right. \\ & + \sum_{j=1}^m \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |\hat{g}_j(\lambda)|_0)^2 d\lambda + \sum_{j=1}^m \int_{-\infty}^{\infty} |g_j(t)|_{2m-m_j}^2 dt \\ & \left. + \sum_{j=1}^m \int_{-\infty}^{\infty} |g_j(t)|_0^2 dt + \int_{-\infty}^{\infty} |u(t)|_0^2 dt \right\}. \end{aligned}$$

Let  $r > 0, \delta > 0$  be such that  $r + \delta < \rho$  and let  $\zeta(x), \psi(t)$  be smooth functions satisfying  $\zeta(x) = 1$  for  $|x| < r, \zeta(x) = 0$  for  $|x| > r + \delta, \psi(t) = 1$  for  $|t| < r, \psi(t) = 0$  for  $|t| > r + \delta$  and for all  $x$  or  $t$

$$|D_x^\kappa \zeta(x)| \leq K \delta^{-|\kappa|}, \quad 1 \leq |\kappa| \leq 2m, \tag{6.8}$$

$$|D_t \psi(t)| \leq K \delta^{-1}. \tag{6.9}$$

If  $u$  is a solution of (6.6)–(6.7), then  $v(x, t) = \psi(t)\zeta(x)u(x, t)$  is a function with a compact support satisfying

$$\begin{aligned} D_t v + A(D_x)v &= F(x, t), \quad x \in \Omega, \\ B_j(D_x)v &= G_j(x, t), \quad x \in \partial\Omega, \quad j=1, \dots, m, \end{aligned}$$

where

$$F = \psi \zeta f + \psi' \zeta u + \psi \sum_{|\alpha|=2m} a_\alpha \sum_{\alpha' < \alpha} D_x^{\alpha-\alpha'} \zeta \cdot D_x^{\alpha'} u, \quad (6.10)$$

$$G_j = \psi \zeta g_j + \psi \sum_{|\beta|=m_j} b_{j,\beta} \sum_{\beta' < \beta} D_x^{\beta-\beta'} \zeta \cdot D_x^{\beta'} u. \quad (6.11)$$

By the same argument as that of section 2 we can prove

$$\begin{aligned} & \left( \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |\hat{G}_j(\lambda)|_0)^2 d\lambda \right)^{1/2} \\ & \leq \left( \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |\psi \hat{g}_j(\lambda)|_{0,r+\delta})^2 d\lambda \right)^{1/2} \\ & + C_{31} M_{0,0} \sum_{k=0}^{m_j-1} \frac{K}{\delta^{m_j-k}} \left( \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |\hat{u} \hat{\psi}(\lambda)|_{k,r+\delta})^2 d\lambda \right)^{1/2}, \end{aligned} \quad (6.12)$$

where  $M_{0,0}$  is a constant such that  $|a_\alpha| \leq M_{0,0}$ ,  $|b_{j,\beta}| \leq M_{0,0}$  for all  $\alpha, \beta$  and  $j$ . Nothing  $|\lambda|^{(2m-m_j)/2m} \leq \varepsilon |\lambda| + \varepsilon^{-(2m-m_j)/m_j}$  and using (6.5) we can easily show that for  $0 \leq k < m_j$

$$\begin{aligned} |\lambda|^{(2m-m_j)/2m} |w|_{k,r+\delta} &\leq c_0 \varepsilon |\lambda|^{(2m-m_j)/2m} |w|_{m_j,r+\delta} \\ &+ (c_0 \varepsilon^{-k/(m_j-k)} \varepsilon_1 + c_1 (\mathbf{r} + \delta)^{-k} \varepsilon_2) |\lambda| |w|_{0,r+\delta} \\ &+ (c_0 \varepsilon^{-k/(m_j-k)} \varepsilon_1^{(2m-m_j)/m_j} + c_1 (\mathbf{r} + \delta)^{-k} \varepsilon_2^{-(2m-m_j)/m_j}) |w|_{0,r+\delta}, \end{aligned} \quad (6.13)$$

$$\begin{aligned} |\lambda|^{(2m-m_j)/2m} |w|_{m_j,r+\delta} &\leq c_0 |w|_{2m,r+\delta} \\ &+ (c_0 + c_1 \varepsilon_3 (\mathbf{r} + \delta)^{-m_j}) |\lambda| |w|_{0,r+\delta} + c_1 (\mathbf{r} + \delta)^{-m_j} \varepsilon^{-2m} \varepsilon_3^{-(2m-m_j)/m_j} |w|_{0,r+\delta}. \end{aligned} \quad (6.14)$$

If we combine (6.13) and (6.14) after a suitable choice of  $\varepsilon_1, \varepsilon_2, \varepsilon_3$ , we get

$$\begin{aligned} |\lambda|^{(2m-m_j)/2m} |w|_{k,r+\delta} &\leq c_0^2 \varepsilon |w|_{2m,r+\delta} + c_2 \varepsilon |\lambda| |w|_{0,r+\delta} \\ &+ \{(c_0 + c_1) \varepsilon^{-(2m-m_j+k)/(m_j-k)} + (c_0 + c_1) c_1 \varepsilon (\mathbf{r} + \delta)^{-2m}\} |w|_{0,r+\delta} \end{aligned} \quad (6.15)$$

which  $c_2 = c_0^2 + c_0 c_1 + c_0 + c_1$ . Using (6.15) with  $\varepsilon = \delta^{m_j-k}/L$ ,  $L \geq 1$ , we obtain

$$\begin{aligned} & \frac{1}{\delta^{m_j-k}} \left( \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |\psi \hat{u}(\lambda)|_{k,r+\delta})^2 d\lambda \right)^{1/2} \\ & \leq \frac{c_0^2}{L} \left( \int_{-\infty}^{\infty} |\psi \hat{u}(\lambda)|_{2m,r+\delta}^2 d\lambda \right)^{1/2} + \frac{c_2}{L} \left( \int_{-\infty}^{\infty} (|\lambda| |\psi \hat{u}(\lambda)|_{0,r+\delta})^2 d\lambda \right)^{1/2} \\ & + \left( (c_0 + c_1) L^{(2m-m_j+k)/(m_j-k)} \delta^{-2m} + \frac{(c_0 + 1) c_1}{L (\mathbf{r} + \delta)^{2m}} \right) \left( \int_{-\infty}^{\infty} |\psi \hat{u}(\lambda)|_{0,r+\delta}^2 d\lambda \right)^{1/2}. \end{aligned} \quad (6.16)$$

Substituting (6.16) in (6.12) and then applying Plancherel's theorem we get

$$\begin{aligned} & \left( \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |\hat{G}_j(\lambda)|_0)^2 d\lambda \right)^{1/2} \\ & \leq \left( \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |\psi \hat{g}_k(\lambda)|_{0,r+\delta})^2 d\lambda \right)^{1/2} \\ & + C_{32} M_{0,0} K L^{-1} \left\{ \left( \int_{-r-\delta}^{r+\delta} |u(t)|_{2m,r+\delta}^2 dt \right)^{1/2} + \left( \int_{-r-\delta}^{r+\delta} |D_t u(t)|_{0,r+\delta}^2 dt \right)^{1/2} \right\} \\ & + C_{32} M_{0,0} K (K(M\delta))^{-1} + L^{2m-1} \delta^{-2m} \left( \int_{-r-\delta}^{r+\delta} |u(t)|_{0,r+\delta}^2 dt \right)^{1/2}. \end{aligned}$$

Similarly noting that if  $L \geq 1$

$$\sum_{k=0}^{2m-m_j} \delta^{-(2m-m_j-k)} |w|_{k,r+\delta} \leq C_{33} (|w|_{2m-m_j,r+\delta} + \delta^{-(2m-m_j)} |w|_{0,r+\delta}), \quad (6.17)$$

$$\sum_{k=0}^{2m-1} \delta^{-(2m-k)} |w|_{k,r+\delta} \leq C_{34} (L^{-1} |w|_{2m,r+\delta} + L^{2m-1} \delta^{-2m} |w|_{0,r+\delta}), \quad (6.18)$$

we obtain

$$\begin{aligned} & \left( \int_{-\infty}^{\infty} |G_j(t)|_{2m-m_j}^2 dt \right)^{1/2} \leq C_{35} K \left\{ \left( \int_{-r-\delta}^{r+\delta} |g_j(t)|_{2m-m_j,r+\delta}^2 dt \right)^{1/2} \right. \\ & + \delta^{-2m+m_j} \left( \int_{-r-\delta}^{r+\delta} |g_j(t)|_{0,r+\delta}^2 dt \right)^{1/2} + M_{0,0} L^{-1} \left( \int_{-r-\delta}^{r+\delta} |u(t)|_{2m,r+\delta}^2 dt \right)^{1/2} \\ & \left. + M_{0,0} L^{2m-1} \delta^{-2m} \left( \int_{-r-\delta}^{r+\delta} |u(t)|_{0,r+\delta}^2 dt \right)^{1/2} \right\}. \end{aligned}$$

$\int_{-\infty}^{\infty} |F(t)|_0^2 dt$  and  $\int_{-\infty}^{\infty} |G_j(t)|_0^2 dt$  can be estimated in a similar manner. Thus with the aid of Lemma 6.1 we obtain

**Lemma 6.2.** *If  $u$  is a solution of (6.6)–(6.7), then for each  $r > 0$ ,  $\delta > 0$  such that  $r + \delta < \rho$  we have*

$$\begin{aligned} & \left( \int_{-r}^r |D_t u(t)|_{0,r}^2 dt \right)^{1/2} + \left( \int_{-r}^r |u(t)|_{2m,r}^2 dt \right)^{1/2} \\ & \leq C_{36} \left[ \left( \int_{-r-\delta}^{r+\delta} |f(t)|_{0,r+\delta}^2 dt \right)^{1/2} + \sum_{j=1}^m \left( \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |\psi \hat{g}_j(\lambda)|_{0,r+\delta})^2 d\lambda \right)^{1/2} \right. \\ & + K \sum_{j=1}^m \left( \int_{-r-\delta}^{r+\delta} |g_j(t)|_{2m-m_j,r+\delta}^2 dt \right)^{1/2} + \sum_{j=1}^m K \delta^{-2m+m_j} \left( \int_{-r-\delta}^{r+\delta} |g_j(t)|_{0,r+\delta}^2 dt \right)^{1/2} \\ & + M_{0,0} K L^{-1} \left\{ \left( \int_{-r-\delta}^{r+\delta} |D_t u(t)|_{0,r+\delta}^2 dt \right)^{1/2} + \left( \int_{-r-\delta}^{r+\delta} |u(t)|_{2m,r+\delta}^2 dt \right)^{1/2} \right\} \\ & \left. + M_{0,0} K (K(L\delta))^{-1} + L^{2m-1} \delta^{-2m} \left( \int_{-r-\delta}^{r+\delta} |u(t)|_{0,r+\delta}^2 dt \right)^{1/2} \right], \end{aligned}$$

where  $\psi$  is the function depending on  $r, \delta$  as was defined after Lemma 6.1.

From now on we shall distinguish the normal variable  $x_n$  from the tangential space variables  $x' = (x_1, \dots, x_{n-1})$  and by  $\nabla^b$  we denote any

derivative of order  $p$  with respect to  $x'$ , For the sake of convenience we express Leibnitz formula as follows :

$$\nabla^p(fg) = \sum_{p'=0}^p \binom{p}{p'} \nabla^{p-p'} f \cdot \nabla^{p'} g,$$

although some comment would be necessary in doing so. We denote by  $A^\#$  and  $B_j^\#$  the principal parts of  $A$  and  $B_j$  respectively :

$$\begin{aligned} A^\#(x, t, D_x) &= \sum_{|\alpha|=2m} a_\alpha(x, t) D_x^\alpha, \\ B_j^\#(x, t, D_r) &= \sum_{|\beta|=m_j} b_{j,\beta}(x, t) D_r^\beta, \quad j=1, \dots, m. \end{aligned}$$

If  $u$  is a solution of (6.1)-(6.2), then for  $D_t^q \nabla^p u$  we have

$$D_t D_t^q \nabla^p u + A^\#(0, 0, D_x) D_t^q \nabla^p u = F_{p,q}(x, t), \quad x \in \Omega, \quad (6.19)$$

$$B_j^\#(0, 0, D_x) D_t^q \nabla^p u = G_{j,p,q}(x, t), \quad x \in \partial\Omega, \quad j=1, \dots, m. \quad (6.20)$$

Here

$$\begin{aligned} F_{p,q} &= D_t^q \nabla^p f + \sum_{|\alpha|=2m} (a_\alpha(0, 0) - a_\alpha(x, t)) D_x^\alpha D_t^q \nabla^p u \\ &\quad - \sum_{|\alpha| < 2m} a_\alpha D_x^\alpha D_t^q \nabla^p u - \sum' \binom{q}{q'} \binom{p}{p'} \sum_{|\alpha| \leq 2m} D_t^{q-q'} \nabla^{p-p'} a_\alpha D_x^\alpha D_t^{q'} \nabla^{p'} u, \end{aligned} \quad (6.21)$$

$$\begin{aligned} G_{j,p,q} &= \sum_{|\beta|=m_j} (b_{j,\beta}(0, 0) - b_{j,\beta}(x, t)) D_r^\beta D_t^q \nabla^p u \\ &\quad - \sum_{|\beta| < m_j} b_{j,\beta} D_r^\beta D_t^q \nabla^p u - \sum' \binom{q}{q'} \binom{p}{p'} \sum_{|\beta| \leq m_j} D_t^{q-q'} \nabla^{p-p'} b_{j,\beta} D_r^\beta D_t^{q'} \nabla^{p'} u, \end{aligned} \quad (6.22)$$

where  $\sum'$  means that the summation extends over all  $(p', q')$  satisfying  $0 \leq p' \leq p$ ,  $0 \leq q' \leq q$  except for  $(p, q) = (p', q')$ . If  $\eta(t)$  is a function such that  $\eta(t) = 1$  for  $|t| \leq r + \delta$ ,  $\eta(t) = 0$  for  $|t| > 2(r + \delta)$ , and  $|D_t^k \eta(t)| \leq K(r + \delta)^{-k}$  for  $k = 1, 2$ , then

$$\begin{aligned} \psi G_{j,p,q} &= \sum_{|\beta|=m_j} \gamma_{j,\beta} D_r^\beta \nabla^p (\psi D_t^q u) - \sum_{|\beta| < m_j} \eta b_{j,\beta} D_r^\beta \nabla^p (\psi D_t^q u) \\ &\quad - \sum' \binom{q}{q'} \binom{p}{p'} \sum_{|\beta| \leq m_j} \eta D_t^{q-q'} \nabla^{p-p'} b_{j,\beta} \cdot D_r^\beta \nabla^{p'} (\psi D_t^{q'} u), \end{aligned}$$

where  $\gamma_{j,\beta}(x, t) = \eta(t)(b_{j,\beta}(0, 0) - b_{j,\beta}(x, t))$ . As in section 3 we can easily show that there exists a constant  $M_1$  such that if  $|x| \leq r + \delta$  and  $m_j > 0$

$$\int_{-\infty}^{\infty} |\hat{\gamma}_{j,\beta}(x, \lambda)| d\lambda \leq M_1(r + \delta) \quad (6.23)$$

$$\int_{-\infty}^{\infty} |\lambda|^{(2m-m_j)/2m} |\hat{\gamma}_{j,\beta}(x, \lambda)| d\lambda \leq M_1(r + \delta)^{m_j/2m}. \quad (6.24)$$

Let  $M_{p,q}$  be positive numbers such that for all  $\alpha, \beta, \kappa, j$  with  $|\alpha| \leq 2m$ ,  $|\beta| \leq m_j$ ,  $|\kappa| \leq 2m - m_j$ ,  $j = 1, \dots, m$

$$\left. \begin{aligned}
 \sup_{x, t} |D_t^q \nabla^p a_\alpha(x, t)| &\leq M_{p, q}, \\
 \sup_{x, t} |D_x^k D_t^q \nabla^p b_{j, \beta}(x, t)| &\leq M_{p, q}, \\
 \sup_x \int_{-\infty}^{\infty} |(\eta D_t^q \nabla^p b_{j, \beta})^\wedge(x, \lambda)| d\lambda &\leq M_{p, q}, \\
 \sup_x \int_{-\infty}^{\infty} |\lambda|^{(2m-m_j)/2m} |(\eta D_t^q \nabla^p b_{j, \beta})^\wedge(x, \lambda)| d\lambda &\leq M_{p, q},
 \end{aligned} \right\} (6.25)$$

where sup is taken over  $\Omega \times (-\infty, \infty)$  or  $\Omega$ . Then it follows from (6.23), (6.24), (6.25) and Lemma 3.2 that

$$\begin{aligned}
 &\left( \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |\psi \widehat{G}_{j, p, q}(\lambda)|_{0, r+\delta})^2 d\lambda \right)^{1/2} \\
 &\leq C_{37} \left[ M_1(r+\delta) \left( \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |(\nabla^p \psi D_t^q u)^\wedge(\lambda)|_{m_j, r+\delta})^2 d\lambda \right)^{1/2} \right. \\
 &\quad + M_1(r+\delta)^{m_j/2m} \left( \int_{-\infty}^{\infty} |(\nabla^p \psi D_t^q u)^\wedge(\lambda)|_{m_j, r+\delta}^2 d\lambda \right)^{1/2} \\
 &\quad + M_{0,0} \left( \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} \|(\nabla^p \psi D_t^q u)^\wedge(\lambda)\|_{m_{j-1}, r+\delta})^2 d\lambda \right)^{1/2} \\
 &\quad + M_{0,0} \left( \int_{-\infty}^{\infty} \|(\nabla^p \psi D_t^q u)^\wedge(\lambda)\|_{m_{j-1}, r+\delta}^2 d\lambda \right)^{1/2} \\
 &\quad + \sum' \binom{q}{q'} \binom{p}{p'} M_{p-p', q-q'} \left\{ \left( \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} \|(\nabla^{p'} \psi D_t^{q'} u)^\wedge(\lambda)\|_{m_j, r+\delta})^2 d\lambda \right)^{1/2} \right. \\
 &\quad \left. + \left( \int_{-\infty}^{\infty} \|(\nabla^{p'} \psi D_t^{q'} u)^\wedge(\lambda)\|_{m_j, r+\delta}^2 d\lambda \right)^{1/2} \right\} \Big].
 \end{aligned} \tag{6.26}$$

The following inequalities follow from (6.5), (6.15), (6.14) :

$$\begin{aligned}
 (r+\delta)^{m_j/2m} |w|_{m_j, r+\delta} &\leq c_0(r+\delta) |w|_{2m, r+\delta} \\
 &\quad + (c_0 + c_1(r+\delta))^{-(2m-1)m_j/2m} |w|_{0, r+\delta}
 \end{aligned} \tag{6.27}$$

$$\begin{aligned}
 |\lambda|^{(2m-m_j)/2m} \|w\|_{m_{j-1}, r+\delta} &+ \|w\|_{m_{j-1}, r+\delta} \\
 &\leq C_{38}(r+\delta) (|w|_{2m, r+\delta} + |\lambda| |w|_{0, r+\delta} + (r+\delta)^{-2m} |w|_{0, r+\delta}),
 \end{aligned} \tag{6.28}$$

$$\begin{aligned}
 |\lambda|^{(2m-m_j)/2m} \|w\|_{m_j, r+\delta} &+ \|w\|_{m_j, r+\delta} \\
 &\leq C_{38} (|w|_{2m, r+\delta} + |\lambda| |w|_{0, r+\delta} + (r+\delta)^{-2m} |w|_{0, r+\delta}).
 \end{aligned} \tag{6.29}$$

We shall use the following notations :

$$\begin{aligned}
 d_{p, q}(u, r) &= \max \left\{ \left( \int_{-r}^r |D_t^{q+1} \nabla^p u(t)|_{0, r}^2 dt \right)^{1/2} + \left( \int_{-r}^r |D_t^q \nabla^p u(t)|_{2m, r}^2 dt \right)^{1/2} \right\} \\
 e_{p, q}(f, r) &= \max \left( \int_{-r}^r |D_t^q \nabla^p f(t)|_{0, r}^2 dt \right)^{1/2}
 \end{aligned}$$

for  $p, q=0, 1, \dots, 0 < r < \rho$ , where the maximum is taken over all deriva-



tives  $\nabla^p$  of order  $p$ .

Applying (6.27), (6.28), (6.29) to the right of (6.26) and then making use of Plancherel theorem we get

$$\begin{aligned} & \left( \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |\widehat{\psi} \widehat{G}_{j,p,q}(\lambda)|_{0,r+\delta})^2 d\lambda \right)^{1/2} \leq C_{39} (M_1 + M_{0,0})(r+\delta) \times \\ & \times \left\{ d_{p,q}(u, r+\delta) + \left( \frac{K}{\delta} + \frac{1}{(r+\delta)^{2m}} \right) \left( \int_{-r-\delta}^{r+\delta} |D_t^q \nabla^p u(t)|_{0,r+\delta}^2 dt \right)^{1/2} \right\} \\ & + C_{39} \sum' \binom{q}{q'} \binom{p}{p'} M_{p-p', q-q'} \left\{ d_{p',q'}(u, r+\delta) \right. \\ & \left. + \left( \frac{K}{\delta} + \frac{1}{(r+\delta)^{2m}} \right) \left( \int_{-r-\delta}^{r+\delta} |D_t^{q'} \nabla^{p'} u(t)|_{0,r+\delta}^2 dt \right)^{1/2} \right\}. \end{aligned}$$

As is easily seen

$$\begin{aligned} & \left( \int_{-r-\delta}^{r+\delta} |G_{j,p,q}(t)|_{2m-m_j, r+\delta}^2 dt \right)^{1/2} \\ & \leq C_{40} (M_1 + M_{0,0})(r+\delta) \left\{ \left( \int_{-r-\delta}^{r+\delta} |D_t^q \nabla^p u(t)|_{2m, r+\delta}^2 dt \right)^{1/2} \right. \\ & \left. + (r+\delta)^{-2m} \left( \int_{-r-\delta}^{r+\delta} |D_t^q \nabla^p(t)|_{0, r+\delta}^2 dt \right)^{1/2} \right\} \quad (6.30) \\ & + C_{40} \sum' \binom{q}{q'} \binom{p}{p'} M_{p-p', q-q'} \left\{ \left( \int_{-r-\delta}^{r+\delta} |D_t^{q'} \nabla^{p'} u(t)|_{2m, r+\delta}^2 dt \right)^{1/2} \right. \\ & \left. + (r+\delta)^{1-2m} \left( \int_{-r-\delta}^{r+\delta} |D_t^{q'} \nabla^{p'} u(t)|_{0, r+\delta}^2 dt \right)^{1/2} \right\}. \end{aligned}$$

It is not difficult to show that  $\left( \int_{-r-\delta}^{r+\delta} |F_{p,q}(t)|_{0, r+\delta}^2 dt \right)^{1/2}$  is dominated by the sum of  $e_{p,q}(f, r+\delta)$  and the right-hand side of (6.30) with  $C_{40}$  possibly replaced by another constant. Similarly

$$\begin{aligned} & \sum_{j=1}^m \delta^{-2m+m_j} \left( \int_{-r-\delta}^{r+\delta} |G_{j,p,q}(t)|_{0, r+\delta}^2 dt \right)^{1/2} \\ & \leq C_{41} (M_1 + M_{0,0})(r+\delta) \left\{ \left( \int_{-r-\delta}^{r+\delta} |D_t^q \nabla^p u(t)|_{2m, r+\delta}^2 dt \right)^{1/2} \right. \\ & \left. + \frac{1}{\delta^{2m}} \left( \int_{-r-\delta}^{r+\delta} |D_t^q \nabla^p u(t)|_{0, r+\delta}^2 dt \right)^{1/2} \right\} \\ & + C_{41} \sum' \binom{q}{q'} \binom{p}{p'} M_{p-p', q-q'} \left\{ \left( \int_{-r-\delta}^{r+\delta} |D_t^{q'} \nabla^{p'} u(t)|_{2m, r+\delta}^2 dt \right)^{1/2} \right. \\ & \left. + \frac{1}{\delta^{2m}} \left( \int_{-r-\delta}^{r+\delta} |D_t^{q'} \nabla^{p'} u(t)|_{0, r+\delta}^2 dt \right)^{1/2} \right\}. \end{aligned}$$

Thus with the aid of Lemma 6.2 we obtain

$$\begin{aligned}
 d_{p,q}(u, r) &\leq C_{42}e_{p,q}(f, r + \delta) + C_{43}(r + \delta + L^{-1})d_{p,q}(u, r + \delta) \\
 &\quad + \frac{C_{43}L^{2m-1}}{\delta^{2m}} \left( \int_{-r-\delta}^{r+\delta} |D_t^q \nabla^p u(t)|_{0, r+\delta}^2 dt \right)^{1/2} \\
 &\quad + C_{42} \sum' \binom{q}{q'} \binom{p}{p'} M_{p-p', q-q'} d_{p', q'}(u, r + \delta) \\
 &\quad + \frac{C_{43}}{\delta^{2m}} \sum' \binom{q}{q'} \binom{p}{p'} M_{p-p', q-q'} \left( \int_{-r-\delta}^{r+\delta} |D_t^{q'} \nabla^{p'} u(t)|_{0, r+\delta}^2 dt \right)^{1/2},
 \end{aligned}$$

where  $C_{43}$  is a constant dependent only on the assumptions (I'), certain smoothness properties of the coefficients of  $A$ ,  $\{B_j\}$  and the boundary of  $\Omega$ , and the constants  $M_1, M_{0,0}, K$ . We introduce the following notations:

$$\begin{aligned}
 N_{p,p,q}(u) &= ((p+q)!)^{-1} \sup_{\rho/2 \leq r < \rho} d_{p,q}(u, r)(\rho-r)^{2m+p+q}, \\
 M_{p,p,q}(f) &= ((p+q)!)^{-1} \sup_{\rho/2 \leq r < \rho} e_{p,q}(f, r)(\rho-r)^{2m+p+q}
 \end{aligned}$$

for  $p, q = 0, 1, 2, \dots$ . Under the present assumptions there exist positive constants  $M_0, M$  such that

$$M_{p,q} \leq M_0 M^{p+q} p! q!, \quad p, q = 0, 1, 2, \dots \tag{6.31}$$

From the definition it follows that if  $p \geq 2m$

$$\left( \int_{-r-\delta}^{r+\delta} |D_t^q \nabla^p u(t)|_{0, r+\delta}^2 dt \right)^{1/2} \leq (p+q-2m)! N_{p,p-2m,q}(u) (\rho-r-\delta)^{-p-q}. \tag{6.34}$$

Due to the previous remark  $u(t)$  is analytic in  $t$  as a function with values in  $H_{4m}(\Omega)$ , hence there exist positive constants  $N_0, N$  such that

$$\left. \begin{aligned} &|D_t^q \nabla^p u(t)|_{0, \rho} \\ &|D_t^q \nabla^p u(t)|_{2m, \rho} \end{aligned} \right\} \leq \|D_t^q u(t)\|_{4m, \Omega} \leq N_0 N^q q! \tag{6.34}$$

for  $0 \leq p \leq 2m, q = 0, 1, 2, \dots$ . We may assume  $N \geq 2M$ . Multiplying by  $(\rho-r)^{2m}/(p+q)!$  both sides of (6.31) with  $\rho/2 \leq r < \rho$  and  $\delta = (\rho-r)/(p+q+1)$  and using (6.32), (6.33), (6.34) we obtain for  $p \geq 2m$

$$\begin{aligned}
 N_{p,p,q}(u) &\leq C_{42}e^2 M_{p,p,q}(f) + C_{43}e^2(\rho + L^{-1})N_{p,p,q}(u) \\
 &\quad + C_{43}e^2 L^{2m-1} \frac{(p+q)^{2m}(p+q+2m)!}{(p+q)!} N_{p,p-2m,q}(u) \\
 &\quad + C_{42}e^2 \sum' \frac{q!}{q'} \frac{p!}{p'} \frac{(p'+q')!}{(p+q)!} M_0(\rho M)^{p-p'+q-q'} N_{p,p',q'}(u) \\
 &\quad + C_{43}e^2 \sum'_{p' \leq 2m} \frac{q!}{q'} \frac{p!}{p'} \frac{(p+q)^{2m}}{(p+q)!} (p'+q'-2m)! M_0(\rho M)^{p-p'+q-q'} N_{p,p'-2m,q'}(u) \\
 &\quad + \sqrt{2} C_{43}e^2 \sum_{p' < 2m} \frac{p! q!}{p'!} \frac{(p+q)^{2m}}{(p+q)!} M_0 M^{p-p'+q-q'} N^{q'} \rho^{p+q+1/2}.
 \end{aligned} \tag{6.35}$$

If we note

$$\begin{aligned} \frac{q!}{q'!} \frac{p!}{p'!} \frac{(p'+q)!}{(p+q)!} &\leq 1 && \text{if } 0 \leq p' \leq p, 0 \leq q' \leq q \\ \frac{(p+q)^{2m} (p+q-2m)!}{(p+q)!} &\leq (2m)^{2m} && \text{if } p \geq 2m, q \geq 0, \\ \frac{q!}{q'!} \frac{p!}{p'!} \frac{(p+q)^{2m}}{(p+q)!} (p'+q'-2m)! &\leq (2m)^{2m} (2m+1)^{p-p'} && \text{if } p \geq p' \geq 2m, \\ \frac{q!}{p'!} \frac{p!}{(p+q)!} &\leq (p+q)^{2m} && \text{if } p' \leq 2m, \end{aligned}$$

we obtain from (6.35)

$$\begin{aligned} N_{\rho, p, q}(u) &\leq C_{42} e^2 M_{\rho, p, q}(f) + C_{43} e^2 (\rho + L^{-1}) N_{\rho, p, q}(u) \\ &\quad + C_{42} e^2 (2m)^{2m} L^{2m-1} N_{\rho, p-2m, q}(u) + C_{42} e^2 M_0 \sum' (\rho M)^{p-p'+q-q'} N_{\rho, p', q'}(u) \\ &\quad + C_{43} e^2 (2m)^{2m} M_0 \sum'_{p' \geq 2m} (2m+1)^{p-p'} (\rho M)^{p-p'+q-q'} N_{\rho, p'-2m, q'}(u) \\ &\quad + \sqrt{2} C_{43} e^2 M_0 N_0 (p+q)^{2m} \rho^{p+q+1/2} \sum'_{p' < 2m} M^{p-p'+q-q'} N^{q'} . \end{aligned}$$

Hence if  $\rho$  is so small and  $L$  is so large that  $C_{43} e^2 (\rho + L^{-1}) \leq 1/2$ , we conclude

$$\begin{aligned} N_{\rho, p, q}(u) &\leq C_{44} \{ M_{\rho, p, q}(f) + L^{2m-1} N_{\rho, p-2m, q}(u) \\ &\quad + M_0 \sum' (\rho M)^{p-p'+q-q'} N_{\rho, p', q'}(u) \\ &\quad + M_0 \sum'_{p' \geq 2m} (2m+1)^{p-p'} (\rho M)^{p-p'+q-q'} N_{\rho, p'-2m, q'}(u) \\ &\quad + M_0 N_0 (p+q)^{2m} \rho^{p+q+1/2} \sum'_{p' < 2m} N^{p-p'+q-q'} N_0^{q'} \} \end{aligned} \quad (6.36)$$

with some constant  $C_{44}$  of the same property as  $C_{43}$ . By assumption we have  $M_{\rho, p, q}(f) \leq R_0 R^{p+q}$  for all  $p, q$  with some constants  $R_0, R$ . We want to show that there exist positive constants  $H_0, H$  such that

$$N_{\rho, p, q}(u) \leq H_0 H^{p+q} \quad (6.37)$$

for all  $p, q=0, 1, 2, \dots$ . It follows from (6.34) that this is the case for  $p \leq 2m, q=0, 1, 2, \dots$ . By induction we can show that the same is the case also for  $q=0, p=0, 1, 2, \dots$ . Hence we can proceed by induction with respect to  $p+q$  to show that (6.37) is true for every  $p, q$  if  $H_0$  and  $H$  are so large that

$$\begin{aligned} H &\geq 2(2m+1)\rho M, \quad 5C_{44}R_0 \leq H_0, \quad R \leq H, \\ 5C_{44}L^{2m-1} &\leq H^{2m}, \quad 30C_{44}M_0\rho M \leq H, \\ 30(2m+1)C_{44}M_0\rho M &\leq H^{2m+1} \\ \log(20mC_{44}M_0N_0\sqrt{\rho}H_0^{-1}) + 2m \log s &\leq s \log H(\rho M)^{-1} \quad \text{for } s=1, 2, \dots \end{aligned}$$

Due to the inequality

$$\int_{|x'| < r} |v(x', 0)|^2 dx' \leq Z_2 r^{-1} \int_{\substack{|x| < r \\ x_n > 0}} |v|^2 dx + Z_2 r \int_{\substack{|x| < r \\ x_n > 0}} |\text{grad } v|^2 dx$$

which may be found in p. 282 of [10] where  $Z_2$  depends only on  $n$ , we have

$$\begin{aligned} \left( \int_{-r}^r \int_{|x'| < r} |v(x', 0, t)|^2 dx' dt \right)^{1/2} &\leq \left( Z_2 r^{-1} \int_{-r}^r \int_{\substack{|x| < r \\ x_n > 0}} |v|^2 dx dt \right)^{1/2} \\ &+ \left( Z_2 r \int_{-r}^r \int_{\substack{|x| < r \\ x_n > 0}} |\text{grad}_x v|^2 dx dt \right)^{1/2}. \end{aligned} \tag{6.38}$$

Applying (6.37) to  $D_{x_n}^i D_t^q \nabla^p u$ ,  $0 \leq i < 2m$ , and using (6.5), (6.38) we get

$$\left( \int_{-r}^r \int_{|x'| < r} |D_{x_n}^i D_t^q \nabla^p u(x', 0, t)|^2 dx' dt \right)^{1/2} \leq \bar{H}_{i,r} \left( \frac{H}{\rho - r} \right)^{p+q} (p+q)! \tag{6.39}$$

for all  $p, q$  where

$$\begin{aligned} \bar{H}_{i,r} &= c_0 ((Z_2 r^{-1})^{1/2} + (Z_2 r)^{1/2}) H_0 (\rho - r)^{-2m} \\ &+ ((Z_2 r^{-1})^{1/2} (c_0 + c_1 r^{-i}) + (Z_2 r)^{1/2} (c_0 + c_1 r^{-i-1})) H_0 H^{-1} (\rho - r)^{-2m+1}. \end{aligned}$$

(6.39) shows that the Cauchy data of  $u$  on the boundary near the origin are analytic, hence with the aid of Holmgren's theorem we get

**Theorem 6.1.** *Under the assumptions (I'), (II'), (III') any solution of (6.1)–(6.2) is analytic in  $(x, t) \in \bar{\Omega} \times (-\infty, \infty)$ .*

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