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SELF DUAL GROUPS OF ORDER p^5 (p AN ODD PRIME)

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1. Introduction

Let G be a finite group, $\mathrm{Irr}(G) = \{\chi_1, \cdots, \chi_k\}$ be the set of all irreducible characters, $\mathrm{Cl}(G) = \{C_1, \cdots, C_k\}$ be the conjugacy classes of G, and x_i be a representative of C_i. We call G self dual if (by renumbering indices)

(*) $|C_j|\chi_i(x_j)/\chi_i(1) = \chi_j(1)\chi_j(x_i)$, for all i, j.

This condition is found in E. Bannai [1]. T. Okuyama [4] proved that self dual groups are nilpotent, and that a nilpotent group is self dual if and only if its all Sylow subgroups are self dual. So if we consider self dual groups we may deal with only p-groups. Obviously abelian groups are self dual. Some examples of self dual groups are discussed in [2].

If G is self dual it is easy to check that $|C_i| = \chi_i(1)^2$ for all i. It is easy to see that non abelian p-groups of order at most p^4 cannot satisfy this condition, and so they are not self dual. By the classification of groups of order 2^5, there is no group of order 2^5 satisfying this condition. For odd p, in classification table of groups of order p^5 [3], we can see that one isoclinism family Φ_6 satisfies this condition. We will show that all of groups in Φ_6 are self dual.

2. Definition of groups

We fix an odd prime p. Let G be a p-group of order p^5 which belongs to Φ_6 defined in [3], namely

$$G = \langle a_1,\ a_2,\ b,\ c_1,\ c_2 \mid [a_1, a_2] = b,\ [a_i, b] = c_i,\ a_i^p = \zeta_i,\ b^p = c_i^p = 1\ (i = 1,\ 2)\rangle,$$

where (ζ_1, ζ_2) is one of the followings:

(1) (c_1, c_2),
(2) (c_1^k, c_2), where $k = g^r$, $r = 1, 2, \cdots, (p-1)/2$,
(3) $(c_2^{-r/4}, c_1^r c_2^r)$, where $r = 1$ or ν,
(4) (c_2, c_1^ν),
(5) $(c_2^k, c_1 c_2)$, where $4k = g^{2r+1} - 1$, $r = 1, 2, \cdots, (p-1)/2$,
(6) $(c_1, 1)$, $p > 3$,
(7) $(1, c_1^r)$, where $r = 1$ or ν, and $p > 3$,
(8) $(1, 1)$,

where g denotes the smallest positive integer which is a primitive root $(\bmod p)$, and ν denotes the smallest positive integer which is a non-quadratic residue $(\bmod p)$.

In this paper, we shall show that

Theorem 2.1. *G is self dual.*

We treat cases (1)–(8) above simultaneously. In any case, $\mathrm{Z}(G)$, the center of G, is $\langle c_1, c_2\rangle$ and $\mathrm{D}(G)$, the derived subgroup of G, is $\langle b, c_1, c_2\rangle$.

3. Irreducible characters and conjugacy classes

First, we consider irreducible characters of G. It is easy to see that $G/\mathrm{Z}(G)$ is isomorphic to the extraspecial group of order p^3 and exponent p. So we know all characters of $G/\mathrm{Z}(G)$. We put

$$\begin{aligned}\mathrm{Irr}^0(G) &= \{\chi \in \mathrm{Irr}(G) \mid \ker\chi \geq \mathrm{D}(G)\},\\ \mathrm{Irr}^1(G) &= \{\chi \in \mathrm{Irr}(G) \mid \ker\chi \geq \mathrm{Z}(G) \text{ and } \ker\chi \not\geq \mathrm{D}(G)\}.\end{aligned}$$

Let χ be an irreducible character of G whose kernel does not contain $\mathrm{Z}(G)$. Then $\ker\chi$ contains some subgroup K of $\mathrm{Z}(G)$ of order p since $\mathrm{Z}(G)$ is not cyclic. So we consider characters of G/K for a fixed K. We put

$$\mathrm{Irr}^2(G|K) = \{\chi \in \mathrm{Irr}(G) \mid \ker\chi \not\geq \mathrm{Z}(G) \text{ and } \ker\chi \geq K\},$$

and

$$\mathrm{Irr}^2(G) = \bigcup_K \mathrm{Irr}^2(G|K),$$

where K runs over subgroups of $\mathrm{Z}(G)$ of order p. Observe that this is a disjoint union. Then obviously

$$\mathrm{Irr}(G) = \mathrm{Irr}^0(G) \cup \mathrm{Irr}^1(G) \cup \mathrm{Irr}^2(G).$$

Let V be a two-dimensional $\mathrm{GF}(p)$-vector space with a nondegenerate skew symmetric form $f : V \times V \to \mathrm{GF}(p)$. That is f is bilinear, $f(u,v) = -f(v,u)$ for all u, $v \in V$, and if $f(u,v) = 0$ for all $u \in V$, then $v = 0$. Note that $f(v,v) = 0$ for all $v \in V$. Let $\alpha : \mathrm{Z}(G) \to V$ be an isomorphism of abelian groups. We define $\gamma : G/\mathrm{D}(G) \longrightarrow \mathrm{Z}(G)$ by $\gamma(\overline{g}) = [g,b]$. Since $[\mathrm{D}(G), b] = 1$, this map is well-defined and γ is an isomorphism as abelian groups by the definition of G. Put $\beta = \alpha\gamma$. Then β is an isomorphism from $G/\mathrm{D}(G)$ to V. For K, choose $x \in G$ such that $\gamma(\langle\overline{x}\rangle) = K$, and define $H = \langle x, \mathrm{D}(G)\rangle$. Then H/K is abelian by the definition. Every character in $\mathrm{Irr}^2(G|K)$ is induced from a linear character of H whose kernel contains K but does not contain $\mathrm{Z}(G)$, and so the character has degree p.

Let ω be a primitive p-th root of unity. For x, we define $\eta_x \in \operatorname{Irr}(Z(G))$ by $\eta_x(z) = \omega^{f(\alpha(z),\beta(\overline{x}))}$. We fix $\chi \in \operatorname{Irr}(G)$ such that $(\chi, \eta_x^G) \neq 0$. Then $\chi \in \operatorname{Irr}^2(G|K)$ since f is nondegenerate skew symmetric. We define $\chi^{(i)}$ by

$$\chi^{(i)}(g) = \chi(g^i).$$

Then $\chi^{(i)}$, $1 \leq i \leq p-1$, is also in $\operatorname{Irr}^2(G|K)$, since it is an algebraic conjugate of χ.

Lemma 3.1. *$\chi^{(i)}(y) = 0$ for $y \notin H$ or $y \in D(G) \setminus Z(G)$, and $\chi^{(i)}(y) \neq 0$ for $y \in H \setminus D(G)$.*

Proof. The first statement holds since $\chi^{(i)}$ is induced from H by the action of G on b. The second assertion holds by the first assertion and the consideration of the inner product with itself. □

Choose $\xi \in \operatorname{Irr}^0(G)$ such that $\ker \xi \not\geq H$. Then

Lemma 3.2. *For $1 \leq i,\ k \leq p-1$ and $0 \leq j,\ l \leq p-1$, $\chi^{(i)}\xi^j = \chi^{(k)}\xi^l$ if and only if $i = k$ and $j = l$.*

Proof. Assume $\chi^{(i)}\xi^j = \chi^{(k)}\xi^l$. Clearly $i = k$ by considering the restriction to $Z(G)$. Then $j = l$ holds by $\chi^{(i)}(x) \neq 0$ and $x \notin \ker \xi$. □

Proposition 3.3. *With the above notation,*

$$\operatorname{Irr}^2(G|K) = \{\chi^{(i)}\xi^j \mid 1 \leq i \leq p-1, 0 \leq j \leq p-1\}.$$

Proof. The result follows by Lemma 3.2, and since $\sum_{\phi \in \operatorname{Irr}(G)} \phi(1)^2 = |G|$. □

Now we are going to consider conjugacy classes of G. Put

$$\begin{aligned} \operatorname{Cl}^0(G) &= \{C \in \operatorname{Cl}(G) \mid C \subset Z(G)\} \\ \operatorname{Cl}^1(G) &= \{C \in \operatorname{Cl}(G) \mid C \subset D(G) \setminus Z(G)\}. \end{aligned}$$

Then $\{c_1^i c_2^j \mid 0 \leq i, j \leq p-1\}$ is a representative set of $\operatorname{Cl}^0(G)$, and $\{b^i \mid 1 \leq i \leq p-1\}$ is a representative set of $\operatorname{Cl}^1(G)$.

As before, we define H, K, and x. Put

$$\operatorname{Cl}^2(G|H) = \{C \in \operatorname{Cl}(G) \mid C \subset H \setminus D(G)\},$$

$$\mathrm{Cl}^2(G) = \bigcup_H \mathrm{Cl}^2(G|H).$$

Then the union is disjoint and

$$\mathrm{Cl}(G) = \mathrm{Cl}^0(G) \cup \mathrm{Cl}^1(G) \cup \mathrm{Cl}^2(G).$$

Choose $z \in \mathrm{Z}(G) \setminus K$. Then

Proposition 3.4. *$\{x^i z^j \mid 1 \le i \le p-1, 0 \le j \le p-1\}$ is a representative set of $\mathrm{Cl}^2(G|H)$.*

Proof. Assume $x^i z^j$ is conjugate to $x^k z^l$. Clearly $i = k$ by considering $G/\mathrm{D}(G)$. For $\chi \in \mathrm{Irr}^2(G|K)$, $\chi(x^i) \neq 0$ and $\chi(z) \neq \chi(1)$. So $\chi(x^i z^j) = \chi(x^i z^l)$ implies $j = l$. Now the result follows. □

4. Self duality for G

In this section, we will define Ψ a correspondence between conjugacy classes and irreducible characters of G and give a proof for Theorem 2.1.

We denote by $C(y)$ the conjugacy class of G containing y. Fix $x \in G \setminus \mathrm{D}(G)$, and put $H = \langle x, \mathrm{D}(G)\rangle$, $K = \gamma(\overline{H})$. Let χ be in $\mathrm{Irr}^2(G|K)$, let z be in $\mathrm{Z}(G) \setminus K$ such that $\chi(z) = \omega\chi(1)$, and let ξ be in $\mathrm{Irr}^0(G)$ such that $\xi(x) = \omega$ (obviously such z and ξ exist). We define $\Psi(C(x^i z^j)) = \chi^{(i)}\xi^j$. By Proposition 3.3, 3.4, this is well-defined. Now we shall show that $\chi^{(i)}\xi^j(x^k z^l) = \chi^{(k)}\xi^l(x^i z^j)$. We have

$$\begin{aligned}\chi^{(i)}\xi^j(x^k z^l) &= \chi^{(i)}(x^k)\chi^{(i)}(z^l)\xi^j(x^k)/\chi^{(i)}(1)\\ &= \chi(x^{ik})\chi(z^{il})\xi(x^{jk})/\chi(1)\\ &= \chi(x^{ik})\omega^{il+jk}.\end{aligned}$$

Similarly $\chi^{(k)}\xi^l(x^i z^j) = \chi(x^{ik})\omega^{il+jk}$. Thus $\chi^{(i)}\xi^j(x^k z^l) = \chi^{(k)}\xi^l(x^i z^j)$.

We extend Ψ to the correspondence between $\mathrm{Cl}^2(G)$ to $\mathrm{Irr}^2(G)$ naturally. If $\chi_1 \in \mathrm{Irr}^2(G|K_1)$ for $K_1 \neq K$, then $\chi_1(x) = 0$. Thus

$$\Psi(C(x_1))(x_2) = \Psi(C(x_2))(x_1)$$

for all $C(x_1)$, $C(x_2) \in \mathrm{Cl}^2(G)$ and (*), denoted in section 1, holds for them.

Now we consider $\mathrm{Cl}^1(G)$ and $\mathrm{Irr}^1(G)$. We know $\{b^i | 1 \le i \le p-1\}$ is a representative set of $\mathrm{Cl}^1(G)$. Fix $\phi \in \mathrm{Irr}^1(G)$ and define $\phi^{(i)}$ similarly as $\chi^{(i)}$. We define $\Psi(C(b^i)) = \phi^{(i)}$. Then obviously $\Psi(C(b^i))(b^j) = \Psi(C(b^j))(b^i)$. It is also clear that $\chi(b^i) = 0$ for $\chi \in \mathrm{Irr}^2(G)$, $\xi(b^i) = 1$ for $\xi \in \mathrm{Irr}^0(G)$, $\phi^{(i)}(x) = 0$ for $x \in G \setminus \mathrm{D}(G)$, and $\phi^{(i)}(z) = p$ for $z \in \mathrm{Z}(G)$. Thus (*) holds for $C(x_1) \in \mathrm{Cl}^1(G)$ and $C(x_2) \in \mathrm{Cl}(G)$.

Finally, we consider $\mathrm{Cl}^0(G)$ and $\mathrm{Irr}^0(G)$. If $z \in Z(G)$ and $\xi \in \mathrm{Irr}^0(G)$ then $\xi(z) = 1$ and $(*)$ holds. It remains to consider the cases $C(x) \in \mathrm{Cl}^2(G)$ and $C(z) \in \mathrm{Cl}^0(G)$. We define $\Psi(C(z)) \in \mathrm{Irr}^0(G)$ by

$$\Psi(C(z))(x) = \omega^{f(\alpha(z),\beta(\overline{x}))}.$$

Then Ψ defines a one-to one correspondence between $\mathrm{Cl}^0(G)$ and $\mathrm{Irr}^0(G)$ since f is nondegenerate. Now

$$\Psi(C(x))(z) = p\omega^{f(\alpha(z),\beta(\overline{x}))}$$

and so $(*)$ holds.

Now Ψ defines a one-to-one correspondence between $\mathrm{Cl}(G)$ and $\mathrm{Irr}(G)$ and $(*)$ holds for all cases. The proof of Theorem 2.1 is complete.

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