

Title	Self-dual groups of order p^5 (p an odd prime)
Author(s)	Hanaki, Akihide
Citation	Osaka Journal of Mathematics. 1997, 34(2), p. 357-361
Version Type	VoR
URL	https://doi.org/10.18910/9798
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

SELF DUAL GROUPS OF ORDER p^5 (p AN ODD PRIME)

AKIHIDE HANAKI

(Received December 14, 1995)

1. Introduction

Let G be a finite group, $\text{Irr}(G) = \{\chi_1, \dots, \chi_k\}$ be the set of all irreducible characters, $\text{Cl}(G) = \{C_1, \dots, C_k\}$ be the conjugacy classes of G , and x_i be a representative of C_i . We call G self dual if (by renumbering indices)

$$(*) \quad |C_j| \chi_i(x_j) / \chi_i(1) = \chi_j(1) \chi_j(x_i), \text{ for all } i, j.$$

This condition is found in E. Bannai [1]. T. Okuyama [4] proved that self dual groups are nilpotent, and that a nilpotent group is self dual if and only if its all Sylow subgroups are self dual. So if we consider self dual groups we may deal with only p -groups. Obviously abelian groups are self dual. Some examples of self dual groups are discussed in [2].

If G is self dual it is easy to check that $|C_i| = \chi_i(1)^2$ for all i . It is easy to see that non abelian p -groups of order at most p^4 cannot satisfy this condition, and so they are not self dual. By the classification of groups of order 2^5 , there is no group of order 2^5 satisfying this condition. For odd p , in classification table of groups of order p^5 [3], we can see that one isoclinism family Φ_6 satisfies this condition. We will show that all of groups in Φ_6 are self dual.

2. Definition of groups

We fix an odd prime p . Let G be a p -group of order p^5 which belongs to Φ_6 defined in [3], namely

$$G = \langle a_1, a_2, b, c_1, c_2 \mid [a_1, a_2] = b, [a_i, b] = c_i, a_i^p = \zeta_i, b^p = c_i^p = 1 \ (i = 1, 2) \rangle,$$

where (ζ_1, ζ_2) is one of the followings:

- (1) (c_1, c_2) ,
- (2) (c_1^k, c_2) , where $k = g^r$, $r = 1, 2, \dots, (p-1)/2$,
- (3) $(c_2^{-r/4}, c_1^r c_2^r)$, where $r = 1$ or ν ,
- (4) (c_2, c_1^ν) ,
- (5) $(c_2^k, c_1 c_2)$, where $4k = g^{2r+1} - 1$, $r = 1, 2, \dots, (p-1)/2$,
- (6) $(c_1, 1)$, $p > 3$,
- (7) $(1, c_1^r)$, where $r = 1$ or ν , and $p > 3$,
- (8) $(1, 1)$,

where g denotes the smallest positive integer which is a primitive root (mod p), and ν denotes the smallest positive integer which is a non-quadratic residue (mod p).

In this paper, we shall show that

Theorem 2.1. *G is self dual.*

We treat cases (1)–(8) above simultaneously. In any case, $Z(G)$, the center of G , is $\langle c_1, c_2 \rangle$ and $D(G)$, the derived subgroup of G , is $\langle b, c_1, c_2 \rangle$.

3. Irreducible characters and conjugacy classes

First, we consider irreducible characters of G . It is easy to see that $G/Z(G)$ is isomorphic to the extraspecial group of order p^3 and exponent p . So we know all characters of $G/Z(G)$. We put

$$\begin{aligned} \text{Irr}^0(G) &= \{\chi \in \text{Irr}(G) \mid \ker \chi \geq D(G)\}, \\ \text{Irr}^1(G) &= \{\chi \in \text{Irr}(G) \mid \ker \chi \geq Z(G) \text{ and } \ker \chi \not\geq D(G)\}. \end{aligned}$$

Let χ be an irreducible character of G whose kernel does not contain $Z(G)$. Then $\ker \chi$ contains some subgroup K of $Z(G)$ of order p since $Z(G)$ is not cyclic. So we consider characters of G/K for a fixed K . We put

$$\text{Irr}^2(G|K) = \{\chi \in \text{Irr}(G) \mid \ker \chi \not\geq Z(G) \text{ and } \ker \chi \geq K\},$$

and

$$\text{Irr}^2(G) = \bigcup_K \text{Irr}^2(G|K),$$

where K runs over subgroups of $Z(G)$ of order p . Observe that this is a disjoint union. Then obviously

$$\text{Irr}(G) = \text{Irr}^0(G) \cup \text{Irr}^1(G) \cup \text{Irr}^2(G).$$

Let V be a two-dimensional $\text{GF}(p)$ -vector space with a nondegenerate skew symmetric form $f : V \times V \rightarrow \text{GF}(p)$. That is f is bilinear, $f(u, v) = -f(v, u)$ for all $u, v \in V$, and if $f(u, v) = 0$ for all $u \in V$, then $v = 0$. Note that $f(v, v) = 0$ for all $v \in V$. Let $\alpha : Z(G) \rightarrow V$ be an isomorphism of abelian groups. We define $\gamma : G/D(G) \rightarrow Z(G)$ by $\gamma(\bar{g}) = [g, b]$. Since $[D(G), b] = 1$, this map is well-defined and γ is an isomorphism as abelian groups by the definition of G . Put $\beta = \alpha\gamma$. Then β is an isomorphism from $G/D(G)$ to V . For K , choose $x \in G$ such that $\gamma(\langle \bar{x} \rangle) = K$, and define $H = \langle x, D(G) \rangle$. Then H/K is abelian by the definition. Every character in $\text{Irr}^2(G|K)$ is induced from a linear character of H whose kernel contains K but does not contain $Z(G)$, and so the character has degree p .

Let ω be a primitive p -th root of unity. For x , we define $\eta_x \in \text{Irr}(Z(G))$ by $\eta_x(z) = \omega^{f(\alpha(z), \beta(\bar{x}))}$. We fix $\chi \in \text{Irr}(G)$ such that $(\chi, \eta_x^G) \neq 0$. Then $\chi \in \text{Irr}^2(G|K)$ since f is nondegenerate skew symmetric. We define $\chi^{(i)}$ by

$$\chi^{(i)}(g) = \chi(g^i).$$

Then $\chi^{(i)}$, $1 \leq i \leq p-1$, is also in $\text{Irr}^2(G|K)$, since it is an algebraic conjugate of χ .

Lemma 3.1. $\chi^{(i)}(y) = 0$ for $y \notin H$ or $y \in D(G) \setminus Z(G)$, and $\chi^{(i)}(y) \neq 0$ for $y \in H \setminus D(G)$.

Proof. The first statement holds since $\chi^{(i)}$ is induced from H by the action of G on b . The second assertion holds by the first assertion and the consideration of the inner product with itself. □

Choose $\xi \in \text{Irr}^0(G)$ such that $\ker \xi \not\supseteq H$. Then

Lemma 3.2. For $1 \leq i, k \leq p-1$ and $0 \leq j, l \leq p-1$, $\chi^{(i)}\xi^j = \chi^{(k)}\xi^l$ if and only if $i = k$ and $j = l$.

Proof. Assume $\chi^{(i)}\xi^j = \chi^{(k)}\xi^l$. Clearly $i = k$ by considering the restriction to $Z(G)$. Then $j = l$ holds by $\chi^{(i)}(x) \neq 0$ and $x \notin \ker \xi$. □

Proposition 3.3. With the above notation,

$$\text{Irr}^2(G|K) = \{\chi^{(i)}\xi^j \mid 1 \leq i \leq p-1, 0 \leq j \leq p-1\}.$$

Proof. The result follows by Lemma 3.2, and since $\sum_{\phi \in \text{Irr}(G)} \phi(1)^2 = |G|$. □

Now we are going to consider conjugacy classes of G . Put

$$\begin{aligned} \text{Cl}^0(G) &= \{C \in \text{Cl}(G) \mid C \subset Z(G)\} \\ \text{Cl}^1(G) &= \{C \in \text{Cl}(G) \mid C \subset D(G) \setminus Z(G)\}. \end{aligned}$$

Then $\{c_1^i c_2^j \mid 0 \leq i, j \leq p-1\}$ is a representative set of $\text{Cl}^0(G)$, and $\{b^i \mid 1 \leq i \leq p-1\}$ is a representative set of $\text{Cl}^1(G)$.

As before, we define H, K , and x . Put

$$\text{Cl}^2(G|H) = \{C \in \text{Cl}(G) \mid C \subset H \setminus D(G)\},$$

$$\text{Cl}^2(G) = \bigcup_H \text{Cl}^2(G|H).$$

Then the union is disjoint and

$$\text{Cl}(G) = \text{Cl}^0(G) \cup \text{Cl}^1(G) \cup \text{Cl}^2(G).$$

Choose $z \in Z(G) \setminus K$. Then

Proposition 3.4. $\{x^i z^j \mid 1 \leq i \leq p-1, 0 \leq j \leq p-1\}$ is a representative set of $\text{Cl}^2(G|H)$.

Proof. Assume $x^i z^j$ is conjugate to $x^k z^l$. Clearly $i = k$ by considering $G/D(G)$. For $\chi \in \text{Irr}^2(G|K)$, $\chi(x^i) \neq 0$ and $\chi(z) \neq \chi(1)$. So $\chi(x^i z^j) = \chi(x^i z^l)$ implies $j = l$. Now the result follows. \square

4. Self duality for G

In this section, we will define Ψ a correspondence between conjugacy classes and irreducible characters of G and give a proof for Theorem 2.1.

We denote by $C(y)$ the conjugacy class of G containing y . Fix $x \in G \setminus D(G)$, and put $H = \langle x, D(G) \rangle$, $K = \gamma(\overline{H})$. Let χ be in $\text{Irr}^2(G|K)$, let z be in $Z(G) \setminus K$ such that $\chi(z) = \omega\chi(1)$, and let ξ be in $\text{Irr}^0(G)$ such that $\xi(x) = \omega$ (obviously such z and ξ exist). We define $\Psi(C(x^i z^j)) = \chi^{(i)} \xi^j$. By Proposition 3.3, 3.4, this is well-defined. Now we shall show that $\chi^{(i)} \xi^j(x^k z^l) = \chi^{(k)} \xi^l(x^i z^j)$. We have

$$\begin{aligned} \chi^{(i)} \xi^j(x^k z^l) &= \chi^{(i)}(x^k) \chi^{(i)}(z^l) \xi^j(x^k) / \chi^{(i)}(1) \\ &= \chi(x^{ik}) \chi(z^{il}) \xi(x^{jk}) / \chi(1) \\ &= \chi(x^{ik}) \omega^{il+jk}. \end{aligned}$$

Similarly $\chi^{(k)} \xi^l(x^i z^j) = \chi(x^{ik}) \omega^{il+jk}$. Thus $\chi^{(i)} \xi^j(x^k z^l) = \chi^{(k)} \xi^l(x^i z^j)$.

We extend Ψ to the correspondence between $\text{Cl}^2(G)$ to $\text{Irr}^2(G)$ naturally. If $\chi_1 \in \text{Irr}^2(G|K_1)$ for $K_1 \neq K$, then $\chi_1(x) = 0$. Thus

$$\Psi(C(x_1))(x_2) = \Psi(C(x_2))(x_1)$$

for all $C(x_1), C(x_2) \in \text{Cl}^2(G)$ and $(*)$, denoted in section 1, holds for them.

Now we consider $\text{Cl}^1(G)$ and $\text{Irr}^1(G)$. We know $\{b^i \mid 1 \leq i \leq p-1\}$ is a representative set of $\text{Cl}^1(G)$. Fix $\phi \in \text{Irr}^1(G)$ and define $\phi^{(i)}$ similarly as $\chi^{(i)}$. We define $\Psi(C(b^i)) = \phi^{(i)}$. Then obviously $\Psi(C(b^i))(b^j) = \Psi(C(b^j))(b^i)$. It is also clear that $\chi(b^i) = 0$ for $\chi \in \text{Irr}^2(G)$, $\xi(b^i) = 1$ for $\xi \in \text{Irr}^0(G)$, $\phi^{(i)}(x) = 0$ for $x \in G \setminus D(G)$, and $\phi^{(i)}(z) = p$ for $z \in Z(G)$. Thus $(*)$ holds for $C(x_1) \in \text{Cl}^1(G)$ and $C(x_2) \in \text{Cl}(G)$.

Finally, we consider $\text{Cl}^0(G)$ and $\text{Irr}^0(G)$. If $z \in Z(G)$ and $\xi \in \text{Irr}^0(G)$ then $\xi(z) = 1$ and $(*)$ holds. It remains to consider the cases $C(x) \in \text{Cl}^2(G)$ and $C(z) \in \text{Cl}^0(G)$. We define $\Psi(C(z)) \in \text{Irr}^0(G)$ by

$$\Psi(C(z))(x) = \omega^{f(\alpha(z), \beta(\bar{x}))}.$$

Then Ψ defines a one-to one correspondence between $\text{Cl}^0(G)$ and $\text{Irr}^0(G)$ since f is nondegenerate. Now

$$\Psi(C(x))(z) = p\omega^{f(\alpha(z), \beta(\bar{x}))}$$

and so $(*)$ holds.

Now Ψ defines a one-to-one correspondence between $\text{Cl}(G)$ and $\text{Irr}(G)$ and $(*)$ holds for all cases. The proof of Theorem 2.1 is complete.

References

- [1] E. Bannai: *Association schemes and fusion algebras (an introduction)*, J. Alg. Comb. **2** (1993), 327–344.
- [2] A. Hanaki and T. Okuyama: *Groups with some combinatorial properties*, Osaka J. Math. **34**, to appear.
- [3] R. James: *The groups of order p^6 (p an odd prime)*, Math. Computation **43** (1980), 613–637.
- [4] T. Okuyama: *Self dual groups are nilpotent*, preprint.

Faculty of Engineering
Yamanashi University
Takeda 4, Kofu 400, Japan

