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**STRUCTURAL OPERATORS AND SEMIGROUPS  
 ASSOCIATED WITH  
 FUNCTIONAL DIFFERENTIAL EQUATIONS  
 IN HILBERT SPACES**

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**1. Introduction**

Let  $\Omega \subset \mathbf{R}^n$  be a bounded domain with smooth boundary  $\partial\Omega$ , and  $H = L^2(\Omega)$  be the usual  $L^2$ -space. A model equation under consideration is described by the following parabolic partial differential equation with time delays

$$(1.1) \quad \frac{\partial u(t, x)}{\partial t} = \mathcal{A}_0 u(t, x) + \mathcal{A}_1 u(t-h, x) + \int_{-h}^0 a(s) \mathcal{A}_2 u(t+s, x) ds + f(t, x), \quad t \geq 0, \quad x \in \Omega,$$

where  $\mathcal{A}_i (i=0, 1, 2)$  are elliptic differential operators of the second order induced by the sesquilinear forms,  $f \in L^2_{\text{loc}}(\mathbf{R}^+; L^2(\Omega))$  is a forcing function,  $h > 0$  is a delay time and  $a(s)$  is a real scalar function on  $[-h, 0]$ . The boundary condition attached to (1.1) is, for simplicity, given by the Dirichlet boundary condition

$$(1.2) \quad u|_{\partial\Omega} = 0, \quad t \geq 0$$

and the initial data is given by

$$(1.3) \quad u(0, x) = g^0(x), \quad u(s, x) = g^1(s, x) \quad \text{a.e. } s \in [-h, 0), \quad x \in \Omega,$$

where  $g^0 \in L^2(\Omega)$ ,  $g^1 \in L^2(-h, 0; H^1_0(\Omega))$ .

Let  $H = L^2(\Omega)$ ,  $V = H^1_0(\Omega)$  and  $g = (g^0, g^1) \in M_2 \equiv H \times L^2(-h, 0; V)$  and  $A_i (i=0, 1, 2)$  be the realization of  $\mathcal{A}_i$  with the boundary condition (1.2) in  $H = L^2(\Omega)$ . Then the system (1.1)-(1.3) can be included in the following general class of abstract functional differential equation (E) in a Hilbert space  $H$ :

$$(1.4) \quad \frac{du(t)}{dt} = A_0 u(t) + A_1 u(t-h) + \int_{-h}^0 a(s) A_2 u(t+s) ds + f(t), \quad t \geq 0$$

$$(1.5) \quad u(0) = g^0, \quad u(s) = g^1(s) \quad \text{a.e. } s \in [-h, 0).$$

This type of equations having the highest order spatial derivatives in time delayed terms has been studied by Ardito and Ricciardi [1], Ardito and Velnole [2], Di Blasio [6], Di Blasio, Kunisch and Sinestrari [7, 8] for linear equations, and by Yong and Pan [26] for quasi-linear equations. However, the structural study as well as the adjoint theory for  $(\mathbf{E})$  as developed in Nakagiri [17], in which general retarded equations with bounded operators in delayed terms are treated, has not been studied except for Tanabe [23], Jeong [10] and Kunisch and Mastinšek [13]. The unbounded operators acting on delayed terms of equations studied in [13] are given by fractional powers of operators which generate analytic semigroups, and the equations do not cover the case studied in [23]. In Tanabe [23] the structural operators are first introduced for  $(\mathbf{E})$  and the basic properties of structural operators and solution semigroups associated with  $(\mathbf{E})$  are announced, but the detail of results is not given there. Our approach to the basic questions such as the existence, uniqueness and wellposedness of solutions is close to that in Di Blasio, Kunisch and Sinestrari [7], but our treatment using the sesquilinear forms gives a more general and convenient way for the semigroup theoretical study of  $(\mathbf{E})$ . In fact, our solution semigroup is defined on a larger product space than that in [7].

The purposes of this paper are the followings; the first is to give complete proofs of theorems in [23], the second is to develop a further and comprehensive study of structural properties of solution semigroups in the product Hilbert space  $M_2$  including adjoint semigroup theory.

We will carry out this work in the following manner. First of all the existence and uniqueness result of solutions is proved in the  $M_2$ -space structure of initial conditions. Next, by introducing the fundamental solution and the structural operator  $F$  which is unbounded and represents the effect of time delays, a simple form of the variation of constants formula for the solutions is established. According to the formula the structural operator  $G(t)$  is introduced to characterize the solution semigroup. It is important to develop the adjoint theory of functional differential equations for practical applications to control problems (cf. Nakagiri and Yamamoto [18, 19], Jeong [11], Salamon [20]). For this the transposed equation is introduced and the adjointness of solutions, fundamental solutions, and transposed solution semigroups is investigated. The surjectivity of the operator  $F$  and the image of the specific operator  $G=G(h)$  are also studied. The structural operators  $F$  and  $G$  are used to give the most useful relations between the transposed solution semigroup and the adjoint solution semigroup. Finally, an application to a practical partial functional differential equation of parabolic type as well as its transposed equation is given.

### 2. Functional Differential Equations

First we shall give an exact description of the following functional differential equation (E) on a Hilbert space  $H$ :

$$(2.1) \quad \frac{du(t)}{dt} = A_0u(t) + A_1u(t-h) + \int_{-h}^0 a(s)A_2u(t+s)ds + f(t), \quad \text{a.e. } t \geq 0$$

$$(2.2) \quad u(0) = g^0, \quad u(s) = g^1(s) \quad \text{a.e. } s \in [-h, 0).$$

Let  $H$  be a ‘pivot’ complex Hilbert space and  $V$  be a complex Hilbert space such that  $V$  is dense in  $H$  and the inclusion map  $i: V \rightarrow H$  is continuous. The norms of  $H, V$  and the inner product of  $H$  are denoted by  $|\cdot|, \|\cdot\|$  and  $\langle \cdot, \cdot \rangle$  respectively. By identifying the antidual of  $H$  with  $H$  we may consider  $V \subset H \subset V^*$ . The norm of the dual space  $V^*$  is denoted by  $\|\cdot\|_*$ . Let  $a(u, v)$  be a bounded sesquilinear form defined in  $V \times V$  satisfying Gårding’s inequality

$$(2.3) \quad \text{Re } a(u, u) \geq c_0\|u\|^2 - c_1|u|^2,$$

where  $c_0 > 0$  and  $c_1 \geq 0$  are real constants. Let  $A_0$  be the operator associated with this sesquilinear form

$$(2.4) \quad \langle v, A_0u \rangle = -a(u, v), \quad u, v \in V,$$

where  $\langle \cdot, \cdot \rangle$  denotes also the duality pairing between  $V$  and  $V^*$ . The operator  $A_0$  is bounded linear from  $V$  into  $V^*$ . The realization of  $A_0$  in  $H$ , which is the restriction of  $A_0$  to the domain  $\mathcal{D}(A_0) = \{u \in V; A_0u \in H\}$  is also denoted by  $A_0$ . It is proved in Tanabe [21; Chap. 3] that  $A_0$  generates an analytic semigroup  $e^{tA_0} = T(t)$  both in  $H$  and  $V^*$  and that  $T(t): V^* \rightarrow V$  for each  $t > 0$ . Throughout this paper it is assumed that each  $A_i (i=1, 2)$  is bounded and linear from  $V$  to  $V^*$  (i.e.,  $A_i \in \mathcal{L}(V, V^*)$ ) such that  $A_i$  maps  $\mathcal{D}(A_0)$  endowed with the graph norm of  $A_0$  to  $H$  continuously. The real valued scalar function  $a(s)$  is assumed to be  $L^2$ -integrable on  $[-h, 0]$ , that is,  $a(\cdot) \in L^2(-h, 0)$ .

For brevity of notations, we introduce a Stieltjes measure  $\eta$  given by

$$(2.5) \quad \eta(s) = -\chi_{(-\infty, -h]}(s)A_1 - \int_s^0 a(\xi)d\xi A_2: V \rightarrow V^*, \quad s \in [-h, 0],$$

where  $\chi_{(-\infty, -h]}$  denotes the characteristic function of  $(-\infty, -h]$ . Then the delayed terms in (2.1) are written simply by  $\int_{-h}^0 d\eta(s)u(t+s)$ .

In this section we give the existence and uniqueness result of solutions and establish a variation of constants formula of solutions in terms of the fundamental solution.

**DEFINITION** A function  $u \in L^2_{\text{loc}}(\mathbf{R}^+; V) \cap W^{1,2}_{\text{loc}}(\mathbf{R}^+; V^*)$  is said to be a solution of (E) if  $u(t)$  satisfies (2.1) and (2.2).

In order to solve the existence and uniqueness problem for (E) we start with the following proposition due to Lions [14; Chapter 3] and Tanabe [21 (see also Lions and Magenes [15; Chapter 1]).

**Proposition 1** *For any  $u_0 \in H$  and  $f \in L^2_{loc}(\mathbf{R}^+; V^*)$  the Cauchy problem*

$$(2.6) \quad \frac{du(t)}{dt} = A_0 u(t) + f(t), \quad \text{a.e. } t > 0, \quad u(0) = u_0$$

*admits a unique solution*

$$(2.7) \quad u \in L^2_{loc}(\mathbf{R}^+; V) \cap W^{1,2}_{loc}(\mathbf{R}^+; V^*) \subset C(\mathbf{R}^+; H)$$

*which is represented by*

$$(2.8) \quad u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds, \quad t \geq 0.$$

*Further, for each  $T > 0$  there exists a constant  $C_T$  depending only on  $T, c_0, c_1$  such that*

$$(2.9) \quad \left( \int_0^T \|u(t)\|^2 dt + \int_0^T \left\| \frac{du(t)}{dt} \right\|_*^2 dt \right) \leq C_T \left( |u_0|^2 + \int_0^T \|f(t)\|_*^2 dt \right).$$

We shall solve the equation (E) by considering the following functional integral equation

$$(2.10) \quad u(t) = \begin{cases} T(t)g^0 + \int_0^t T(t-s)f(s)ds \\ + \int_0^t T(t-s) \int_{-h}^0 d\gamma(\xi)u(\xi+s)ds, & t \geq 0 \\ g^1(t) & \text{a.e. } t \in [-h, 0). \end{cases}$$

In view of Proposition 1, the solution  $u(t)$  of the integral equation (2.10) satisfies the equation (2.1) in the sense of vectorial distribution (cf. Barbu [3; Chap.1]).

**Theorem 1** *Let  $f \in L^2_{loc}(\mathbf{R}^+; V^*)$  and  $g = (g^0, g^1) \in H \times L^2(-h, 0; V)$ . Then there exists a unique solution  $u(t) = u(t; f, g)$  of (E) satisfying*

$$(2.11) \quad u \in L^2_{loc}(-h, \infty; V) \cap W^{1,2}_{loc}(\mathbf{R}^+; V^*) \subset C([0, \infty); H).$$

*Further, for each  $T > 0$  there is a constant  $K_T$  such that*

$$(2.12) \quad \left( \int_0^T \|u(t)\|^2 dt + \int_0^T \left\| \frac{du(t)}{dt} \right\|_*^2 dt \right) \leq K_T \left( |g^0|^2 + \int_{-h}^0 \|g^1(s)\|^2 ds + \int_0^T \|f(t)\|_*^2 dt \right).$$

Proof. Let  $b > 0$  be fixed and define the operator  $\mathcal{E}: L^2(-h, b; V) \rightarrow L^2(0, b; V^*)$  by

$$(2.13) \quad (\mathcal{E}w)(s) = \int_{-h}^0 d\eta(\xi)w(s+\xi) \quad \text{a.e. } s \in [0, b]$$

for  $w \in L^2(-h, b; V)$ . Using Schwartz inequality, from  $A_1, A_2 \in \mathcal{L}(V, V^*)$  and  $a \in L^2(-h, 0)$  we have the following estimate:

$$(2.14) \quad \begin{aligned} & \|\mathcal{E}w\|_{L^2(0, b; V^*)} \\ & \leq \begin{cases} \|a\|_{L^2(-h, 0)} \|A_2\|_{\mathcal{L}(V, V^*)} b^{1/2} \|w\|_{L^2(-h, b; V)} & \text{if } b < h \\ (\|A_1\|_{\mathcal{L}(V, V^*)} + \|a\|_{L^2(-h, 0)} \|A_2\|_{\mathcal{L}(V, V^*)} b^{1/2}) \|w\|_{L^2(-h, b; V)} & \text{if } b \geq h, \end{cases} \end{aligned}$$

where  $\|a\|_{L^2(-h, 0)} = (\int_{-h}^0 |a(s)|^2 ds)^{1/2}$ .

For any  $v \in L^2(0, b; V)$  we define the operator  $\mathcal{S}: L^2(0, b; V) \rightarrow L^2(0, b; V)$  by

$$(2.15) \quad (\mathcal{S}v)(t) = T(t)g^0 + \int_0^t T(t-s)(f(s) + \mathcal{E}\bar{v}(s))ds, \quad t \in [0, b],$$

where the extension  $\bar{v} \in L^2(-h, b; V)$  of  $v$  is given by

$$\bar{v}(t) = \begin{cases} v(t) & \text{a.e. } t \in [0, b] \\ g^1(t) & \text{a.e. } t \in [-h, 0]. \end{cases}$$

It is easily verified, by Proposition 1 and the estimate (2.14), that  $\mathcal{S}$  is into and the image  $\text{Im } \mathcal{S}$  is contained in  $W^{1,2}(0, b; V^*)$ . Let  $v_1, v_2 \in L^2(0, b; V)$ . If  $b < h$ , then by (2.9) and (2.14)

$$\begin{aligned} \|\mathcal{S}v_1 - \mathcal{S}v_2\|_{L^2(0, b; V)} & \leq \left( \int_0^b \left\| \int_0^t T(t-s)\mathcal{E}(\bar{v}_1 - \bar{v}_2)(s)ds \right\|^2 dt \right)^{1/2} \\ & \leq C_b \|\mathcal{E}(\bar{v}_1 - \bar{v}_2)\|_{L^2(0, b; V^*)} \\ & \leq C_b \|a\|_{L^2(-h, 0)} \|A_2\|_{\mathcal{L}(V, V^*)} b^{1/2} \|v_1 - v_2\|_{L^2(0, b; V)}. \end{aligned}$$

This implies that the operator  $\mathcal{S}$  is a contraction and hence has a unique fixed point  $u \in L^2(0, b; V) \cap W^{1,2}(0, b; V^*)$  for sufficiently small  $b > 0$ . This function  $u$  gives a unique solution to the equation (2.10) on the interval  $[0, b]$ . Combining the estimates (2.9) and (2.14) via the equation (2.10) and using that  $b$  is small, we can verify that the estimate (2.12) for the solution holds true. By the inclusion  $L^2(0, b; V) \cap W^{1,2}(0, b; V^*) \subset C([0, b]; H)$ , we see easily that  $u(b) \in H$ . Hence by the step by step method using Proposition 1 and the estimate (2.14) we have the existence and uniqueness of a global solution  $u(t)$  of (2.10) on  $\mathbf{R}^+$  satisfying the estimate (2.12) for each  $T > 0$  as shown in Di Blasio, Kunisch and Sinestrari [7; Theorem 3.3].  $\square$

Using Theorem 1 and repeating the similar argument in [7; Theorem 3.4], we can obtain the following theorem on the ‘strong’ solution of (E).

**Theorem 2.** *If  $f \in W_{loc}^{1,2}(\mathbf{R}^+; V^*)$  and  $g = (g^0, g^1)$  satisfy*

$$(2.16) \quad g^1 \in W^{1,2}(-h, 0; V), \quad g^1(0) = g^0, \quad A_0 g^0 + \int_{-h}^0 d\eta(s) g^1(s) ds \in H,$$

*then the solution  $u$  of (E) satisfies*

$$(2.17) \quad u \in W_{loc}^{1,2}(-h, \infty; V) \cap W_{loc}^{2,2}(\mathbf{R}^+; V^*) \subset C^1([0, \infty); H).$$

According to Nakagiri [17], we define the fundamental solution  $W(t)$  of (E) by

$$(2.18) \quad W(t)g^0 = \begin{cases} u(t; 0, (g^0, 0)), & t \geq 0 \\ O & t < 0 \end{cases} \quad \text{for } g^0 \in H.$$

In other words,  $W(t)$  is defined as a unique solution of

$$(2.19) \quad W(t) = \begin{cases} T(t) + \int_0^t T(t-s) \int_{-h}^0 d\eta(\xi) W(\xi+s) ds, & t \geq 0 \\ O & t < 0 \end{cases}$$

in  $\mathcal{L}(H)$ . By Theorem 1 we see easily that  $W(t)$  is strongly continuous in  $H$  and  $W(\cdot)g^0 \in L_{loc}^2(\mathbf{R}^+; V) \cap W_{loc}^{1,2}(\mathbf{R}^+; V^*)$  for each  $g^0 \in H$ .

If we want to get a stronger regularity of  $W(t)$  like as that for  $T(t)$ , we have to solve the equation (2.19) for  $W(t)$  in the space  $\mathcal{L}(V^*)$ . For this sake we assume the following additional condition on  $a(s)$ , which is used essentially in Tanabe [22, 24].

$$(2.20) \quad a(s) \text{ is H\"older continuous on } [-h, 0].$$

Modifying Theorem 1 in Tanabe [23] on our Hilbert space setting, we have the following lemma.

**Lemma 1.** *Under the condition (3.20), the fundamental solution  $W(t)$  is strongly continuous in  $V^*$ ,  $W(t): V^* \rightarrow V$  for each  $t > 0$  and satisfies*

$$(2.21) \quad \frac{d}{dt} W(t) = A_0 W(t) + \int_{-h}^0 d\eta(s) W(t+s) \quad \text{a.e. } t > 0.$$

*The functions  $\frac{d}{dt} W(t)$  and  $(A_0 + c_1) W(t)$  are strongly continuous in  $V^*$  on each  $(nh, (n+1)h]$ ,  $n = 0, 1, 2, \dots$ , and the following estimates hold for some constants  $C_n$ :*

$$(2.22) \quad \begin{cases} \|(A_0 + c_1)W(t)\|_{\mathcal{L}(V^*)} \leq \frac{C_n}{t-nh} \\ \|\frac{d}{dt} W(t)\|_{\mathcal{L}(V^*)} \leq \frac{C_n}{t-nh} \\ \|\int_{nh}^t (A_0 + c_1)W(s)ds\|_{\mathcal{L}(V^*)} \leq C_n \end{cases} \quad \text{for } t \in (nh, (n+1)h].$$

By virtue of Lemma 1 we can derive the next proposition which is crucial for the variation of constants formula of solutions.

**Proposition 2.** *Let  $f \in L^2_{loc}(\mathbf{R}; V^*)$ . Then the function  $v(t) = \int_0^t W(t-s)f(s)ds$  solves the integral equation (2.10) with  $g^0=0, g^1=0$  and satisfies*

$$(2.23) \quad v \in L^2_{loc}(\mathbf{R}^+; V) \cap W^{1,2}_{loc}(\mathbf{R}^+; V^*) \subset C([0, \infty); H).$$

Hence  $v(t)$  gives a unique solution of (E) with zero initial data.

Proof. Since  $W(t)$  is strongly continuous in  $V^*$ ,  $v(t)$  makes sense as a Bochner integral in  $V^*$  for each  $t > 0$ . Noting that  $W(t): V^* \rightarrow V$  and  $W(t)$  satisfies (2.19) in  $\mathcal{L}(V^*)$ , we have for  $t > 0$

$$(2.24) \quad \begin{aligned} v(t) &= \int_0^t W(t-s)f(s)ds \\ &= \int_0^t T(t-s)f(s)ds \\ &\quad + \int_0^t \left( \int_0^{t-s} T(t-s-\xi)A_1W(\xi-h)d\xi \right) f(s)ds \\ &\quad + \int_0^t \left( \int_0^{t-s} T(t-s-\xi) \int_{-h}^0 a(\tau)A_2W(\xi+\tau)d\tau d\xi \right) f(s)ds \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Using Fubini's theorem several times and noting that  $W(t) = O$  for  $t < 0$ , we transform the integrals  $I_2, I_3$  in (2.24) as

$$(2.25) \quad \begin{aligned} I_2 &= \int_0^t \int_s^t T(t-\beta)A_1W(\beta-s-h)d\beta f(s)ds \quad (\text{by } s+\xi \rightarrow \beta) \\ &= \int_0^t T(t-\beta)A_1 \left( \int_0^\beta W(\beta-s-h)f(s)ds \right) d\beta \\ &= \int_0^t T(t-s)A_1 \left( \int_0^{s-h} W(s-h-\tau)f(\tau)d\tau \right) ds; \\ &\hspace{15em} (\text{by } \beta \rightarrow s, s \rightarrow \tau) \end{aligned}$$

$$(2.26) \quad \begin{aligned} I_3 &= \int_0^t \int_s^t T(t-\beta) \int_{-h}^0 a(\tau)A_2W(\beta-s+\tau)d\tau d\beta f(s)ds \\ &= \int_0^t T(t-\beta) \left( \int_0^\beta \int_{-h}^0 a(\tau)A_2W(\beta-s+\tau)d\tau f(s)ds \right) d\beta \\ &= \int_0^t T(t-\beta) \left( \int_{-h}^0 a(\tau)A_2 \int_0^\beta W(\beta-s+\tau)f(s)d\tau ds \right) d\beta \\ &= \int_0^t T(t-s) \left( \int_{-h}^0 a(\tau)A_2 \left[ \int_0^{s+\tau} W(s+\tau-\xi)f(\xi)d\xi \right] d\tau \right) ds. \\ &\hspace{15em} (\text{by } \beta \rightarrow s, s \rightarrow \xi) \end{aligned}$$

Hence by (2.24)-(2.26), we can verify that  $v(t)$  satisfies the integral equation



$$(2.27) \quad v(t) = \begin{cases} \int_0^t T(t-s)f(s)ds + \int_0^t T(t-s) \int_{-h}^0 d\eta(\xi)v(\xi+s)ds, & t \geq 0 \\ 0 & \text{a.e. } t \in [-h, 0) \end{cases}$$

in the space  $V$ . Next we shall prove (2.23). To this end we use the density argument and Lemma 1. Let  $T > 0$  and assume further that  $f \in C^\rho([0, T]; V^*)$ ,  $0 < \rho < 1$ , the space of Hölder continuous functions in  $[0, T]$  of exponent  $\rho$  with values in  $V^*$ . Let  $\|f\|_\rho = \sup_{0 \leq s < t \leq T} \|f(t) - f(s)\|_* |t - s|^{-\rho}$ . Then by the estimates (2.22), we have for  $t \in (nh, (n+1)h]$ ,  $\leq T$

$$(2.28) \quad \begin{aligned} \|(A_0 + c_1)v(t)\|_* &= \left\| \int_0^t (A_0 + c_1)W(s)f(t-s)ds \right\|_* \\ &\leq \sum_{j=0}^{n-1} \left\| \int_{jh}^{(j+1)h} (A_0 + c_1)W(s)(f(t-s) - f(t-jh))ds \right\|_* \\ &\quad + \left\| \int_{nh}^t (A_0 + c_1)W(s)(f(t-s) - f(t-nh))ds \right\|_* \\ &\quad + \sum_{j=0}^{n-1} \left\| \int_{jh}^{(j+1)h} (A_0 + c_1)W(s)ds f(t-jh) \right\|_* \\ &\quad + \left\| \int_{nh}^t (A_0 + c_1)W(s)ds f(t-nh) \right\|_* \\ &\leq \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} \|(A_0 + c_1)W(s)\|_{\mathcal{L}(V^*)} \|f(t-s) - f(t-jh)\|_* ds \\ &\quad + \int_{nh}^t \|(A_0 + c_1)W(s)\|_{\mathcal{L}(V^*)} \|f(t-s) - f(t-nh)\|_* ds \\ &\quad + \sum_{j=0}^{n-1} \left\| \int_{jh}^{(j+1)h} (A_0 + c_1)W(s)ds \right\|_{\mathcal{L}(V^*)} \|f(t-jh)\|_* \\ &\quad + \left\| \int_{nh}^t (A_0 + c_1)W(s)ds \right\|_{\mathcal{L}(V^*)} \|f(t-nh)\|_* \\ &\leq \sum_{j=0}^{n-1} C_j \|f\|_\rho \int_{jh}^{(j+1)h} (s-jh)^{-1} (s-jh)^\rho ds \\ &\quad + C_n \|f\|_\rho \int_{nh}^{(n+1)h} (s-nh)^{-1} (s-nh)^\rho ds \\ &\quad + \sum_{j=0}^{n-1} C_j \|f(t-jh)\|_* + C_n \|f(t-nh)\|_* \\ &\leq \frac{K_1 h^\rho}{\rho} + K_2 \left( \sup_{t \in [0, T]} \|f(t)\|_* \right); \end{aligned}$$

$$(2.29) \quad \begin{aligned} &\left\| \int_0^t \left( \frac{d}{dt} W(t-s) \right) f(s) ds \right\|_* \\ &\leq \sum_{j=0}^{n-1} \left\| \int_{jh}^{(j+1)h} \frac{d}{ds} W(s)(f(t-s) - f(t-jh)) ds \right\|_* \\ &\quad + \left\| \int_{nh}^t \frac{d}{ds} W(s)(f(t-s) - f(t-nh)) ds \right\|_* \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=0}^{n-1} \left\| \int_{jh}^{(j+1)h} \left( \frac{d}{ds} W(s) ds \right) f(t-jh) \right\|_* \\
 & + \left\| \int_{nh}^t \left( \frac{d}{ds} W(s) ds \right) f(t-nh) \right\|_* \\
 \leq & \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} \left\| \left( \frac{d}{ds} W(s) \right) \right\|_{\mathcal{L}(V^*)} \|f(t-s) - f(t-jh)\|_* ds \\
 & + \int_{nh}^t \left\| \left( \frac{d}{ds} W(s) \right) \right\|_{\mathcal{L}(V^*)} \|f(t-s) - f(t-nh)\|_* ds \\
 & + \sum_{j=0}^{n-1} \|W((j+1)h) - W(jh)\|_{\mathcal{L}(V^*)} \|f(t-jh)\|_* \\
 & + \|W(t) - W(nh)\|_{\mathcal{L}(V^*)} \|f(t-nh)\|_* \\
 \leq & \frac{K_1 h^p}{\rho} + K_2 \left( \sup_{t \in [0, T]} \|f(t)\|_* \right)
 \end{aligned}$$

for some constants  $K_1, K_2$ . Here in (2.29) we use the uniform boundedness

$$\left( \sup_{t \in [0, T]} \|W(t)\|_{\mathcal{L}(V^*)} \right) < \infty$$

because of the strong continuity of  $W(t)$  in  $V^*$ . Since  $A_0 + c_1$  is an isomorphism from  $V$  to  $V^*$ , the estimate (2.28) implies  $v \in L^2(0, T; V)$ . In view of Theorem 2 in Tanabe [22] and the estimate (2.29),  $v(t)$  is strongly differentiable in  $V^*$  and its derivative is given by

$$(2.30) \quad \frac{dv(t)}{dt} = f(t) + \int_0^t \frac{d}{dt} W(t-s) f(s) ds, \quad t > 0,$$

and hence

$$(2.31) \quad \left\| \frac{dv(t)}{dt} \right\|_* \leq \|f(t)\|_* + \frac{K_1 h^p}{\rho} + K_2 \left( \sup_{t \in [0, T]} \|f(t)\|_* \right).$$

This estimate implies  $v(\cdot) \in W^{1,2}(0, T; V^*)$ . Therefore, by differentiating the integral equation (2.27) we can verify that  $v(t)$  gives a solution of (E) with the data  $f \in C^p([0, T]; V^*)$  and  $g = 0$ .

Now let  $f \in L^2_{loc}(\mathbf{R}^+; V^*)$ . Since the Hölder space  $C^p([0, T]; V^*)$  is dense in  $L^2(0, T; V^*)$ , there exists a sequence  $\{f_n\} \subset C^p([0, T]; V^*)$  such that  $f_n \rightarrow f$  in  $L^2(0, T; V^*)$ . Set

$$(2.32) \quad v_n(t) = \int_0^t W(t-s) f_n(s) ds.$$

It is clear by the strong continuity of  $W(t)$  in  $V^*$  that

$$(2.33) \quad v_n(t) \rightarrow v(t) \quad \text{in } V^* \quad \text{for each } t > 0.$$

Since  $v_n(t) - v_m(t)$  give a solution of (E) with  $f = f_n - f_m$  and  $g = 0$ , we have by the boundedness (2.12) in Theorem 1

$$(2.34) \quad \left( \int_0^T \|v_n(t) - v_m(t)\|^2 dt + \int_0^T \left\| \frac{dv_n(t)}{dt} - \frac{dv_m(t)}{dt} \right\|_*^2 dt \right) \leq K_T \left( \int_0^T \|f_n(t) - f_m(t)\|_*^2 dt \right),$$

i.e.,  $\{v_n\}$  and  $\{dv_n/dt\}$  are Cauchy sequences in  $L^2(0, T; V)$  and in  $L^2(0, T; V^*)$ , respectively. Then by the closedness of differential operator  $d/dt$  in  $L^2(0, T; V^*)$ , there exists a function  $v_0 \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$  such that

$$(2.35) \quad v_n \rightarrow v_0 \text{ in } L^2(0, T; V), \quad \frac{dv_n}{dt} \rightarrow \frac{dv_0}{dt} \text{ in } L^2(0, T; V^*).$$

Combining (2.33) and the easy consequence  $v_n(t) \rightarrow v_0(t)$  in  $H$  from (2.35), we have  $v_0(t) = v(t)$  for each  $t \in [0, T]$ . This proves that  $v \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$ , so that  $v$  gives a solution of (E) on  $[0, T]$  with the data  $f \in L^2_{loc}(\mathbf{R}^+; V^*)$  and  $g = 0$ . This completes the proof of Proposition 2.  $\square$

Here we note an important consequence from Proposition 2. Let  $f \in L^2_{loc}(\mathbf{R}^+; V^*)$ . Then by Lemma 1, we have

$$(2.36) \quad \int_0^t \frac{d}{dt} W(t-s) f(s) ds = A_0 \int_0^t W(t-s) f(s) ds + A_1 \int_0^t W(t-s-h) f(s) ds + A_2 \int_0^t \int_{-h}^0 a(\xi) W(t-s+\xi) d\xi f(s) ds.$$

It is easily verified that all integrals in the right hand side of (2.36) make sense as integrals in  $V$ . Hence for each  $t \geq 0$ , the integral  $\int_0^t \frac{d}{dt} W(t-s) f(s) ds$  makes sense as an integral in  $V^*$ . Thus from Proposition 2 it is calculated by standard manipulations using the Lebesgue density theorem that

$$(2.37) \quad \frac{d}{dt} \int_0^t W(t-s) f(s) ds = f(t) + \int_0^t \frac{d}{dt} W(t-s) f(s) ds \quad \text{a.e. } t \geq 0.$$

For each  $t > 0$ , we introduce the operator valued function  $U_t(\cdot)$  defined by

$$(2.38) \quad U_t(s) = W(t-s-h)A_1 + \int_{-h}^s W(t-s+\xi)a(\xi)A_2 d\xi = \int_{-h}^s W(t-s+\xi)d\eta(\xi): V \rightarrow V, \quad \text{a.e. } s \in [-h, 0].$$

Let  $T > 0$  be arbitrary fixed. Associated with  $U_t(\cdot)$ , we consider the operator  $\mathcal{U}: L^2(-h, 0; V) \rightarrow L^2(0, T; V)$  defined by

$$(2.39) \quad (\mathcal{U}g^1)(t) = \int_{-h}^0 U_t(s)g^1(s)ds, \quad t \in [0, T]$$

for  $g^1 \in L^2(-h, 0; V)$ . We want to show that  $\mathcal{U}$  is into and bounded for each  $T > 0$ . For this sake it is convenient to introduce the structural operator  $F_1: L^2(-h, 0; V) \rightarrow L^2(-h, 0; V^*)$  defined by

$$(2.40) \quad [F_1g^1](s) = \int_{-h}^s d\eta(\xi)g^1(\xi-s) \\ = A_1g^1(-h-s) + \int_{-h}^s a(\xi)A_2g^1(\xi-s)d\xi \quad \text{a.e. } s \in [-h, 0].$$

By simple calculations using Schwartz inequality we see that  $F_1$  is into and bounded. Then by using Fubini's theorem as in Nakagiri [17; Lemma 4.1], we have

$$(2.41) \quad (\mathcal{U}g^1)(t) = \int_{-h}^0 W(t+s)[F_1g^1](s)ds \\ = \int_0^t W(t-s)\phi(s)ds \quad \text{a.e. } s \in [-h, 0],$$

where  $\phi(\cdot) = \chi_{[0, h]}[F_1g^1](-\cdot) \in L^2(0, T; V^*)$ . Hence by Proposition 2,  $\mathcal{U}$  is into and bounded for any  $T > 0$ . At the same time we know  $\mathcal{U}g^1 \in W^{1,2}(0, T; V^*)$ ,  $T > 0$ .

The following variation of constants formula of solutions are fundamental in our structural study for (E) (cf. Nakagiri [17]). Recently Yong and Pan [26] have derived a similar formula in a slightly different form under some abstract assumptions on the delay terms.

**Theorem 3.** For  $f \in L^2_{loc}(\mathbf{R}^+; V^*)$  and  $g = (g^0, g^1) \in H \times L^2(-h, 0; V)$ , the solution  $u(t) = u(t; f, g)$ ,  $t \geq 0$  of (E) is represented by

$$(2.42) \quad u(t; f, g) = W(t)g^0 + \int_{-h}^0 U_t(s)g^1(s)ds + \int_0^t W(t-s)f(s)ds.$$

*Proof.* Now it is obvious from the definition of fundamental solution and Proposition 2 that, by the uniqueness of solutions in the class  $L^2_{loc}(\mathbf{R}^+; V) \cap W^{1,2}_{loc}(\mathbf{R}^+; V^*)$

$$(2.43) \quad u(t; f, (g^0, 0)) = W(t)g^0 + \int_0^t W(t-s)f(s)ds.$$

Thus, it is left to prove that for  $g^1 \in L^2(-h, 0; V)$

$$(2.44) \quad u(t; 0, (0, g^1)) = \int_{-h}^0 U_t(s)g^1(s)ds = (\mathcal{U}g^1)(t).$$

Since  $\mathcal{U}g^1 \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$  for each  $T > 0$ , it is sufficient to verify that

$$v(t) = \begin{cases} (\mathcal{U}g^1)(t), & t \geq 0 \\ g^1(t) & \text{a.e. } t \in [-h, 0) \end{cases}$$

satisfies the integral equation (2.10) with  $g^0=0, f=0$ . Let us set  $\phi(s) = \chi_{[0,h]}[F_1g^1](-s)$ . Then we can write  $v(t)$  as  $\int_0^t W(t-s)\phi(s)ds$  for  $t \geq 0$ . If we set  $\vartheta(t) = \begin{cases} v(t), & t \geq 0 \\ 0 & \text{a.e. } t \in [-h, 0) \end{cases}$ , then as proved in Proposition 2,  $\vartheta(t)$  satisfies

$$\vartheta(t) = \begin{cases} \left( \int_0^t T(t-s)\phi(s)ds + \int_0^t T(t-s)A_1\vartheta(s-h)ds \right. \\ \left. + \int_0^t T(t-s) \left( \int_{-h}^0 a(\tau)A_2\vartheta(s+\tau)d\tau \right) ds, \right. & t \geq 0 \\ 0 & \text{a.e. } t \in [-h, 0). \end{cases}$$

Hence for  $t \geq 0$ , by defining  $T(t) = O$  for  $t < 0$  for simplicity, we have

$$\begin{aligned} v(t) &= \int_0^h T(t-s)[F_1g^1](-s)ds \\ &\quad + \int_0^h T(t-s) \left( \int_{-s}^0 a(\tau)A_2v(s+\tau)d\tau \right) ds \\ &\quad + \int_h^t T(t-s)(A_1v(s-h) + \int_{-h}^0 a(\tau)A_2v(s+\tau)d\tau)ds \\ &= \int_0^h T(t-s)(A_1g^1(s-h) + \int_{-h}^{-s} a(\tau)A_2g^1(s+\tau)d\tau)ds \\ &\quad + \int_0^h T(t-s) \left( \int_{-s}^0 a(\tau)A_2v(s+\tau)d\tau \right) ds \\ &\quad + \int_h^t T(t-s) \left( \int_{-h}^0 d\eta(\tau)v(s+\tau)d\tau \right) ds \\ &= \int_0^t T(t-s) \left( \int_{-h}^0 d\eta(\tau)v(s+\tau)d\tau \right) ds, \quad t \geq 0; \\ v(t) &= g^1(t) \quad \text{a.e. } t \in [-h, 0). \end{aligned}$$

This proves (2.44). Thus, we show the formula (2.42).  $\square$

### 3. Solution Semigroups and Structural Operators

As in previous researches [4–8, 16, 17–19, 20, 22, 23, 25] we take a product state space approach to study the properties of solution semigroups and the adjoint semigroups associated with the functional differential equations. Our main concern is to establish the relations between the semigroups and structural operators  $F, G(t)$  introduced in Nakagiri [17] and Tanabe [23] (see also Bernir

and Manitius [4], Delfour and Manitius [5], Manitius [16] for those in Euclidean spaces and Kunisch and Mastinšek [13] in Hilbert spaces). In what follows we assume the condition (2.20) on  $a(s)$ .

Let  $M_2 = H \times L^2(-h, 0; V)$  be the state space of the equation (E).  $M_2$  is a product Hilbert space with the inner product

$$\langle g, k \rangle_{M_2} = \langle g^0, k^0 \rangle + \int_{-h}^0 \langle g^1(s), k^1(s) \rangle_V ds,$$

$$g = (g^0, g^1), \quad k = (k^0, k^1) \in M_2,$$

where  $\langle \cdot, \cdot \rangle_V$  denotes the inner product of  $V$ .

Let  $g \in M_2$  and  $u(t; g)$  be the solution of (E) with  $f=0$ . The segment  $u_t$  is given by  $u_t(s; g) = u(t+s; g)$ ,  $s \in [-h, 0]$ . The solution semigroup  $S(t)$  associated with (E) is defined by

$$(3.1) \quad S(t)g = (u(t; g), u_t(\cdot; g)), \quad t \geq 0, \quad g \in M_2.$$

Then we have the following theorem which follows from Theorem 1 and Theorem 2 as shown in Di Blasio, Kunisch and Sinestrari [7].

**Theorem 4.** (i) *The family of operators  $\{S(t); t \geq 0\}$  is a  $C_0$ -semigroup on  $M_2$ .*

(ii) *The infinitesimal generator  $A$  of  $S(t)$  is characterized by*

$$(3.2) \quad \mathcal{D}(A) = \{g = (g^0, g^1); g^1 \in W^{1,2}(-h, 0; V),$$

$$g^1(0) = g^0, \quad A_0 g^0 + \int_{-h}^0 d\eta(s) g^1(s) ds \in H\}$$

$$(3.3) \quad Ag = (A_0 g^0 + \int_{-h}^0 d\eta(s) g^1(s) ds, \dot{g}^1) \quad \text{for } g = (g^0, g^1) \in \mathcal{D}(A),$$

where  $\dot{g}^1$  denotes the distribution derivative  $dg^1/ds$ .

In order to decompose the semigroup  $S(t)$  on the basis of the variation of constants formula of solutions, we introduce the structural operators  $F$  and  $G(t)$ . The structural operator  $F: M_2 \rightarrow M_2^*$  is defined by  $F = \begin{pmatrix} I & O \\ O & F_1 \end{pmatrix}$  i.e.,

$$(3.4) \quad [Fg]^0 = g^0, \quad [Fg]^1 = F_1 g^1, \quad \text{for } g = (g^0, g^1) \in M_2,$$

where  $F_1$  is given in (2.40). As is seen before  $F$  is into and bounded.

For  $t \geq 0$ , the structural operator  $G(t): M_2^* \rightarrow M_2$  is defined by

$$(3.5) \quad [G(t)f]^1(s) = W(t+s)f^0 + \int_{-h}^0 W(t+s+\xi)f^1(\xi)d\xi \quad s \in [-h, 0],$$

$$[G(t)f]^0 = [G(t)f]^1(0), \quad f = (f^0, f^1) \in M_2^*.$$

In view of Proposition 2 we see that  $G(t)$  is also into and bounded. We define

the structural operator  $G: M_2^* \rightarrow M_2$  by

$$(3.6) \quad G = G(h).$$

**Proposition 3.** *The semigroup  $S(t)$  is represented by*

$$(3.7) \quad S(t) = G(t)F + \kappa(t), \quad t \geq 0,$$

where  $\kappa(t); M_2 \rightarrow M_2$  is given by

$$(3.8) \quad [\kappa(t)g]^0 = 0, \quad [\kappa(t)g]^1(s) = \chi_{[-h, -t]}(s)g^1(t+s) \quad \text{a.e. } s \in [-h, 0].$$

In particular,  $S(h)$  has the following decomposition

$$(3.9) \quad S(h) = GF.$$

*Proof.* This proposition follows immediately from the equality

$$(3.10) \quad u(t+s; g) = \begin{cases} W(t+s)g^0 + \int_{-h}^0 W(t+s+\xi)[Fg]^1(\xi)d\xi, & t+s \geq 0 \\ g^1(t+s) & t+s < 0, \end{cases}$$

which is proved by Theorem 3 and (2.41), via the definition (3.5).

Using the trivial relation  $AS(t) = S(t)A$  on  $\mathcal{D}(A)$  and Proposition 3, we can derive the following useful commutative relation between  $W(t)$  and  $A_0, \eta$ . A direct proof of the relation is given in Tanabe [23].

**Corollary 1.** *For any  $g^0 \in V$ , the following relation holds :*

$$(3.11) \quad \begin{aligned} & A_0W(t)g^0 + \int_{-h}^0 d\eta(s)W(t+s)g^0 \\ &= W(t)A_0g^0 + \int_{-h}^0 W(t+s)d\eta(s)g^0 \quad \text{for all } t \in \mathbf{R}^+. \end{aligned}$$

*Proof.* By improving the proof of Bernir and Manitius [4; Lemma 4.1], we shall show this corollary. For each  $g \in W^{1,2}(-h, 0; V)$  it is calculated that

$$(3.12) \quad \frac{d}{ds}[Fg]^1(s) = a(s)A_2g^1(0) - [F\dot{g}]^1(s) \quad \text{a.e. } s \in [-h, 0]$$

and  $\frac{d}{ds}[Fg]^1, a(\cdot)A_2g^1(0), [F\dot{g}]^1 \in L^2(-h, 0; V^*)$ . Let  $g = (g^0, g^1) \in \mathcal{D}(A)$ . Then by (3.3), (3.5), (3.7) and noting that  $W(s) = 0$  if  $s < 0$ ,

$$(3.13) \quad \begin{aligned} & [AS(t)g]^0 \\ &= A_0[S(t)g]^0 + \int_{-h}^0 d\eta(s)[S(t)g]^1(s) \\ &= A_0W(t)g^0 + A_0 \int_{-h}^0 W(t+\xi)[Fg]^1(\xi)d\xi \end{aligned}$$





$$\begin{aligned}
(3.16) \quad & \int_{-h}^0 W(t+s) \left( \frac{d}{ds} [Fg]^1(s) \right) ds \\
= & \left\{ \int_{-t}^0 W(t+s) \left( \frac{d}{ds} [Fg]^1(s) \right) ds \quad \text{if } 0 \leq t < h \right. \\
& \left. \int_{-h}^0 W(t+s) \left( \frac{d}{ds} [Fg]^1(s) \right) ds \quad \text{if } t \geq h \right\} \\
= & \left\{ [W(t+s)[Fg]^1(s)]_{-t}^0 - \int_{-t}^0 \frac{d}{dt} W(t+s)[Fg]^1(s) ds \quad \text{if } 0 \leq t < h \right. \\
& \left. [W(t+s)[Fg]^1(s)]_{-h}^0 - \int_{-h}^0 \frac{d}{dt} W(t+s)[Fg]^1(s) ds \quad \text{if } t \geq h \right\} \\
= & \left\{ W(t) \int_{-h}^0 d\eta(s)g^1(s) - \int_{-h}^{-t} d\eta(s)g^1(t+s) \quad \text{if } 0 \leq t < h \right. \\
& \left. W(t) \int_{-h}^0 d\eta(s)g^1(s) - W(t-h)A_1g^1(0) \quad \text{if } t \geq h \right\} \\
- & \left\{ \int_{-t}^0 \frac{d}{dt} W(t+s)[Fg]^1(s) ds \quad \text{if } 0 \leq t < h \right. \\
& \left. \int_{-h}^0 \frac{d}{dt} W(t+s)[Fg]^1(s) ds \quad \text{if } t \geq h \right\}.
\end{aligned}$$

From (3.15) and (3.16), we have

$$\begin{aligned}
(3.17) \quad & [S(t)Ag]^0 \\
= & W(t)A_0g^0 + \int_{-h}^0 W(t+s)a(s)A_2g^0 ds \\
= & \left\{ \int_{-h}^{-t} d\eta(s)g^1(t+s) + \int_{-t}^0 \frac{d}{dt} W(t+s)[Fg]^1(s) ds \quad \text{if } 0 \leq t < h \right. \\
& \left. W(t-h)A_1g^1(0) + \int_{-h}^0 \frac{d}{dt} W(t+s)[Fg]^1(s) ds \quad \text{if } t \geq h \right\} \\
= & W(t)A_0g^0 + \int_{-h}^0 W(t+s)d\eta(s)g^0 ds \\
= & \left\{ \int_{-t}^0 \frac{d}{dt} W(t+s)[Fg]^1(s) ds \quad \text{if } 0 \leq t < h \right. \\
& \left. \int_{-h}^0 \frac{d}{dt} W(t+s)[Fg]^1(s) ds \quad \text{if } t \geq h \right\} \\
& + \left\{ \int_{-h}^{-t} d\eta(s)g^1(t+s) \quad \text{if } 0 \leq t < h \right. \\
& \left. 0 \quad \text{if } t \geq h \right\}.
\end{aligned}$$

Here we note that all integrals containing  $[Fg]^1$  in (3.16) and (3.17) make sense by  $[Fg]^1 \in L^2(-h, 0; V^*)$  and Lemma 1. Since  $W(t)$  satisfies the equation (2.21)

the sum of the last two terms in (3.17) equals the sum of the last two terms of (3.15). Consequently, the trivial equality  $[AS(t)g]^0 = [S(t)Ag]^0$  for  $g \in \mathcal{D}(A) \subset W^{1,2}(-h, 0; V)$  implies the commutative relation (3.11) for any  $g^0 \in V$ .  $\square$

The following useful ‘quasi-semigroup’ property for  $W(t)$  is verified by the semigroup property  $S(t_1+t_2) = S(t_1)S(t_2)$  (cf. Nakagiri [17]).

**Corollary 2.** *The fundamental solution  $W(t)$  satisfies*

$$(3.18) \quad \begin{aligned} &W(t_1+t_2) \\ &= W(t_1)W(t_2) + \int_{-h}^0 W(t_1+s)[F_1W(t_2+\cdot)](s)ds \quad t_1, t_2 \geq 0, \end{aligned}$$

where

$$(3.19) \quad [F_1W(t_2+\cdot)](s)g^0 = \int_{-h}^s d\eta(\xi)W(t_2+\xi-s)g^0d\xi, \quad g^0 \in H.$$

Proof. Let  $g^0 \in H$ . Then by (3.7) the relation  $[S(t_1+t_2)(g^0, 0)]^0 = [S(t_1)S(t_2)(g^0, 0)]^0$  implies the equality (3.18).  $\square$

Next we shall study the adjoint semigroup  $S^*(t)$  on the adjoint space of  $M_2$ , which plays important role in the control theory involving functional differential equations. The adjoint space  $M_2^*$  of  $M_2$  can be identified with the product space  $H \times L^2(-h, 0; V^*)$  via the duality pairing (we use the same bracket as that of inner product)

$$(3.20) \quad \begin{aligned} \langle g, f \rangle_{M_2} &= \langle g^0, f^0 \rangle + \int_{-h}^0 \langle g^1(s), f^1(s) \rangle ds, \\ g &= (g^0, g^1) \in M_2, \quad f = (f^0, f^1) \in M_2^*. \end{aligned}$$

By identifying the second dual  $V^{**}$  of  $V$  with itself, the second adjoint space  $M_2^{**}$  is identified with  $M_2$ . That is,  $M_2$  is reflexive in this identification. Hence, by the reflexiveness, the adjoint semigroup  $S^*(t)$  is strongly continuous in  $M_2^*$  (cf. Hille and Phillips [9]). Further the infinitesimal generator of  $S^*(t)$  is given by the adjoint  $A^*$  and is characterized precisely by the following theorem. For relevant results concerning abstract functional differential equations we refer to Webb [25], Nakagiri [17] and Kunisch and Mastinšek [13].

**Theorem 5.** *The infinitesimal generator  $A^*$  of  $S^*(t)$  is given by*

$$(3.21) \quad \begin{aligned} \mathcal{D}(A^*) &= \{f = (f^0, f^1); f^0 \in V, f^1 \in W^{1,2}(-h, 0; V^*), \\ &A_0^*f^0 + f^1(0) \in H, f^1(-h) = A_1^*f^0\} \end{aligned}$$

$$(3.22) \quad A^*f = (A_0^*f^0 + f^1(0), a(\cdot)A_2^*f^0 - f^1(\cdot)) \quad \text{for } f = (f^0, f^1) \in \mathcal{D}(A^*).$$

Proof. We follow the argument in [17; Prop. 3.3]. Let  $(f^0, f^1) \in \mathcal{D}(A^*)$

and  $A^*(f^0, f^1) = (k^0, k^1) \in M_2^*$ . Then for each  $(g^0, g^1) \in \mathcal{D}(A)$ ,

$$(3.23) \quad \langle A(g^0, g^1), (f^0, f^1) \rangle_{M_2} = \langle (g^0, g^1), (k^0, k^1) \rangle_{M_2},$$

that is, by (3.3)

$$(3.24) \quad \langle A_0g^0 + A_1g^1(-h) + \int_{-h}^0 a(s)A_2g^1(s)ds, f^0 \rangle + \int_{-h}^0 \langle \dot{g}^1(s), f^1(s) \rangle ds \\ = \langle g^0, k^0 \rangle + \int_{-h}^0 \langle g^1(s), k^1(s) \rangle ds.$$

First we assume  $f^0 \in V$ . Let  $\varphi$  be a scalar function such that

$$(3.25) \quad \varphi \in W^{1,2}(-h, 0), \quad \varphi(-h) = 0, \quad \varphi(0) = 1, \quad \int_{-h}^0 a(s)\varphi(s)ds = 0.$$

For  $g^0 \in \mathcal{D}(A_0)$  it is clear that  $\tilde{g} = (g^0, \varphi(\cdot)g^0) \in \mathcal{D}(A)$ . We substitute this  $\tilde{g}$  into (3.24) and obtain

$$(3.26) \quad \langle A_0g^0, f^0 \rangle \\ = \langle g^0, k^0 \rangle + \langle g^0, \int_{-h}^0 \varphi(s)k^1(s)ds \rangle - \langle g^0, \int_{-h}^0 \dot{\varphi}(s)f^1(s)ds \rangle.$$

The linear form  $l(g^0) = \langle A_0g^0, f^0 \rangle$  on  $\mathcal{D}(A_0)$  defined by the right hand side of (3.26) can be extended to a bounded form  $\tilde{l}$  on  $V$  by the density of  $\mathcal{D}(A_0)$  in  $V$  (note that  $A_0 + c_1: V \rightarrow V^*$  is an isomorphism), and then  $\tilde{l} = A_0^*f^0 \in V^*$ . This means

$$(3.27) \quad f^0 \in V \text{ and } \langle A_0g^0, f^0 \rangle = \langle g^0, A_0^*f^0 \rangle \quad \text{for any } g^0 \in V.$$

Next we set

$$M(s) = \int_{-h}^s k^1(\xi)d\xi, \quad N(s) = A_1^*f^0 + \int_{-h}^s a(\xi)A_2^*f^0d\xi, \quad s \in [-h, 0].$$

It is easy to see  $M \in W^{1,2}(-h, 0; V^*)$ ,  $N \in C^1([-h, 0]; V^*)$ . For any  $y \in W^{1,2}(-h, 0; \mathcal{D}(A_0))$ , we have by using integration by parts

$$(3.28) \quad \int_{-h}^0 \langle \dot{y}(s), M(s) \rangle ds \\ = \langle y(0), M(0) \rangle - \int_{-h}^0 \langle y(s), k^1(s) \rangle ds;$$

$$(3.29) \quad \int_{-h}^0 \langle \dot{y}(s), N(s) \rangle ds \\ = \langle y(0), N(0) \rangle - \langle y(-h), N(-h) \rangle - \int_{-h}^0 \langle y(s), \dot{N}(s) \rangle ds \\ = \langle y(0), N(0) \rangle - \langle y(-h), A_1^*f^0 \rangle - \int_{-h}^0 \langle y(s), a(s)A_2^*f^0 \rangle ds.$$

Then by (3.28), (3.29) and taking adjoints, we find

$$\begin{aligned} & \int_{-h}^0 \langle \dot{y}(s), f^1(s) - N(s) + M(s) \rangle ds + \langle A_0 y(0), f^0 \rangle \\ = & \int_{-h}^0 \langle \dot{y}(s), f^1(s) \rangle ds - \langle y(0), N(0) \rangle + \langle y(-h), A_1^* f^0 \rangle \\ & + \int_{-h}^0 \langle y(s), a(s) A_2^* f^0 \rangle ds + \langle y(0), M(0) \rangle \\ & - \int_{-h}^0 \langle y(s), k^1(s) \rangle ds + \langle A_0 y(0), f^0 \rangle \\ = & \langle A_0 y(0) + A_1 y(-h) + \int_{-h}^0 a(s) A_2 y(s) ds, f^0 \rangle + \int_{-h}^0 \langle \dot{y}(s), f^1(s) \rangle ds \\ & - \langle y(0), k^0 \rangle - \int_{-h}^0 \langle y(s), k^1(s) \rangle ds \\ & + \langle y(0), k^0 - N(0) + M(0) \rangle. \end{aligned}$$

Hence if  $y \in \mathcal{D}(A)$ , then by (3.24)

$$(3.30) \quad \begin{aligned} & \int_{-h}^0 \langle \dot{y}(s), f^1(s) - N(s) + M(s) \rangle ds + \langle A_0 y(0), f^0 \rangle \\ & = \langle y(0), k^0 - N(0) + M(0) \rangle. \end{aligned}$$

Let  $g^0 \in \mathcal{D}(A_0)$ . Applying  $y(s) \equiv g^0 \in W^{1,2}(-h, 0; \mathcal{D}(A_0)) \cap \mathcal{D}(A)$  to (3.30), we get

$$\langle A_0 g^0, f^0 \rangle = \langle g^0, k^0 - A_1^* f^0 - \int_{-h}^0 a(\xi) A_2^* f^0 d\xi + \int_{-h}^0 k^1(\xi) d\xi \rangle,$$

so that by the extension and (3.27),

$$(3.31) \quad k^0 = A_0^* f^0 + A_1^* f^0 + \int_{-h}^0 a(\xi) A_2^* f^0 d\xi - \int_{-h}^0 k^1(\xi) d\xi \in H.$$

Since  $W^{1,2}(-h, 0; \mathcal{D}(A_0)) \cap \mathcal{D}(A)$  is dense in  $W^{1,2}(-h, 0; V)$ , it follows from (3.30) and (3.31) that

$$(3.32) \quad \begin{aligned} f^1(s) & = N(s) - M(s) \\ & = A_1^* f^0 + \int_{-h}^s a(\xi) A_2^* f^0 d\xi - \int_{-h}^s k^1(\xi) d\xi, \quad s \in [-h, 0], \end{aligned}$$

and

$$(3.33) \quad f^1 \in W^{1,2}(-h, 0; V^*), \quad f^1(-h) = A_1^* f^0.$$

Therefore by (3.31)-(3.33),  $\mathcal{D}(A^*)$  is contained in the right hand side of (3.21). The reverse inclusion can be verified analogously. So the proof is complete.  $\square$

Finally in this section we note that the condition (2.20) on  $a(s)$  is not nec-

sary in Theorem 4 and Theorem 5.

### 4. Adjoint Theory for Transposed Equations

The purpose of this section is to give an elementary adjoint theory for transposed equations and study the relations between transposed semigroups, adjoint semigroups and structural operators.

The ‘transposed’ equation  $(E_T)$  in  $H$  with an initial data  $(\varphi^0, \varphi^1) \in M_2$  and a forcing function  $h \in L^2_{loc}(\mathbf{R}^+; V^*)$  is defined by

$$(4.1) \quad \frac{dv(t)}{dt} = A_0^*v(t) + A_1^*v(t-h) + \int_{-h}^0 a(s)A_2^*v(t+s)ds + h(t), \quad \text{a.e. } t \geq 0$$

$$(4.2) \quad v(0) = \varphi^0, \quad v(s) = \varphi^1(s) \quad \text{a.e. } s \in [-h, 0),$$

where  $A_i^*$  denote the adjoint operator of  $A_i$  ( $i=0, 1, 2$ ).

It is well known (cf. Hille and Phillips [9] and Tanabe [21]) that the adjoint operator  $A_0^*$  generates an analytic semigroup  $T^*(t)$ , which is the adjoint of  $T(t)$ , both in  $H$  and  $V^*$  and that  $T^*(t): V^* \rightarrow V$  for each  $t > 0$ . Then we can construct the fundamental solution  $W_T(t)$  of  $(E_T)$ , which is strongly continuous both in  $H$  and  $V^*$  and  $W_T(t): V^* \rightarrow V, t > 0$  satisfying the same estimates in Lemma 1. For simplicity we use the adjoint Stieltjes measure  $\eta^*(s)$  defined by

$$(4.3) \quad \eta^*(s) = -\chi_{(-\infty, -h]}(s)A_1^* - \int_s^0 a(\xi)d\xi A_2^*: V \rightarrow V^*, \quad s \in [-h, 0].$$

Hence, as shown in Theorem 1 and Theorem 3 the solution  $v(t)$  of  $(E_T)$  exists uniquely and is represented by

$$(4.4) \quad v(t) = W_T(t)\varphi^0 + \int_{-h}^0 V_i(s)\varphi^1(s)ds + \int_0^t W_T(t-s)h(s)ds \quad t \geq 0,$$

where

$$(4.5) \quad V_i(s) = \int_{-h}^0 W_T(t-s+\xi)d\eta^*(\xi): V \rightarrow V, \quad \text{a.e. } s \in [-h, 0].$$

Now we denote the adjoint of  $W(t)$  by  $W^*(t)$ . First we shall show the following important relationship in the adjoint theory.

**Lemma 2.** For  $t \in \mathbf{R}$ ,

$$(4.6) \quad W_T(t) = W^*(t).$$

Proof. By the identification of  $V^{**}$  with  $V$ , it is obvious that  $W^*(t): V^* \rightarrow V$  for  $t > 0$ . The commutative relation (3.11) in Corollary 1 implies, by taking adjoint, the adjoint relation

$$(4.7) \quad A_0^*W^*(t) + \int_{-h}^0 d\eta^*(s)W^*(t+s) = W(t)^*A_0^* + \int_{-h}^0 W(t+s)^*d\eta^*(s).$$

On the other hand, taking adjoint of (2.21), we have that  $W^*(t)$  satisfies

$$(4.8) \quad \frac{d}{dt}W^*(t) = W^*(t)A_0^* + \int_{-h}^0 W^*(t+s)d\eta^*(s), \quad \text{a.e. } t > 0,$$

so by (3.11),

$$(4.9) \quad \frac{d}{dt}W^*(t) = A_0^*W^*(t) + \int_{-h}^0 d\eta^*(s)W^*(t+s), \quad \text{a.e. } t > 0.$$

This shows that  $W^*(t)$  is the fundamental solution of  $(E_T)$ . Hence by uniqueness of solutions the assertion (4.6) follows.  $\square$

Calculating the adjoints of  $F$  and  $G(t)$  straightforwardly by means of the pairing (3.20) and using Lemma 2, we can deduce the following representation formulas.

**Proposition 4.** (i) *The adjoint  $F^*: M_2 \rightarrow M_2^*$  of  $F$  is given by*

$$(4.10) \quad [F^*g]^0 = g^0, \quad [F^*g]^1 = F_1^*g, \quad \text{for } g = (g^0, g^1) \in M_2,$$

where  $F_1^*: L_2(-h, 0; V) \rightarrow L_2(-h, 0; V^*)$  denotes the adjoint of  $F_1$  and is represented by

$$(4.11) \quad [F_1^*g^1](s) = \int_{-h}^s d\eta^*(\xi)g^1(\xi-s) \quad \text{a.e. } s \in [-h, 0].$$

(ii) *The adjoint  $G^*(t): M_2^* \rightarrow M_2$  of  $G(t)$  is given by*

$$(4.12) \quad \begin{aligned} [G^*(t)f]^1(s) &= W_T(t+s)f^0 + \int_{-h}^0 W_T(t+s+\xi)f^1(\xi)d\xi \quad s \in [-h, 0], \\ [G^*(t)f]^0 &= [G^*(t)f]^1(0), \quad f = (f^0, f^1) \in M_2^*. \end{aligned}$$

By Proposition 4, we see that the solution  $v(t) = v(t; \varphi)$  of  $(E_T)$  with  $h=0$  is written as

$$(4.13) \quad v(t; \varphi) = W_T(t)\varphi^0 + \int_{-h}^0 W_T(t+s)[F^*\varphi]^1(s)ds.$$

Let us denote by  $\{S_T(t)\}_{t \geq 0}$  the transposed  $C_0$ -semigroup on  $M_2$  associated with the transposed equation  $(E_T)$  and by  $A_T$  its infinitesimal generator. Now the following proposition is immediate from (4.13) and Proposition 4.

**Proposition 5.** *The semigroup  $S_T(t)$  is expressed by*

$$(4.14) \quad S_T(t) = G^*(t)F^* + \kappa(t), \quad t \geq 0,$$

where  $\kappa(t): M_2 \rightarrow M_2$  is given by (3.8). In particular,  $S_T(h)$  has the following decomposition

$$(4.15) \quad S_T(h) = G^*F^* .$$

The adjoint semigroup of  $S_T(t)$  and its infinitesimal generator are denoted by  $S_T^*(t)$  and  $A_T^*$ , respectively. Taking adjoints of  $S(t)$  and  $S_T(t)$  and using Proposition 3 and Proposition 5, we have the following result.

**Proposition 6.** *For  $t \geq 0$ , the semigroups  $S^*(t)$  and  $S_T^*(t)$  are represented by*

$$(4.16) \quad S^*(t) = F^*G^*(t) + \kappa^*(t), \quad S_T^*(t) = FG(t) + \kappa^*(t),$$

where the adjoint  $\kappa^*(t): M_2^* \rightarrow M_2^*$  of  $\kappa(t)$  is given by

$$(4.17) \quad [\kappa^*(t)f]^0 = 0, \quad [\kappa^*(t)f]^1(s) = f^1(s-t)\chi_{[0, s+h]}(s) \quad \text{a.e. } s \in [-h, 0]$$

for  $f = (f^0, f^1) \in M_2^*$ . In particular,

$$(4.18) \quad S^*(h) = F^*G^*, \quad S_T^*(h) = FG .$$

On the ranges and the invertibility of the operators  $G, G^*$ , we have the following characterizations.

**Proposition 7.** (i) *Im  $G$  equals Im  $G^*$  and is given by*

$$(4.19) \quad \mathcal{W}(-h, 0) = \{\varphi \in M_2; \varphi^1 \in W^{1,2}(-h, 0; V^*) \cap L^2(-h, 0; V), \\ \varphi^1(0) = \varphi^0 \in H\} .$$

(ii)  *$G: M_2^* \rightarrow \mathcal{W}(-h, 0)$  is bijective and the inverse  $G^{-1}: \mathcal{W}(-h, 0) \rightarrow M_2^*$  is given explicitly by*

$$(4.20) \quad [G^{-1}\varphi]^1(s) = \varphi^1(-s-h) - A_0\varphi^1(-s-h) - \int_s^0 d\eta(\xi)\varphi^1(\xi-s-h) \\ \text{a.e. } s \in [-h, 0] \\ [G^{-1}\varphi]^0 = \varphi^1(-h), \quad \varphi = (\varphi^0, \varphi^1) \in \mathcal{W}(-h, 0) .$$

*$G^*: M_2^* \rightarrow \mathcal{W}(-h, 0)$  is also bijective and the inverse  $(G^*)^{-1}: \mathcal{W}(-h, 0) \rightarrow M_2^*$  is given by the formula (4.20) in which  $A_0, \eta(\xi)$  are replaced by  $A_0^*, \eta^*(\xi)$ , respectively. In particular,*

$$(4.21) \quad \text{Ker } G = \text{Ker } G^* = \{0\}; \quad \text{Cl}(\text{Im } G) = \text{Cl}(\text{Im } G^*) = M_2 .$$

**Proof.** Since the inclusion  $\text{Im } G \subset \mathcal{W}(-h, 0)$  is clear from Theorem 1 and Proposition 2, it suffices to show the reverse inclusion  $\text{Im } G \supset \mathcal{W}(-h, 0)$ . Let  $\varphi = (\varphi^1(0), \varphi^1) \in \mathcal{W}(-h, 0)$  and define  $k = (k^0, k^1)$  by the right hand side of (4.20). It is easy to see  $k \in M_2^*$ . If we set  $u(t) = \varphi^1(t-h), t \in [0, h]$  and  $u(0) = \varphi^1(-h), u(s) = 0$  a.e.  $s \in [-h, 0]$ , then from (4.20)  $u(\cdot) \in W^{1,2}(0, h; V^*) \cap L^2(0, h; V)$  gives a solution of (E) on  $[0, h]$  with  $g^0 = \varphi^1(-h), g^1 \equiv 0, f(t) = k^1(-t)$  a.e.  $t \in [0, h]$

(note that  $\varphi^1(-h) \in H, f \in L^2(0, h; V^*)$ ). Thus, by the uniqueness of solution and the variation of constants formula,

$$\begin{aligned} u(t) &= W(t)\varphi^1(-h) + \int_0^t W(t-s)f^1(-s)ds \\ &= [Gk]^1(t-h) = \varphi^1(t-h), \quad t \in [0, h], \\ u(h) &= [Gk]^1(0) = [Gk]^0 = \varphi^1(0). \end{aligned}$$

This implies  $Gk = \varphi$ , and hence  $\text{Im } G = \mathcal{W}(-h, 0)$ . At the same time we see that  $G$  is bijective and the inverse  $G^{-1}$  is given by (4.20). The proof for  $G^*$  is quite same. Now the first relation in (4.21) is obvious and the second one follows readily from  $\mathcal{D}(A) \subset \mathcal{W}(-h, 0)$ .  $\square$

Related to the operators  $F, F^*$  we have the following Proposition

**Proposition 8.** (i) *If  $A_1: V \rightarrow V^*$  has a bounded inverse  $A_1^{-1}: V^* \rightarrow V$ , then*

$$(4.22) \quad \text{Im } F = \text{Im } F^* = M_2^*$$

and

$$(4.23) \quad \text{Ker } F = \text{Ker } F^* = \{0\}.$$

(ii) *If  $A_1 = 0, a \in C^1[-h, 0], a(-h) \neq 0$  and  $A_2: V \rightarrow V^*$  has a bounded inverse  $A_2^{-1}: V^* \rightarrow V$ , then*

$$(4.24) \quad \text{Ker } F = \text{Ker } F^* = \{0\}.$$

**Proof.** (i) Let  $f = (f^0, f^1) \in M_2^*, g = (g^0, g^1) \in M_2$ . In order to show  $\text{Im } F = M_2^*$ , we shall solve the equation  $Fg = f$ . Since  $A_1^{-1}$  exists, this equation is equivalent to  $g^0 = f^0$  and

$$(4.25) \quad g^1(-h-s) + \int_{-h}^s a(\xi)A_1^{-1}A_2g^1(\xi-s)d\xi = A_1^{-1}f^1(s) \quad \text{a.e. } s \in [-h, 0].$$

The equation (4.25) for  $g^1$  is a Volterra integral equation of the second kind, so by the contraction mapping principle, we have a unique solution  $g^1 \in L^2(-h, 0; V)$ . Hence we can construct a unique solution  $g \in M_2$  of the equation  $Fg = f$ . This proves  $\text{Im } F = M_2^*$  and  $\text{Ker } F = \{0\}$ . Since  $A_1$  is boundedly invertible if and only if the adjoint  $A_1^*$  is boundedly invertible (cf. Kato [12; p. 169]), as seen as above we have  $\text{Im } F^* = M_2^*$  and  $\text{Ker } F^* = \{0\}$ .

(ii) Assume  $A_1 = 0$  and let  $g = (g^0, g^1) \in M_2$  satisfy  $Fg = 0$ . Then  $g^0 = 0$  and  $g^1$  satisfies the Volterra integral equation of the first kind

$$(4.26) \quad \int_{-h}^s a(\xi)A_2g^1(\xi-s)d\xi = 0 \quad \text{a.e. } s \in [-h, 0].$$

Since  $a(s)$  is continuously differentiable, we differentiate the equality (4.26) and



apply  $A_2^{-1}$  to obtain

$$(4.27) \quad a(-h)g^1(-h-s) + \int_{-h}^s \frac{da(\xi)}{d\xi} g^1(\xi-s) d\xi = 0 \quad \text{a.e. } s \in [-h, 0].$$

The Volterra integral equation (4.28) is of the second kind by  $a(-h) \neq 0$ , and so this equation admits a trivial solution  $g^1(s) = 0$ , and hence  $g = (g^0, g^1) = 0$ . This shows  $\text{Ker } F = \{0\}$ . Noting Kato [12; p. 169] again, we see  $\text{Ker } F^* = \{0\}$  readily.  $\square$

The next Proposition extends the key relation  $G(t+h) = S(t)G$  proved in Nakagiri [17; Prop. 3.7], which plays a central role in the structural study of retarded functional differential equation (E).

**Proposition 9.** (i) For  $t_1, t_2 \geq 0$ ,

$$(4.28) \quad S(t_1)G(t_2) = G(t_1+t_2)E(t_2), \quad S_T(t_1)G^*(t_2) = G^*(t_1+t_2)E(t_2),$$

where  $E(t_2): M_2^* \rightarrow M_2^*$  is given by

$$(4.29) \quad [E(t_2)f]^0 = f^0, \quad [E(t_2)f]^1(s) = \chi_{(-t_2, 0]}(s)f^1(s) \quad \text{a.e. } s \in [-h, 0].$$

(ii) For  $t_1, t_2 \geq 0$ ,

$$(4.30) \quad G^*(t_1)S^*(t_2) = E^*(t_1)G^*(t_1+t_2), \quad G(t_1)S_T^*(t_2) = E^*(t_1)G(t_1+t_2),$$

where  $E^*(t_1): M_2 \rightarrow M_2$  is given by

$$(4.31) \quad [E^*(t_1)g]^0 = g^0, \quad [E^*(t_1)g]^1(s) = \chi_{(-t_1, 0]}(s)g^1(s) \quad \text{a.e. } s \in [-h, 0].$$

(ii) For  $t_1 \geq 0$  and  $t_2 \geq h$ ,

$$(4.32) \quad S(t_1)G(t_2) = G(t_2)S_T^*(t_1) \quad S_T(t_1)G^*(t_2) = G^*(t_2)S^*(t_1).$$

*Proof.* (i) Let  $f = (f^0, f^1) \in M_2^*$ . By definitions of  $S(t)$ ,  $G(t)$  and (3.10), we have for  $t_1 + s \geq 0$ ,

$$\begin{aligned} (4.33) \quad & [S(t_1)G(t_2)f]^1(s) \\ &= W(t_1+s)[G(t_2)f]^0 + \int_{-h}^0 W(t_1+s+\xi) \left[ F[G(t_2)f]^1 \right] (\xi) d\xi \\ &= \left( W(t_1+s)W(t_2)f^0 + \int_{-h}^0 W(t_1+s+\xi)[F_1W(t_2+\cdot)]f^0(\xi) d\xi \right) \\ & \quad + \int_{-h}^0 W(t_1+s)W(t_2+\xi)f^1(\xi) d\xi \\ & \quad + \int_{-h}^0 W(t_1+s+\xi) \left[ F_1 \left[ \int_{-h}^0 W(t_2+\cdot+\alpha)f^1(\alpha) d\alpha \right] \right] (\xi) d\xi \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By the change of variables  $\alpha \rightarrow \xi$ , the term  $I_3$  is calculated as

$$\begin{aligned}
 (4.34) \quad I_3 &= \int_{-h}^0 \int_{-h}^{\xi} W(t_1+s+\xi) d\eta(\tau) \int_{-h}^0 W(t_2+\alpha+\tau-\xi) f^1(\alpha) d\alpha d\xi \\
 &= \int_{-h}^0 \int_{-h}^0 W(t_1+s+\alpha) \int_{-h}^{\alpha} d\eta(\tau) W(t_2+\xi+\tau-\alpha) d\alpha f^1(\xi) d\xi \\
 &= \int_{-h}^0 \left( \int_{-h}^0 W(t_1+s+\alpha) [F_1 W(t_2+\xi+\cdot)](\alpha) d\alpha \right) f^1(\xi) d\xi,
 \end{aligned}$$

so that

$$\begin{aligned}
 (4.35) \quad I_2 + I_3 &= \int_{-h}^0 (W(t_1+s)W(t_2+\xi) \\
 &\quad + \int_{-h}^0 W(t_1+s+\alpha) [F_1 W(t_2+\xi+\cdot)](\alpha) d\alpha) f^1(\xi) d\xi.
 \end{aligned}$$

Noting that  $W(t_2+\xi) = O$  if  $t_2+\xi < 0$  and applying the quasi-semigroup property (3.18) of  $W(t)$  we obtain that

$$\begin{aligned}
 (4.36) \quad [S(t_1)G(t_2)f]^1(s) &= W(t_1+t_2+s)f^0 + \int_{-t_2}^0 W(t_1+t_2+s+\xi) f^1(\xi) d\xi \\
 &= W(t_1+t_2+s)f^0 + \int_{-h}^0 W(t_1+t_2+s+\xi) \mathcal{X}_{(-t_2,0]} f^1(\xi) d\xi \\
 &= [G(t_1+t_2)E(t_2)f]^1(s).
 \end{aligned}$$

On the other hand, we have for  $t_1+s < 0$ ,

$$\begin{aligned}
 (4.37) \quad [S(t_1)G(t_2)f]^1(s) &= [G(t_2)f]^1(t_1+s) \\
 &= W(t_1+t_2+s)f^0 + \int_{-h}^0 W(t_2+t_1+s+\xi) f^1(\xi) d\xi \\
 &= W(t_1+t_2+s)f^0 + \int_{-h}^0 W(t_2+t_1+s+\xi) \mathcal{X}_{(-t_2,0]} f^1(\xi) d\xi \\
 &= [G(t_1+t_2)E(t_2)f]^1(s).
 \end{aligned}$$

Substituting  $s=0$  in (4.36) we have  $[S(t_1)G(t_2)f]^0 = [G(t_1+t_2)E(t_2)f]^0$ . Hence  $S(t_1)G(t_2) = G(t_1+t_2)E(t_2)$  is proved. Similarly we have  $S_T(t_1)G^*(t_2) = G^*(t_1+t_2)E(t_2)$ .

(ii) Taking adjoints of the equalities in (4.28) and changing  $t_1$  and  $t_2$ , we have the relations (4.30).

(iii) The assertion (4.32) follows from (i), (ii) and that  $E(h)$  is the identity operator on  $M^*$ .

The following result gives the interconnected property between the struc-

tural operator  $G$  and the semigroup  $S(t)$ .

**Theorem 6.** *The following relations on  $G$  hold :*

$$(4.38) \quad S(t)G = GS_T^*(t), \quad G^*S^*(t) = S_T(t)G^*, \quad t \geq 0;$$

$$(4.39) \quad G\mathcal{D}(A_T^*) \subset \mathcal{D}(A) \quad \text{and} \quad AG = GA_T^* \quad \text{on} \quad \mathcal{D}(A_T^*);$$

$$(4.40) \quad G^*\mathcal{D}(A^*) \subset \mathcal{D}(A_T) \quad \text{and} \quad G^*A^* = A_TG^* \quad \text{on} \quad \mathcal{D}(A^*).$$

Proof. Substituting  $t_1=t$  and  $t_2=h$  in (4.32) we get (4.38). It is well known that the first and the second equalities in (4.38) are equivalent to (4.39) and (4.40), respectively (cf. Salamon [20; Lemma I. 3.8]).  $\square$

**Theorem 7.** *The following relations on  $F$  hold :*

$$(4.41) \quad FS(t) = S_T^*(t)F, \quad S^*(t)F^* = F^*S_T(t), \quad t \geq 0;$$

$$(4.42) \quad F\mathcal{D}(A) \subset \mathcal{D}(A_T^*) \quad \text{and} \quad FA = A_T^*F \quad \text{on} \quad \mathcal{D}(A);$$

$$(4.43) \quad F^*\mathcal{D}(A_T) \subset \mathcal{D}(A^*) \quad \text{and} \quad A^*F^* = F^*A_T \quad \text{on} \quad \mathcal{D}(A_T).$$

Proof. It is sufficient to prove (4.41). By (4.38) and (3.9),

$$G(S_T^*(t)F) = S(t)GF = GFS(t) = G(FS(t))$$

and, by  $\text{Ker } G = \{0\}$  in (6.21), the equality  $S_T^*(t)F = FS(t)$  follows. The second equality in (4.41) can be proved similarly.  $\square$

### 5. Examples

In this section we give an application of the results obtained in the preceding sections to practical partial functional differential equations.

Let  $\Omega \subset \mathbf{R}^n$  be a bounded domain with smooth boundary  $\partial\Omega$ . We set  $H = L^2(\Omega)$  and  $V = H_0^1(\Omega)$ . Let  $a(u, v)$  be the sesquilinear form in  $H_0^1(\Omega) \times H_0^1(\Omega)$  defined by

$$(5.1) \quad a(u, v) = \int_{\Omega} \left\{ \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \overline{\frac{\partial v}{\partial x_j}} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} v + c(x)uv \right\} dx, \quad x \in \Omega.$$

Here in (5.1) we assume that the real valued coefficients  $a_{ij}, b_i, c$  satisfy

$$a_{ij} \in C^1(\overline{\Omega}), \quad b_i \in C^1(\overline{\Omega}), \quad c \in L^\infty(\Omega),$$

$a_{ij} = a_{ji}, (1 \leq i, j \leq n)$  and the uniform ellipticity

$$\sum_{i,j=1}^n a_{ij}(x) y_i y_j \geq \nu |y|^2 \quad y = (y_1, \dots, y_n) \in \mathbf{R}^n$$

for some positive  $\nu$ . As is well known (see e.g. Tanabe [21; Chap.2]) this sesquilinear form is bounded and satisfies the Gårding's inequality (2.3). The

operator  $A_0: H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$  defined through (2.4) in this case has the following realization in  $L^2(\Omega)$ . Let

$$(5.2) \quad \mathcal{A} = - \sum_{i,j=1}^n \frac{\partial}{\partial x_j} (a_{ij}(x) \frac{\partial}{\partial x_j}) + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} + c(x), \quad x \in \Omega$$

be the associated uniformly elliptic differential operator of second order. Then the realization of  $-\mathcal{A}$  in  $L^2(\Omega)$  under the Dirichlet boundary condition is exactly  $A_0$ , i.e.,

$$\begin{aligned} \mathcal{D}(A_0) &= W^{2,2}(\Omega) \cap H_0^1(\Omega), \\ A_0 u &= -\mathcal{A}u \quad \text{for } u \in \mathcal{D}(A_0). \end{aligned}$$

It is not difficult to verify that  $A_0 u = -\mathcal{A}u$  for  $u \in H_0^1(\Omega)$  in the sense of distribution and  $u|_{\partial\Omega} = 0$  for  $u \in H_0^1(\Omega)$  also in the sense of distribution (cf. Lions and Magenes [15]).

Next, let  $A_\iota$ ,  $\iota = 1, 2$ , be the restriction to  $H_0^1(\Omega)$  of the second order differential operator  $\mathcal{A}_\iota$ ,  $\iota = 1, 2$ , given by

$$(5.3) \quad \mathcal{A}_\iota = - \sum_{i,j=1}^n \frac{\partial}{\partial x_j} (a_{ij}^\iota(x) \frac{\partial}{\partial x_j}) + \sum_{i=1}^n b_i^\iota(x) \frac{\partial}{\partial x_i} + c^\iota(x) \quad x \in \Omega,$$

where

$$a_{ij}^\iota = a_{ji}^\iota \in C^1(\bar{\Omega}), \quad b_i^\iota \in C^1(\bar{\Omega}), \quad c^\iota \in L^\infty(\Omega).$$

It is clear that each  $A_\iota: H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$  is bounded without the ellipticity condition. The kernel function  $a(s)$  is assumed to be an element of  $L^2(-h, 0)$ . Now we consider the following parabolic partial functional differential equation

$$(5.4) \quad \begin{aligned} \frac{\partial u(t, x)}{\partial t} &= \mathcal{A}_0 u(t, x) + \mathcal{A}_1 u(t-h, x) \\ &+ \int_{-h}^0 a(s) \mathcal{A}_2 u(t+s, x) ds + f(t, x), \quad 0 < t < T, \quad x \in \Omega, \end{aligned}$$

having the Dirichlet boundary condition

$$(5.5) \quad u|_{\partial\Omega} = 0, \quad 0 < t < T,$$

and the initial data given by

$$(5.6) \quad u(0, x) = g^0(x), \quad u(s, x) = g^1(s, x) \quad \text{a.e. } s \in [-h, 0), \quad x \in \Omega.$$

Here in (5.4) and (5.6) we suppose that

$$(5.7) \quad f \in L_{loc}^2(\mathbf{R}^+; H^{-1}(\Omega))$$

$$(5.8) \quad g^0 \in L^2(\Omega), \quad g^1 \in L^2(-h, 0; H_0^1(\Omega)).$$

Under the above conditions the system (5.4)-(5.6) takes the abstract equation

form (E). Thus by applying Theorem 1, we have the following result.

**Theorem 8.** *Under the conditions (5.7) and (5.8) there exists a unique solution  $u(t, x)$  of the system (5.4)-(5.6) such that  $t \rightarrow u(t, \cdot)$  belongs to  $L^2_{loc}(-h, \infty; H^1_0(\Omega)) \cap W^{1,2}_{loc}(\mathbf{R}^+; H^{-1}(\Omega))$  and satisfies the equation (5.4) and the boundary condition (5.5) in the sense of distribution and also satisfies the initial data (5.6). Further, for each  $T > 0$  there is a constant  $K_T$  such that*

$$(5.9) \quad \left( \int_0^T \|u(t, \cdot)\|_{W^{1,2}(\Omega)}^2 dt + \int_0^T \left\| \frac{\partial u(t, \cdot)}{\partial t} \right\|_{H^{-1}(\Omega)}^2 dt \right) \leq K_T \left( \|g^0\|_{L^2(\Omega)}^2 + \int_{-h}^0 \|g^1(s)\|_{W^{1,2}(\Omega)}^2 ds + \int_0^T \|f(t)\|_{H^{-1}(\Omega)}^2 dt \right).$$

Next, we shall introduce the transposed system of (5.4)-(5.6) in the space  $L^2(\Omega)$ . Let  $a(u, v)$  be the sesquilinear form given in (5.1) and the adjoint operator  $A_0^*: H^1_0(\Omega) \rightarrow H^{-1}(\Omega)$  be defined through the equation

$$(5.10) \quad \langle u, A_0^* v \rangle = -a(u, v), \quad u, v \in V.$$

The realization of  $A_0^*$  in  $L^2(\Omega)$  is characterized as follows. Let us denote the formal adjoint of  $\mathcal{A}$  in (5.2) by  $\mathcal{A}^*$ , that is,

$$(5.11) \quad \mathcal{A}^* = - \sum_{i,j=1}^n \frac{\partial}{\partial x_j} (a_{ij}(x) \frac{\partial}{\partial x_j}) - \sum_{i=1}^n \frac{\partial}{\partial x_i} (b_i(x) \cdot) + c(x) \quad x \in \Omega.$$

The realization of  $-\mathcal{A}^*$  in  $L^2(\Omega)$  under the Dirichlet boundary condition coincides with the realization of  $A_0^*$  in  $L^2(\Omega)$ . That is to say,

$$\begin{aligned} \mathcal{D}(A_0^*) &= W^{2,2}(\Omega) \cap H^1_0(\Omega). \\ A_0^* u &= -\mathcal{A}^* u \quad \text{for } u \in \mathcal{D}(A_0). \end{aligned}$$

The adjoint operators  $A_i^*$  of  $A_i, i=1, 2$ , are given by the restrictions to  $H^1_0(\Omega)$  of the following formal adjoint operators of  $\mathcal{A}_i, i=1, 2$ , respectively:

$$(5.12) \quad \mathcal{A}_i^* = - \sum_{i,j=1}^n \frac{\partial}{\partial x_j} (a'_{ij}(x) \frac{\partial}{\partial x_j}) - \sum_{i=1}^n \frac{\partial}{\partial x_i} (b'_i(x) \cdot) + c'(x) \quad x \in \Omega.$$

Evidently,  $A_i^*: H^1_0(\Omega) \rightarrow H^{-1}(\Omega)$  is bounded.

The transposed system of (5.4)-(5.6) is now given by the following system of equations

$$(5.13) \quad \frac{\partial v(t, x)}{\partial t} = \mathcal{A}_0^* v(t, x) + \mathcal{A}_1^* v(t-h, x) + \int_{-h}^0 a(s) \mathcal{A}_2^* v(t+s, x) ds + h(t, x), \quad t \geq 0, \quad x \in \Omega,$$

$$(5.14) \quad v|_{\partial\Omega} = 0, \quad t \geq 0,$$

$$(5.15) \quad v(0, x) = \varphi^0(x), \quad v(s, x) = \varphi^1(s, x) \quad \text{a.e. } s \in [-h, 0], \quad x \in \Omega.$$

In (5.13) and (5.15) it is assumed that

$$h \in L^2_{\text{loc}}(\mathbf{R}^+; H^{-1}(\Omega)), \quad \varphi^0 \in L^2(\Omega), \quad \varphi^1 \in L^2(-h, 0; H^1_0(\Omega)).$$

Under the above condition the system (5.13)-(5.15) can be considered as the transposed equation ( $\mathbf{E}_T$ ). Hence as shown in Theorem 8, we have the similar existence, uniqueness and well posedness result for the transposed system (5.13)-(5.15) as for the system (5.4)-(5.6).

Finally we give an explicit form of the fundamental solution for some special case. Let  $b_i=0$ ,  $i=1, \dots, n$  in (5.2) and assume that  $\mathcal{A}_1=\mathcal{A}$  and  $\mathcal{A}_2=O$ . Then it is easy to see that  $A_0$  is selfadjoint with compact resolvent in  $L^2(\Omega)$ , and generates an analytic semigroup  $T(t)$  both in  $L^2(\Omega)$  and  $H^{-1}(\Omega)$ . Further, there exists a set of eigenvalues and eigenfunctions  $\{\lambda_n, \phi_n\}_{n \geq 1}$  of  $A_0$  such that

$$\begin{aligned} \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \rightarrow -\infty \quad n \rightarrow \infty; \\ A\phi_n = \lambda_n \phi_n, \quad \langle \phi_n, \phi_m \rangle_{L^2} = \delta_{nm} \quad n, m = 1, 2, \dots; \end{aligned}$$

and

$$T(t)u = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle u, \phi_n \rangle_{L^2} \phi_n, \quad t \geq 0, \quad u \in L^2(\Omega),$$

where  $\langle u, \phi \rangle_{L^2} = \int_{\Omega} u(x)\phi(x)dx$  and  $\delta_{mn}$  denotes the Kronecker's delta. Hence, in this special case the fundamental solution  $W(t)$  is given by

$$\begin{aligned} W(t)u = \sum_{n=1}^{\infty} \sum_{k=1}^m \frac{(t-(k-1)h)^{k-1}}{(k-1)!} \lambda_n^{k-1} e^{\lambda_n(t-(k-1)h)} \langle u, \phi_n \rangle_{L^2} \phi_n, \\ t \in [(m-1)h, mh], \quad u \in \mathcal{D}(A_0^m), \quad m = 1, 2, \dots. \end{aligned}$$

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