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Direct, Subdirect Decompositions and Congruence Relations

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1. Introduction

On the direct decompositions of (universal) algebras, applying to all of groups, rings, linear algebras and lattices, many researches have been made, but almost all of them are concerned with the decompositions into a finite number of factors. In the present paper we attempt to extend those earlier results to the case of infinite factors and clarify the structure of some algebras.

By an *algebra* A, we shall mean below a set of elements, together with a number of finitary operations f_α. Each f_α is a single-valued function assigning for some finite $n=n(\alpha)$ to every sequence $(x_1, \cdots, x_n)$ of n elements of A, a value $f_\alpha(x_1, \cdots, x_n)$ in A. A *congruence relation* θ on an algebra A is an equivalence relation $x\equiv y(\theta)$ with the substitution property for each f_α: If $x_i\equiv y_i(\theta)$, then $f_\alpha(x_1, \cdots, x_n)\equiv f_\alpha(y_1, \cdots, y_n)$ (θ). A congruence relation θ on A generates a homomorphism of A onto the algebra $\theta(A)$ of subsets $C(a, \theta)=\{x\,;\, x\equiv a(\theta)\}$ of A, which we denote by the same notation θ. If we define $\theta\leq\varphi$ to mean that $x\equiv y(\theta)$ implies $x\equiv y(\varphi)$, then all congruence relations on A form a complete, upper continuous lattice $\Theta(A)$, which we shall call the *structure lattice* of A. By the (complete) *direct union* $A=\Pi_{\omega\in\Omega}A_\omega$ of algebras A_ω having the same operations f_α is meant the algebra whose elements are the sets $\{x_\omega\,;\, \omega\in\Omega\}$ with $x_\omega\in A_\omega$, in which algebraic combination is performed component by component: If $x^i=\{x^i_\omega\,;\, \omega\in\Omega\}$, then $f_\alpha(x^1, \cdots, x^n)=\{f_\alpha(x^1_\omega, \cdots, x^n_\omega)\,;\, \omega\in\Omega\}$.

Direct factorizations of an algebra A are correlated with the lattice-theoretic properties of congruence relations on A; for instance

THEOREM 1.1. *The representations of an algebra A as a direct union $A=A_1\times\cdots\times A_n$ correspond one-one with the sets of permutable congruence relations $\theta_1, \cdots, \theta_n$ on A satisfying*

$$\theta_1\frown\cdots\frown\theta_n=0 \quad\text{and}\quad (\theta_1\frown\cdots\frown\theta_{i-1})\smile\theta_i=I \ [i=2, \cdots, n].$$

But this theorem does not hold for infinite n. We first intend to obtain the corresponding theorem for infinite n, by introducing the concept of *L-restricted factorizations*, which includes complete direct factorizations and discrete or finitely-restricted direct factorizations.[1]

We shall state in § 2 some important preliminaries on congruence relations, and define in § 3 L-restricted factorizations and show the first main theorems mentioned above. In § 4 we shall especially discuss the relations between finitely restricted factorizations and congruence relations, and obtain a unique factorization theorem. In § 5 we shall deal with factorizations into simple factors and give a condition for some algebras to be completely reducible. Further we prove in § 6 the unicity of factorizations for algebras with distributive structure lattices and derive some other properties of such algebras, in referrence to which we shall clarify in § 7, 8 the structure of some lattices.

All the basic theories and notations employed throughout this work may be found in Birkhoff [1], Ore [5] and Shoda [6].

2. Preliminaries on congruence relations

A *semilattice* is a partially ordered set K any two of whose elements have a least upper bound $x \cup y$. In the present paper we shall assume that any semilattice contains the least element 0. An *ideal* of a semilattice K is a non-void subset J of K satisfying (1) $x \in J$ and $y \in J$ imply $x \cup y \in J$, and (2) $x \in J$ and $t \leq x$ imply $t \in J$. It is easily shown that all ideals of a semilattice K form a complete, upper continuous lattice $J(K)$, which includes K by identifying an element $a \in K$ with the principal ideal $(a]$ generated by a.

An element a of a complete lattice L is called *inaccessible* if any *covering* $\{x_\alpha\}$ of a, i.e., a set of elements x_α satifying $\bigvee x_\alpha \geq a$, contains a finite covering $\{x_{\alpha_i}\}$ of a. Let a and b be inaccessible and suppose that $\bigvee x_\alpha \geq a \cup b$. Then $\{x_\alpha\}$ contains finite coverings $\bigvee x_{\alpha_i} \geq a$ and $\bigvee x_{\alpha_j} \geq b$, and then $\bigvee x_{\alpha_i} \cup \bigvee x_{\alpha_j} \geq a \cup b$. Hence $a \cup b$ is also inaccessible. Accordingly,

LEMMA 2.1. *The inaccessible elements of any complete lattice L form a semilattice $K(L)$.*

Now an ideal J of a semilattice K is written $J = \bigvee_{a \in J}(a]$. If J is inaccessible in $J(K)$, then $J = (a_1] \cup \cdots (a_n] = (a_1 \cup \cdots \cup a_n]$. Conversely, if $(a] \leq J_\nu$, namely $a \in \bigvee J_\nu$, then we can find a finite number of $a_i \in J_{\nu_i}$ such that $a \leq a_1 \cup \cdots \cup a_n$; hence $(a] \leq J_{\nu_1} \cup \cdots \cup J_{\nu_n}$. So inaccessible elements in $J(K)$ are nothing but principal ideals. Hence

1) On rings, discrete direct sums in the sense of N. H. McCoy, *Subdirect sums of rings*, Bull. Amer. Math. Soc. 53 (1947).

LEMMA 2.2. *If K is any semilattice, $K(J(K))=K$.*

If a subset S of a complete lattice L contains $\inf X$ for every subset $X\leqq S$, then we call S *$\wedge$-closed*. Dually if S always contains $\sup X$, then we call S *$\vee$-closed*. If S is $\wedge$-closed and $\vee$-closed, then S is called a *closed sublattice*. A closed sublattice is a complete lattice as a sublattice of L.

Now let A be an algebra, $\Theta(A)$ the lattice of congruence relations on A and Σ a closed sublattice of $\Theta(A)$. If P is a set of pairs (a, b) of elements of A, we define the congruence relation $\theta_\Sigma(P)$ *generated by P* in Σ, as the least of elements θ of Σ satisfying $a\theta b$ for every pair $(a, b)\in P$. It is easily seen that $\theta_\Sigma(P)=\bigvee_{(a,b)\in P}\theta_\Sigma(a, b)$, where $\theta_\Sigma(a, b)$ is the congruence relation generated by one pair (a, b).

LEMMA 2.3. *A congruence relation θ is inaccessible in Σ if and only if it is generated by a finite set of pairs of elements.*

PROOF. Let θ be inaccessible and P the set of all pairs (a_α, b_α) satisfying $a_\alpha\theta b_\alpha$. Then evidently $\theta=\theta_\Sigma(P)=\bigvee_\alpha\theta_\Sigma(a_\alpha, b_\alpha)$. Since θ is inaccessible, there exist a finite number of pairs $(a_i, b_i)\in P$ such that $\theta=\bigvee_i\theta_\Sigma(a_i, b_i)$. Conversely, if $\theta=\bigvee_i\theta_\Sigma(a_i, b_i)$ for a finite number of pairs (a_i, b_i) and $\{\theta_\alpha\}$ is a covering of θ in Σ, then $\bigvee\theta_\alpha\geqq\theta_\Sigma(a_i, b_i)$ and $a_i\equiv b_i\ (\bigvee\theta_\alpha)$; hence we can find a finite number of $\theta_{ij}\in\{\theta_\alpha\}$ and $a_{ij}\in A$ such that $a_i=a_{i0}\theta_{i1}a_{i1}\theta_{i2}\cdots\theta_{in}a_{in}=b_i$. Therefore $\bigvee\theta_{ij}\geqq\theta_\Sigma(a_i, b_i)$ and we get a finite covering $\{\theta_{ij}\}$ of θ.

Now inaccessible elements of Σ form a semilattice $K=K(\Sigma)$. If we denote by θ an element of Σ and by J an ideal in K, then we can consider (natural) mappings $J(\theta)=\{\theta_\alpha;\ \theta_\alpha\leqq\theta,\ \theta_\alpha\in K\}$ of Σ into the lattice $J(K)$ formed by all ideals of K and $\theta(J)=\bigvee_{\theta_\alpha\in J}\theta_\alpha$ of $J(K)$ into Σ. It is easy to show that $\theta(J(\theta))\leqq\theta$ and $J\leqq J(\theta(J))$. $x\theta y$ implies $\theta_\Sigma(x, y)\leqq\theta$, $\theta_\Sigma(x, y)\in J(\theta)$, $\theta_\Sigma(x, y)\leqq\theta(J(\theta))$ and $x\equiv y(\theta(J(\theta)))$; hence $\theta\leqq\theta(J(\theta))$. If $\varphi\in J(\theta(J))$, then $\varphi\leqq\theta(J)=\bigvee_{\theta_\alpha\in J}\theta_\alpha$ and we can find a finite number of $\theta_i\in J$ satisfying $\varphi\leqq\bigvee\theta_i$, since φ is inaccessible. Since J is an ideal, we have $\bigvee\theta_i\in J$, $\varphi\in J$ and hence $J(\theta(J))\leqq J$. Thus $\theta(J(\theta))=\theta$, $J(\theta(J))=J$ and evidently this correspondence $\theta\to J$ is order-preserving; hence we have

THEOREM 2.1. *If Σ is a closed sublattice of $\Theta(A)$, then $\Sigma\cong J(K(\Sigma))$.*

In this meaning we call $K(\Sigma)$ the *kernel* of Σ.[2]

Again we deal with permutability of congruence relations. Two

2) By the way, the lattice $L(A)$ of all subalgebras of A has the same structure: $L\cong J(K(L))$. Furthermore it can be proved that any ideal lattice $J(K)$ is isomorphic with the lattice of all subalgebras of some algebra.

congruence relations θ_1 and θ_2 are called *permutable*, if $x\equiv y(\theta_1\cup\theta_2)$ implies $x\theta_1\theta_2 y$, i.e., $x\theta_1 z\theta_2 y$ for some z.

LEMMA 2.4. *Let $\theta_1, \cdots, \theta_n$ be congruence relations on A and put $\varphi_i=\theta_1\frown\cdots\frown\theta_{i-1}\frown\theta_{i+1}\frown\cdots\frown\theta_n$. If $\varphi_1, \cdots, \varphi_n$ are permutable and $x_1, \cdots, x_n\in A$ satisfy $x_i\equiv x_j(\varphi_i\cup\varphi_j)$ for every pair (i, j), then we can find an element $x\in A$ satisfying $x\equiv x_i$ (θ_i).*

PROOF. Since $x_i\equiv x_{i+1}(\varphi_i\cup\varphi_{i+1})$, we can find $x_{i,i+1}\in A$ such that $x_{i,i+1}\varphi_{i+1}x_i$ and $x_{i,i+1}\varphi_i x_{i+1}$ for $i=1, \cdots, n-1$. Suppose that

$$(1)\qquad x_{i,i+1,\cdots,i+p}\varphi_{i+p}x_{i,\cdots,i+p-1}$$
$$\text{and}\quad x_{i,i+1,\cdots,i+p}\varphi_i x_{i+1,\cdots,i+p}\quad\text{for}\quad i=1, \cdots, n-p.$$

If $i+p+1\leq n$, we have $x_{i+1,\cdots,i+p+1}\varphi_{i+p+1}x_{i+1,\cdots,i+p}$ and $x_{i,\cdots,i+p}\equiv x_{i+1,\cdots,i+p+1}$ $(\varphi_i\cup\varphi_{i+p+1})$; hence we can find $x_{i,\cdots,i+p+1}\in A$ such that $x_{i,\cdots,i+p+1}\varphi_{i+p+1}x_{i,\cdots,i+p}$ and $x_{i,\cdots,i+p+1}\varphi_i x_{i+1,\cdots,i+p+1}$. So that we can always find $x_{i,\cdots,i+p}\in A$ satisfying (1). Then

$$x_i\varphi_{i+1}x_{i,i+1}\varphi_{i+2}x_{i,i+1,i+2}\varphi_{i+3}\cdots\varphi_n x_{i,i+1,\cdots,n}$$
$$\varphi_{i-1}x_{i-1,i,\cdots,n}\varphi_{i-2}x_{i-2,\cdots,n}\varphi_{1-3}\cdots\varphi_1 x_{1,\cdots,n}.$$

Since $\varphi_j\leq\theta_i$ for $j\neq i$; we have $x_i\theta_i x_{1,\cdots,n}$.

This proposition however does not hold when $\{\theta_i\}$ is not a finite set. We call a set S of congruence relations *completely permutable* if and only if any subset $\{\theta_\nu\}\leq S$ satisfies the following condition: If $x_\lambda\equiv x_\mu(\varphi_\lambda\cup\varphi_\mu)$, where $\varphi_\lambda=\bigwedge_{\nu\neq\lambda}\theta_\nu$, there exists $x\in A$ such that $x\equiv x_\nu(\theta_\nu)$. If S is completely permutable, then all congruence relations in S are permutable. Further it is easily shown that the joins $\bigvee\theta_\nu$ of any number of permutable congruence relations θ_ν are permutable; hence

LEMMA 2.5. *Let Σ be a closed sublattice of $\Theta(A)$. If the kernal $K(\Sigma)$ is completely permutable, then Σ is permutable.*

We give next some useful examples of completely permutable congruence relations.

THEOREM 2.2. *Let S be a $\bigwedge$-closed set of permutable congruence relations. If S satisfies the descending chain condition, then S is completely permutable.*

PROOF. Let $X=\{\theta_\nu\}$ be any subset of S and choose $\theta_i\in X$ so that $\theta_i\ngeq\theta_1\frown\cdots\frown\theta_{i-1}$. Since S satisfies the descending chain condition, the chain $\theta_1>\theta_1\frown\theta_2>\theta_1\frown\theta_2\frown\theta_3>\cdots$ must end at $\theta_1\frown\cdots\frown\theta_n$; namely every

$\theta_\nu \in X$ satisfies $\theta_\nu \geq \theta_1 \cap \cdots \cap \theta_n$. Now assume that $x_\lambda \equiv x_\mu(\varphi_\lambda \cup \varphi_\mu)$, where $\varphi_\lambda = \bigwedge_{\nu \neq \lambda} \theta_\nu$. If we set $Y = \{\theta_1, \cdots, \theta_n\}$, $Z = X - Y$, $\varphi = \bigwedge_{\theta_\nu \in Z} \theta_\nu$ and $\theta_i' = \theta_i \cap \varphi$, then $\varphi_i = \theta_1' \cap \cdots \cap \theta_{i-1}' \cap \theta_{i+1}' \cap \cdots \cap \theta_n'$ and $x_i \equiv x_j(\varphi_i \cup \varphi_j)$. Since φ_i are permutable, we can find x such that $x \equiv x_i(\theta_i')$ and hence $x \equiv x_i(\theta_i)$ for $i = 1, \cdots, n$, by Lemma 2.4. If $\theta_\lambda \in Z$, then $\varphi_\lambda \leq \theta_1 \cap \cdots \cap \theta_n = \bigwedge_{\theta_\nu \in X} \theta_\nu \leq \varphi_i$ and hence $x_\lambda \equiv x_i(\varphi_i)$. Since $\theta_i' \leq \varphi \leq \theta_\lambda$ and $\varphi_i \leq \varphi \leq \theta_\lambda$, we have $x \equiv x_\lambda(\theta_\lambda)$, completing the proof.

THEOREM 2.3. *Let A be a Boolean algebra. Then $K(\Theta(A))$ is completely permutable if and only if A is complete.*

PROOF. All congruence relations and all ideals of a Boolean algebra A correspond one-one, and it follows from Lemma 2.2 that congruence relations θ_ν in $K(\Theta(A))$ correspond to principal ideals $(a_\nu]$ so that $\{x;\ x\theta_\nu 0\} = \{x;\ x \leq a_\nu\}$. It is easy to see that $x\theta_\nu y$ is equivalent to $(x \cap y') \cup (x' \cap y)\theta_\nu 0$, $(x \cap y') \cup (x' \cap y) \leq a_\nu$ and $x \cap a_\nu' \leq y \leq x \cup a_\nu$.

Assume that A is complete, $\{\theta_\nu\}$ is any subset of $K(\Theta(A))$ and $\{a_\nu\}$ is the set of elements corresponding to θ_ν. If we put $\bigwedge_{\nu \neq \lambda} a_\nu = b_\lambda$, then $x_\lambda \equiv x_\mu(\bigwedge_{\nu \neq \lambda} \theta_\nu \cup \bigwedge_{\nu \neq \mu} \theta_\nu)$ implies $(x_\lambda \cap x_\mu') \cup (x_\lambda' \cap x_\mu) \leq b_\lambda \cup b_\mu$. Set $x = \bigwedge_\lambda (\bigvee_{\nu \neq \lambda} b_\nu \cup x_\lambda)$. Then $x \cap x_\mu' \leq (\bigvee_{\nu \neq \mu} b_\nu \cup x_\mu) \cap x_\mu' = (\bigvee_{\nu \neq \mu} b_\nu) \cap x_\mu' \leq a_\mu$, since $b_\nu \leq a_\mu$ for $\nu \neq \mu$. $x' \cap x_\mu = (\bigvee_\lambda (\bigwedge_{\nu \neq \lambda} b_\nu' \cap x_\lambda')) \cap x_\mu = \bigvee_\lambda (\bigwedge_{\nu \neq \lambda} b_\nu' \cap x_\lambda' \cap x_\mu)$ $\leq \bigvee_{\lambda \neq \mu} (\bigwedge_{\nu \neq \lambda} b_\nu' \cap (b_\lambda \cup b_\mu)) \leq \bigvee_{\lambda \neq \mu} (b_\mu' \cap b_\lambda) \leq a_\mu$. Hence $(x \cap x_\mu') \cup (x' \cap x_\mu) \leq a_\mu$; namely $x \equiv x_\mu(\theta_\mu)$.

Conversely if $K(\Theta(A))$ is completely permutable, we shall show that A contains $\sup X$ for any subset $X = \{x_\mu;\ \mu \in M\}$ of A. We first prove in the case $x_\lambda \cap x_\mu = 0$, hence $x_\lambda \leq x_\mu'$, for $\lambda \neq \mu$. Let $\{a_\nu;\ \nu \in N\}$ be the set of upper bounds of X and put $x_\nu = 0$ for $\nu \in N$ and $a_\mu = x_\mu'$ for $\mu \in M$. Then we have $x_\xi \leq a_\eta$ for $\xi, \eta \in M + N$ provided $\xi \neq \eta$; hence $(x_\xi \cap x_\eta') \cup (x_\xi' \cap x_\eta) \in \bigwedge_{\zeta \neq \xi} (a_\zeta] \cup \bigwedge_{\zeta \neq \eta} (a_\zeta]$. If follows from complete permutability that we can find $x \in A$ such that $x \equiv x_\xi(a_\xi)$ for $\xi \in M + N$, whence $x \equiv 0(a_\nu)$, meaning $x \leq a_\nu$, for $\nu \in N$ and $x \equiv x_\mu(x_\mu')$, meaning $x \geq x_\mu \cap (x_\mu')' = x_\mu$, for $\mu \in M$. Thus $x = \sup X$. Next we prove generally by using transfinite induction. Assume that A contains $y_\mu = \bigvee_{\nu < \mu} x_\nu$ for $\mu < \kappa$ and put $z_1 = x_1$, $z_\mu = y_\mu' \cap x_\mu$. Then A contains $u_\lambda = \bigvee_{\mu < \lambda} z_\mu$ for $\lambda \leq \kappa$, since $z_\mu \cap z_\nu = 0$ for $\nu < \mu < \lambda \leq \kappa$. Evidently $u_n = y_n$ for any finite n. If $u_\mu = y_\mu$ for $\mu < \lambda \leq \kappa$, then $z_\mu \leq x_\mu$ and $u_\lambda \geq u_\mu \cup z_\mu = y_\mu \cup (y_\mu' \cap x_\mu) \geq x_\mu$ for $\mu < \lambda$, whence we have $u_\lambda = \bigvee_{\mu < \lambda} x_\mu$ for $\lambda \leq \kappa$, especially $u_\kappa = \bigvee_{\mu < \kappa} x_\mu$, completing the proof.

3. *L*-restricted factorizations

If A is a finite algebra, then by factoring again and again each factor of the direct factorization of A we can obtain a representation of A as a direct union of directly indecomposable algebras, but an infinite algebra has not necessarily such a factorization. Again in the theory of many algebras, such as groups, rings etc., we often consider the finitely restricted direct union, i.e., the subsystem of elements of ΠA_ω whose components excluding some finite number are identical with those of a special element 0. In this view we feel the necessity to extend the meaning of direct unions.

Let Ω be a set of indices ω and L a family of subsets of Ω. Then it is natural to define an *L-restricted direct union* $\Pi_{(L)}A_\omega$ to mean a subalgebra S of the complete direct union $A=\Pi_{\omega\in\Omega}A_\omega$ such that

(I) $x=\{x_\omega;\ \omega\in\Omega\}$, $y=\{y_\omega;\ \omega\in\Omega\}$ *and* $x, y\in S$ *imply* $\{\omega;\ x_\omega\neq y_\omega\}\in L$,

(II) $x=\{x_\omega;\ \omega\in\Omega\}$, $y=\{y_\omega;\ \omega\in\Omega\}$, $x\in S$ *and* $\{\omega;\ x_\omega\neq y_\omega\}\in L$ *imply* $y\in S$.

While this is an extended interpretation of direct unions, for any partition $\Omega=\Omega_1+\Omega_2$, $S=\Pi_{(L)}A_\omega$ must be decomposed into a direct union $\Pi_{(L_1)}A_\omega\times\Pi_{(L_2)}A_\omega$ so that the components of $x=\{x_\omega;\ \omega\in\Omega\}$ be $x_1=\{x_\omega;\ \omega\in\Omega_1\}$, $x_2=\{x_\omega;\ \omega\in\Omega_2\}$. If $x=(x_1, x_2)$ and $y=(y_1, y_2)$ under the above decomposition, then $z=(y_1, x_2)\in S$ must have the components z_ω such that $z_\omega=y_\omega$ for $\omega\in\Omega_1$ and $z_\omega=x_\omega$ for $\omega\in\Omega_2$, and then $\{\omega;\ x_\omega\neq y_\omega\}=M$ implies $\{\omega;\ x_\omega\neq z_\omega\}=M\cap\Omega_1$. Hence L must contain N with M provided $N\leq M$. If we omit the factors A_ω which consist of only one element from any direct union, we get an isomorphic system; hence we shall assume, without loss of generality, that every factor A_ω contains two or more elements. Then it follows from the condition (II) that for $M, N\in L$ we can find $x, y, z\in S$ so that $\{\omega;\ x_\omega\neq y_\omega\}=M$, $\{\omega;\ x_\omega\neq z_\omega\}=N\cap(\Omega-M)$ and hence $\{\omega;\ y_\omega\neq z_\omega\}=M\cup N$. In summary we must assume that L is an ideal of the Boolean algebra $\mathbf{2}^\Omega$ of all subsets of Ω.

If L coincides with the whole Boolean algebra $\mathbf{2}^\Omega$, then $\Pi_{(L)}A_\omega$ coincides with the complete direct union $\Pi_{\omega\in\Omega}A_\omega$, and if L is the ideal of all finite subsets of Ω, then $\Pi_{(L)}A_\omega$ becomes a finitely restricted direct union.

We can define an *L-restricted subdirect union* of A_ω in the same way. A subdirect union of A_ω is a subalgebra S of the direct union $\Pi_{\omega\in\Omega}A_\omega$ such that all elements of A_ω appear as ω-components of elements of S. If S is a subdirect union of A_ω satisfying the condition (I) mentioned above, then we say that S is an L-restricted subdirect union of A_ω.

We shall now correlate L-restricted factorizations of an algebra A with congruence relations on A. If A is decomposed into a subdirect union of $\{A_\omega;\ \omega\in\Omega\}$, then the correspondence from each element $x=\{x_\omega;\ \omega\in\Omega\}$ to its ω-component x_ω is a homomorphism θ^*_ω of A onto A_ω, and hence generates a congruence relation, which may be denoted by the same notation θ^*_ω. Hence every subdirect factorization of A is written in the form $\{\theta^*_\omega(A);\ \omega\in\Omega\}$ which means the subdirect factorization where the ω-component of x is $\theta^*_\omega(x)$.

Let L be an ideal of the Boolean algebra $\mathbf{2}^\Omega$. All ideals of L form a complete atomic distributive lattice $J(L)$, which includes L as a sublattice by identifying an element $M\in L$ with the principal ideal of L generated by M, which we shall denote by the same notation M. Again the subset $\{M;\ M\in L,\ M\not\ni\omega\}$ of L forms a prime ideal P_ω. Associating P_ω with θ^*_ω, we can characterize an L-restricted subdirect factorization as follows.

THEOREM 3.1. *Let $\{\theta^*_\omega;\ \omega\in\Omega\}$ be a set of congruence relations on an algebra A and L an ideal of the Boolean algebra $\mathbf{2}^\Omega$. Then A is decomposed into an L-restricted subdirect union $\{\theta^*_\omega(A);\ \omega\in\Omega\}$ if and only if for every ideal J of L there exists a congruence relation $\theta(J)$ on A which satisfies*

(1) $\theta(\bigwedge J_\nu)=\bigwedge\theta(J_\nu)$, (2) $\theta(0)=0$, (3) $\bigvee_{M\in L}\theta(M)=I$,
(4) $\theta^*_\omega=\theta(P_\omega)$, *where* $P_\omega=\{M;\ M\in L,\ M\not\ni\omega\}$.

PROOF. Suppose that A is decomposed into an L-restricted subdirect union of $\{\theta^*_\omega(A);\ \omega\in\Omega\}$. By x_ω we denote the ω-component of $x\in A$, i.e., $x_\omega=\theta^*_\omega(x)$. Associated with $M\in L$, we define the congruence relation $\theta(M)=\bigwedge_{\omega\in\Omega-M}\theta^*_\omega$. It is easily seen that $M\leq N$ implies $\theta(M)\leq\theta(N)$ and the relation $x\equiv y(\theta(M))$ is equivalent to $\{\omega;\ x_\omega\neq y_\omega\}\leq M$. If J is an ideal of L, we set $\theta(J)=\bigvee_{M\in J}\theta(M)$. Then it is obvious that $\{\omega;\ x_\omega\neq y_\omega\}\in J$ implies $x\equiv y(\theta(J))$. Conversely if $x\equiv y(\theta(J))$, there exist a finite number of elements $x_0=x,\ x_1,\ \cdots,\ x_n=y$ of A and congruence relations $\theta(M_i)$ with $M_i\in J$ such that $x_{i-1}\equiv x_i(\theta(M_i))$. Then $x\equiv y(\theta(M))$, where $M=\bigvee M_i\in J$; hence the relation $x\equiv y(\theta(J))$ is equivalent to $\{\omega;\ x_\omega\neq y_\omega\}\in J$. Now the relation $x\equiv y(\bigwedge\theta(J_\nu))$ is equivalent to $\{\omega;\ x_\omega\neq y_\omega\}\in J_\nu$ for all ν, $\{\omega;\ x_\omega\neq y_\omega\}\in\bigwedge J_\nu$ and $x\equiv y(\theta(\bigwedge J_\nu))$; hence $\theta(\bigwedge J_\nu)=\bigwedge\theta(J_\nu)$. Since $\theta(0)=\bigwedge_{\omega\in\Omega}\theta^*_\omega$, $x\equiv y(\theta(0))$ implies $x_\omega=y_\omega$ for all ω; hence $\theta(0)=0$. By the definition of L-restricted factorizations, any two elements $x,\ y$ satisfy $M=\{\omega;\ x_\omega\neq y_\omega\}\in L$ and $x\equiv y(\theta(M))$; so $\bigvee_{M\in L}\theta(M)=I$. The relation $x\equiv y(\theta^*_\eta)$, where $\eta\in\Omega$, is equivalent to $\{\omega;\ x_\omega\neq y_\omega\}\not\ni\eta$, $\{\omega;\ x_\omega\neq y_\omega\}\in P_\eta$ and $x\equiv y(\theta(P_\eta))$; hence $\theta^*_\eta=\theta(P_\eta)$.

Conversely suppose that congruence relations $\theta(J)$ satisfy the conditions (1)–(4). Putting $x_\omega=\theta^*_\omega(x)$ and $x^*=\{x_\omega;\ \omega\in\Omega\}$, we have the homomorphism $x\to x^*$ of A into $\Pi\theta^*_\omega(A)$. It follows from (1) and (2) that $\bigwedge_{\omega\in\Omega}\theta_\omega^*=\bigwedge_{\omega\in\Omega}\theta(P_\omega)=\theta(\bigwedge_{\omega\in\Omega}P_\omega)=\theta(0)=0$, and $x^*=y^*$ implies $x\equiv y$ (θ_ω^*) for all $\omega\in\Omega$ and $x=y$; hence $\{\theta_\omega^*(A)\,;\,\omega\in\Omega\}$ is a subdirect factorization of A. It follows from (3) that any $x, y\in A$ satisfy $x\equiv y(\bigvee_{M\in L}\theta(M))$ and we can find a finite sequence of elements $x_0=x, x_1, \cdots, x_n=y$ and congruence relations $\theta(M_i)\,;\,M_i\in L$ so that $x_{i-1}\equiv x_i(\theta(M_i))$. If $M=\bigvee M_i$, then $M\in L$ and $\theta(M_i)\leq\theta(M)$; hence $x\equiv y(\theta(M))$. Since $\theta(M)=\theta(\bigwedge_{\omega\in\Omega-M}P_\omega)=\bigwedge_{\omega\in\Omega-M}\theta_\omega^*$, $x\equiv y(\theta(M))$ means $\{\omega\,;\,x_\omega\neq y_\omega\}\leq M$, whence $\{\omega;\,x_\omega\neq y_\omega\}\in L$, completing the proof.

The correspondence $J\rightarrow\theta(J)$, mentioned in the above theorem, is not necessarily one-one. It is natural to call a component A_η in a subdirect factorization $\{A_\omega\,;\,\omega\in\Omega\}$ of A *redundant*, if the correspondence $\{x_\omega\,;\,\omega\in\Omega\}\rightarrow\{x_\omega\,;\,\omega\in\Omega-\eta\}$ is an isomorphism of A into $\Pi_{\omega\in\Omega-\eta}A_\omega$, and a subdirect factorization *irredundant*, if none of its component is redundant. If we denote by θ_η the congruence relation corresponding to the one-element subset $\{\eta\}\in L$, namely $\theta_\eta=\theta(\eta)=\bigwedge_{\omega\neq\eta}\theta_\omega^*$, then it is obviously seen that A_η is redundant if and only if $\theta_\eta=0$. If M is any subset of Ω[3], by $\theta(M)$ we denote the congruence relation corresponding to the ideal $J(M)$ of L which consists of elements of L included in M. Since $J(M)=\bigwedge_{\omega\in\Omega-M}P_\omega$, $\theta(M)=\bigwedge_{\omega\in\Omega-M}\theta(P_\omega)$. If $\theta(M)=\theta(N)$ is compatible with $M-N\in\eta$, then $\theta_\eta\leq\theta(M)=\theta(N)\leq\theta_\eta^*$ and $\theta_\eta=0$, whence A_η is redundant. So we infer that the correspondence $M\rightarrow\theta(M)$ is one-one if and only if the factorization is irredundant. However even if $M\rightarrow\theta(M)$ is one-one, $J\rightarrow\theta(J)$ is not necessarily one-one. We can prove that if $\{A_\omega\,;\,\omega\in\Omega\}$ is the L-restricted direct decomposition, the above correspondence is isomorphic. Namely, denoting by Σ the closed sublattice of $\Theta(A)$ generated by $\{\theta_\omega^*\}$, we can deduce the following theorem.

THEOREM 3.2. *The representations of an algebra A as an L-resticted direct union of* $\{A_\omega\,;\,\omega\in\Omega\}$ *correspond one-one with closed sublattices* Σ *of* $\Theta(A)$ *satisfying that*

(1) $\Sigma\ni 0, I$, (2) $K(\Sigma)\cong L$, (3) $K(\Sigma)$ *is completely permutable,*

where $K(\Sigma)$ *denotes the kernel of* Σ.

PROOF. Suppose that $A=\Pi_{(L)}A_\omega$ and define the congruence relations $\theta(J)$, corresponding to the ideals J of L, in the same way as in Theorem 3.1. We shall show that the set Σ of the congruence relations $\theta(J)$ is a closed sublattice of $\Theta(A)$. Suppose that $x\equiv y(\theta(\bigvee J_\nu))$ and put $M=\{\omega\,;\,x_\omega\neq y_\omega\}$. Then $M\in\bigvee J_\nu$ and hence we can choose a finite number of subsets M_i of Ω so that $M=\bigvee M_i$, $M_i\cap M_j=0$ $(i\neq j)$; $i, j=1, \cdots, n$

3) The set M need not be contained in L; hence $J(M)$ is not necessarily a principal ideal.

and each M_i belongs to some one of J_ν. We define formally the elements $x^0=x, x^1, \cdots, x^n=y$, as follows:

$x^i=\{x^i_\omega;\ \omega\in\Omega\}$ with $x^i_\omega=x^{i-1}_\omega$ for $\omega\overline{\in}M_i$ and $x^i_\omega=y_\omega$ for $\omega\in M_i$. Since $x\in A$ and $\{\omega;\ x^i_\omega\neq x_\omega\}=M_1\cup\cdots\cup M_i\in L$, $x^i\in A$. Again since $\{\omega;\ x^{i-1}_\omega\neq x^i_\omega\}$ belongs to some J_ν, $x^{i-1}\equiv x^i$ mod some $\theta(J_\nu)$; hence $x\equiv y$ $(\bigvee\theta(J_\nu))$, proving $\theta(\bigvee J_\nu)\leq\theta(J_\nu)$. It is obvious that $\theta(\bigvee J_\nu))\geq\bigvee\theta(J_\nu)$, so we have $\theta(\bigvee J_\nu)=\bigvee\theta(J_\nu)$. Combining this with Theorem 3.1, we infer that Σ is a closed sublattice of $\Theta(A)$ generated by $\{\theta^*_\omega\}$ and the correspondence $J\to\theta(J)$ is a homomorphism. If $J_1-J_2\ni M$, given $x\in A$, we set y so that $y_\omega=x_\omega$ for $\omega\overline{\in}M$ and $y_\omega\neq x_\omega$ for $\omega\in M$. Then since $\{\omega;\ x_\omega\neq y_\omega\}=M\in L$, $y\in A$ and $x\equiv y(\theta(J_1))$, $x\not\equiv y(\theta(J_2))$. Hence Σ is isomorphic with $J(L)$ and accordingly $K(\Sigma)\cong L$, by Lemma 2.2. It remains to prove (3). If $\{\theta_\nu\}\leq K(\Sigma)$ and $\varphi_\lambda=\bigwedge_{\nu\neq\lambda}\theta_\nu$, then θ_ν is written $\theta_\nu=\theta(M_\nu)$ with $M_\nu\in L$ and so $\varphi_\lambda=\theta(\bigvee_{\nu\neq\lambda}M_\nu)$. If $x^\lambda\equiv x^\mu(\varphi_\lambda\cap\varphi_\mu)$, then $\{\omega;\ x^\lambda_\omega\neq x^\mu_\omega\}\leq\bigwedge_{\nu\neq\lambda}M_\nu\cup\bigwedge_{\nu\neq\mu}M_\nu$, $\{\omega;\ x^\lambda_\omega=x^\mu_\omega\}\geq\bigvee_{\nu\neq\lambda}(\Omega-M_\nu)\cap\bigvee_{\nu\neq\mu}(\Omega-M_\nu)\geq(\Omega-M_\mu)\cap(\Omega-M_\lambda)$ and hence $\omega\in(\Omega-M_\lambda)\cap(\Omega-M_\mu)$ implies $x^\lambda_\omega=x^\mu_\omega$. We set $x=\{x_\omega;\ \omega\in\Omega\}$ so that if $\omega\in\bigwedge M_\nu$, x_ω be arbitrary, and if ω is contained in some $\Omega-M_\lambda$, then $x_\omega=x^\lambda_\omega$. Then $x_\omega=x^\lambda_\omega$ for any λ satisfying $\Omega-M_\lambda\ni\omega$, since $\omega\in(\Omega-M_\lambda)\cap(\Omega-M_\mu)$ implies $x^\lambda_\omega=x^\mu_\omega$. Thus we obtained $x\in A$ such that $x\equiv x^\nu(\theta_\nu)$.

Conversely suppose that Σ is a closed sublattice of $\Theta(A)$ satisfying (1)–(3). Then it follows from Theorem 2.1 that $\Sigma\cong J(L)$. Let $\theta(J)$ be the element of Σ corresponding to $J\in J(L)$ and put $\theta^*_\omega=\theta(P_\omega)$, where $P_\omega=\{M;\ M\in L,\ M\overline{\ni}\omega\}$. Then it follows from Theorem 3.1 that A is decomposed into an L-restricted subdirect union of $\{\theta^*_\omega(A);\ \omega\in\Omega\}$ and $x\equiv y(\theta(M))$ is equivalent to $\{\omega;\ x_\omega\neq y_\omega\}\leq M$. Now suppose that $x=\{x_\omega;\ \omega\in\Omega\}\in A$ and $\{y_\omega;\ \omega\in\Omega\}\in\Pi\theta^*_\omega(A)$ satisfy $\{\omega:\ x_\omega\neq y_\omega\}\leq M\in L$. Since $y_\eta\in\theta^*_\eta(A)$, we can find $y^\eta\in A$ such that $y^\eta_\eta=\theta^*_\eta(y^\eta)=y_\eta$. Since θ^*_η and $\theta(\eta)$ are permutable, by Lemma 2.5, and $\theta^*_\eta\cup\theta(\eta)=I$, we can find $x^\eta\in A$ such that $x^\eta\equiv x(\theta(\eta))$ and $x^\eta\equiv y^\eta(\theta^*_\eta)$, accordingly $x^\eta_\omega=x_\omega$ for $\omega\neq\eta$ and $x^\eta_\eta=y^\eta_\eta=y_\eta$. Now put $\theta_\omega=\theta(M-\omega)$ for $\omega\in M$. Then $\theta_\omega\in K(\Sigma)$ and $\bigwedge_{\omega\neq\eta}\theta_\omega=\theta(\eta)$. Since $x^\xi=x^\eta(\theta(\xi)\cup\theta(\eta))$, we can find $y^*\in A$ such that $y^*\equiv x^\omega(\theta(M-\omega))$ for $\omega\in M$, by the condition (3). If $\omega\in M$, $y^*_\omega=x^\omega_\omega=y_\omega$, and if $\omega\overline{\in}M$, $y^*_\omega=x^\eta_\omega$ for any $\eta\in M$ and hence $y^*_\omega=x_\omega=y_\omega$. Thus $y^*=\{y_\omega;\ \omega\in\Omega\}$ and $\{\theta^*_\omega(A);\ \omega\in\Omega\}$ becomes the L-restricted direct factorization. So the proof is completed.

If $\bigwedge_{\omega\in\Omega}\theta^*_\omega=0$, then A is decomposed into a subdirect union of $\{\theta^*_\omega(A);\ \omega\in\Omega\}$ and congruence relations $\theta(M)$ are defined. If $\theta(\eta)\cup\theta^*_\eta=I$ and $\theta(\eta)$ and θ^*_η are permutable, then for $x, y^\eta\in A$ we can find $x^\eta\in A$ such that $x^\eta\equiv x(\theta(\eta))$ and $x^\eta\equiv y^\eta(\theta^*_\eta)$. Then the condition (II) of L-

restricted direct decompositions follows from complete permutability of $\{\theta(M)\,;\, M\in L\}$, as is shown above. Hence

COROLLARY. *An algebra A is decomposed into the complete direct union of* $\{\theta^*_\omega(A)\,;\,\omega\in\Omega\}$ *if and only if*

(1) $\bigwedge_{\omega\in\Omega}\theta^*_\omega=0$, (2) $(\bigwedge_{\omega\neq\eta}\theta^*_\omega)\cup\theta^*_\eta=I$,

(3) $\{\bigwedge_{\omega\in M}\theta^*_\omega\,;\, M\in \mathbf{2}^\Omega\}$ *is completely permutable.*

Now L is an atomic, relatively complemented, distributive lattice. Conversely if $K(\Sigma)$ is such a lattice and $\{\theta_\omega\,;\,\omega\in\Omega\}$ is the set of atoms of $K(\Sigma)$, then corresponding $\theta\in K(\Sigma)$ to $M=\{\omega\,;\,\theta_\omega\leq\theta\}$, we obtain an isomorphism between $K(\Sigma)$ and a lattice of subsets of Ω. If θ^*_ω is the congruence relation corresponding to the prime ideal $P_\omega=\{\theta\,;\theta\not\geq\theta_\omega,\ \theta\in K(\Sigma)\}$, then $\theta(M)\in K(\Sigma)$, which corresponds to M, is written $\theta(M)=\bigwedge_{\omega\in\Omega-M}\theta^*_\omega$, since $(\theta(M)]=\bigwedge_{\omega\in\Omega-M}P_\omega$. Moreover if $\theta(M)\in K(\Sigma)$ and $\omega\in M$, then $K(\Sigma)$ contains $\theta(M-\omega)$ which is the complement of θ_ω in the interval $[0,\,\theta(M)]$. Now if Σ contains 0 and I, A is decomposed into subdirect union of $\{\theta^*_\omega(A)\,;\,\omega\in\Omega\}$. Assume $\theta(M)\in K(\Sigma)$ and $N\leq M$, and choose $x\in A$ and $y\in\Pi\theta^*_\omega(A)$ so that $\{\omega\,;\,x_\omega\neq y_\omega\}=N$. If $K(\Sigma)$ is completely permutable, then we can show $y\in A$ in the same way as in the last part of the proof of the above theorem. Then it is easy to show that $\theta_\Sigma(x,y)\in K(\Sigma)$ is written in the form $\theta(N)$. Hence $K(\Sigma)$ is isomorphic with an ideal L of $\mathbf{2}^\Omega$, and A can be decomposed into an L-restricted direct union. So we can infer

THEOREM 3.3. *The representations of an algebra A as a (restricted) direct union of* $\{A_\omega\,;\,\omega\in\Omega\}$ *correspond one-one withc losed sublattices* Σ *of* $\Theta(A)$ *satisfying that* (1) $\Sigma\ni 0,\, I$, (2) $K(\Sigma)$ *is an atomic, relatively complemented, distributive lattice as a sublattice of* $\Theta(A)$, (3) $K(\Sigma)$ *is completely permutable.*

Complete direct factorizations correspond to Σ such that $K(\Sigma)\ni I$. Now congruence relations on a Boolean algebra A are permutable, but $K(\Theta(A))$ is not completely permutable unless A is complete, as is shown in Theorem 2.3. Hence the above condition (3) cannot be replaced by usual permutability. Moreover Theorem 2.3 indicates that the condition that $K(\Sigma)$ is atomic cannot dispense with. But it seems that complete Boolean algebras yield fundamental models of direct factorizations of all algebras.

4. Fintely restricted factorizations

In Theorem 3.3 finitely restricted factorizations correspond to Σ in which $K(\Sigma)$ satisfies the descending chain condition. On that occasion Σ becomes a Boolean algebra. Further we can show generally

LEMMA 4.1. *The ideal lattice $J(K)$ of a semilattice K is complemented and modular if and only if K is a relatively complemented modular lattice satisfying the descending chain condition.*

PROOF. If a_i' denotes a relative complement of $a_i \cup a'_{i-1}$ in $[a'_{i-1}, I]$ in a complemented modular lattice, then $I = a_0 > a_1 > a_2 > \cdots$ implies $0 = a_0' < a_1' < a_2' < \cdots$; hence a complemented modular lattice satisfying one chain condition has a finite length. Let K be a relatively complemented modular lattice satisfying the descending chain condition and J any ideal in K. If $\Gamma = \{J_\nu\}$ is a maximal chain of ideals satisfying $J_\nu \cap J = 0$, then the set-sum J' of all J_ν is also an ideal satisfying $J' \cap J = 0$. Given $a \in K$, the principal ideal $(a]$ has a finite length; hence $(a] \cap (J \cup J')$ becomes a principal ideal $(b]$ and any $x \in J$, $y \in J'$ satisfy $a \cap (x \cup y) \leq b$. If b' is a relative complement of b such that $b \cap b' = 0$, $b \cup b' = a$, then $b' \cap (x \cup y) \leq a \cap (x \cup y) \leq b$ and $b' \cap (x \cup y) \leq b' \cap b = 0$. Hence $((y \cup b') \cap x) \cup y = (y \cup b') \cap (x \cup y) = y \cup (b' \cap (x \cup y)) = y$ and so $((y \cup b') \cap x \leq y \cap x = 0$; namely $(J' \cup (b']) \cap J = 0$. Since Γ is a maximal chain, it follows that $b' \in J'$ and $a = b \cup b' \in J \cup J'$. Thus $J \cup J' = K$ and $J(K)$ is complemented. Conversely if $J(K)$ is complemented and modular and $J \leq (a]$, then J' exists such that $J \cap J' = 0$ and $J \cup J' = (a]$, whence we have $b \in J$, $b' \in J'$ such that $b \cup b' = a$. It follows from the modular law that $J = (b]$, $J' = (b']$ and hence $(a]$ satisfies the ascending chain condition. Further $(x] \cap (y]$ must be a principal ideal and hence K contains $x \cap y$. We conclude altogether that K is a relatively complemented modular lattice satisfying the descending chain condition.

Further if $K(\Sigma)$ is a lattice satisfying the descending chain condition, it follows from Theorem 2.2 that complete permutability can be replaced by usual permutability; hence we infer

THEOREM 4.1. *The representations of an algebra A as a finitely restricted direct union $\{A_\omega;\ \omega \in \Omega\}$ correspond one-one with closed sublattices Σ of $\Theta(A)$ satisfying that* (1) $\Sigma \in 0, I$, (2) *Σ is a Boolean algebra,* (3) *all congruence relations in Σ are permutable.*

Furthermore we can show that a $\bigvee$-closed sublattice Σ of $\Theta(A)$ which is complemented and modular and contains 0, I generates a finitely restricted factorization.

LEMMA 4 2. *Let Σ be a $\bigvee$-closed modular sublattice of a compelete, upper continuous lattice Θ and $S(\Omega) = \{\theta_\omega:\ \omega \in \Omega\}$ a subset of elements θ_ω*

of Σ *different from* 0 *and satisfying that* $\bigvee_{\omega\in\Omega}\theta_\omega=I$ *and* $\theta_\eta\frown\bigwedge_{\omega\in\Omega-\eta}\theta_\omega=0$ *for all* $\eta\in\Omega$. *Then the closed sublattice of* Θ *generated by* $S(\Omega)$ *is isomorphic with* $\mathbf{2}^\Omega$.

PROOF. Put $\theta(M)=\bigvee_{\omega\in M}\theta_\omega$ and $\theta(0)=0$. If $\theta(M)=\theta(N)$ and $\eta\in M-N$, then $\bigvee_{\omega\in\Omega-\eta}\theta_\omega\geq\theta(N)=\theta(M)\geq\theta_\eta$ and $\theta_\eta=\theta_\eta\frown\bigvee_{\omega\in\Omega-\eta}\theta_\omega=0$. Hence $\theta(M)=\theta(N)$ implies $M=N$. Put $\theta_\eta^*=\bigvee_{\omega\in\Omega-\eta}\theta_\omega$, $\varphi(M)=\bigwedge_{\omega\in\Omega-M}\theta_\omega^*$ and $\varphi(\Omega)=I$. Then $\varphi(M)\geq\theta(M)$. Since $\varphi(M)\frown\theta(M)=\theta(M)$, M is contained in a maximal chain $\{N_\alpha\}$ of subsets of Ω such that $\varphi(M)\frown\theta(N_\alpha)=\theta(M)$. Then $N=\bigvee N_\alpha$ also satisfies $\varphi(M)\frown\theta(N)=\theta(M)$, since $\theta(N_\alpha)\uparrow\theta(N)$ and Θ is upper continuous. Suppose that $\omega\in\Omega-N\leq\Omega-M$. Then $\theta_\omega^*\geq\varphi(M)$, $\theta_\omega^*\geq\theta(N)$ and θ_ω, θ_ω^* and $\theta(N)$ are elements of a modular lattice Σ; hence we have $\varphi(M)\frown\theta(N+\omega)\leq\theta_\omega^*\frown(\theta_\omega\smile\theta(N))=(\theta_\omega^*\frown\theta_\omega)\smile\theta(N)=\theta(N)$ and $\varphi(M)\frown\theta(N+\omega)=\varphi(M)\frown\theta(N)=\theta(M)$, which contradicts that $\{N_\alpha\}$ is a maximal chain. Therefore $\varphi(M)\frown\theta(\Omega)=\theta(M)$ and $\varphi(M)=\theta(M)$, since $\theta(\Omega)=I\geq\varphi(M)$. Thus $\bigvee\theta(M_\alpha)=\theta(\bigvee M_\alpha)$, $\bigwedge\theta(M_\alpha)=\theta(\bigwedge M_\alpha)$ and the closed sublattice generated by $\{\theta_\omega;\ \omega\in\Omega\}$ is a Boolean algebra isomorphic with $\mathbf{2}^\Omega$.

LEMMA 4.3. *If a* $\bigvee$*-closed sublattice* Σ *of* $\Theta(A)$ *is complemented and modular, then there exists a subset* $S(\Omega)=\{\theta_\omega;\ \omega\in\Omega\}$ *of points of* Σ *such that the closed sublattice of* $\Theta(A)$ *generated by* $S(\Omega)$ *is a Boolean algebra and contains* 0, I *of* Σ.

PROOF. We shall first show that Σ is atomic. Let θ be any element of Σ different from 0 and θ' a complement of θ. Choose two elements x, y of A so that $x\not\equiv y(\theta')$, and consider the partially ordered set $C(x, y)$ of all congruence relations $\theta_\alpha\in\Sigma$ such that $x\not\equiv y(\theta_\alpha)$. Then θ' is contained in a maximal chain Γ in $C(x, y)$. Put $\sigma=\sup\Gamma$. $x\equiv y(\sigma)$ means $x\equiv y$ (θ_α) for some $\theta_\alpha\in\Gamma$; hence $x\not\equiv y(\sigma)$ and besides $\sigma\in\Sigma$. If $\sigma<\tau<I$ and $\tau\in\Sigma$, then $\tau'\in\Sigma$ exists such that $\tau\frown\tau'=\sigma$, $\tau\smile\tau'=I$. Since Γ is a maximal chain, $x\equiv y(\tau)$, $x\equiv y(\tau')$ and hence $x\equiv y(\sigma)$. Therefore σ is a maximal element of Σ. Then its relative complement σ', satisfying $\sigma\frown\sigma'=\theta'$ and $\sigma\smile\sigma'=I$, covers θ' and hence $\theta\frown\sigma'$ covers 0, by Dedekind's transposition principle. Thus Σ is atomic. Now let $\{\theta_\omega;\ \omega_\omega\in\Omega^*\}$ be the set of all points of Σ and consider a maximal chain $C(M)$ of subsets M of indices ω satisfying that $\theta_\eta\frown\bigvee_{\omega\in M-\eta}\theta_\omega=0$ for all $\eta\in M$. If Ω is the set-sum of $M\in C(M)$, then we have $\bigvee_{\omega\in M-\eta}\theta_\omega\uparrow\bigvee_{\omega\in\Omega-\eta}\theta_\omega$ and $\theta_\eta\frown\bigvee_{\omega\in\Omega-\eta}\theta_\omega=0$ for all $\eta\in\Omega$, since $\Theta(A)$ is upper continuous. Put $\bigvee_{\omega\in\Omega-\eta}\theta_\omega=\theta_\eta^*$. Then $\theta_\eta^*\in\Sigma$. If $\varphi=\theta_\eta\smile\theta_\eta^*=\bigvee_{\omega\in\Omega}\theta_\omega\neq I$, then a complement φ' of φ contains a point θ_ξ of Σ, which satisfies $\theta_\xi\frown\bigvee_{\omega\in\Omega}\theta_\omega=0$ and $\theta_\omega\frown(\theta_\xi\smile\theta_\omega^*)=\theta_\omega\frown\varphi\frown(\theta_\xi\smile\theta_\omega^*)=\theta_\omega\frown((\varphi\frown\theta_\xi)\smile\theta_\omega^*)=\theta_\omega\frown\theta_\omega^*=0$; namely $\theta_\eta\frown\bigvee_{\omega\in\Omega+\xi-\eta}\theta_\omega$

$=0$ for all $\eta \in \Omega + \xi$, which contradicts that $C(M)$ is a maximal chain. Hence $\theta_\omega \cap \theta^*_\omega = I$ and $S(\Omega) = \{\theta_\omega ; \omega \in \Omega\}$ satisfies the conditions in Lemma 42. Thus the closed sublattice generated by $S(\Omega)$ is a Boolean algebra isomorphic with $\mathbf{2}^\Omega$.

Again θ^*_ω are maximal elements in Σ and the above Boolean algebra $\{\theta(M)\}$ is considered to be generated by $\{\theta^*_\omega ; \omega \in \Omega\}$. Then we can deduce immediately

THEOREM 4.2. *If a $\bigvee$-closed sublattice Σ of $\Theta(A)$ containing* 0 *and* I *is complemented and modular, then we can choose a set $\{\theta^*_\omega ; \omega \in \Omega\}$ of maximal elements of Σ so that A be decomposed into an irredundant, finitely restricted, subdirect union of $\{\theta^*_\omega(A) ; \omega \in \Omega\}$. If all congruence relations of Σ are permutable moreover, then the above factorization is a finitely restricted direct factorization.*

We may choose many systems of maximal elements θ^*_ω satisfying the condition of Theorem 4.2 from Σ, and A may be factorized in many ways. What sorts of relations are there among those factorizations? In order to clarify this we first prove some lemmas.

LEMMA 4.4. *Let Σ be a $\bigvee$-closed modular sublattice of a complete, upper continuous lattice Θ and $S(\Omega) = \{\theta_\omega ; \omega \in \Omega\}$ a system of points θ_ω of Σ such that the closed sublattice of Θ generated by $S(\Omega)$ is isomorphic with $\mathbf{2}^\Omega$ and contains $0, I$ of Σ. Then θ_η can be replaced by any element θ of Σ satisfying $\theta \cap \theta^*_\eta = 0$ and $\theta \cup \theta^*_\eta = I$, where $\theta^*_\eta = \bigvee_{\omega \in \Omega - \eta} \theta_\omega$.*

PROOF. It is sufficient to prove that the closed sublattice generated by $S'(\Omega) = \{\varphi_\omega ; \omega \in \Omega\}$, where $\varphi_\omega = \theta_\omega$ for $\omega \neq \eta$ and $\varphi_\eta = \theta$, is isomorphic with $\mathbf{2}^\Omega$ and contains $0, I$. Put $\varphi^*_\xi = \bigvee_{\omega \in \Omega - \xi} \varphi_\omega$. If $\xi \neq \eta$, then $\varphi^*_\xi = \bigvee_{\omega \in \Omega - \xi - \eta} \theta_\omega \cup \theta = (\theta^*_\xi \cap \theta^*_\eta) \cup \theta$ and $\varphi^*_\xi \cap \varphi^*_\eta = ((\theta^*_\xi \cap \theta^*_\eta) \cup \theta) \cap \theta^*_\eta = (\theta^*_\xi \cap \theta^*_\eta) \cup (\theta \cap \theta^*_\eta) = \theta^*_\xi \cap \theta^*_\eta$; hence $\varphi_\xi \cap \varphi^*_\xi = \varphi_\xi \cap \varphi^*_\eta \cap \varphi^*_\xi = \theta_\xi \cap \theta^*_\eta \cap \theta^*_\xi = 0$. Since $\varphi_\eta \cap \varphi^*_\eta = \theta \cap \theta^*_\eta = 0$, we have $\varphi_\omega \cap \varphi^*_\omega = 0$ for all $\omega \in \Omega$ and $\bigvee_{\omega \in \Omega} \varphi_\omega = \theta \cup \theta^*_\eta = I$. Then it follows from Lemma 4.2 that the closed sublattice generated by $S'(\Omega)$ is isomorphic with $\mathbf{2}^\Omega$.

LEMMA 4.5. *Let Θ, Σ and $S(\Omega)$ be the same as are defined in Lemma 4.4 and $\theta(M) = \bigvee_{\omega \in M} \theta_\omega$. If $\theta \in \Sigma$ satisfies $\theta \cap \theta(N) = 0$ for some $N \leq \Omega$, then there exists $M \leq \Omega$ satisfying that $\theta \cap \theta(M) = 0$, $\theta \cup \theta(M) = I$ and $M \geq N$; hence θ and $\theta(\Omega - M)$ are perspective.*

PROOF. N is contained in a maximal chain Γ of subsets M_γ satisfying $\theta \cap \theta(M_\gamma) = 0$ and $M = \bigvee M_\gamma \geq N$ also satisfies $\theta \cap \theta(M) = 0$, since Θ is upper continuous. Put $\varphi = \theta \cup \theta(M)$. Then $\varphi \in \Sigma$ and $\varphi \neq I$ implies $\varphi \cap \theta_\omega = 0$ for some $\omega \in \Omega$, since θ_ω are points of Σ and $\bigvee_{\omega \in \Omega} \theta_\omega = I$. Then we have $\theta\cap\theta(M+\omega)=\theta\cap\varphi\cap(\theta_\omega\cup\theta(M))=\theta\cup((\varphi\cap\theta_\omega)\cup\theta(M))=\theta\cup\theta(M)=0$, which contradicts that Γ is a maximal chain. Thus $\theta\cup\theta(M)=I$, completing the proof.

Now let A have a one-element subalgebra e and $e=(e_1, e_2)$ in a direct decomposition $A=A_1\times A_2$ of A. If θ is the congruence relation generated by the homomorphism from $x\in A$ to its A_2-component x_2, then the congruence class $C(\theta)=\{x ; x\equiv e(\theta)\}$ is the subalgebra $A_1^*=\{(x_1, e_2) ; x_1\in A_1\}$ of A which is isomorphic with A_1. Hence any direct factor $\theta_\omega^*(A)$ may be replaced by the congruence class $C(\theta_\omega)$. Further, as is already known, the projectivity between two congruence relations yields an isomorphism between their congruence classes; namely

LEMMA 4.6. *Let A be an algebra with a one-element subalgebra e and Σ a sublattice of $\Theta(A)$ such that $\Sigma\in 0, I$ and all congruence relations in Σ are permutable. If $[0, \theta]$ and $[0, \varphi]$ are projective intervals in Σ, then the congruence classes $C(\theta)$ and $C(\varphi)$ are isomorphic.*

It is obviously seen that the congruence class $C(\theta(M))$ (the homorphic image of A mod $\theta(\Omega-M)$ in general) is isomorphic with the finitely restricted direct union of $\{C(\theta_\omega) ; \omega\in M\}$ $(\{\theta_\omega^*(A) ; \omega\in M\})$ in the factorization mentioned in Theorem 4.1; hence from Lemma 4.5 we can deduce

THEOREM 4.3. *Let A be an algebra with a one-element subalgebra and Σ a $\bigvee$-closed sublattice of $\Theta(A)$ such that $\Sigma\ni 0, I$ and all congruence relations in Σ are permutable. If the closed sublattice of $\Theta(A)$ generated by a set $\{\theta_\omega ; \omega\in\Omega\}$ of points of Σ forms a Boolean algebra and contains $0, I$, then any congruence class $C(\theta)$ with $\theta\in\Sigma$ is isomorphic with the finitely restricted direct union of $\{C(\theta_\omega) ; \omega\in M\}$, where M is a subset of Ω.*

LEMMA 4.7. *Let Σ be a $\bigvee$-closed modular sublattice of a complete, upper continuous lattice Θ and $S(\Omega)=\{\theta_\omega ; \omega\in\Omega\}$, $S(H)=\{\varphi_\eta ; \eta\in H\}$ two systems of points of Σ satisfying the conditions in Lemma 4.4. Then there is a one-one correspondence from $S(\Omega)$ into $S(H)$, under which corresponding two elements are perspective.*

PROOF. Using the axiom of choice, we can well-order Ω so that $\Omega=\{1, \cdots, \alpha, \cdots, \lambda, \cdots\}$ and $S(\Omega)\cap S(H)=\{\theta_\mu ; \mu<\alpha\}$, and put $\varphi_\mu=\theta_\mu$ for $\mu<\alpha$. If φ_μ is defined for each $\mu<\lambda$ so that $\varphi_\mu\in S(H)$ and $\varphi_\mu\neq\varphi_\nu$ for $\mu\neq\nu$, then we may regard the subset $C_\lambda=\{\mu ; \mu<\lambda\}\leq\Omega$ as a subset of H. Let D_λ be a subset of H such that $C_\lambda\leq D_\lambda\leq H$, and put $E_\lambda=\Omega-C_\lambda$, $\Omega_\lambda=D_\lambda+E_\lambda$. If we put $D_\alpha=C_\alpha$, then the closed sublattice generated by

$$S(\Omega_\alpha)=\{\varphi_\eta ; \eta\in D_\alpha\}+\{\theta_\mu ; \mu\in E_\alpha\}=S(\Omega)$$

is an atomic Boolean algebra and contains $0, I$ of Σ. Now let λ be an ordinal number, and $\{\varphi_\nu; \nu<\mu\}$, D_μ and $S(\Omega_\mu)$ be defined for all $\mu<\lambda$ so that

(1) φ_ν and θ_ν be perspective and $\varphi_\nu \in \{\varphi_\eta; \eta \in H-D_\nu\}$,

(2) $C_\mu \leq D_\mu \leq H$ and $D_\nu \leq D_\mu$ for $\nu \leq \mu$,

(3) the closed sublattice generate by

$$S(\Omega_\mu)=\{\varphi_\eta; \eta \in D_\mu\}+\{\theta_\nu; \nu \in E_\mu\}$$

be an atomic Boolean algebra containing $0, I$ of Σ. Accordingly $\varphi(D_\mu) \cap \theta(E_\mu)=0$ and $\varphi(D_\mu) \cup \theta(E_\mu)=I$, where $\varphi(D_\mu)=\bigvee_{\eta \in D_\mu}\varphi_\eta$ and $\theta(E_\mu)=\bigvee_{\nu \in E_\mu}\theta_\nu$. Then we shall construct $S(\Omega_\lambda)$ satisfying the above conditions.

Case I: $\lambda-1=\mu$ exists. Let ψ_μ^* be the complement of θ_μ, which is contained in $\{\theta_\nu; \nu \in E_\mu\}$, in the Boolean algebra generated by $S(\Omega_\mu)$. Then $\psi_\mu^* \geq \varphi(D_\mu)$. If $\psi_\mu^* \geq \varphi_\eta$ for all $\eta \in H-D_\mu$, ψ_μ^* must coincide with I; therefore $\varphi_\eta \cap \psi_\mu^*=0$ and $\varphi_\eta \cup \psi_\mu^*=I$ for some $\eta \in H-D_\mu$, since φ_η is a point and ψ_μ^* a maximal element of Σ. Put $\varphi_\mu=\varphi_\eta$ and $D_{\mu+1}=D_\mu+\mu$, i.e., $\{\varphi_\eta; \eta \in D_\lambda\}=\{\varphi_\eta; \eta \in D_\mu\}+\varphi_\mu$. Then it follows from Lemma 4.4 that the system $S(\Omega_\lambda)=\{\varphi_\eta; \eta \in D_\lambda\}+\{\theta_\nu; \nu \in E_\lambda\}$, which is obtained by replacing θ_μ of $S(\Omega_\mu)$ by φ_μ, satisfies the condition (3) mentioned above. The other conditions (1), (2) are obvious.

Case II: λ is a limit-ordinal. In this case $\{\varphi_\mu; \mu<\lambda\}$ are all defined. Put $D=\bigvee_{\mu<\lambda}D_\mu$. Then $\varphi(D_\mu) \uparrow \varphi(D)$ and $\varphi(D_\mu) \cap \theta(E_\lambda) \leq \varphi(D_\mu) \cap \theta(E_\mu)=0$ for all $\mu<\lambda$; hence $\varphi(D) \cap \theta(E_\lambda)=0$. It follows from Lemma 4.5 that there exists $D_\lambda \leq H$ such that $\varphi(D_\lambda) \cap \theta(E_\lambda)=0$, $\varphi(D_\lambda) \cup \theta(E_\lambda)=I$ and $D_\lambda \geq D$. The conditions (1), (2) are satisfied. We show that $S(\Omega_\lambda) =\{\varphi_\eta; \eta \in D_\lambda\}+\{\theta_\nu; \nu \in E_\lambda\}$ satisfies the condition (3). If we set $\psi_\eta=\varphi_\eta$ for $\eta \in D_\lambda$, $\psi_\nu=\theta_\nu$ for $\nu \in E_\lambda$ and $\psi_\xi^*=\bigvee_{\zeta \in \Omega_\lambda-\xi}\psi_\zeta$, then $\bigvee_{\zeta \in \Omega_\lambda}\psi_\zeta=\varphi(D_\lambda) \cup \theta(E_\lambda)=I$ and we may write $\psi_\xi^*=\varphi((\Omega_\lambda-\xi) \cap D_\lambda) \cup \theta((\Omega_\lambda-\xi) \cap E_\lambda)$. For $\xi=\eta \in D_\lambda$, $\psi_\eta \cap \psi_\eta^*=\psi_\eta \cap \varphi(D_\lambda) \cap (\varphi(D_\lambda-\eta) \cup \theta(E_\lambda))=\psi_\eta \cap ((\varphi(D_\lambda) \cap \theta(E_\lambda)) \cup \varphi(D_\lambda-\eta))=\varphi_\eta \cap \varphi(D_\lambda-\eta)=0$ and it is similar for $\xi=\nu \in E_\lambda$. Hence $\psi_\xi \cap \psi_\xi^*=0$ for all $\xi \in \Omega_\lambda$ and it follows from Lemma 4.2 that the closed sublattice generated by $S(\Omega_\lambda)$ is an atomic Boolean algebra.

If for some λ, $H-D_\lambda$ is exhausted, then $\varphi(D_\lambda)=I$ and hence $\theta(E_\lambda)=0$; namely E_λ is void. So we can find φ_λ for all $\lambda \in \Omega$ and the proposition is thus proved.

THEOREM 4.4. *Let Σ be a $\bigvee$-closed modular sublattice of a complete, upper continuous lattice Θ and $S(\Omega)=\{\theta_\omega; \omega \in \Omega\}$ and $S(H)=\{\varphi_\eta; \eta \in H\}$ two systems of points of Σ such that both the closed sublattices of Θ*

generated by them are atomic Boolean algebras and contain 0, I *of* Σ. *Then there is a one-one correspondence between* $S(\Omega)$ *and* $S(H)$ *and the* θ_ω *and* θ_η *are projective in pairs.*

PROOF. Let θ_ω be any element of $S(\Omega)$ and $P(\omega)$ the class of all elements of $S(\Omega)$ projective to θ_ω in Σ. Again let θ_ω correspond to φ_η under the correspondence mentioned in Lemma 4.7. Then the images of the elements in $P(\omega)$ under this correspondence are projective to φ_η in Σ and containd in the class $P(\eta)$ of all elements of $S(H)$ projective to φ_η. So $P(\omega)$ corresponds one-one with a subset of $P(\eta)$. Conversely we may consider a simliar correspondence from $S(H)$ into $S(\Omega)$, under which $P(\eta)$ corresponds one-one with a subset of $P(\omega)$. Then there exists a one-one correspondence between $P(\omega)$ and $P(\eta)$. Thus we can find a one-one correspondence between all classes $P(\omega)$ and $P(\eta)$, and the theorem is proved.

Referring Lemma 4.6, we infer that the factorization of an algebra with a one-element subalgebra mentioned in Theorem 4.2 is not influenced by the selection of the system $S(\Omega)$ to within isomorphism.

COROLLARY. *Let* $A=\Pi_{\omega\in\Omega}C(\theta_\omega)=\Pi_{\eta\in H}C(\varphi_\eta)$ *be two representations of an algebra* A *with a one-element subalgebra as a finitely restricted direct union. If there exists a* $\bigvee$*-closed sublattice* Σ *of* $\Theta(A)$ *such that* θ_ω *and* φ_η *are points of* Σ *and all congruence relations in* Σ *are permutable, then there is a one-one correspondence between* Ω *and* H, *and the* $C(\theta_\omega)$ *and* $C(\varphi_\eta)$ *are pairwise isomorphic.*

A congruence relation θ on A is called a *decomposition congruence relation* if and only if θ has a complement θ' which is permutable with θ; that is to say, A is decomposed into $\theta(A)\times\theta'(A)$. All congruence relations $\theta(M)$ mentioned in Theorem 3.2 are decomposition congruence relations. If θ and φ are two dccomposition congruence relations and $\theta<\varphi$, then it is easy to see $\theta(A)\cong\varphi(A)\times\varphi^*(A)$, where $\varphi^*=\theta\cup\varphi'$; hence if $\theta(A)$ is an indecomposable direct factor, θ is a maximal decomposition congruence relation. Conversely if a direct factor $\theta(A)$ is decomposed into a direct union $\varphi\theta(A)\times\varphi'\theta(A)$, then evidently $\varphi\theta$ is a decomposition congruence relation on A and $\varphi\theta>\theta$; hence if θ is a maximal decomposition congruence relation, then $\theta(A)$ is directly indecomposable.

Now let A be an algebra on which all congruence relations are permutable and all decomposition congruence relations form a sublattice $\Theta_0(A)$ of $\Theta(A)$. If $\Theta_0(A)$ is $\bigvee$-closed in $\Theta(A)$, then it follows from Theorem 4.2 that A is decomposed into a finitely restricted direct union of directly indecomposable factors. Conversely suppose that A is decomposed into a finitely restricted direct union of indecomposable factors $\{\theta_\omega^*(A)\,;\ \omega\in\Omega\}$, and define θ_ω and $\theta(M)$ as before. Given a subset $\{\varphi_\nu\}$ of $\Theta_0(A)$, we set $\psi_\mu=\bigvee_{\nu<\mu}\varphi_\nu$ and assume $\psi_\mu\in\Theta_0$ for $\mu<\lambda$. If $\lambda-1$ exists, then $\psi_\lambda=\psi_{\lambda-1}\cup\varphi_{\lambda-1}\in\Theta_0$. If λ is a limit ordinal, then $\psi_\mu\uparrow\psi_\lambda$. We can find a maximal subset M of Ω satisfying $\psi_\lambda\cap\theta(M)=0$. $\psi_\lambda\cup\theta(M)\not\geq\theta_\omega$ implies $\psi_\mu\cup\theta(M)\not\geq\theta_\omega$ and hence $(\psi_\mu\cup\theta(M))\cap\theta_\omega=0$ for all $\mu<\lambda$, since $(\psi_\mu\cup\theta(M))\cap\theta_\omega\in\Theta_0$ and θ_ω is a point in Θ_0. Since $\psi_\mu\cup\theta(M)\uparrow\psi_\lambda\cup\theta(M)$, we have $(\psi_\lambda\cup\theta(M))\cap\theta_\omega=0$, $(\psi_\lambda\cup\theta(M))\cap(\theta(M)\cup\theta_\omega)=((\psi_\lambda\cup\theta(M))\cap\theta_\omega)\cup\theta(M))=\theta(M)$ and $\psi_\lambda\cap\theta(M+\omega)=\psi_\lambda\cap\theta(M)=0$, which is a contradiction. Hence $\psi_\lambda\cup\theta(M)\geq\theta_\omega$ for all $\omega\in\Omega-M$ and so $\psi_\lambda\cup\theta(M)=I$; acchrdingly $\psi_\lambda\in\Theta_0$ and thus Θ_0 is $\bigvee$-closed.

THEOREM 4.5. *Let A be an algebra on which all congruence relations are permutable and decomposition congruence relations form a sublattice $\Theta_0(A)$ of $\Theta(A)$. Then A is decomposed into a finitely restricted direct union of directly indecomposable factors if and only if $\Theta_0(A)$ is $\bigvee$-closed in $\Theta(A)$.*

If A contains a one-element subalgebra moreover, then $\Theta_0(A)$ satisfies the conditions in Cor. of Theorem 4.4; hence

COROLLARY. *Let $A=\Pi_{\omega\in\Omega}A_\omega=\Pi_{\eta\in H}B_\eta$ be any two representations of an algebra A as a finitely restricted direct union of indecomposable factors, where* (1) *A has a one-element subalgebra,* (2) *all congruence relations on A are permutable,* (3) *all decomposition congruence relations on A form a sublattice of $\Theta(A)$. Then there is a one-one correspondence between Ω and H, and the A_ω and B_η are pairwise isomorphic.*

5. Factorizations into simple factors

Almost all of well-known algebras, such as groups, rings etc., have modular structure lattices. Applying the results in the last paragraph, we shall discuss the finitely restricted factorizations of such algebras into simple factors.

If θ^* is a maximal element of $\Theta(A)$, then the homomorphic image $\theta^*(A)$ of A is simple. Therefore if $\Theta(A)$ is complemented and modular, then we can infer, by using Theorem 4.2, that A is decomposed into an irredundant, finitely restricted subdirect union of simple factors.

We shall prove the converse of this fact. Let A be an algebra with a modular stıucture lattice $\Theta(A)$ and decomposed into an irredundant, finitely restricted, subdirect union of simple algebras $\{A_\omega\,;\ \omega\in\Omega\}$. Define θ_ω^* and $\theta(M)$ in the same way as in Theorem 3.1; namely $\theta(M)=\bigwedge_{\omega\in\Omega-M}\theta_\omega^*$

and $x \equiv y(\theta(M))$ means $\{\omega; x_\omega \neq y_\omega\} \leq M$. Since $\theta^*_\omega(A) = A_\omega$ is simple, θ^*_ω is maximal, and since the factorization is irredundant, $\theta_\omega = \theta(\omega) \not\leq \theta^*_\omega$; hence $\theta_\omega \cup \theta^*_\omega = I$. We shall first prove $\theta(M) = \bigvee_{\omega \in M} \theta_\omega$. If we put $\bigvee_{\omega \in M} \theta_\omega = \varphi(M)$, then evidently $\varphi(M) \leq \theta(M)$. Suppose that $x \equiv y(\theta(M))$ and put $\{\omega; x_\omega \neq y_\omega\} = N$. Then $N \leq M$ and N is a finite set, so we put $N = \{\omega_1, \cdots, \omega_n\}$ and $N_i = \{\omega_1, \cdots, \omega_i\}$ for $i = 1, \cdots, n$. Evidently $\theta(N_1) = \theta(\omega_1) \leq \varphi(M)$. From $\theta(N_{i-1}) \leq \varphi(M)$ we can deduce $\theta(N_i) = (\theta_{\omega_i} \cup \theta^*_{\omega_i}) \cap \theta(N_i) = \theta_{\omega_i} \cup (\theta^*_{\omega_i} \cap \theta(N_i)) = \theta_{\omega_i} \cup \theta(N_{i-1}) \leq \varphi(M)$; hence we have $\theta(N) \leq \varphi(M)$ and $x \equiv y(\varphi(M))$. Thus $\theta(M) \leq \varphi(M)$. Now let θ be any congruence relation on A and consider a maximal chain Γ of subsets M_γ of Ω satisfying $\theta \cap \theta(M_\gamma) = 0$. It follows from the above argument that $M = \sup \Gamma$ satisfies $\theta(M_\gamma) \uparrow \theta(M)$ and hence $\theta \cap \theta(M) = 0$. As a complement θ_ω of a maximal element θ^*_ω is a point, $\varphi = \theta \cup \theta(M) \not\geq \theta_\omega$ implies $\varphi \cap \theta_\omega = 0$ and $\theta \cap \theta(M + \omega) = \theta \cap \varphi \cap (\theta_\omega \cup \theta(M)) = \theta \cap ((\varphi \cap \theta_\omega) \cup \theta(M)) = \theta \cap \theta(M) = 0$, which is impossible since Γ is a maximal chain. Then $\theta \cup \theta(M) \geq \bigvee_{\omega \in \Omega} \theta_\omega = \theta(\Omega) = I$. Thus θ has a complement $\theta(M)$. In summary

THEOREM 5.1. *Let A be an algebra with a modular structure lattice $\Theta(A)$. Then A can be decomposed into an irredundant, finitely restricted, subdirect union of simple factors if and only if $\Theta(A)$ is complemented.*

This is a generalization of the following result which has been guessed by the author and proved by Tanaka [7].

COROLLARY. *Congruence relations on a lattibe L form a Boolean algebra if and only if L can be decomposed into an irredundant, finitely restricted, subdirect union of simple factors.*

Further referring Theorem 4.5, we have

THEOREM 5.2. *Let A be an algebra on which all congruence relations are permutable. Then A can be decomposed into a finitely restricted direct union of simple factors if and only if $\Theta(A)$ is complemented. Moreover if A contains a one-element subalgebra, then the factorization is unique to within isomorphism.*

This yields the condition in order that a group or a ring be completely reducible; namely

COROLLARY. *A group G (with or without operators) can be decomposed into a finitely restricted direct product of simple groups if and only if every normal subgroup H has a complementary normal subgroup H' satisfying $H \cap H' = \{e\}$, $HH' = G$*[4].

4) For instance, vector spaces satisfy this condition.

This includes the results of Blair [2] on the decomposition of rings into simple rings or minimal ideals.

6. Algebras with distributive structure lattices

We shall first show an extended unique factorization theorem for an algebra A on which congruence relations form a distributive lattice.

Let an algebra A be decomposed into an L-restricted direct union of $\{A_\xi ; \xi \in \Xi\}$ and each factor A_ξ decomposed into an L_ξ-restricted direct union of $\{A_{\eta(\xi)} ; \eta(\xi) \in H_\xi\}$. If L^* is the family of subsets M^* of $\Omega = \{\eta(\xi) ; \eta(\xi) \in H_\xi, \xi \in \Xi\}$ written in the forms $M^* = \bigvee_{\xi \in M} M_\xi$ with $M \in L$ and $M_\xi \in L_\xi$, then it is obviously seen that L^* is an ideal of the Boolean algebra $\mathbf{2}^\Omega$ and A is isomorphic with the L^*-restricted direct union of $\{A_{\eta(\xi)} ; \eta(\xi) \in \Omega\}$. We call the L^*-restricted direct decomposition $A = \Pi_{L^*} A_{\eta(\xi)}$ just mentioned a *refinement* of the decomposition $A = \Pi_L A_\xi$.

THEOREM 6.1. *Let Ξ, H be two sets of indices ξ, η and X, Y ideals of the Boolean algebras $\mathbf{2}^\Xi$, $\mathbf{2}^H$ respectively. If an algebra A with a distributive structure lattice is directly decomposed in two ways $A = \Pi_X A_\xi = \Pi_Y A_\eta$, then there exists a common refinement $A = \Pi_L A_{\xi\eta}$ such that $A_\xi = \Pi_Y A_{\xi\eta}$ and $A_\eta = \Pi_X A_{\xi\eta}$.*

REMARK. We assume, without loss of generality, that each of A_ξ and A_η contains two or more elements, but some of $A_{\xi\eta}$ may be one-element algebras.

PROOF. Define the congruence relations θ_ξ^*, φ_η^*, θ_ξ and φ_η as before: $\theta_\xi^*(x) = x_\xi \in A_\xi$, $\varphi_\eta^*(x) = x_\eta \in A_\eta$, $\theta_\xi \cup \theta_\xi^* = I$, $\theta_\xi \cap \theta_\xi^* = 0$, $\varphi_\eta \cup \varphi_\eta^* = I$, $\varphi_\eta \cap \varphi_\eta^* = 0$. Put $\psi_{\xi\eta}^* = \theta_\xi^* \cup \varphi_\eta^*$. If $x_\xi = y_\xi$, then $\theta_\xi^*(x) = \theta_\xi^*(y)$ and hence $\psi_{\xi\eta}^*(x) = \psi_{\xi\eta}^*(y)$ for all $\eta \in H$. Hence defining $\sigma_\eta^*(x_\xi) = \psi_{\xi\eta}^*(x) = x_{\xi\eta}$ (similarly $\tau_\xi^*(x_\eta) = x_{\xi\eta}$), we obtain a homomorphism σ_η^* from A_ξ onto $A_{\xi\eta} = \psi_{\xi\eta}^*(A)$. Suppose that $x_{\xi\eta} = y_{\xi\eta}$ for all $\eta \in H$, in other words $x \equiv y(\bigwedge_{\eta \in H} \psi_{\xi\eta}^*)$. Then since $\theta_\xi \cap \psi_{\xi\eta}^* = \theta_\xi \cap (\theta_\xi^* \cup \varphi_\eta^*) = \theta_\xi \cap \varphi_\eta^*$, $\theta_\xi \cap \bigwedge_{\eta \in H} \psi_{\xi\eta}^* \leq \bigwedge_{\eta \in H} \varphi_\eta^* = 0$ and $\bigwedge_{\eta \in H} \psi_{\xi\eta}^* = \bigwedge_{\eta \in H} \psi_{\xi\eta}^* \cap (\theta_\xi \cup \theta_\xi^*) = (\bigwedge_{\eta \in H} \psi_{\xi\eta}^* \cap \theta_\xi) \cup \theta_\xi^* = \theta_\xi^*$, we infer that $x \equiv y(\theta_\xi^*)$, namely $x_\xi = y_\xi$; hence the mapping $x_\xi \to \{x_{\xi\eta} ; \eta \in H\}$ is an isomorphism of A_ξ into $\Pi_{\eta \in H} A_{\xi\eta}$ and A_ξ is decomposed into a subdiret union of $\{A_{\xi\eta} ; \eta \in H\}$. Now if x and y are any two elements of A and ξ is a fixed index in Ξ, then it is evident that $\{\eta ; x_{\xi\eta} \neq y_{\xi\eta}\} \leq \{\eta ; x_\eta \neq y_\eta\} \in Y$. Conversely suppose that $x_\xi = \{x_{\xi\eta} ; \eta \in H\}$, $y_\xi = \{y_{\xi\eta} ; \eta \in H\}$ with $x_\xi = \theta_\eta^*(x) \in A_\xi$, $y_\xi \in \Pi_{\eta \in H} A_{\xi\eta}$ and $M = \{\eta ; x_{\xi\eta} \neq y_{\xi\eta}\} \in Y$. We can choose a set of elements $z^\eta \in A$ so that $\psi_{\xi\eta}^*(z^\eta) = y_{\xi\eta}$, and an elements $z \in A$ so that $z_\eta = x_\eta$ for $\eta \bar{\in} M$ and $z_\eta = z_\eta^\eta$ for $\eta \in M$, since $\{\eta ; z_\eta \neq x_\eta\} \leq M \in Y$.

Then if $\eta \bar{\in} M$, we have $z_{\xi_\eta} = \tau_\xi^*(z_\eta) = \tau_\xi^*(x_\eta) = x_{\xi_\eta} = y_{\xi_\eta}$, and if $\eta \in M$, $z_{\xi_\eta} = \tau_\xi^*(z_\eta^\eta) = \psi_{\xi_\eta}^*(z^\eta) = y_{\xi_\eta}$; hence $y_\xi = z_\xi \in A_\xi$. Thus A_ξ is isomorphic with a Y-restricted direct union of $\{A_{\xi_\eta};\ \eta \in H\}$.

Now if A_ξ is directly indecomposable, then A_{ξ_η} are one-element algebras except some one $A_{\xi,\eta(\xi)}$, so $\theta_\xi^* \cup \varphi_\eta^* = I$ for $\eta \neq \eta(\xi)$. Then $\theta_\xi^* = \bigwedge_{\eta \in H}(\theta_\xi^* \cup \varphi_\eta^*) = \theta_\xi^* \cup \varphi_{\eta(\xi)}^* \geq \varphi_{\eta(\xi)}^*$. Moreover if $A_{\eta(\xi)}$ is directly indecomposable, we infer similary $\varphi_{\eta(\xi)}^* \geq \theta_{\xi(\eta(\xi))}^*$ and hence $\theta_\xi^* \geq \theta_{\xi(\eta(\xi))}^*$; then $\xi(\eta(\xi)) = \xi$, for otherwise $\theta_\xi^* = \theta_\xi^* \cup \theta_{\xi(\eta(\xi))}^* = I$. Hence if all of A_ξ and A_η are directly indecomposable, there is a one-one correspondence between Ξ and H under which the corresponding congruence relations θ_ξ^* and φ_η^* are identical. Further observing that $\{\xi;\ x_\xi \neq y_\xi\}$ corresponds to $\{\eta;\ x_\eta \neq y_\eta\}$ under this correspondence, we conclude

COROLLARY. *Let A be an algebra on which congruence relations form a distributive lattice. Then for any two direct factorizations of A into directly indecomposable factors, $A = \Pi_X A_\xi = \Pi_Y A_\eta$, there exists a one-one correspondence $\xi \to \eta(\xi)$ between factors under which $X \cong Y$ and $A_\xi \cong A_{\eta(\xi)}$.*

It follows from Theorem 6.1 that two direct decompositions $A = \theta(A) \times \theta'(A) = \varphi(A) \times \varphi'(A)$ yield a refinement $A = \psi_1(A) \times \psi_2(A) \times \psi_3(A) \times \psi_4(A)$, where $\psi_1 = \theta \cup \varphi$, $\psi_2 = \theta \cup \varphi'$, $\psi_3 = \theta' \cup \varphi$, $\psi_4 = \theta' \cup \varphi'$. Putting together the three factors, we have a decomposition $A = \psi(A) \times \psi'(A)$ with $\psi = \theta \cup \varphi$ and $\psi' = \theta' \cap \varphi'$; hence all decomposition congruence relations from a sublattice $\Theta_0(A)$ of $\Theta(A)$. Further if θ is a decomposition congruence relation; there exists for any pair x, y an element z such that $x\theta z\theta' y$. So $x \equiv y(\theta \cup \varphi)$ implies $z \equiv x \equiv y(\theta \cup \varphi)$ and $z \equiv y \bmod \theta' \cap (\theta \cup \varphi) = \theta' \cap \varphi$; hence θ is permutable with any congruence relation φ.

THEOREM 6.2. *All decomposition congruence relations on an algebra A with a distributive structure lattice $\Theta(A)$ form a Boolean algebra as a sublattice of $\Theta(A)$ and they are permutable with any congruence relation.*

Referring Theorem 4.5 and its proof, we have

COROLLARY. *An algebra A with a distributive structure lattice $\Theta(A)$ is decomposed into a finitely restricted direct union of indecomposable factors if and only if the set of decomposition congruence relations is $\bigvee$-closed in $\Theta(A)$.*

We shall next show an application of this corollary. We say that a lattice L satisfies the *restricted chain condition* if every closed interval of L satisfies either one of chain conditions. Let θ_α be decomposition congruence relations on a lattice L satisfying the restricted chain condition and put $\theta = \bigvee \theta_\alpha$. If an interval $[x, y]$ satisfies the ascending condition, then we can find a maximal element z satisying $x \equiv z(\theta)$ and $x \leq z \leq y$. Since θ_ω is a decomposition congruence relation, we can find an element u such that $z\theta_\omega u\theta_\omega' y$ and $z \leq u \leq y$. Then $u \equiv z \equiv x(\theta)$ and hence $u = z$, since z is a maximal element satisfying $x \equiv z(\theta)$. Thus $z \equiv y(\theta_\omega')$ for every θ_ω'. If we put $\theta' = \bigwedge \theta_\omega'$, then every interval $[x, y]$ satisfies $x \equiv y(\theta \cup \theta')$; hence $\theta \cup \theta' = I$. It is easy to show $\theta' \cap \theta = \bigvee (\theta' \cap \theta_\omega) = 0$. Since θ' is permutable with every θ_ω, θ and θ' are permutable. Thus $\theta \in \Theta_0(A)$ and $\Theta_0(A)$ is $\bigvee$-closed.

THEOREM 6.3. *Any lattice satisfying the restricted chain condition is a finitely restricted direct union of directly indecomposable lattices.*

COROLLARY. *If a lattice L with* 0, I *satisfies either one of chain conditions, then L is decomposed into a direct union of a finite number of indecomposable factors.*

Now let an algebra A with a distributive structure lattice be decomposed into an L-restricted direct union of indecomposable factors $\{\theta_\omega^*(A)\,;\ \omega \in \Omega\}$ and define θ_ω, $\theta(M)$ as before. Given $\theta \in \Theta_0(A)$, we set $M = \{\omega\,;\ \theta_\omega \leq \theta\}$. Since θ_ω is a point in Θ_0, we get for $\omega \in \Omega - M$, $\theta_\omega \cap \theta = 0$, $\theta_\omega^* = \theta_\omega^* \cup (\theta_\omega \cap \theta) = \theta_\omega^* \cup \theta \geq \theta$ and $\theta(M) = \bigwedge_{\omega \in \Omega - M} \theta_\omega^* \geq \theta$. Again the complement θ' of θ satisfies $\theta_\omega \cap \theta' = 0$ for $\omega \in M$, where $\theta(\Omega - M) \geq \theta'$. So we conclude altogether $\theta = \theta(M)$. Accordingly

THEOREM 6.4. *If an algebra A with a distributive structure lattice* $\Theta(A)$ *is decomposed into a (restricted) direct union of indecomposable factors, then* $\Theta_0(A)$ *is atomic and* $\bigwedge$*-closed in* $\Theta(A)$.

7. Direct decomposition of complete lattices

Congruence relations on a lattice L form a distributive lattice $\Theta(L)$ and decomposition congruence relations on L form a Boolean algebra $\Theta_0(L)$ as a sublattice of $\Theta(L)$. So applying the results in the last paragraph, we shall discuss the direct decompositions of complete lattices.

THEOREM 7.1. *For a complete lattice L the following conditions are equivalent*:

(1) *L is a direct union of directly indecomposable lattices,*

(2) $\Theta_0(L)$ *is atomic and* $\bigwedge$*-closed in* $\Theta(L)$,

(3) $\Theta_0(L)$ *is atomic and* $\bigwedge_{\omega \in \Omega} \theta_\omega^* = 0$, *where* $\{\theta_\omega^*\,;\ \omega \in \Omega\}$ *is the set of all maximal elements in* $\Theta_0(L)$.

PROOF. (1) implies (2) by Theorem 6.4 and it is easy to see that

(2) implies (3). If (3) holds, then L is decomposed into a subdirect union of directly indecomposable lattices $\{\theta_\omega^*(L)\,;\ \omega\in\Omega\}$. Given $\{x_\omega\,;\ \omega\in\Omega\}$ with $x_\omega\in\theta_\omega^*(L)$, we can find $x^\omega\in L$ such that $\theta_\omega^*(x^\omega)=x_\omega$ and $y^\omega\in L$ such that $0\leq y^\omega\leq x^\omega$ and $0\theta_\omega y^\omega\theta_\omega^* x^\omega$, where θ_ω is the complement of θ_ω^*. Put $\bigvee_{\omega\in\Omega}y^\omega=y$. We can find $x^\eta\in L$ such that $y^\eta\leq z^\eta\leq y$ and $y^\eta\theta_\eta^* z^\eta\theta_\eta y$. Since $y^\omega\theta_\omega 0$ and so $y^\omega\theta_\eta^* 0$ for $\omega\neq\eta$, we have $y^\eta\theta_\eta^* y^\eta\cup y^\omega\theta_\eta^* z^\eta\cup y^\omega\theta_\eta y\cup y^\omega=y$. Then $z^\eta\cup y^\omega\theta_\eta^* y^\eta\theta_\eta^* z^\eta$ and $z^\eta\cup y^\omega\theta_\eta y\theta_\eta z^\eta$; accordingly $z^\eta\cup y^\omega=z^\eta$ and $z^\eta\geq y^\omega$ for all $\omega\in\Omega$. Thus we have $z^\eta=y$, $y\theta_\eta^* y^\eta\theta_\eta^* x^\eta$ and $\theta_\eta^*(y)=\theta_\eta^*(x^\eta)=x_\eta$ for all $\eta\in\Omega$. So L is isomorphic with the complete direct union of $\{\theta_\omega^*(L)\,;\ \omega\in\Omega\}$.

Further let us consider the case that L is atomic. If p is a point of L and $\theta\in\Theta_0(L)$, then it is easy to see that $p\equiv 0(\theta)$ or $p\equiv 0(\theta')$ holds. Let $\{\theta_\alpha\}$ be all elements of $\Theta_0(L)$ satisfying $p\equiv 0(\theta_\alpha)$. If $\varphi\in\Theta_0(L)$ satisfies $0\leq\varphi<\theta=\bigwedge\theta_\alpha$, then we have $p\not\equiv 0(\varphi)$, $p\equiv 0(\varphi')$ and $p\equiv 0(\theta\frown\varphi')$. Hence if $\Theta_0(L)$ is $\bigwedge$-closed in $\Theta(L)$, then $\theta\frown\varphi'\in\Theta_0(L)$, $\varphi'\geq\theta$ and $\varphi=0$, so we can show that $\Theta_0(L)$ is atomic.

COROLLARY. *A complete atomic lattice L is a direct union of directly indecomposable lattices if and only if $\Theta_0(L)$ is $\bigwedge$-closed in $\Theta(L)$.*

It is well-known that a complete atomic Boolean algebra is decomposed into a direct union of the two-element lattices. Then it may be a natural inquiry whether a complete atomic distributive lattice is directly decomposed into indecomposable factors. But the answei is negative, as shown below.

In a complete lattice L decomposition congruence relations θ_α correspond one-one to elements c_α in the center $C(L)$ so that $C(\theta_\alpha)=\{x\,;\ x\equiv 0(\theta_\alpha)\}=(c_\alpha]$ and $C'(\theta_\alpha)=\{x\,;\ x\equiv I(\theta_\alpha)\}=[c_\alpha')$. Further $C(\bigwedge\theta_\alpha)=\bigwedge C(\theta_\alpha)=\bigwedge(c_\alpha]=(\bigwedge c_\alpha]$ and $C'(\bigwedge\theta_\alpha)=[\bigvee c_\alpha')$; hence

LEMMA 7.1. *Let L be a complete lattice. If $\Theta_0(L)$ is $\bigwedge$-closed in $\Theta(L)$, then the center $C(L)$ is a closed sublattice of L.*

Let Ω be a set which consists of a directed set of points p_μ and another point p, and introduce a topology in the space Ω so that if M contains a cofinal subset of $\{p_\mu\}$, then $\bar{M}=M\cup p$, otherwise $\bar{M}=M$. Then the lattice L of all closed subsets of Ω is complete, atomic and distributive, and moreover $\Theta(L)$, $\Theta_0(L)$ and $C(L)$ are atomic; nevertheless L cannot be directly decomposed into indecomposable factors. We can show more generally

THEOREM 7.2. *Let L be the lattice of all closed subsets of a totally disconnected T_2-space Ω. Then L cannot be decomposed into a direct union* *of indecomposable factors, unless Ω is discrete.*

PROOF. If Ω is not discrete, then there exists a point p such that $\overline{\Omega - p} \ni p$. Since Ω is totally disconnected, we can find for every point $q \neq p$ an open and closed subset M_q so that $M_q \ni p$ and $M_q \not\ni q$. Then $M_q \in C(L)$ and $\bigwedge M_q = p \bar{\in} C(L)$; hence $C(L)$ is not closed in L and the theorem is proved with Lemma 7.1 and Theorem 7.1.

As is shown in a previous paper [4], the lattice $\Theta(L)$ of all congruence relations of a distributions lattice L is isomorphic with the lattice of all open subsets of a totally-disconnected locally-compact T_2-space.

COROLLARY. *Let L be any distributive lattice. Then $\Theta(L)$ cannot be decomposed into a direct union of indecomposable factors, unless every closed interval of L has a finite length.*

8. The structure of relatively complemented lattices

Dilworth [3] has proved that a relatively complemented lattice with $0, I$ satisfying the asceding chain condition is a direct union of a finite number of simple lattices. We intend to generalize this result for atomic relatively complemented lattices. In the present section we shall call a lattice L *atomic* if and only if every interval of L contains a prime interval and *uniserial* if and only if all prime intervals of L are projective.

First it is easy to show

LEMMA 8.1. *All congruence relations on a relatively complemented lattice are permutable.*

LEMMA 8.2. *Let x, y, z be any three elements of a relatively complemented lattice satisfying $x < y$. Then any subinterval of the interval $[x \frown z, y \frown z]$ (or $[x \smile z, y \smile z]$) is projective to a subinterval of $[x, y]$.*

PROOF. Suppose $x \frown z \leq u < v \leq y \frown z$. Let t be a relative complement of u in $[x \frown z, v]$. Then it is easy to see $x < x \smile t \leq y$ and that $[u, v]$, $[x \frown z, t]$ and $[x, x \smile t]$ are successive transposes.

It follows from this lemma that any subinterval of an interval $[x \frown z, z]$ is projective to a subinterval of its transpose $[x, x \smile z]$; hence we infer

COROLLARY. 1. *Let $[x, y]$ and $[a, b]$ be projective intervals in a relatively complemented lattice. Then any subinterval $[u, v]$ of $[x, y]$ is projective to some subinterval $[c, d]$ of $[a, b]$.*

COROLLARY 2. *Let an interval* $[x, y]$ *be projective to a prime interval* $[p, q]$ *in a relatively complemented lattice. Then any subinterval of* $[x, y]$ *is projective to* $[p, q]$.

LEMMA 8.3. *Let* x, y, z *be elements of a relatively complemented lattice satisfying* $x \leq y \leq z$. *If* $[x, z]$ *contains a prime interval* $[p, q]$, *then either* $[x, y]$ *or* $[y, z]$ *contains a subinterval projective to* $[p, q]$.

PROOF. Let t be a relative complement of p in $[x, q]$ and put $t \frown y = u$. If $u \neq x$, then $[x, y]$ contains $[x, u]$ projective to $[p, q]$, and if $u = x$, then $[y, z]$ contains $[y, y \smile t]$ projective to $[p, q]$

LEMMA 8.4. *Let* L *be an atomic, relatively complemented lattice. Then with every prime interval* $[p, q]$, *a congruence relation* $\theta^* = \theta^*[p, q]$ *can be associated so that*

(1) $p \not\equiv q(\theta^*)$, (2) $p \not\equiv q(\theta)$ *imply* $\theta \leq \theta^*$,
(3) *the homomorphic image* $\theta^*(L)$ *be uniserial,*
(4) $\theta^*[p, q] = \theta^*[r, s]$ *if and anly if* $[p, q]$ *and* $[r, s]$ *are projective.*

PROOF. We define $x \equiv y(\theta^*)$ to mean that the interval $[x \frown y, x \smile y]$ contains no interval projective to $[p, q]$. If a, b are two elements in $[x \frown y, x \smile y]$ and $x \equiv y(\theta^*)$, then $a \equiv b(\theta^*)$. It follows from Lemma 8.2 that $x \equiv y(\theta^*)$ implies $(x \frown y) \frown z \equiv (x \smile y) \frown z(\theta^*)$ and $x \frown z \equiv y \frown z(\theta^*)$, and similary $x \smile z \equiv y \smile z (\theta^*)$. Then from $x \equiv y$, $y \equiv z (\theta^*)$ we can deduce $x \frown y \frown z \equiv y \frown z$, $y \frown z \equiv z$, $z \equiv y \smile z$, $y \smile z \equiv x \smile y \smile z(\theta^*)$ and that $[x \frown y \frown z, x \smile y \smile z]$ cannot contain any interval projective to $[p, q]$, by Lemma 8.3; hence $x \equiv z(\theta^*)$. Thus θ^* is a congruence relation. Now suppose that $p \not\equiv q(\theta)$ and $x \equiv y(\theta)$. If $[x \frown y, x \smile y]$ contains an interval projective to $[p, q]$, then $p \equiv q(\theta)$ follows; hence $[x \frown y, x \smile y]$ cannot contain such a subinterval and $x \equiv y(\theta^*)$, proving (2). As (1) and (4) are obvious, it remains to prove (3). If $\theta^*(x) < \theta^*(y)$, then the interval $[x \frown y, x \smile y]$ must contain a prime interval $[r, s]$ projective to $[p, q]$. It is obvious that $[\theta^*(r), \theta^*(s)]$ is a prime interval in $\theta^*(L)$ and $\theta^*(x) \leq \theta^*(r) < \theta^*(s) \leq \theta^*(y)$; hence $\theta^*(L)$ is atomic. Further if $[\theta^*(x), \theta^*(y)]$ is any prime interval, then it coincides with $[\theta^*(r), \theta^*(s)]$ which is projective to $[\theta^*(p), \theta^*(q)]$.

Further if we define $x \equiv y(\theta[p, q])$ to mean that all prime intervals contained in $[x \frown y, x \smile y]$ are projective to $[p, q]$, then it is similary shown that $\theta[p, q]$ is a congruence relation and $\theta^*[p, q] \frown \theta[p, q] = 0$.

LEMMA 8.5. *The congruence relation* $\theta^*[p, q]$ *defined in Lemma* 8.4 *is a maximal decomposition congruence relation if either*: (1) *L satisfies the restricted chain condition, or* (2) *L is conditionally complete.*

PROOF. Let $[a, b]$ be any interval of L and S the set of elements x satisfying $x \equiv a(\theta^*[p, q])$ and $a \leq x \leq b$. (1) If $[a, b]$ satisfies the ascending chain condition, then S contains a maximal element c. (2) If $[a, b]$ is complete, put $c = \sup S$. Then $c \equiv a(\theta^*[p, q])$. Indeed if $[a, c]$ contains a subinterval $[r, s]$ projective to $[p, q]$, then the relative complement t of s in $[r, c]$ satisfies $t < t \cup x \leq c$ for some $x \in S$ and hence $[a, x]$ contains a subinterval $[x \cap t, x]$ projective to $[p, q]$. So in either case, c is a maximal element in S. Suppose that $[c, b]$ contains a prime interval $[r, s]$ which is not projective to $[p, q]$. If t is a relative complement of r in $[c, s]$, then we have $r \equiv s$ and $c \equiv t(\theta^*)$, which contradicts that c is a maximal element in S. Hence $c \equiv b(\theta[p, q])$. Thus we get $a \equiv b(\theta^*[p, q] \cup \theta[p, q])$ and $\theta^*[p, q] \cup \theta[p, q] = I$, whence $\theta^*[p, q]$ is a decomposition congruence relation. Any congruence relation $\varphi > \theta^*[p, q]$ annuls every prime interval; hence every congruence relation $\theta \neq 0$ satisfies $r \equiv s(\varphi \cap \theta)$ for some prime interval $[r, s]$ and $\varphi \cap \theta \neq 0$. So $\theta^*[p, q]$ is maximal in $\Theta_0(L)$.

Now let $\{\theta^*_\omega ; \omega \in \Omega\}$ be the set of all congruence relations $\theta^*_\omega = \theta^*[p, q]$ associated with prime intervals. Then it is obvious that $\bigwedge_{\omega \in \Omega} \theta^*_\omega = 0$; hence from Theorem 7.1 we can infer

THEOREM 8.1. *A complete, atomic, relatively complemented lattice is a direct union of uniserial, relatively complemented lattices.*

A uniserial, relatively complemented lattice is directly indecomposable but not always simple. We can show that if it satisfies the restricted chain condition it is simple. Indeed a congruence relation $\theta \neq 0$ on such a lattice L annuls every prime interval; hence if c is an element in any interval $[a, b]$ satisfying $c \equiv a(\theta)$ and $[p, q]$ is a prime interval contained in $[c, b]$, then the relative complement d of p in $[c, q]$ satisfies $d \equiv c \equiv a(\theta)$, and so $a \equiv b(\theta)$.

Therfore combining Theorem 6.3 and Lemma 8.5, we have

THEOREM 8.2. *A relatively complemented lattice satisfying the restricted chain condition is a finitely restricted direct union of simple, relatively complemented lattices.*

COROLLARY. *All congruence relations on a relatively complemented lattice satisfying the restricted chain condition form a Boolean algebra.*

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