<table>
<thead>
<tr>
<th>Title</th>
<th>Note on generalised uniserial algebras. I</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Yoshii, Tensho</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Mathematical Journal. 6(1) P.105–P.107</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1954–06</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/9809">https://doi.org/10.18910/9809</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/9809</td>
</tr>
<tr>
<td>Note</td>
<td></td>
</tr>
</tbody>
</table>
Note on Generalised Uniserial Algebras, I

By Tensho Yoshii

An associative algebra with unit element is called \textit{generalized uniserial} if every primitive left ideal as well as every primitive right ideal possesses only one composition series, and it is called \textit{uniserial} if it is generalized uniserial and moreover it is primarily decomposable.

The structure of absolutely uniserial algebras i.e. of those uniserial algebras which remain so after any coefficient field extension, was studied by T. Nakayama and G. Azumaya\(^2\). In the present note we shall consider absolutely generalized uniserial algebras i.e. those generalized uniserial algebras which remain so after any coefficient field extension.

Let \( A \) be an (associative and finite dimensional) algebra with unit element over a field \( K \), and let \( N \) be its radical. Let \( \bar{A} = A/N = \bar{A}_1 + \cdots + \bar{A}_k \) be the direct decomposition of the semi simple residue class algebra \( \bar{A} = A/N \) into simple components. Let \( E_\kappa \) be the unit element of \( \bar{A}_\kappa \). Then \( E_\kappa \) is a sum of mutually orthogonal primitive idempotent elements \( \bar{e}_{\kappa 1}, \bar{e}_{\kappa 2}, \ldots, \bar{e}_{\kappa f(\kappa)} \). There are then mutually orthogonal primitive idempotent elements \( e_{\kappa i}(\kappa = 1, \ldots, k; i = 1, \ldots, f(\kappa)) \) in \( A \) such that \( e_{\kappa i} \equiv \bar{e}_{\kappa i} \) (mod \( N \)) and \( \sum e_{\kappa i} = \) unit element in \( A \). We put \( E_\kappa = \sum e_{\kappa i} \) for each \( \kappa = 1, \ldots, k \). \( A^0 = E'AE' \) is called a basic algebra of \( A \), were \( E' = \sum e_{\kappa 1} \). It is clear that the radical of \( A^0 \) is \( N^0 = E'NE' \).

The following lemma is easily seen.\(^3\)

\textbf{Lemma 1.} \( A \) is a generalized uniserial algebra if and only if \( N^0 = CA^0 = A^0C \), \( C \in A^0 \).

\textbf{Lemma 2.} If \( A \) is generalized uniserial and \( A/N \) is separable, then \( A \) is absolutely generalized uniserial.

---

1) A primitive left (right) ideal is a left (right) ideal generated by a primitive idempotent element. Cf. M. Thrall [1].
2) See G. Azumaya and T. Nakayama [2].
3) See K. Morita [3], Theorem 1.
PROOF. If $A$ is generalized uniserial, the radical $N^0$ of $A^0$ is expressible as a principal ideal by Lemma 1. We put $N^0 = CA^0 = A^0C$.

Since the radical of $A_L$ is $N_L$ from the assumption, the radical of $A_{L^0}$ is $N_{L^0} = CA_{L^0} = A_{L^0}C$. Therefore $A_{L^0}$ is generalized uniserial.

Now let $A^*$ be a basic algebra of $A_{L^0}$. Then $A^*$ is also a basic algebra of $A_L$. On the other hand the radical of $A^*$ is expressible as a principal ideal. Therefore $A_L$ is generalized uniserial. Thus our lemma is proved.

From this lemma, we get the following theorem.

**Theorem.** Let $A$ be an absolutely generalized uniserial algebra. Then $A$ is a direct sum of two subalgebras $A_1$ and $A_2$, where $A_1$ is a generalized uniserial algebra with the separable residue class algebra over its radical and $A_2$ is an absolutely uniserial algebra. The converse is also true.

**PROOF.** (i) Suppose that $A$ is an absolutely generalized uniserial algebra and, in the direct decomposition of $A$, $A_i$ ($i = 1, \ldots, r$) is a separable simple algebra and $A_j$ ($j = r + 1, \ldots, k$) is an inseparable simple algebra.

Now let $\Gamma$ be an algebraically closed field and let $M$ be a radical of $A_\Gamma$. Then $A_\Gamma = A_\Gamma/N_\Gamma$ is the direct sum of some matric algebras over $\Gamma$ and some primary algebras with non-zero radicals. Here the former ones are obtained from $A_1, \ldots, A_r$ and the latter ones are obtained from $A_{r+1}, \ldots, A_k$. We put $A_\Gamma = A_\Gamma/N_\Gamma = A_{\lambda}' + \cdots + A_{r} + A_{r+1} + \cdots + A_{k}$, where $A_{\lambda}', \ldots, A_k$ are matric algebras over $\Gamma$ and $A_{r+1}, \ldots, A_k$ are primary algebras. Further let $E_{\lambda}'$ be the unit element of $A_\lambda$ ($\kappa = 1, \ldots, s$) and let $E_{\lambda}$ be the unit element of $A_{\lambda}$ ($\lambda = s + 1, \ldots, t$). Then $E_{\lambda}'$ is a sum of mutually orthogonal idempotent elements $\bar{e}_{\lambda 1}, \ldots, \bar{e}_{\lambda s\phi(\lambda)}$ and $E_{\lambda}$ is a sum of mutually orthogonal primitive idempotent elements $f_{\lambda 1}, \ldots, f_{\lambda s\phi(\lambda)}$. There are then mutually orthogonal primitive idempotent elements $e_{\lambda s} (\kappa = 1, \ldots, s, i = 1, \ldots, \phi(\kappa))$ and $f_{\lambda j} (\lambda = s + 1, \ldots, t, j = 1, \ldots, \phi(\lambda))$ in $A_\Gamma$ such that $e_{\lambda s} (\mod N_\Gamma) = \bar{e}_{\lambda i}$, $f_{\lambda j} (\mod N_\Gamma) = \bar{f}_{\lambda j}$ and $\sum_{i} e_{\lambda s} + \sum_{j} f_{\lambda j} = \text{unit element in } A_\Gamma$. Furthermore we put $E^* = \sum_{i} e_{\lambda s}^*$, $F^* = \sum_{j} f_{\lambda j}^*$.

From the assumption of $A_{\lambda}'$ ($j = s + 1, \ldots, t$),

$$f_{\lambda j}^* \overline{M} = 0, \quad f_{\lambda j}^* \overline{M} / f_{\lambda j}^* \overline{M}^2 \cong f_{\lambda j}^* A_\Gamma / f_{\lambda j}^* \overline{M}, \quad f_{\lambda j}^* A_\Gamma = f_{\lambda j}^* \overline{A}_\lambda.$$}

Now the right hand side is simple from our assumption. Therefore the left hand side is also simple. On the other hand, $f_{\lambda j}^* A_\Gamma$ has a unique composition series from the assumption of $A_\Gamma$. Therefore
there exists an $n$ such that $f^*_{\lambda_j}N_{\Gamma} = f^*_{\lambda_j}M^n (n \geq 2)$ and

$$f^*_{\lambda_j}M / f^*_{\lambda_j}M^2 \simeq f^*_{\lambda_j}M / f^*_{\lambda_j}M^2 \simeq f^*_{\lambda_j}A_{\Gamma}/f^*_{\lambda_j}M,$$

consequently

$$f^*_{\lambda_j}M^l / f^*_{\lambda_j}M^{l+1} \simeq f^*_{\lambda_j}A_{\Gamma}/f^*_{\lambda_j}M \quad (l = 1, \ldots, \rho - 1, f^*_{\lambda_j}M^\rho = 0).$$

Therefore $e^*_{\lambda_j}A_{\Gamma}f^*_{\lambda_j} = 0$ for each $e^*_{\lambda_j}$. In the same way $f^*_{\lambda_j}A_{\Gamma}e^*_{\lambda_j} = 0$ for each $e^*_{\lambda_j}$. Therefore $A_{\Gamma} = (E^* + F^*)A_{\Gamma}(E^* + F^*) = E^*A_{\Gamma}E^* + F^*A_{\Gamma}F^*$ and $F^*A_{\Gamma}F^*$ is a uniserial algebra from the above proof and $E^*A_{\Gamma}E^*$ is the direct sum of matric algebras over $\Gamma$. If we put $A = A_1 + A_2$, where $A_{1\Gamma} = E^*A_{\Gamma}E^*$, $A_{2\Gamma} = F^*A_{\Gamma}F^*$, it is clear that $A$ is the direct sum of $A_1$ and $A_2$ which satisfy the conditions of this theorem.

(ii) Conversely suppose that $A$ is generalized uniserial and $A = A_1 + A_2$ such that $A_1$ is a generalized uniserial algebra with the separable residue class algebra over its radical and $A_2$ is an absolutely uniserial algebra. Then $A_{\Gamma} = A_{1\Gamma} + A_{2\Gamma}$ and since $A_2$ is absolutely uniserial, $A_{2\Gamma}$ is uniserial and $A_{1\Gamma}$ is generalized uniserial from Lemma 2. Therefore $A_{\Gamma}$ is generalized uniserial and $A$ is absolutely generalized uniserial.

From this theorem, it follows readily

**Corollary.** If $A$ is an algebra such that the radical of $A$ and that of $A_{\Gamma}$ are expressible as principal ideals for any coefficient field extension, then $A$ is a direct sum of $A_1$ and $A_2$, where $A_1$ has a radical expressible as a principal ideal with the separable residue class algebra over its radical and $A_2$ is an absolutely uniserial algebra. The converse is also true.

(Received March 24, 1954)

Bibliography


