<table>
<thead>
<tr>
<th>Title</th>
<th>Notes on signatures on rings</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kanzaki, Teruo</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 22(2) P.327-P.338</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1985</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/9811">https://doi.org/10.18910/9811</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/9811</td>
</tr>
<tr>
<td>rights</td>
<td></td>
</tr>
</tbody>
</table>
NOTES ON SIGNATURES ON RINGS

Dedicated to Professor Hirosi Nagao on his 60th birthday

TERUO KANZAKI

(Received June 21, 1984)

0. Introduction

The notion of infinite prime introduced by Harrison [3] was investigated in [1], [2], [7] and [9] which were concerned with ordering on a field. In this note, we study about signatures on rings as some generalization of infinite primes and signatures of fields in [2]. In the section 1, we introduce notions of $U$-prime and signature of a ring which are generalizations of infinite prime and signature of field. In the section 2, we show that a $U$-prime of a commuative ring defines a signature on the ring. In the sections 3 and 4, we consider the category of signatures and a space of signatures on a ring which include notions of extension of signature and space of ordering on fields (cf. [2] and [8]), and investigate them. Throughout this paper, we assume that every ring has identity 1.

1. Preliminaries, definitions and notations

Let $S$ be a multiplicative semigroup, and $T$ a normal subsemigroup of $S$, (cf. [6], p. 195), denoted by $T \triangleleft S$, that is, $T$ is a subsemigroup of $S$ which satisfies 1) for $x, y \in S$, $xy \in T$ implies $yx \in T$, 2) if there is an $x \in T$ with $xy \in T$, then $y \in T$, and 3) for every $x \in S$, there exists an $x' \in S$ with $x'x \in T$. We can define a binary relation $\sim$ on $S$; for $x, y \in S$, $x \sim y$ if and only if there is a $z \in S$ such that both $zx$ and $zy$ are contained in $T$. Then, the relation $\sim$ is an equivalence relation on $S$, and is compatible with the multiplication of $S$, so the quotient set $S/\sim$, denoted by $S/T$, makes a group such that the canonical map $\psi: S \to S/T; x \mapsto [x]$ is a homomorphism with $\ker \psi = T$.

Let $R$ be any ring with identity 1, and $P$ a preprime of $R$ ([3]), that is, $P$ is closed under addition and multiplication of $R$ and $-1 \notin P$. We put $p(P) = P \cap -P$, $R_p = \{x \in R \mid xp(P) \cup p(P)x \subset p(P)\}$, $R_p^+ = R_p \setminus p(P)$ ($:= \{x \in R_p \mid x \notin p(P)\}$), $P^+ = P \setminus p(P) (= P \setminus -P)$. We shall say a preprime $P$ to be complete quasi-prime, if it satisfies the following conditions:

1) $p(P)$ is an ideal of $R_p$ such that $R_p/p(P)$ is an integral domain,
2) $P^+ \triangleleft R_p^+$ under the multiplication of $R_p$. 

Kanzaki, T.
Osaka J. Math.
22 (1985), 327–338
3) \( P \) is complete in \( R_p \), that is, for \( x \in R_p \), \( x^2 \in P \) implies \( x \in P \cup -P \).

A multiplicative semigroup \( F \) with unit element 1 and zero element 0 will be called a \( f \)-semigroup, if \( F^* = F \setminus \{0\} \) makes a group with a unique element of order 2, denoted by \(-1\), under the multiplication of \( F \). If \( P \) is a complete quasi-prime of \( R \), then the quotient group \( G(P) = R_p/P^+ \) has a unique element \([-1]\) of order 2, and the formally composed semigroup \( F(P) = G(P) \cup \{0\} \) makes an \( f \)-semigroup under the multiplication of \( G(P) \) and \( \alpha 0 = 0 \alpha = 00 = 0 \) for \( \alpha \in G(P) \). Furthermore, we can define a map \( \sigma: R_P \to F(P) \) by \( \sigma(a) = 0 \) or \([a]\) for \( a \in p(P) \) or \( a \in R^* \), respectively. Then, it can be verified that 1) \( \sigma(-1) = [-1] \), 2) \( \sigma(ab) = \sigma(a)\sigma(b) \) for every \( a, b \in R_p \), and 3) for \( a, b \in R_p \), either \( \sigma(a) = 0 \) or \( \sigma(a) = \sigma(b) \) implies \( \sigma(a+b) = \sigma(b) \).

Let \( \pi \) be a set of prime numbers, and suppose \( 2 \in \pi \). A complete quasi-prime \( P \) will be called a \( \pi \)-complete quasi-prime, if for each \( q \in \pi \), there is a \( \xi_q \in R^* \) such that \( \xi_q^q \in P \) and for any \( x \in R_P \) with \( x^q \in P \), \( \sigma(x) = 0 \) or \( \sigma(x) = 1 \) for some \( y \in P^+ \).

**Remark 1.1.** If \( R \) is a commutative ring and \( P \) is a \( \pi \)-complete quasi-prime, then for each \( q \in \pi \), the \( q \)-torsion subgroup \( G(P)_q = \{ \alpha \in G(P) | \alpha^n = 0 \} \) of \( G(P) \) is isomorphic to a subgroup of \( \mathbb{Z}(q^n) \). Because, since \( G(P)_q \) has a unique minimal non-trivial subgroup \( \langle [\xi_q] \rangle \), \( G(P)_q \) is indecomposable, so by [4], p. 22, Theorem 10, \( G(P)_q \) is isomorphic to \( \mathbb{Z}(q^n) \) or \( \mathbb{Z}(q^n) \).

Let \( R \) be a ring with identity 1, and \( F \) an abelian \( f \)-semigroup. A partial map \( \sigma: R \to F \) will be called a signature of \( R \) with domain of definition \( R_\sigma \), if \( \sigma \) is a map of a subset \( R_\sigma \) of \( R \) into \( F \) satisfying the following conditions;

1. \( \sigma(-1) = -1 \),
2. \( \sigma(ab) = \sigma(a)\sigma(b) \) for every \( a, b \in R_\sigma \),
3. \( \sigma(a+b) = \sigma(a) + \sigma(b) \) for \( a, b \in R_\sigma \),
4. \( \sigma(ab) = \sigma(0) \) or \( \sigma(ab) = 1 \) for \( a, b \in R_\sigma \),

Let \( \sigma: R \to F \) be a signature. For \( \alpha \in F \), we put \( P_\sigma(\alpha) = \{ x \in R_\sigma | \sigma(x) = \alpha \} \), \( P(\sigma) = P_\sigma(0) \cup P_\sigma(1) \) and \( G(\sigma) = \text{Im } \sigma \cap F^* \).

**Lemma 1.2.** Let \( \sigma: R \to F \) be a signature of a ring \( R \).

1. \( R_\sigma \) is a subring of \( R \) with prime ideal \( p_\sigma(\alpha) \) such that \( R_\sigma/p_\sigma(\alpha) \) is an integral domain.
2. \( P(\sigma) \) is a preprime of \( R_\sigma \), and \( R_\sigma = R_{P(\sigma)} \).
3. If \( G(\sigma) \) is a subgroup of \( F^* \), then \( P(\sigma) \) is a complete quasi-prime of \( R_\sigma \), and \( G(P(\sigma)) \) and \( G(\sigma) \) are group isomorphic.

Proof. 1) If \( R_\sigma \) is closed under the addition of \( R \), then it is easy to see...
that \( R_\sigma \) is a subring of \( R \). Suppose \( a + b^R \in R_\sigma \) for some \( a \) and \( b \) in \( R_\sigma \). There is a \( c \in R_\sigma \) such that \( \sigma(c) = 0 \), and \( \sigma((a + b)c) = 1 \). Since \( \sigma(ac) = \sigma(a)\sigma(c) = 0 \) and \( \sigma(cb) = \sigma(bc) = 0 \), we get \( \sigma(ac + bc) = \sigma(ac + bc) = 0 \) which is a contradiction. Hence, we get \( R_\sigma + R_\sigma \subseteq R_\sigma \). It is easy to see that \( p_0(\sigma) \) is an ideal of \( R_\sigma \), and \( R_\sigma/p_0(\sigma) \) is an integral domain.

2) From the definition of signature, it follows that \( p(\sigma) \) is a preprime of \( R \) and \( p_0(\sigma) = P(\sigma) \cap -P(\sigma) \). We shall show \( R_\sigma = R_{P(\sigma)} \). Since \( R_\sigma \subseteq R_{P(\sigma)} \) is clear, it suffices to show \( R_{P(\sigma)} \subseteq R_\sigma \). If \( x \in R \setminus R_\sigma \), then there is a \( y \in p_0(\sigma) \) with \( xy \in p_1(\sigma) \) or \( yx \in p_1(\sigma) \), so \( x \in R_{p_0(\sigma)} \), that is, \( x \in R_{P(\sigma)} \).

3) If \( G(\sigma) \) is a group, then it is easy to see that \( Y(\sigma) + = p_1(\sigma), P(\sigma) + < R_{P(\sigma)} \), \( \sigma(R_{P(\sigma)}) = G(\sigma) \), and \( P(\sigma) \) is complete. Furthermore, a map \( G(P(\sigma)) = R_{P(\sigma)}^+/\sigma(\cdot) \) is a group isomorphism.

**Remark.**
1) If \( R \) is a field, then a signature \( \sigma : R \rightarrow F \) with \( p_0(\sigma) = \{0\} \) and \( F = \mu \cup \{0\} \) coincides with the notion of signature defined by Becker, Harman and Rosenberg [2], where \( \mu \) is the group of all roots of unity in the complices. 2) Let \( F \) be a finite field with characteristic \( \neq 2 \). The multiplicative semigroup \( F \) is an abelian \( f \)-semigroup. For a signature \( \sigma : R \rightarrow F \), let \( \tau \) be the set of all prime factors of order \( |G(\sigma)| \). Then, it is easy to see that \( P(\sigma) \) is a \( \tau \)-complete quasiprime of \( R \).

Let \( R \) be a ring with identity 1, and \( U \) a non empty multiplicatively closed subset of \( R \) satisfying \( U \cap -U = \phi \). A preprime \( P \) of \( R \) will be called a \( U \)-preprime of \( R \), if \( U \subseteq P \) and \( P \cap -U = \phi \). A maximal \( U \)-preprime of \( R \) will be called a \( U \)-prime of \( R \). Any Harrison’s infinite prime is a \{1\}-prime.

**Lemma 1.3.** Let \( U \) a non empty multiplicatively closed subset of \( R \) with \( U \cap -U = \phi \), and \( P \) a \( U \)-prime of \( R \). If either \( R \) is commutative or \( Px = xP \) and \( Ux = xU \) hold for every \( x \in R^+ \), then \( P \) is a complete quasi-prime of \( R \).

The proof of this lemma is obtained by checking the following facts;

1. \( U + P \subseteq P^+ \).
2. For \( x \in R_p \) (\( x \in R \), if \( R \) is commutative), if there are \( u \in U \) and \( y \in P \) with \( (u + y)x \in P \), then \( x \in P \). Hence \( 1 \in P \).
3. For \( x \in R_p \) (\( x \in R \), if \( R \) is commutative), if \( x \in p(P) \), then there is an \( x' \in (\pm P)[x] \) with \( x'x \in U + P \), where \( (\pm P)[x] = \{ \sum a_i x^i \in R | a_i \in P \cap -P \} \).
4. \( R_p/p(\sigma) \) is an integral domain.
5. For \( x, y \in R_p \), \( xy \in P^+ \) implies \( yx \in P^+ \).
6. \( P \) is complete in \( R_p \).
7. For any \( x \in P^+ \), there is an \( x' \in P^+ \) with \( x'x \in U + P \).
8. For \( x \in R_p \) (\( x \in R \), if \( R \) is commutative), if there is a \( y \in P^+ \) with \( yx \in P^+ \), then \( x \in P^+ \).

The proofs of these statements are obtained similarly to the case of Harrison’s.
infinite prime; (1.3.1): Since \( U \cap -P = \phi \), it follows that \( U \subset P^+ \) and \( U+P \subset P^+ \).

(1.3.2): A subset \( P' = \{ x \in R_p \mid x \in U, \exists y \in P; (u+y)x \in P \} \) of \( R \) is closed under addition and multiplication. Because, if \( x_1, x_2 \in P' \), there are \( u_i \in U \) and \( y_i \in P \) with \( (u_i+y_i)x_i \in P, i = 1, 2 \). If either \( x_1 \) or \( x_2 \) belongs to \( p(P) \), then it is trivial that \( x_1+x_2 \) and \( x_1x_2 \) belong to \( P' \). Otherwise, by assumption, there are \( u_2 \in U \) and \( y_2 \in P \) such that \( x_1u_2 = u_2x_1 \) and \( x_1y_2 = y_2x_1 \). Then \((u_1+y_1)(u_2+y_2)\) and \((u_1+y_1)(u_2+y_2)\) belong to \( U+P \), and \((u_1+y_1)(u_2+y_2)(x_1+x_2)\) and \((u_1+y_1)(u_2+y_2)\) are in \( P \). Furthermore, it is immediately seen that \( P \subset P' \) and \( P' \cap -U = \phi \), so we get \( P = P' \).

(1.3.3): For \( x \in R_p \), if \( x \in p(P) \), then either \( x \notin P \) or \( x \in P \). By assumption, a subset \( P[x] = \{ x \in R \mid x \in U+P \} \) (resp. \( P[-x] = P+P(-x)+P(-x)^2+\cdots \) of \( R \) is closed under addition and multiplication. Since \( P \subseteq P[x] \) or \( P \subseteq P[-x] \), we get \( P[x] \cap -U = \phi \) or \( P[-x] \cap -U = \phi \), so we can find an element \( y \in (\pm P)[x] \) such that \( xy \in U+P \) holds. (1.3.4): For \( x, y \in R_p \), suppose that \( xy \in p(P) \) and \( x \in p(P) \). By (1.3.3), there is an \( x' \in (\pm P)[x] \) (\( \subset R_p \)) with \( x'x \in U+P \), and (1.3.2) derives that \( x'y \in p(P) \) implies \( y \in p(P) \).

(1.3.5): For \( x, y \in R_p \), suppose \( xy \in P^+ \). (xy)x is in \( P \setminus -x \), and for an element \( x' \) in \( (\pm P)[x] \), also in \( R_p \), with \( x'x \in U+P \), we get \( (x'x)y \in x'xP \subset P \), so \( xy \in P^+ \) by (1.3.2) and (1.3.4). (1.3.6) is easy. (1.3.7): If \( x \in p(P) \), then \( P[-x] = P-Px \) is closed under addition and multiplication, and \( P \subseteq P[-x] \). Hence, there are \( u \in U \) and \( x', y \in P \) with \( -u = y - x'x \), so we get \( x'x = u + y \in U+P \) and \( x' \in P^+ \).

(1.3.8) is immediately obtained from (1.3.2) and (1.3.7).

2. The connection between \( U \)-prime and signature

Theorem 2.1. Let \( R \) be a commutative ring with identity 1, and \( U \) any non-empty multiplicatively closed subset of \( R \) with \( U \cap -U = \phi \). If \( P \) is a \( U \)-prime of \( R \), then there exists a signature \( \sigma : R \to F \) with \( P(\sigma) = P \) and group \( G(\sigma) = G(P) \).

Proof. By Lemma 1.3, \( U \)-prime \( P \) is a complete quasi-prime of \( R \), so it defines a map \( \sigma : R_p \to F(P) \). Then, we put \( R_p = R_p \) and \( F = F(P) \). The conditions (S 1), (S 2) and (S 3) of signature were verified. (S 4) is proved in the following proposition. Then we have a signature \( \sigma : R \to F \) with \( P = p(\sigma) \) and \( G(\sigma) = G(P) = R_f / P^+ \).

Proposition 2.2. Let \( P \) be a \( U \)-prime of a commutative ring \( R \), and let \( A_p = \{ a \in R \mid a = u \in U+P, z \in P \cap -P, i = 1, 2, \ldots, n; \Sigma_i z_i b_i a^{-i} = 0 \} \).

1) \((R_p, p(P))\) is a valuation pair of \( R \), (cf. [3], Proposition 2.5).
2) If \( x \in R_p \setminus p(P) \) then there is an \( a \in A_p \) with \( ax \in U+P \).
3) If \( x \) and \( y \) are elements of \( R \) with \( xy \in U+P \), then \( x \in p(P) \) implies \( y \in A_p \).
4) \( R_p = A_p \).

Proof. The proof of 1) is quite similar to [3], Proposition 2.5. 2) If \( x \in R \setminus p(P) \), by (1.3.3) there is an \( a \in (\pm P)[x] \) with \( ax \in U+P \), then \( a \) can be
SIGNATURES ON RINGS

represented as \(-(b_1 + b_2 x + \cdots + b_n x^{n-1})\) for some \(b_i \in P \cup -P\). If we put \(ax = b_0\), then \(a\) satisfies an equation \(b_0 a^x + b_1 a b_0 x^{n-1} + \cdots + b_n b_0 = 0\) with \(b_0 \in U + P\) and \(b_i b_0 \in P - P\), \(i = 1, 2, \cdots, n\), so \(a \in A_p\).

3) Suppose that \(x \) and \(y \) are in \(R\) and \(x \notin p(P)\), \(y \in A_p\). Suppose that \(x \notin P\) and \(y \notin P\), \(i = 1, 2, \cdots, n\), with \(\sum_i a_i x^{m-i} = 0\). Put \(xy = b_0\) and \(zz = c_0\), so we get that \(\sum_i a_i c_0 x^{m-i} = 0\), with \(b_0 \notin U + P\) and \(c_0 \notin P\). Hence \(y \in A_p\).

4) In the first place, we show \(A_p \supset R_p\): Let \(x \) be any element in \(R_p\). If \(x \notin p(P), x \notin A_p\) is obvious. Otherwise, by (1.3.3) there is a \(y \in (\pm P)[x]\) with \(xy \in U + P\), so \(y \notin p(P)\) and by 3) we get \(x \in A_p\). Now, we show \(A_p = R_p\): Let \((U + P)^{-1}R\) be the ring of quotients of \(R\) with respect to \(U + P\), and \(\psi: R \rightarrow (U + P)^{-1}R\) the canonical ring homomorphism. Then, \((U + P)^{-1}R_p\) may be regarded as a subring of \((U + P)^{-1}R\). By \(B'\), we denote the integral closure of \((U + P)^{-1}R_p\) in \((U + P)^{-1}R\). There is a prime ideal \(Q'\) of \(B'\) which lies over \((U + P)^{-1}R_p\) \(p(P)\), (cf. [5], (10.8)). It follows that \(B = \psi^{-1}(B')\) is a subring of \(R\) with \(B \supset A_p \supset R_p\), and \(Q = \psi^{-1}(Q')\) is a prime ideal of \(B\) with \(Q \cap R_p = p(P)\). By 1), we get \(B = A_p = R_p\).

Lemma 2.3. Let \(R\) be a commutative ring, and \(\sigma: R \rightarrow F\) a signature. If \(G(\sigma)\) is a torsion group, then \(R_\sigma = \{a \in R|a^n \in P(\sigma)\text{ for some integer } n > 0\}\).

Proof. Since \(G(\sigma)\) is a torsion group, it is clear that any element \(a\) in \(R_\sigma\) has a positive integer \(n\) with \(a^n \in P(\sigma)\). Conversely, suppose that an element \(a \in R\) does not belong to \(R_\sigma\). There is a \(b \in p_0(\sigma)\) with \(ab \in p_0(\sigma)\). Then \(a^n\) is not contained in \(P(\sigma)\) for every positive integer \(n\). Because, if \(a^n \in P(\sigma)\) for some \(n > 0\), it derives a contradiction \(1 = \sigma((ab)^n) = \sigma(a^n)\sigma(b^n) = 0\).

Let \(R\) be a ring with identity 1. By [1], a preprime \(P\) is called a torsion preprime (resp. 2-torsion preprime) of \(R\), if for each \(a \in R\) there exists a positive integer \(n\) such that \(a^n \in P\) (resp. \(a^2 \in P\)) holds. From Theorem 2.1 and Lemma 2.3, the following corollaries immediately follow;

Corollary 2.4. Let \(R\) be a commutative ring with 1 and \(U\) a non empty multiplicatively closed subset of \(R\) with \(1 \in U\) and \(U \cap -U = \phi\).

1) If \(P\) is a torsion \(U\)-prime of \(R\), then \(p(P)\) is an ideal of \(R\), i.e. \(R_\sigma = R\), so there is a signature \(\sigma: R \rightarrow F\) such that \(P = P(\sigma)\), \(R = R_\sigma\) and \(G(\sigma)\) is a torsion group.

2) If \(P\) is a 2-torsion \(U\)-prime of \(R\), then there is a signature \(\sigma: R \rightarrow F\) such that \(P = P(\sigma)\), \(R = R_\sigma\) and \(F^* \cong \mathbb{Z}(2^n)\).

In particular, on a field, we have

Corollary 2.5. Let \(K\) be a field.

1) For any signature \(\sigma: K \rightarrow F, K_\sigma\) is a valuation ring of \(K\) with maximal ideal \(p_0(\sigma)\), and the residue field \(k(\sigma) = K_\sigma/p_0(\sigma)\) has an induced signature \(\sigma: k(\sigma)\)
$\rightarrow F$ with $k(\sigma) = k(\sigma)$ and $v_0(\sigma) = \{0\}$, and $P(\sigma)$ is a preordering on $k(\sigma)$.

2) Let $U$ be a non-empty multiplicatively closed subset of $K$ with $U \cap -U = \phi$. If $P$ is a $U$-prime of $K$, $K_P$ is a valuation ring of $K$ with maximal ideal $p(P)$. If $P$ is a torsion $U$-prime of $K$, then $K = K_P$, $p(P) = \{0\}$, and $P$ is a preordering, i.e. $P^+ = P \setminus \{0\}$ is a subgroup of $K^* = K \setminus \{0\}$, (cf. [1], (3.3)).

3) If $O$ is a real valuation ring of $K$ with maximal ideal $p$, i.e. the residue field $O/p$ is a formally real field, then there is a signature $\sigma: K \rightarrow GF(3)$ with $K_\sigma = O$ and $p_0(\sigma) = p$, where $GF(3) = \{0, 1, -1\}$ is a multiplicative semigroup of prime field with characteristic 3.

Theorem 2.6. Let $R$ be a ring with identity 1, and $\sigma: R \rightarrow F$ a signature of $R$. Assume that $G(\sigma)$ is a torsion group and $xp_\alpha(\sigma) = p_\alpha(\sigma)x$ holds for all $x \in R_\sigma \setminus p_\alpha(\sigma)$ and $\alpha \in G(\sigma) \cup \{0\}$. Then, there exists a signature $\tau: R \rightarrow F'$ of $R$ satisfying the following conditions;

1) $P(\tau)$ is a $p_\alpha(\sigma)$-prime of $R$ and $P(\tau) \subseteq P(\sigma)$,

2) $R = R_\sigma$ and $p_\alpha(\tau) = p_\alpha(\sigma)$,

3) there is a subgroup $H$ of $G(\sigma)$ such that $p_\alpha(\tau) = \sigma^{-1}(H), -1 \in H$ and $G(\sigma)/H \cong G(\tau)$ hold.

Proof. Since $P(\sigma)$ is a $p_\alpha(\sigma)$-preprime of $R$, by Zorn's Lemma there exists a $p_\alpha(\sigma)$-preprime $P$ of $R$ containing $P(\sigma)$. From the facts that $P \cap -P(\sigma) = \phi$ and $p_\alpha(\sigma) \subseteq p(P)$, we can derive that $p_\alpha(\sigma) = p(P)$ and $R_p = R_\sigma$. If there is an element $x \in R \setminus R_\sigma$, then there exists a $y \in p_\alpha(\sigma)$ such that either $xy$ or $yx$ belongs to $p_\alpha(\sigma)$. However, $xy$ and $yx$ are also contained in $p(P)$, so these are contrary to $p_\alpha(\sigma) \cap p(P) = \phi$. Hence, we get $R_p \subseteq R_\sigma$. Furthermore, if there is an element $x \in p(P) \setminus p_\alpha(\sigma)$, we have $x \in p(\sigma) \cap p(P)$ for some integer $n > 0$, which is a contradiction. Therefore, we get $p_\alpha(\sigma) = p(P)$ and $R_p = R_\sigma$. Now, we put $H = \sigma(P^+)$, so $H$ is a subgroup of $G(\sigma)$. We shall show $P^+ = \sigma^{-1}(H)$. If $x \in \sigma^{-1}(H)$ then there is a $y \in P^+$ with $\sigma(x) = \sigma(y)$. Since $y \in p_\alpha(\sigma)$ for some integer $n > 0$, we have $xy^n = (xy)^n \in p_\alpha(\sigma) \cap P^+$. Hence, for any $x \in R$, it follows that $x \in \sigma^{-1}(H)$ if and only if $xp_\alpha(\sigma) \cap P^+ \neq \phi$. On the other hand, we can show that $P = \{x \in R_\sigma | xp_\alpha(\sigma) \cap P^+ \neq \phi\}$; The set $P' = \{x \in R_\sigma | xp_\alpha(\sigma) \cap P^+ \neq \phi\}$ is closed under addition and multiplication: Because, for $x, y \in P'$, there are $x_i, y_i \in p_\alpha(\sigma)$ such that both $xx_i$ and $yy_i$ are in $P$. Since we may suppose that $y$ is not in $p_\alpha(\sigma)$, there is an $x_i \in p_\alpha(\sigma)$ with $x_i y = xy_i$, and it follows that both $(x+y)(x_i y_i)$ and $(xy)(x_i y_i)$ are contained in $P$. Hence, both $x+y$ and $xy$ belong to $P'$. Furthermore, it is derived that $P \subseteq P'$ and $P' \cap -P(\sigma) = \phi$, because of $P \cap -P(\sigma) = \phi$. Hence, we get $P = P'$. Accordingly, we conclude that $\sigma^{-1}(H) = P^+ = \bigcup_{\alpha \in H} p_\alpha(\sigma)$.

From the assumption $xp_\alpha(\sigma) = p_\alpha(\sigma)x$ for $x \in R_\sigma \setminus p_\alpha(\sigma)$ and $\alpha \in G(\sigma) \cup \{0\}$, $P$ is a complete quasi-prime of $R$. Therefore, we can define a signature $\tau: R \rightarrow F(P)$ such that $R_\tau = R_P = R_\sigma$, $p_\alpha(\tau) = p(P) = p_\alpha(\sigma)$ and $G(\tau) = G(P) \cong G(\sigma)/H$. 

It is easy to check the conditions of signature for $\tau$.

**Corollary 2.7.** Let $R$ be a commutative ring with identity $1$. If $\sigma: R \to F$ is a signature of $R$ such that $G(\sigma)$ is a 2-torsion group, then $P(\sigma)$ is a $p_i(\sigma)$-prime of $R$.

Proof. Since $G(\sigma)$ is a 2-torsion group, by Remark 1.1 every non-trivial subgroup $H$ of $G(\sigma)$ contains $-1$. By Theorem 1.7, $P(\sigma)$ is a $p_i(\sigma)$-prime of $R$.

**Corollary 2.8.** Let $S$ be a commutative ring with identity $1$, and $R$ a subring of $S$ containing $1$. If $\sigma: R \to F$ a signature of $R$ such that $G(\sigma)$ is 2-torsion group, then $\sigma$ can be extended to a signature $\tau: S \to F$ of $S$, i.e. $S \cap R = R_\sigma$ and $P(\tau) \cap R = P(\sigma)$ hold.

Proof. A signature $\tau: S \to F'$ is defined by a $p_i(\sigma)$-prime $P$ of $S$ containing $P(\sigma)$. Then, $\tau$ is an extension of $\sigma$.

### 3. Category of signatures

Let $\sigma_1: R_1 \to F_1$ and $\sigma_2: R_2 \to F_2$ be signatures of rings $R_1$ and $R_2$. Suppose that $f: R_1 \to R_2$ is a ring homomorphism such that $f(1) = 1$ and $f(R_{1\sigma_1}) \subseteq R_{2\sigma_2}$, and that $\xi: F_1 \to F_2$ is a partial homomorphism which is defined on $G(\sigma_1)$ and satisfies $\xi(0) = 0$, $\xi(-1) = -1$ and $\xi(\alpha \beta) = \xi(\alpha) \xi(\beta)$ if $\xi$ is defined on $\alpha$, $\beta$ and $\alpha \beta$ for $\alpha, \beta \in F_1$. Then, the pair $(f, \xi)$ will be called a morphism of signatures of $\sigma_1$ to $\sigma_2$, denoted by $(f, \xi): \sigma_1 \to \sigma_2$, if it satisfies $\xi(\sigma_1(x)) = \sigma_2(f(x))$ for all $x \in R_{1\sigma_1}$. Let $\sigma_i: R_i \to F_i$ and $\sigma'_i: R'_i \to F'_i$ be signatures of rings for $i = 1, 2$, and $(f, \xi): \sigma_1 \to \sigma_2$ and $(f', \xi'): \sigma_1 \to \sigma'_2$ morphisms of signatures. We define the equality of morphisms that $(f, \xi) = (f', \xi')$ if and only if $\sigma_i = \sigma'_i$ (i.e. $R_i = R'_i$, $R_{1\sigma_i} = R_{1\sigma'_i}$, $F_i = F'_i$ and $\sigma_i(x) = \sigma'_i(x)$ for all $x \in R_{1\sigma_i}$) for $i = 1, 2$, $f = f'$ and for every $\alpha \in G(\sigma_1)$, $\xi(\alpha) = \xi'(\alpha)$ hold. By $C_{\text{sig}}$, we denote the category of signatures in which objects are signatures of rings and morphisms are morphisms of signatures.

**Proposition 3.1.** Let $R$ and $S$ be rings with identity $1$, and $f: R \to S$ a ring homomorphism with $f(1) = 1$.

1) If $\tau: S \to F$ is a signature of ring $S$ with $\text{Im } f \supseteq p_0(\tau)$, then there exists a signature $\sigma: R \to F$ of ring $R$ with a morphism $(f, I_F): \sigma \to \tau$ in $C_{\text{sig}}$.

2) If $f: R \to S$ is surjective, and if $\sigma: R \to F$ is a signature of ring $R$ with $\text{Ker } f \subseteq p_0(\sigma)$, then there exists a signature $\tau: S \to F$ of ring $S$ with a morphism $(f, I_F): \sigma \to \tau$ in $C_{\text{sig}}$.

Proof. 1) Suppose that $\tau: S \to F$ is a signature of ring $S$ and $f: R \to S$ is a ring homomorphism with $f(1) = 1$ and $\text{Im } f \supseteq p_0(\tau)$. On a subring $R_\tau = \{x \in R \mid f(x) \in S_\tau\}$ of $R$, a map $\sigma: R_\tau \to F$; $x \mapsto \tau(f(x))$ is defined. The condition
Im \( f \ni p_0(\tau) \) derives that a signature \( \sigma: R \to F \) of ring \( R \) and a morphism \( (f, I_R): \sigma \to \tau \) in \( C_{\text{sig}} \) are defined. 2) Suppose that \( f: R \to S \) is a surjective ring homomorphism, and \( \sigma: R \to F \) is a signature of ring \( R \) with \( \text{Ker } f \subseteq p_0(\sigma) \). For a subring \( S = f(R_\sigma) \), we can define a map \( \tau: S \to F \) as follows: For any \( a \in S \), there is a \( b \in R_\sigma \) with \( f(b) = a \), then we put \( \tau(a) = \sigma(b) \). From the condition \( \text{Ker } f \subseteq p_0(\sigma) \), it is known that the map \( \tau: S \to F \) is well defined. Then, it is easy to see that a signature \( \tau: S \to F \) of ring \( S \) and a morphism \( (f, I_R): \sigma \to \tau \) in \( C_{\text{sig}} \) are defined.

Concerning commutative rings, the situation of Proposition 3.1, 2) is reformed as follows;

**Theorem 3.2.** Let \( f: R \to S \) be a ring homomorphism of a commutative ring \( R \) into a commutative ring \( S \) with \( f(1) = 1 \). If \( \sigma: R \to F \) is a signature of \( R \) such that \( \text{G}(\sigma) \) is a torsion group and \( \text{Ker } f \subseteq p_0(\sigma) \), then there exists a signature \( \tau: S \to F' \) of ring \( S \) with a morphism \( (f, \xi): \sigma \to \tau \) in \( C_{\text{sig}} \).

Proof. Suppose that \( f: R \to S \) is a ring homomorphism with \( f(1) = 1 \), and \( \sigma: R \to F \) is a signature of \( R \) with torsion group \( \text{G}(\sigma) \) and satisfying \( \text{Ker } f \subseteq p_0(\sigma) \). By Proposition 3.1, 2), for the surjective ring homomorphism \( f: R \to \text{Im } f \), there exists a signature \( \sigma': \text{Im } f \to F \) of the subring \( \text{Im } f \) of \( S \) with a morphism \( (f, I_R): \sigma' \to \sigma \) in \( C_{\text{sig}} \). Hence, we may assume that \( R \) is a subring of \( S \) with common identity, and it is sufficient to show that there exists a signature \( \tau: S \to F' \) of \( S \) with a morphism \( (f, \xi): \sigma \to \tau \) in \( C_{\text{sig}} \), where \( \iota \) denotes the inclusion map \( R \to S \).

By Theorem 2.6, there exists a signature \( \sigma: R \to F'' \) of \( R \) such that \( R_\sigma = R_\tau \), \( p_0(\sigma') = p_0(\sigma) \) and \( \text{G}(\sigma') \cong \text{G}(\sigma)/H \) for some subgroup \( H \) of \( \text{G}(\sigma) \) hold, and \( P(\sigma) \) is a \( p_0(\sigma') \)-prime of \( R \) containing \( P(\sigma) \). Then, we can define a partial homomorphism \( \xi_1: F'' \to F'' \) such that \( \xi_1 \) induces a group homomorphism \( \text{G}(\sigma') \to \text{G}(\sigma) \) and the pair \( (I_R, \xi_1) \) defines a morphism \( (I_R, \xi_1): \sigma \to \sigma \) in \( C_{\text{sig}} \). On the other hand, by Zorn's Lemma, there exists a \( p_0(\sigma') \)-prime \( P \) of \( S \) containing \( P(\sigma) \), and by Theorem 2.1 the \( p_1(\sigma') \)-prime \( P \) defines a signature \( \tau: S \to F(P) \) of \( S \) such that \( P(\tau) = P, S_\tau = S_P, F(P) = G(P) \cup \{0\} \) and \( G(P) = S_P^+ \) hold, and \( \tau \) is induced from the canonical map \( S_P^+ \to G(P) \). From the fact that \( P(\sigma) \) is a \( p_0(\sigma') \)-prime of \( R_\sigma \) and \( P \ni P(\sigma) \), it follows that \( P \cap R = P(\sigma), p(P) \cap R = p_0(\sigma) \) and \( P^+ \cap R = P(\sigma)^+( = p_1(\sigma)) \) hold. Since \( \text{G}(\sigma) \) is a torsion group, so is also \( \text{G}(\sigma) \), and by Lemma 2.3 and Proposition 2.2, it is derived that \( R_{p(\sigma)}( = R_\sigma) = \{a \in R | a^n \in P(\sigma) \text{ for some integer } n > 0\} \) is included in \( S_{p(\sigma)} = \{a \in S | \sum_{i=1}^n b_i a^{n-i} = 0 \text{ for some } n > 0\} \). Hence we have that \( R_{p(\sigma)} \subseteq S_P^+ \), and the natural homomorphism \( G(P(\sigma)) = R_{p(\sigma)}/P(\sigma)^+ \to G(P) = S_P^+/P^+; [a] \mapsto [a] \) defines a partial homomorphism \( \xi_2: F'' \to F(P) \) such that \( (I_R, \xi_2): \sigma \to \tau \) is a morphism in \( C_{\text{sig}} \). Thus, we obtain a signature \( \tau: S \to F' = F(P) \) of ring \( S \) and a morphism \( (I_R, \xi_2 \circ \xi_1) = (I_R, \xi_2) \circ (I_R, \xi_1): \sigma \to \tau \) in \( C_{\text{sig}} \).
ideal $p_0(\sigma)$, that is, every element in $R_\sigma \setminus p_0(\sigma)$ is invertible in $R_\sigma$. Then, $a \in p_0(\sigma)$ if and only if $a^{-1} \in R_\sigma$.

Proof. 1) For elements $x, y \in R$, we suppose that $xR_\sigma y \subset p_0(\sigma)$ and $x \in p_0(\sigma)$. If $x \in R_\sigma$, then there is an $x' \in p_0(\sigma)$ with $x'x \in p_1(\sigma)$ or $xx' \in p_1(\sigma)$. Since both $x'xR_\sigma y$ and $xx'R_\sigma y$ are included in $p_0(\sigma)$, we may assume that $x \in R_\sigma$, and similarly $y \in R_\sigma$. Then, $y \in p_0(\sigma)$ follows. 2) Suppose that $a \in R_\sigma \setminus p_0(\sigma)$. If $a^{-1} \in R_\sigma$, then there is a $b \in p_0(\sigma)$ with $a^{-1}b \in p_1(\sigma)$ or $ba^{-1} \in p_1(\sigma)$, so it means either $a(a^{-1}b)$ or $(ba^{-1})a$ belongs to $p_0(\sigma)$, that is, $a \in p_0(\sigma)$, which is contrary to $a \notin p_0(\sigma)$. Hence, we get $a^{-1} \in R_\sigma \setminus p_0(\sigma)$. 3) First, we suppose that $R$ is commutative. It is easy to see the “only if” part. If $a^{-1} \notin R_\sigma$, there is a $b \in p_0(\sigma)$ with $a^{-1}b \in p_1(\sigma)$ or $ba^{-1} \in p_1(\sigma)$, so it follows that $a(a^{-1}b)$ or $(ba^{-1})a$ belongs to $p_0(\sigma)$, that is, $a \in p_0(\sigma)$, which is contrary to $a \notin p_0(\sigma)$. Hence, we get $a^{-1} \in R_\sigma \setminus p_0(\sigma)$.

Lemma 4.2. For a $\sigma \in X(R, F)$, put $q(\sigma) = \{a \in R \mid RaR \subset p_0(\sigma)\}$. Then, the following properties hold:

1) $q(\sigma)$ is a prime ideal of $R$, and $q(\sigma) \subset p_0(\sigma)$.
2) If $R$ is a local ring with maximal ideal $q(\sigma)$ then so is $R_\sigma$ with maximal ideal $p_0(\sigma)$. If $R$ is commutative, then the converse also holds.
3) If $p_0(\sigma) = \{0\}$, then $R = R_\sigma$, and $\mathcal{P}(\sigma)$ gives a partial ordering on the ring $R$.

Proof. 1) It is easy to see that $q(\sigma)$ is an ideal of $R$, and $q(\sigma) \subset p_0(\sigma)$. For $x, y \in R$, we suppose that $xR_\sigma y \subset q(\sigma)$ and $x \in q(\sigma)$. We can find elements $a$ and $b$ in $R$ with $ab \in p_0(\sigma)$, so it follows that $axbR_\sigma (RyR) \subset p_0(\sigma)$ and $RyR \subset p_0(\sigma)$ by Lemma 4.1, 1), i.e. $y \in q(\sigma)$. 2) If $R$ is a local ring with maximal ideal $q(\sigma)$, then every element in $R_\sigma \setminus p_0(\sigma)$ is invertible in $R$, and by Lemma 4.1, 2), so is also in $R_\sigma$. Hence, $R_\sigma$ is a local ring with maximal ideal $p_0(\sigma)$. If $R$ is commutative and $R_\sigma$ a local ring with maximal ideal $p_0(\sigma)$, then for any element $x \in R \setminus q(\sigma)$, we can find an element $a \in R$ such that $ax \in R_\sigma \setminus p_0(\sigma)$, that is, $ax$ is invertible in $R_\sigma$, so $x$ is invertible in $R$. 3) is easy.

Corollary 4.3. Assume that $R$ is a division ring, then the following hold.

1) For any $\sigma \in X(R, F)$, $R_\sigma$ is a local ring with maximal ideal $p_0(\sigma)$.
2) $X(R, F)$ is a Hausdorff and totally disconnected space.
3) If $F$ is a finite set, then $X(R, F)$ is compact, that is, a Boolean space.

Proof. 1) is obtained by Lemma 4.2, 2). 2) By Lemma 4.1, 3), it follows that $H_0(a) = H_\gamma(a^{-1})$ is a clopen set of $X(R, F)$ for any $a \neq 0$ in $R$, and so is also $H_\gamma(a)$ for any $\gamma \in F \cup \{\infty\}$ and $a \in R$. Hence, $X(R, F)$ is Hausdorff and totally disconnected. 3) Suppose that $F$ is finite, then $(F \cup \{\infty\})^\kappa$ is compact. Whenever $F \cup \{\infty\}$ is a discrete space, the subset $X(R, F)$ becomes a closed subset of $(F \cup \{\infty\})^\kappa$. Hence, under our topology on $F \cup \{\infty\}$, $X(R, F)$ is also
SIGNATURES ON RINGS

Remark 3.3. Let \( \sigma: R \to F \) and \( \tau: S \to F' \) be signatures of rings \( R \) and \( S \). If \( (f, \xi): \sigma \to \tau \) is a morphism in \( C_{\text{sig}} \), then the following identities hold; 1) \( R_{\sigma} = f^{-1}(S_{\tau}) \), 2) if \( G(\sigma) \) is a group, then \( p_{\sigma}(\alpha) = f^{-1}(p_{\tau}(\alpha)) \) and \( \bigcup \ p_{\sigma}(\alpha) = f^{-1}(\bigcup \ p_{\tau}(\alpha)) \) for each \( \beta \in G(\tau) \).

Proof. 1) It is easy that \( R_{\sigma} \subseteq f^{-1}(S_{\tau}) \). To prove the opposite, we suppose that there is an \( x \in R \setminus R_{\sigma} \) such that \( xy \in p_{\sigma}(\sigma) \). However, \( xy \in p_{\sigma}(\sigma) \) (resp. \( xy \in p_{\sigma}(\sigma) \)) implies \( \tau(f(xy)) = \xi(\tau(xy)) = 1 \) (resp. \( \tau(f(xy)) = \xi(\tau(xy)) = 0 \)) which is contrary to that \( \tau(f(xy)) = \xi(\tau(xy)) = 1 \) (resp. \( \tau(f(xy)) = 0 \)). Hence, we get \( R_{\sigma} = f^{-1}(S_{\tau}) \). 2) It is also easy that \( p_{\sigma}(\sigma) \subseteq f^{-1}(p_{\tau}(\alpha)) \). If \( x \in f^{-1}(p_{\sigma}(\alpha)) \), then we have \( \xi(\sigma(x)) = \tau(f(x)) = 0 \) and \( \sigma(x) = 0 \), i.e. \( x \in p_{\sigma}(\sigma) \), since \( G(\sigma) \) is a group and \( \xi(1) = 1 \). Hence, we get \( p_{\sigma}(\sigma) = f^{-1}(p_{\tau}(\alpha)) \). Since \( R_{\sigma} = f^{-1}(S_{\tau}) \) and \( p_{\sigma}(\sigma) = f^{-1}(p_{\tau}(\alpha)) \), it follows that \( R_{\sigma} \setminus p_{\sigma}(\sigma) = \bigcup \ p_{\sigma}(\sigma) = f^{-1}(S_{\tau} \setminus p_{\tau}(\alpha)) = \bigcup \ f^{-1}(p_{\tau}(\alpha)) \). Since \( \bigcup \ p_{\sigma}(\sigma) \subseteq f^{-1}(p_{\tau}(\alpha)) \) holds for every \( \beta \in G(\tau) \), we get \( \bigcup \ p_{\sigma}(\sigma) = f^{-1}(p_{\tau}(\alpha)) \) for every \( \beta \in G(\tau) \).

4. Space of signatures

In this section, we assume that \( F \) is a f-semigroup with abelian torsion group \( F^{*} \). Let \( R \) be any ring with identity \( 1 \), and \( X(R, F) \) denote the set of signatures \( \sigma: R \to F \) of the ring \( R \) over the f-semigroup \( F \). We consider a set \( F \cup \{ \infty \} \), which is added a formal symbol \( \infty \) to \( F \). We make the set \( F \cup \{ \infty \} \) a topological space such that \( \{ \alpha \} \) and \( \{ \infty \} \) are open subsets for every \( \alpha \in F^{*} \). Then, for any subset \( H \subseteq F \cup \{ \infty \} \), \( H \) is a closed subset if and only if \( 0 \in H \). Considering \( R \) as a discrete space, we make the power space \( (F \cup \{ \infty \})^{\alpha} \) have a weak topology. We can introduce a topology on \( X(R, F) \) as a subspace of \( (F \cup \{ \infty \})^{\alpha} \). For any \( \alpha \in F \) and \( a \in R \), we put \( H_{a}(a) = \{ \sigma \in X(R, F) | \sigma(a) = \alpha \} \) and \( H_{a}(a) = \{ \sigma \in X(R, F) | a \in R_{\sigma} \} \). Then, for every finite subsets \( \{ a_{1}, a_{2}, \ldots, a_{n} \} \subseteq R \) and \( \{ \gamma_{1}, \gamma_{2}, \ldots, \gamma_{n} \} \subseteq F^{*} \cup \{ \infty \} \), the intersections \( H_{a}(a_{1}) \cap H_{a}(a_{2}) \cap \cdots \cap H_{a}(a_{n}) \) construct an open basis of the space \( X(R, F) \). Furthermore, for a subset \( H \subseteq F \cup \{ \infty \} \) and \( a \in R \), we have that \( \bigcup \ H_{a}(a) \) is a closed subset of \( X(R, F) \) if and only if \( 0 \in H \).

In the following lemma and corollary, we need not assume that \( F^{*} \) is a torsion group.

Lemma 4.1. For a \( \sigma \in X(R, F) \) and an invertible element \( a \) in \( R \), the following statements hold; 1) For any \( x, y \in R \), \( xR_{\sigma}y \subseteq p_{\sigma}(\alpha) \) implies either \( x \in p_{\sigma}(\alpha) \) or \( y \in p_{\sigma}(\alpha) \). 2) \( a \in R_{\sigma} \setminus p_{\sigma}(\alpha) \) if and only if \( a^{-1} \in R_{\sigma} \setminus p_{\sigma}(\alpha) \). 3) Assume that either \( R \) is commutative or \( R_{\sigma} \) is a local ring with maximal
Proposition 4.4. Assume that $R$ is a commutative ring and $\sigma, \tau \in X(R, F)$. If $P(\sigma) \subset P(\tau)$ holds, then there is a subgroup $H$ of $G(\sigma)$ and a homomorphism $\psi: H \to G(\tau)$ such that $p_\beta(\tau) \cap R_\sigma \subset \bigcup_{\alpha \in \Phi^{-1}(\beta)} p_\alpha(\sigma) \cup p_\beta(\tau)$ holds for every $\beta \in G(\tau)$, and $R_\sigma \subset R$, holds.

Proof. Suppose that $P(\sigma) \subset P(\tau)$. Since $G(\sigma)$ and $G(\tau)$ are torsion groups, by Lemma 2.3, we get $R^c_{\sigma} = R_{\tau}$. We put $H = \{a^G(\sigma) | \tau(a) = \tau(x)\}$, then $H$ is a subgroup of $G(\tau)$. We can define a homomorphism $\psi: H \to G(\tau)$ as follows; for any $a^H$, we can find an element $a$ in $p_\sigma(\sigma) \backslash p_\beta(\tau)$, which satisfies $\sigma(ab) = \sigma(xb) = 1$ for every $x \in p_\sigma(\sigma) \backslash p_\beta(\tau)$. The condition $P(\sigma) \subset P(\tau)$ means that for every $x \in p_\sigma(\sigma) \backslash p_\beta(\tau)$, $\tau(ab) = \tau(xb) = 1$ holds, so $\tau(a) = \tau(x)$. Therefore, we can define the image $\psi(a)$ of $a$ as $\tau(a)$ for $a^H$. Then, it is easy to see that the map $\psi: H \to G(\tau)$ is a group homomorphism. Further, for any $\alpha \in H$ and $\beta \in G(\tau)$ with $\psi(\alpha) = \beta$, from the definition of $\psi$, $p_\alpha(\sigma) \subset p_\beta(\tau) \cup p_\beta(\tau)$ follows. Hence, we get $\bigcup_{\alpha \in \Phi^{-1}(\beta)} p_\alpha(\sigma) \subset p_\beta(\tau) \cup p_\beta(\tau)$.

Remark 4.5. Let $R$ be a commutative ring, and $\sigma: R \to F$ a signature of $R$. By $\sigma$, a topology on affine $n$-space $R^n$ is introduced as follows; for any $\gamma_i \in G(\sigma) \cup \{\infty\}$ and $f_i(X_1, X_2, \ldots, X_n)$ in polynomial ring $R[X_1, X_2, \ldots, X_n]$, $i = 1, 2, \ldots, m$, we put $U(f_1, f_2, \ldots, f_m, \gamma_1, \gamma_2, \ldots, \gamma_m) = \{\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}^n | \sigma(f_i(\alpha_1, \alpha_2, \ldots, \alpha_n)) = \gamma_i, i = 1, 2, \ldots, n\}$, where $\sigma(f_i(\alpha_1, \alpha_2, \ldots, \alpha_n)) = \infty$ whenever $f_i(\alpha_1, \alpha_2, \ldots, \alpha_n) \notin R_\sigma$. Then, the sets $U(f_1, f_2, \ldots, f_m, \gamma_1, \gamma_2, \ldots, \gamma_m)$ form an open basis on $R^n$. We can define a continuous map $\psi_\sigma$ of the topologic space $R^n$ into $X(R[X_1, X_2, \ldots, X_n], F)$; Let $(a_1, a_2, \ldots, a_n)$ be any element in $R^n$, and let $\psi(a_1, a_2, \ldots, a_n): R[X_1, X_2, \ldots, X_n] \to R; (X_1, X_2, \ldots, X_n) \mapsto f(a_1, a_2, \ldots, a_n)$ a natural ring homomorphism. By Proposition 3.1, 1), there exists a signature $\sigma(a_1, a_2, \ldots, a_n): R[X_1, X_2, \ldots, X_n] \to F$ with a morphism $(\psi(a_1, a_2, \ldots, a_n), \sigma(a_1, a_2, \ldots, a_n)) \to \sigma$ in $C_{\text{sis}}$. Thus, we get a map $\psi_\sigma: R^n \to X(R[X_1, X_2, \ldots, X_n], F)$; $(a_1, a_2, \ldots, a_n) \mapsto \sigma(a_1, a_2, \ldots, a_n)$, which is continuous, because of $\psi_\sigma^{-1}(H_\gamma(f)) = U(f, \gamma)$ for $f \in R[X_1, X_2, \ldots, X_n]$ and $\gamma \in G(\sigma) \cup \{\infty\}$.

References


Osaka Women’s University
Daisen-cho, 2–1
Sakai, Osaka 590
Japan