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Kanzaki, T.  
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## NOTES ON SIGNATURES ON RINGS

Dedicated to Professor Hirosi Nagao on his 60th birthday

TERUO KANZAKI

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### 0. Introduction

The notion of infinite prime introduced by Harrison [3] was investigated in [1], [2], [7] and [9] which were concerned with ordering on a field. In this note, we study about signatures on rings as some generalization of infinite primes and signatures of fields in [2]. In the section 1, we introduce notions of  $U$ -prime and signature of a ring which are generalizations of infinite prime and signature of field. In the section 2, we show that a  $U$ -prime of a commutative ring defines a signature on the ring. In the sections 3 and 4, we consider the category of signatures and a space of signatures on a ring which include notions of extension of signature and space of ordering on fields (cf. [2] and [8]), and investigate them. Throughout this paper, we assume that every ring has identity 1.

### 1. Preliminaries, definitions and notations

Let  $S$  be a multiplicative semigroup, and  $T$  a normal subsemigroup of  $S$ , (cf. [6], p. 195), denoted by  $T \triangleleft S$ , that is,  $T$  is a subsemigroup of  $S$  which satisfies 1) for  $x, y \in S$ ,  $xy \in T$  implies  $yx \in T$ , 2) if there is an  $x \in T$  with  $xy \in T$ , then  $y \in T$ , and 3) for every  $x \in S$ , there exists an  $x' \in S$  with  $x'x \in T$ . We can define a binary relation  $\sim$  on  $S$ ; for  $x, y \in S$ ,  $x \sim y$  if and only if there is a  $z \in S$  such that both  $zx$  and  $zy$  are contained in  $T$ . Then, the relation  $\sim$  is an equivalence relation on  $S$ , and is compatible with the multiplication of  $S$ , so the quotient set  $S/\sim$ , denoted by  $S/T$ , makes a group such that the canonical map  $\psi: S \rightarrow S/T; x \mapsto [x]$  is a homomorphism with  $\text{Ker } \psi = T$ .

Let  $R$  be any ring with identity 1, and  $P$  a preprime of  $R$  ([3]), that is,  $P$  is closed under addition and multiplication of  $R$  and  $-1 \notin P$ . We put  $p(P) = P \cap -P$ ,  $R_p = \{x \in R \mid xp(P) \cup p(P)x \subset p(P)\}$ ,  $R_p^+ = R_p \setminus p(P)$  ( $:= \{x \in R_p \mid x \notin p(P)\}$ ),  $P^+ = P \setminus p(P)$  ( $= P \setminus -P$ ). We shall say a preprime  $P$  to be *complete quasi-prime*, if it satisfies the following conditions;

- 1)  $p(P)$  is an ideal of  $R_p$  such that  $R_p/p(P)$  is an integral domain,
- 2)  $P^+ \triangleleft R_p^+$  under the multiplication of  $R_p$ .

3)  $P$  is complete in  $R_P$ , that is, for  $x \in R_P$ ,  $x^2 \in P$  implies  $x \in P \cup -P$ .

A multiplicative semigroup  $F$  with unit element 1 and zero element 0 will be called a *f-semigroup*, if  $F^* = F \setminus \{0\}$  makes a group with a unique element of order 2, denoted by  $-1$ , under the multiplication of  $F$ . If  $P$  is a complete quasi-prime of  $R$ , then the quotient group  $G(P) = R_P^+ / P^+$  has a unique element  $[-1]$  of order 2, and the formally composed semigroup  $F(P) = G(P) \cup \{0\}$  makes an f-semigroup under the multiplication of  $G(P)$  and  $\alpha 0 = 0\alpha = 00 = 0$  for  $\alpha \in G(P)$ . Furthermore, we can define a map  $\sigma: R_P \rightarrow F(P)$  by  $\sigma(a) = 0$  or  $[a]$  for  $a \in p(P)$  or  $a \in R_P^+$ , respectively. Then, it can be verified that 1)  $\sigma(-1) = [-1]$ , 2)  $\sigma(ab) = \sigma(a)\sigma(b)$  for every  $a, b \in R_P$ , and 3) for  $a, b \in R_P$ , either  $\sigma(a) = 0$  or  $\sigma(a) = \sigma(b)$  implies  $\sigma(a+b) = \sigma(b)$ .

Let  $\pi$  be a set of prime numbers, and suppose  $2 \in \pi$ . A complete quasi-prime  $P$  will be called a  $\pi$ -complete quasi-prime, if for each  $q \in \pi$ , there is a  $\zeta_q \in R_P \setminus P$  such that  $\zeta_q^q \in P$  and for any  $x \in R_P$  with  $x^q \in P$ ,  $yx \in \bigcup_{1 \leq i \leq q} \zeta_q P^i$  for some  $y \in P^+$ .

**REMARK 1.1.** If  $R$  is a commutative ring and  $P$  is a  $\pi$ -complete quasi-prime, then for each  $q \in \pi$ , the  $q$ -torsion subgroup  $G(P)_q = \{\alpha \in G(P) \mid \exists n > 0; \alpha^{q^n} = [1]\}$  of  $G(P)$  is isomorphic to a subgroup of  $\mathbb{Z}(q^\infty)$ . Because, since  $G(P)_q$  has a unique minimal non trivial subgroup  $\langle [\zeta_q] \rangle$ ,  $G(P)_q$  is indecomposable, so by [4], p. 22, Theorem 10,  $G(P)_q$  is isomorphic to  $\mathbb{Z}(q^n)$  or  $\mathbb{Z}(q^\infty)$ .

Let  $R$  be a ring with identity 1, and  $F$  an abelian f-semigroup. A partial map  $\sigma: R \rightarrow F$  will be called a *signature* of  $R$  with domain of definition  $R_\sigma$ , if  $\sigma$  is a map of a subset  $R_\sigma$  of  $R$  into  $F$  satisfying the following conditions;

- (S 1)  $-1 \in R_\sigma$  and  $\sigma(-1) = -1$ ,
- (S 2)  $a, b \in R_\sigma$  implies  $ab \in R_\sigma$  and  $\sigma(ab) = \sigma(a)\sigma(b)$ ,
- (S 3) for  $a, b \in R_\sigma$ , if  $\sigma(a) = 0$  or  $\sigma(a) = \sigma(b)$  then  $a+b \in R_\sigma$  and  $\sigma(a+b) = \sigma(b)$ ,
- (S 4) for  $a \in R$ , if  $a \notin R_\sigma$ , then there exists a  $b \in R_\sigma$  such that  $\sigma(b) = 0$  and either  $\sigma(ab) = 1$  or  $\sigma(ba) = 1$ .

Let  $\sigma: R \rightarrow F$  be a signature. For  $\alpha \in F$ , we put  $p_\alpha(\sigma) = \{x \in R_\sigma \mid \sigma(x) = \alpha\}$ ,  $P(\sigma) = p_0(\sigma) \cup p_1(\sigma)$  and  $G(\sigma) = \text{Im } \sigma \cap F^*$ .

**Lemma 1.2.** Let  $\sigma: R \rightarrow F$  be a signature of a ring  $R$ .

- 1)  $R_\sigma$  is a subring of  $R$  with prime ideal  $p_0(\sigma)$  such that  $R_\sigma/p_0(\sigma)$  is an integral domain.
- 2)  $P(\sigma)$  is a preprime of  $R$ , and  $R_\sigma = R_{P(\sigma)}$ .
- 3) If  $G(\sigma)$  is a subgroup of  $F^*$ , then  $P(\sigma)$  is a complete quasi-prime of  $R$ , and  $G(P(\sigma))$  and  $G(\sigma)$  are group isomorphic.

**Proof.** 1) If  $R_\sigma$  is closed under the addition of  $R$ , then it is easy to see

that  $R_\sigma$  is a subring of  $R$ . Suppose  $a+b \notin R_\sigma$  for some  $a$  and  $b$  in  $R_\sigma$ . There is a  $c \in R_\sigma$  such that  $\sigma(c) = 0$ , and  $\sigma(c(a+b)) = 1$  or  $\sigma((a+b)c) = 1$ . Since  $\sigma(ca) = \sigma(ac) = \sigma(a)\sigma(c) = 0$  and  $\sigma(cb) = \sigma(bc) = 0$ , we get  $\sigma(ca+cb) = \sigma(ac+bc) = 0$  which is a contradiction. Hence, we get  $R_\sigma + R_\sigma \subset R_\sigma$ . It is easy to see that  $p_0(\sigma)$  is an ideal of  $R_\sigma$ , and  $R_\sigma/p_0(\sigma)$  is an integral domain. 2) From the definition of signature, it follows that  $P(\sigma)$  is a preprime of  $R$  and  $p_0(\sigma) = P(\sigma) \cap -P(\sigma)$ . We shall show  $R_\sigma = R_{P(\sigma)}$ . Since  $R_\sigma \subset R_{P(\sigma)}$  is clear, it suffices to show  $R_\sigma \supset R_{P(\sigma)}$ . If  $x \in R \setminus R_\sigma$ , then there is a  $y \in p_0(\sigma)$  with  $xy \in p_1(\sigma)$  or  $yx \in p_1(\sigma)$ , so  $xp_0(\sigma) \cup p_0(\sigma)x \subset p_0(\sigma)$ , that is,  $x \notin R_{P(\sigma)}$ . 3) If  $G(\sigma)$  is a group, then it is easy to see that  $P(\sigma)^+ = p_1(\sigma)$ ,  $P(\sigma)^+ \triangleleft R_{P(\sigma)}^+$ ,  $\sigma(R_{P(\sigma)}^+) = G(\sigma)$ , and  $P(\sigma)$  is complete. Furthermore, a map  $G(P(\sigma)) = R_{P(\sigma)}^+/P(\sigma)^+ \rightarrow G(\sigma)$ ;  $[x] \mapsto \sigma(x)$  is a group isomorphism.

**REMARK.** 1) If  $R$  is a field, then a signature  $\sigma: R \rightarrow F$  with  $p_0(\sigma) = \{0\}$  and  $F = \mu \cup \{0\}$  coincides with the notion of signature defined by Becker, Harman and Rosenberg [2], where  $\mu$  is the group of all roots of unity in the complices. 2) Let  $F$  be a finite field with characteristic  $\neq 2$ . The multiplicative semigroup  $F$  is an abelian f-semigroup. For a signature  $\sigma: R \rightarrow F$ , let  $\pi$  be the set of all prime factors of order  $|G(\sigma)|$ . Then, it is easy to see that  $P(\sigma)$  is a  $\pi$ -complete quasiprime of  $R$ .

Let  $R$  be a ring with identity 1, and  $U$  a non empty multiplicatively closed subset of  $R$  satisfying  $U \cap -U = \phi$ . A preprime  $P$  of  $R$  will be called a  $U$ -preprime of  $R$ , if  $U \subset P$  and  $P \cap -U = \phi$ . A maximal  $U$ -preprime of  $R$  will be called a  $U$ -prime of  $R$ . Any Harrison's infinite prime is a  $\{1\}$ -prime.

**Lemma 1.3.** *Let  $U$  a non empty multiplicatively closed subset of  $R$  with  $U \cap -U = \phi$ , and  $P$  a  $U$ -prime of  $R$ . If either  $R$  is commutative or  $Px = xP$  and  $Ux = xU$  hold for every  $x \in R_P^+$ , then  $P$  is a complete quasi-prime of  $R$ .*

The proof of this lemma is obtained by checking the following facts;

(1.3.1)  $U + P \subset P^+$ .

(1.3.2) For  $x \in R_P$  ( $x \in R$ , if  $R$  is commutative), if there are  $u \in U$  and  $y \in P$  with  $(u+y)x \in P$ , then  $x \in P$ . Hence  $1 \in P$ .

(1.3.3) For  $x \in R_P$  ( $x \in R$ , if  $R$  is commutative), if  $x \notin p(P)$ , then there is an  $x' \in (\pm P)[x]$  with  $x'x \in U + P$ , where  $(\pm P)[x] = \{\sum_i a_i x^i \in R \mid a_i \in P \cup -P\}$ .

(1.3.4)  $R_P/p(P)$  is an integral domain.

(1.3.5) For  $x, y \in R_P$ ,  $xy \in P^+$  implies  $yx \in P^+$ .

(1.3.6)  $P$  is complete in  $R_P$ .

(1.3.7) For any  $x \in P^+$ , there is an  $x' \in P^+$  with  $x'x \in U + P$ .

(1.3.8) For  $x \in R_P$  ( $x \in R$ , if  $R$  is commutative), if there is a  $y \in P^+$  with  $yx \in P^+$ , then  $x \in P^+$ .

The proofs of these statements are obtained similarly to the case of Harrison's

infinite prime; (1.3.1): Since  $U \cap -P = \phi$ , it follows that  $U \subset P^+$  and  $U + P \subset P^+$ . (1.3.2): A subset  $P' = \{x \in R_P \mid {}^x u \in U, {}^x y \in P; (u+y)x \in P\}$  of  $R$  is closed under addition and multiplication. Because, if  $x_1, x_2 \in P'$ , there are  $u_i \in U$  and  $y_i \in P$  with  $(u_i+y_i)x_i \in P$ ,  $i = 1, 2$ . If either  $x_1$  or  $x_2$  belongs to  $p(P)$ , then it is trivial that  $x_1 + x_2$  and  $x_1 x_2$  belong to  $P'$ . Otherwise, by assumption, there are  $u'_2 \in U$  and  $y'_2 \in P$  such that  $x_1 u_2 = u'_2 x_1$  and  $x_1 y_2 = y'_2 x_1$ . Then  $(u_1+y_1)(u_2+y_2)$  and  $(u_1+y_1)(u'_2+y'_2)$  belong to  $U+P$ , and  $(u_1+y_1)(u_2+y_2)(x_1+x_2)$  and  $(u_1+y_1)(u'_2+y'_2)x_1 x_2$  are in  $P$ . Furthermore, it is immeadiately seen that  $P \subset P'$  and  $P' \cap -U = \phi$ , so we get  $P = P'$ . (1.3.3): For  $x \in R_P$ , if  $x \notin p(P)$ , then either  $x \in P$  or  $-x \in P$ . By assumption, a subset  $P[x] = P + Px + Px^2 + \dots$ , (resp.  $P[-x] = P + P(-x) + P(-x)^2 + \dots$ ) of  $R$  is closed under addition and multiplication. Since  $P \subseteq P[x]$  or  $P \subseteq P[-x]$ , we get  $P[x] \cap -U \neq \phi$  or  $P[-x] \cap -U \neq \phi$ , so we can find an element  $y \in (\pm P)[x]$  such that  $yx \in U+P$  holds. (1.3.4): For  $x, y \in R_P$ , suppose that  $xy \in p(P)$  and  $x \notin p(P)$ . By (1.3.3), there is an  $x' \in (\pm P)[x] \subset R_P$  with  $x'x \in U+P$ , and (1.3.2) derives that  $x'xy \in p(P)$  implies  $y \in p(P)$ . (1.3.5): For  $x, y \in R_P$ , suppose  $xy \in P^+$ .  $(xy)x$  is in  $Px = xP$ , and for an element  $x'$  in  $(\pm P)[x]$ , also in  $R_P$ , with  $x'x \in U+P$ , we get  $(x'x)yx \in x'xP \subset P$ , so  $yx \in P^+$  by (1.3.2) and (1.3.4). (1.3.6) is easy. (1.3.7): If  $x \in P^+$ , then  $P[-x] = P - Px$  is closed under addition and multiplication, and  $P \subseteq P[-x]$ . Hence, there are  $u \in U$  and  $x', y \in P$  with  $-u = y - x'x$ , so we get  $x'x = u + y \in U+P$  and  $x' \in P^+$ . (1.3.8) is immeadiately obtained from (1.3.2) and (1.3.7).

## 2. The connection between $U$ -prime and signature

**Theorem 2.1.** *Let  $R$  be a commutative ring with identity 1, and  $U$  any non empty multiplicatively closed subset of  $R$  with  $U \cap -U = \phi$ . If  $P$  is a  $U$ -prime of  $R$ , then there exists a signature  $\sigma: R \rightarrow F$  with  $P(\sigma) = P$  and group  $G(\sigma) = G(P)$ .*

**Proof.** By Lemma 1.3,  $U$ -prime  $P$  is a complete quasi-prime of  $R$ , so it defines a map  $\sigma: R_P \rightarrow F(P)$ . Then, we put  $R_\sigma = R_P$  and  $F = F(P)$ . The conditions (S 1), (S 2) and (S 3) of signature were verified. (S 4) is proved in the following proposition. Then we have a signature  $\sigma: R \rightarrow F$  with  $P = P(\sigma)$  and  $G(\sigma) = G(P) = R_P^+ / P^+$ .

**Proposition 2.2.** *Let  $P$  be a  $U$ -prime of a commutative ring  $R$ , and let  $A_P = \{a \in R \mid {}^a b_0 \in U+P, {}^a b_i \in P \cup -P, i = 1, 2, \dots, n; \sum_{i=0}^n b_i a^{n-i} = 0\}$ .*

- 1)  $(R_P, p(P))$  is a valuation pair of  $R$ , (cf. [3], Proposition. 2.5).
- 2) If  $x \in R \setminus p(P)$  then there is an  $a \in A_P$  with  $ax \in U+P$ .
- 3) If  $x$  and  $y$  are elements of  $R$  with  $xy \in U+P$ , then  $x \notin p(P)$  implies  $y \in A_P$ .
- 4)  $R_P = A_P$ .

**Proof.** The proof of 1) is quite similar to [3], Proposition 2.5. 2) If  $x \in R \setminus p(P)$ , by (1.3.3) there is an  $a \in (\pm P)[x]$  with  $ax \in U+P$ , then  $a$  can be

represented as  $-(b_1 + b_2x + \dots + b_nx^{n-1})$  for some  $b_i \in P \cup -P$ . If we put  $ax = b_0$ , then  $a$  satisfies an equation  $b_0a^n + b_1b_0a^{n-1} + \dots + b_nb_0^n = 0$  with  $b_0 \in U + P$  and  $b_i b_0^i \in P \cup -P$ ,  $i = 1, 2, \dots, n$ , so  $a \in A_P$ . 3) Suppose that  $x$  and  $y$  are in  $R$  and  $xy \in U + P$ . If  $x \notin p(P)$ , by 2), there is a  $z \in A_P$  with  $zx \in U + P$ . Since  $z \in A_P$ , there are  $a_0 \in U + P$  and  $a_i \in P \cup -P$ ,  $i = 1, 2, \dots, m$ , with  $\sum_{i=0}^m a_i z^{m-i} = 0$ . Put  $xy = b_0$  and  $zx = c_0$ , so we get that  $\sum_{i=0}^m (a_i c_0^{m-i} b_0^i) y^{m-i} = (\sum_{i=0}^m a_i z^{m-i}) b_0^m = 0$ ,  $a_i c_0^{m-i} b_0^i \in U + P$  and  $a_i c_0^{m-i} b_0^i \in P \cup -P$ , hence  $y \in A_P$ . 4) In the first place, we show  $A_P \supset R_P$ : Let  $x$  be any element in  $R_P$ . If  $x \in p(P)$ ,  $x \in A_P$  is obvious. Otherwise, by (1.3.3) there is a  $y \in (\pm P)[x]$  with  $xy \in U + P$ , so  $y \notin p(P)$  and by 3) we get  $x \in A_P$ . Now, we show  $A_P = R_P$ : Let  $(U + P)^{-1}R$  be the ring of quotients of  $R$  with respect to  $U + P$ , and  $\psi: R \rightarrow (U + P)^{-1}R$  the canonical ring homomorphism. Then,  $(U + P)^{-1}R_P$  may be regarded as a subring of  $(U + P)^{-1}R$ . By  $B'$ , we denote the integral closure of  $(U + P)^{-1}R_P$  in  $(U + P)^{-1}R$ . There is a prime ideal  $Q'$  of  $B'$  which lies over  $(U + P)^{-1}R_P p(P)$ , (cf. [5], (10.8)). It follows that  $B = \psi^{-1}(B')$  is a subring of  $R$  with  $B \supset A_P \supset R_P$ , and  $Q = \psi^{-1}(Q')$  is a prime ideal of  $B$  with  $Q \cap R_P = p(P)$ . By 1), we get  $B = A_P = R_P$ .

**Lemma 2.3.** *Let  $R$  be a commutative ring, and  $\sigma: R \rightarrow F$  a signature. If  $G(\sigma)$  is a torsion group, then  $R_\sigma = \{a \in R \mid a^n \in P(\sigma) \text{ for some integer } n > 0\}$ .*

Proof. Since  $G(\sigma)$  is a torsion group, it is clear that any element  $a$  in  $R_\sigma$  has a positive integer  $n$  with  $a^n \in P(\sigma)$ . Conversely, suppose that an element  $a \in R$  does not belong to  $R_\sigma$ . There is a  $b \in p_0(\sigma)$  with  $ab \in p_1(\sigma)$ . Then  $a^n$  is not contained in  $P(\sigma)$  for every positive integer  $n$ . Because, if  $a^n \in P(\sigma)$  for some  $n > 0$ , it derives a contradiction  $1 = \sigma((ab)^n) = \sigma(a^n)\sigma(b^n) = 0$ .

Let  $R$  be a ring with identity 1. By [1], a preprime  $P$  is called a torsion preprime (resp. 2-torsion preprime) of  $R$ , if for each  $a \in R$  there exists a positive integer  $n$  such that  $a^n \in P$  (resp.  $a^{2^n} \in P$ ) holds. From Theorem 2.1 and Lemma 2.3, the following corollaries immediately follow;

**Corollary 2.4.** *Let  $R$  be a commutative ring with 1 and  $U$  a non empty multiplicatively closed subset of  $R$  with  $1 \in U$  and  $U \cap -U = \emptyset$ .*

1) *If  $P$  is a torsion  $U$ -prime of  $R$ , then  $p(P)$  is an ideal of  $R$ , i.e.  $R_P = R$ , so there is a signature  $\sigma: R \rightarrow F$  such that  $P = P(\sigma)$ ,  $R = R_\sigma$  and  $G(\sigma)$  is a torsion group.*

2) *If  $P$  is a 2-torsion  $U$ -prime of  $R$ , then there is a signature  $\sigma: R \rightarrow F$  such that  $P = P(\sigma)$ ,  $R = R_\sigma$  and  $F^* \cong \mathbb{Z}(2^\infty)$ .*

In particular, on a field, we have

**Corollary 2.5.** *Let  $K$  be a field.*

1) *For any signature  $\sigma: K \rightarrow F$ ,  $K_\sigma$  is a valuation ring of  $K$  with maximal ideal  $p_0(\sigma)$ , and the residue field  $k(\sigma) = K_\sigma/p_0(\sigma)$  has an induced signature  $\bar{\sigma}: k(\sigma)$*

$\rightarrow F$  with  $k(\sigma)_{\bar{\sigma}} = k(\sigma)$  and  $p_0(\bar{\sigma}) = \{\bar{0}\}$ , and  $P(\bar{\sigma})$  is a preordering on  $k(\sigma)$ .

2) Let  $U$  be a non empty multiplicatively closed subset of  $K$  with  $U \cap -U = \phi$ . If  $P$  is a  $U$ -prime of  $K$ ,  $K_P$  is a valuation ring of  $K$  with maximal ideal  $p(P)$ . If  $P$  is a torsion  $U$ -prime of  $K$ , then  $K = K_P$ ,  $p(P) = \{0\}$ , and  $P$  is a preordering, i.e.  $P^+ = P \setminus \{0\}$  is a subgroup of  $K^* = K \setminus \{0\}$ , (cf. [1], (3.3)).

3) If  $O$  is a real valuation ring of  $K$  with maximal ideal  $p$ , i.e. the residue field  $O/p$  is a formally real field, then there is a signature  $\sigma: K \rightarrow \text{GF}(3)$  with  $K_\sigma = O$  and  $p_0(\sigma) = p$ , where  $\text{GF}(3) = \{0, 1, -1\}$  is a multiplicative semigroup of prime field with characteristic 3.

**Theorem 2.6.** Let  $R$  be a ring with identity 1, and  $\sigma: R \rightarrow F$  a signature of  $R$ . Assume that  $G(\sigma)$  is a torsion group and  $xp_\alpha(\sigma) = p_\alpha(\sigma)x$  holds for all  $x \in R_\sigma \setminus p_0(\sigma)$  and  $\alpha \in G(\sigma) \cup \{0\}$ . Then, there exists a signature  $\tau: R \rightarrow F'$  of  $R$  satisfying the following conditions;

- 1)  $P(\tau)$  is a  $p_1(\sigma)$ -prime of  $R$  and  $P(\tau) \supset P(\sigma)$ ,
- 2)  $R_\tau = R_\sigma$  and  $p_0(\tau) = p_0(\sigma)$ ,
- 3) there is a subgroup  $H$  of  $G(\sigma)$  such that  $p_1(\tau) = \sigma^{-1}(H)$ ,  $-1 \notin H$  and  $G(\sigma)/H \cong G(\tau)$  hold.

Proof. Since  $P(\sigma)$  is a  $p_1(\sigma)$ -preprime of  $R$ , by Zorn's Lemma there exists a  $p_1(\sigma)$ -prime  $P$  of  $R$  containing  $P(\sigma)$ . From the facts that  $P \cap -p_1(\sigma) = \phi$  and  $p_0(\sigma) \subset p(P)$ , we can derive that  $p_0(\sigma) = p(P)$  and  $R_P = R_\sigma$ ; If there is an element  $x \in R_P \setminus R_\sigma$ , then there exists a  $y \in p_0(\sigma)$  such that either  $xy$  or  $yx$  belongs to  $p_1(\sigma)$ . However,  $xy$  and  $yx$  are also contained in  $p(P)$ , so these are contrary to  $p_1(\sigma) \cap p(P) = \phi$ . Hence, we get  $R_P \subset R_\sigma$ . Furthermore, if there is an element  $x \in p(P) \setminus p_0(\sigma)$ , we have  $x^n \in p_1(\sigma) \cap p(P)$  for some integer  $n > 0$ , which is a contradiction. Therefore, we get  $p_0(\sigma) = p(P)$  and  $R_P = R_\sigma$ . Now, we put  $H = \sigma(P^+)$ , so  $H$  is a subgroup of  $G(\sigma)$ . We shall show  $P^+ = \sigma^{-1}(H)$ ; If  $x \in \sigma^{-1}(H)$  then there is a  $y \in P^+$  with  $\sigma(x) = \sigma(y)$ . Since  $y^n \in p_1(\sigma)$  for some integer  $n > 0$ , we have  $xy^n = (xy^{n-1})y \in xp_1(\sigma) \cap P^+$ . Hence, for any  $x \in R$ , it follows that  $x \in \sigma^{-1}(H)$  if and only if  $xp_1(\sigma) \cap P^+ \neq \phi$ . On the other hand, we can show that  $P = \{x \in R_\sigma \mid xp_1(\sigma) \cap P \neq \phi\}$ ; The set  $P' = \{x \in R_\sigma \mid xp_1(\sigma) \cap P \neq \phi\}$  is closed under addition and multiplication: Because, for  $x, y \in P'$ , there are  $x_1, y_1 \in p_1(\sigma)$  such that both  $xx_1$  and  $yy_1$  are in  $P$ . Since we may suppose that  $y$  is not in  $p_0(\sigma)$ , there is an  $x'_1 \in p_1(\sigma)$  with  $x_1y = yx'_1$ , and it follows that both  $(x+y)(x_1y_1)$  and  $(xy)(x'_1y_1)$  are contained in  $P$ . Hence, both  $x+y$  and  $xy$  belong to  $P'$ . Furthermore, it is derived that  $P \subset P'$  and  $P' \cap -p_1(\sigma) = \phi$ , because of  $P \cap -p_1(\sigma) = \phi$ . Hence, we get  $P = P'$ . Accordingly, we conclude that  $\sigma^{-1}(H) = P^+ = \bigcup_{\alpha \in H} p_\alpha(\sigma)$ .

From the assumption  $xp_\alpha(\sigma) = p_\alpha(\sigma)x$  for  $x \in R_\sigma \setminus p_0(\sigma)$  and  $\alpha \in G(\sigma) \cup \{0\}$ ,  $P$  is a complete quasi-prime of  $R$ . Therefore, we can define a signature  $\tau: R \rightarrow F(P)$  such that  $R_\tau = R_P = R_\sigma$ ,  $p_0(\tau) = p(P) = p_0(\sigma)$  and  $G(\tau) = G(P) \cong G(\sigma)/H$ .

It is easy to check the conditions of signature for  $\tau$ .

**Corollary 2.7.** *Let  $R$  be a commutative ring with identity 1. If  $\sigma: R \rightarrow F$  is a signature of  $R$  such that  $G(\sigma)$  is a 2-torsion group, then  $P(\sigma)$  is a  $p_1(\sigma)$ -prime of  $R$ .*

Proof. Since  $G(\sigma)$  is a 2-torsion group, by Remark 1.1 every non-trivial subgroup  $H$  of  $G(\sigma)$  contains  $-1$ . By Theorem 1.7,  $P(\sigma)$  is a  $p_1(\sigma)$ -prime of  $R$ .

**Corollary 2.8.** *Let  $S$  be a commutative ring with identity 1, and  $R$  a subring of  $S$  containing 1. If  $\sigma: R \rightarrow F$  a signature of  $R$  such that  $G(\sigma)$  is 2-torsion group, then  $\sigma$  can be extended to a signature  $\tau: S \rightarrow F'$  of  $S$ , i.e.  $S_\tau \cap R = R_\sigma$  and  $P(\tau) \cap R = P(\sigma)$  hold.*

Proof. A signature  $\tau: S \rightarrow F'$  is defined by a  $p_1(\sigma)$ -prime  $P$  of  $S$  containing  $P(\sigma)$ . Then,  $\tau$  is an extension of  $\sigma$ .

### 3. Category of signatures

Let  $\sigma_1: R_1 \rightarrow F_1$  and  $\sigma_2: R_2 \rightarrow F_2$  be signatures of rings  $R_1$  and  $R_2$ . Suppose that  $f: R_1 \rightarrow R_2$  is a ring homomorphism such that  $f(1)=1$  and  $f(R_{1\sigma_1}) \subset R_{2\sigma_2}$ , and that  $\xi: F_1 \rightarrow F_2$  is a partial homomorphism which is defined on  $G(\sigma_1)$  and satisfies  $\xi(0)=0$ ,  $\xi(-1)=-1$  and  $\xi(\alpha\beta)=\xi(\alpha)\xi(\beta)$  if  $\xi$  is defined on  $\alpha, \beta$  and  $\alpha\beta$  for  $\alpha, \beta \in F_1$ . Then, the pair  $(f, \xi)$  will be called a morphism of signatures of  $\sigma_1$  to  $\sigma_2$ , denoted by  $(f, \xi): \sigma_1 \rightarrow \sigma_2$ , if it satisfies  $\xi(\sigma_1(x))=\sigma_2(f(x))$  for all  $x \in R_{1\sigma_1}$ . Let  $\sigma_i: R_i \rightarrow F_i$  and  $\sigma'_i: R'_i \rightarrow F'_i$  be signatures of rings for  $i=1, 2$ , and  $(f, \xi): \sigma_1 \rightarrow \sigma_2$  and  $(f', \xi'): \sigma'_1 \rightarrow \sigma'_2$  morphisms of signatures. We define the equality of morphisms that  $(f, \xi)=(f', \xi')$  if and only if  $\sigma_i=\sigma'_i$  (i.e.  $R_i=R'_i$ ,  $R_{i\sigma_i}=R'_{i\sigma'_i}$ ,  $F_i=F'_i$  and  $\sigma_i(x)=\sigma'_i(x)$  for all  $x \in R_{i\sigma_i}$ ) for  $i=1, 2$ ,  $f=f'$  and for every  $\alpha \in G(\sigma_1)=G(\sigma'_1)$ ,  $\xi(\alpha)=\xi'(\alpha)$  hold. By  $C_{\text{sig}}$ , we denote the category of signatures in which objects are signatures of rings and morphisms are morphisms of signatures.

**Proposition 3.1.** *Let  $R$  and  $S$  be rings with identity 1, and  $f: R \rightarrow S$  a ring homomorphism with  $f(1)=1$ .*

1) *If  $\tau: S \rightarrow F$  is a signature of ring  $S$  with  $\text{Im } f \supset p_0(\tau)$ , then there exists a signature  $\sigma: R \rightarrow F$  of ring  $R$  with a morphism  $(f, I_F): \sigma \rightarrow \tau$  in  $C_{\text{sig}}$ .*

2) *If  $f: R \rightarrow S$  is surjective, and if  $\sigma: R \rightarrow F$  is a signature of ring  $R$  with  $\text{Ker } f \subset p_0(\sigma)$ , then there exists a signature  $\tau: S \rightarrow F$  of ring  $S$  with a morphism  $(f, I_F): \sigma \rightarrow \tau$  in  $C_{\text{sig}}$ .*

Proof. 1) Suppose that  $\tau: S \rightarrow F$  is a signature of ring  $S$  and  $f: R \rightarrow S$  is a ring homomorphism with  $f(1)=1$  and  $\text{Im } f \supset p_0(\tau)$ . On a subring  $R_\sigma = \{x \in R \mid f(x) \in S_\tau\}$  of  $R$ , a map  $\sigma: R_\sigma \rightarrow F; x \mapsto \tau(f(x))$  is defined. The condition

$\text{Im } f \supset p_0(\tau)$  derives that a signature  $\sigma: R \rightarrow F$  of ring  $R$  and a morphism  $(f, I_F): \sigma \rightarrow \tau$  in  $\mathbf{C}_{\text{sig}}$  are defined. 2) Suppose that  $f: R \rightarrow S$  is a surjective ring homomorphism, and  $\sigma: R \rightarrow F$  is a signature of ring  $R$  with  $\text{Ker } f \subset p_0(\sigma)$ . For a subring  $S_\tau = f(R_\sigma)$ , we can define a map  $\tau: S_\tau \rightarrow F$  as follows: For any  $a \in S_\tau$ , there is a  $b \in R_\sigma$  with  $f(b) = a$ , then we put  $\tau(a) = \sigma(b)$ . From the condition  $\text{Ker } f \subset p_0(\sigma)$ , it is known that the map  $\tau: S_\tau \rightarrow F$  is well defined. Then, it is easy to see that a signature  $\tau: S \rightarrow F$  of ring  $S$  and a morphism  $(f, I_F): \sigma \rightarrow \tau$  in  $\mathbf{C}_{\text{sig}}$  are defined.

Concerning commutative rings, the situation of Proposition 3.1, 2) is reformed as follows;

**Theorem 3.2.** *Let  $f: R \rightarrow S$  be a ring homomorphism of a commutative ring  $R$  into a commutative ring  $S$  with  $f(1) = 1$ . If  $\sigma: R \rightarrow F$  is a signature of  $R$  such that  $G(\sigma)$  is a torsion group and  $\text{Ker } f \subset p_0(\sigma)$ , then there exists a signature  $\tau: S \rightarrow F'$  of ring  $S$  with a morphism  $(f, \xi): \sigma \rightarrow \tau$  in  $\mathbf{C}_{\text{sig}}$ .*

Proof. Suppose that  $f: R \rightarrow S$  is a ring homomorphism with  $f(1) = 1$ , and  $\sigma: R \rightarrow F$  is a signature of  $R$  with torsion group  $G(\sigma)$  and satisfying  $\text{Ker } f \subset p_0(\sigma)$ . By Proposition 3.1, 2), for the surjective ring homomorphism  $f: R \rightarrow \text{Im } f$ , there exists a signature  $\sigma': \text{Im } f \rightarrow F$  of the subring  $\text{Im } f$  of  $S$  with a morphism  $(f, I_P): \sigma \rightarrow \sigma'$  in  $\mathbf{C}_{\text{sig}}$ . Hence, we may assume that  $R$  is a subring of  $S$  with common identity, and it is sufficient to show that there exists a signature  $\tau: S \rightarrow F'$  of  $S$  with a morphism  $(\iota, \xi): \sigma \rightarrow \tau$  in  $\mathbf{C}_{\text{sig}}$ , where  $\iota$  denotes the inclusion map  $R \hookrightarrow S$ . By Theorem 2.6, there exists a signature  $\bar{\sigma}: R \rightarrow F''$  of  $R$  such that  $R_{\bar{\sigma}} = R_\sigma$ ,  $p_0(\bar{\sigma}) = p_0(\sigma)$  and  $G(\bar{\sigma}) \cong G(\sigma)/H$  for some subgroup  $H$  of  $G(\sigma)$  hold, and  $P(\bar{\sigma})$  is a  $p_1(\sigma)$ -prime of  $R$  containing  $P(\sigma)$ . Then, we can define a partial homomorphism  $\xi_1: F \rightarrow F''$  such that  $\xi_1$  induces a group homomorphism  $G(\sigma) \rightarrow G(\bar{\sigma})$  and the pair  $(I_R, \xi_1)$  defines a morphism  $(I_R, \xi_1): \sigma \rightarrow \bar{\sigma}$  in  $\mathbf{C}_{\text{sig}}$ . On the other hand, by Zorn's Lemma, there exists a  $p_1(\sigma)$ -prime  $P$  of  $S$  containing  $P(\bar{\sigma})$ , and by Theorem 2.1 the  $p_1(\sigma)$ -prime  $P$  defines a signature  $\tau: S \rightarrow F(P)$  of  $S$  such that  $P(\tau) = P$ ,  $S_\tau = S_P$ ,  $F(P) = G(P) \cup \{0\}$  and  $G(P) = S_P^+ / P^+$  hold, and  $\tau$  is induced from the canonical map  $S_P^+ \rightarrow G(P)$ . From the fact that  $P(\bar{\sigma})$  is a  $p_1(\sigma)$ -prime of  $R$ , and  $P \supset P(\bar{\sigma})$ , it follows that  $P \cap R = P(\bar{\sigma})$ ,  $P(P) \cap R = p_0(\bar{\sigma})$  and  $P^+ \cap R = P(\bar{\sigma})^+ (= p_1(\bar{\sigma}))$  hold. Since  $G(\sigma)$  is a torsion group, so is also  $G(\bar{\sigma})$ , and by Lemma 2.3 and Proposition 2.2, it is derived that  $R_{P(\bar{\sigma})} (= R_{\bar{\sigma}}) = \{a \in R \mid a^n \in P(\bar{\sigma}) \text{ for some integer } n > 0\}$  is included in  $S_P = \{a \in S \mid {}^g b_0 \in p_1(\sigma) + P, {}^g b_i \in P \cup -P, i = 1, 2, \dots, n; \sum_i b_i a^{n-i} = 0 \text{ for some } n > 0\}$ . Hence we have that  $R_{P(\bar{\sigma})} \subset S_P^+$ , and the natural homomorphism  $G(P(\bar{\sigma})) = R_{P(\bar{\sigma})} / P(\sigma)^+ \rightarrow G(P) = S_P^+ / P^+$ ;  $[a] \rightsquigarrow [a]$  defines a partial homomorphism  $\xi_2: F'' \rightarrow F(P)$  such that  $(\iota, \xi_2): \bar{\sigma} \rightarrow \tau$  is a morphism in  $\mathbf{C}_{\text{sig}}$ . Thus, we obtain a signature  $\tau: S \rightarrow F' = F(P)$  of ring  $S$  and a morphism  $(\iota, \xi_2 \circ \xi_1) = (\iota, \xi_2) \circ (I_R, \xi_1): \sigma \rightarrow \tau$  in  $\mathbf{C}_{\text{sig}}$ .

ideal  $p_0(\sigma)$ , that is, every element in  $R_\sigma \setminus p_0(\sigma)$  is invertible in  $R_\sigma$ . Then,  $a \in p_0(\sigma)$  if and only if  $a^{-1} \notin R_\sigma$ .

Proof. 1) For elements  $x, y \in R$ , we suppose that  $xR_\sigma y \subset p_0(\sigma)$  and  $x \notin p_0(\sigma)$ . If  $x \notin R_\sigma$ , then there is an  $x' \in p_0(\sigma)$  with  $x'x \in p_1(\sigma)$  or  $xx' \in p_1(\sigma)$ . Since both  $x'xR_\sigma y$  and  $xx'R_\sigma y$  are included in  $p_0(\sigma)$ , we may assume that  $x \in R_\sigma$ , and similarly  $y \in R_\sigma$ . Then,  $y \in p_0(\sigma)$  follows. 2) Suppose that  $a \in R_\sigma \setminus p_0(\sigma)$ . If  $a^{-1} \notin R_\sigma$ , then there is a  $b \in p_0(\sigma)$  with  $a^{-1}b \in p_1(\sigma)$  or  $ba^{-1} \in p_1(\sigma)$ , so it means either  $a(a^{-1}b)$  or  $(ba^{-1})a$  belongs to  $p_0(\sigma)$ , that is,  $a \in p_0(\sigma)$ , which is contrary to  $a \notin p_0(\sigma)$ . Hence, we get  $a^{-1} \in R_\sigma \setminus p_0(\sigma)$ . 3) First, we suppose that  $R$  is commutative. It is easy to see the “only if” part. If  $a^{-1} \notin R_\sigma$ , there is a  $b \in p_0(\sigma)$  with  $a^{-1}b \in p_1(\sigma)$ , so by 1)  $a(a^{-1}b) \in p_0(\sigma)$  implies  $a \in p_0(\sigma)$ . Next, we suppose that  $R_\sigma$  is a local ring with maximal ideal  $p_0(\sigma)$ . If  $a^{-1} \notin R_\sigma$  then there is a  $b \in p_0(\sigma)$  with  $a^{-1}b \in p_1(\sigma)$  or  $ba^{-1} \in p_1(\sigma)$ , so either  $a^{-1}b$  or  $ba^{-1}$  is invertible in  $R_\sigma$ . Hence, we get  $a \in p_0(\sigma)$ .

**Lemma 4.2.** For a  $\sigma \in X(R, F)$ , put  $q(\sigma) = \{a \in R \mid RaR \subset p_0(\sigma)\}$ . Then, the following properties hold;

- 1)  $q(\sigma)$  is a prime ideal of  $R$ , and  $q(\sigma) \subset p_0(\sigma)$ .
- 2) If  $R$  is a local ring with maximal ideal  $q(\sigma)$  then so is  $R_\sigma$  with maximal ideal  $p_0(\sigma)$ . If  $R$  is commutative, then the converse also holds.
- 3) If  $p_0(\sigma) = \{0\}$ , then  $R = R_\sigma$ , and  $P(\sigma)$  gives a partial ordering on the ring  $R$ .

Proof. 1) It is easy to see that  $q(\sigma)$  is an ideal of  $R$ , and  $q(\sigma) \subset p_0(\sigma)$ . For  $x, y \in R$ , we suppose that  $xRy \subset q(\sigma)$  and  $x \notin q(\sigma)$ . We can find elements  $a$  and  $b$  in  $R$  with  $axb \in p_0(\sigma)$ , so it follows that  $axbR_\sigma(RyR) \subset p_0(\sigma)$  and  $RyR \subset p_0(\sigma)$  by Lemma 4.1, 1), i.e.  $y \in q(\sigma)$ . 2) If  $R$  is a local ring with maximal ideal  $q(\sigma)$ , then every element in  $R_\sigma \setminus p_0(\sigma) (\subset R \setminus q(\sigma))$  is invertible in  $R$ , and by Lemma 4.1, 2), so is also in  $R_\sigma$ . Hence,  $R_\sigma$  is a local ring with maximal ideal  $p_0(\sigma)$ . If  $R$  is commutative and  $R_\sigma$  a local ring with maximal ideal  $p_0(\sigma)$ , then for any element  $x \in R \setminus q(\sigma)$ , we can find an element  $a \in R$  such that  $ax \in R_\sigma \setminus p_0(\sigma)$ , that is,  $ax$  is invertible in  $R_\sigma$ , so  $x$  is invertible in  $R$ . 3) is easy.

**Corollary 4.3.** Assume that  $R$  is a division ring, then the following hold.

- 1) For any  $\sigma \in X(R, F)$ ,  $R_\sigma$  is a local ring with maximal ideal  $p_0(\sigma)$ .
- 2)  $X(R, F)$  is a Hausdorff and totally disconnected space.
- 3) If  $F$  is a finite set, then  $X(R, F)$  is compact, that is, a Boolean space.

Proof. 1) is obtained by Lemma 4.2, 2). 2) By Lemma 4.1, 3), it follows that  $H_0(a) = H_\infty(a^{-1})$  is a clopen set of  $X(R, F)$  for any  $a \neq 0$  in  $R$ , and so is also  $H_\gamma(a)$  for any  $\gamma \in F \cup \{\infty\}$  and  $a \in R$ . Hence,  $X(R, F)$  is Hausdorff and totally disconnected. 3) Suppose that  $F$  is finite, then  $(F \cup \{\infty\})^R$  is compact. Whenever  $F \cup \{\infty\}$  is a discrete space, the subset  $X(R, F)$  becomes a closed subset of  $(F \cup \{\infty\})^R$ . Hence, under our topology on  $F \cup \{\infty\}$ ,  $X(R, F)$  is also

REMARK 3.3. Let  $\sigma: R \rightarrow F$  and  $\tau: S \rightarrow F'$  be signatures of rings  $R$  and  $S$ . If  $(f, \xi): \sigma \rightarrow \tau$  is a morphism in  $C_{\text{sig}}$ , then the following identities hold; 1)  $R_\sigma = f^{-1}(S_\tau)$ , 2) if  $G(\sigma)$  is a group, then  $p_0(\sigma) = f^{-1}(p_0(\tau))$  and  $\bigcup_{\alpha \in \xi^{-1}(\beta)} p_\alpha(\sigma) = f^{-1}(p_\beta(\tau))$  for each  $\beta \in G(\tau)$ .

Proof. 1) It is easy that  $R_\sigma \subset f^{-1}(S_\tau)$ . To prove the opposite, we suppose that there is an  $x \in R \setminus R_\sigma$  with  $f(x) \in S_\tau$ . Then, there is a  $y \in p_0(\sigma)$  such that  $xy \in p_1(\sigma)$  or  $yx \in p_1(\sigma)$  hold. However,  $xy \in p_1(\sigma)$  (resp.  $yx \in p_1(\sigma)$ ) implies  $\tau(f(xy)) = \xi(\sigma(xy)) = 1$  (resp.  $\tau(f(yx)) = 1$ ) which is contrary to that  $\tau(f(xy)) = \tau(f(x))\tau(f(y)) = \tau(f(x))\xi(\sigma(y)) = \tau(f(x))\xi(0) = \tau(f(x))0 = 0$  (resp.  $\tau(f(yx)) = 0$ ). Hence, we get  $R_\sigma = f^{-1}(S_\tau)$ . 2) It is also easy that  $p_0(\sigma) \subset f^{-1}(p_0(\tau))$ . If  $x \in f^{-1}(p_0(\tau))$ , then we have  $\xi(\sigma(x)) = \tau(f(x)) = 0$  and  $\sigma(x) = 0$ , i.e.  $x \in p_0(\sigma)$ , since  $G(\sigma)$  is a group and  $\xi(1) = 1$ . Hence, we get  $p_0(\sigma) = f^{-1}(p_0(\tau))$ . Since  $R_\sigma = f^{-1}(S_\tau)$  and  $p_0(\sigma) = f^{-1}(p_0(\tau))$ , it follows that  $R_\sigma \setminus p_0(\sigma) = \bigcup_{\alpha \in G(\sigma)} p_\alpha(\sigma) = f^{-1}(S_\tau \setminus p_0(\tau)) = \bigcup_{\beta \in G(\tau)} f^{-1}(p_\beta(\tau))$ . Since  $\bigcup_{\alpha \in \xi^{-1}(\beta)} p_\alpha(\sigma) \subset f^{-1}(p_\beta(\tau))$  holds for every  $\beta \in G(\tau)$ , we get  $\bigcup_{\alpha \in \xi^{-1}(\beta)} p_\alpha(\sigma) = f^{-1}(p_\beta(\tau))$  for every  $\beta \in G(\tau)$ .

#### 4. Space of signatures

In this section, we assume that  $F$  is a f-semigroup with abelian torsion group  $F^*$ . Let  $R$  be any ring with identity 1, and  $X(R, F)$  denote the set of signatures  $\sigma: R \rightarrow F$  of the ring  $R$  over the f-semigroup  $F$ . We consider a set  $F \cup \{\infty\}$  which is added a formal symbol  $\infty$  to  $F$ . We make the set  $F \cup \{\infty\}$  a topological space such that  $\{\alpha\}$  and  $\{\infty\}$  are open subsets for every  $\alpha \in F^*$ . Then, for any subset  $H \subset F \cup \{\infty\}$ ,  $H$  is a closed subset if and only if  $0 \in H$ . Considering  $R$  as a discrete space, we make the power space  $(F \cup \{\infty\})^R$  have a weak topology. We can introduce a topology on  $X(R, F)$  as a subspace of  $(F \cup \{\infty\})^R$ . For any  $\alpha \in F$  and  $a \in R$ , we put  $H_\alpha(a) = \{\sigma \in X(R, F) \mid \sigma(a) = \alpha\}$  and  $H_\infty(a) = \{\sigma \in X(R, F) \mid a \notin R_\sigma\}$ . Then, for every finite subsets  $\{a_1, a_2, \dots, a_n\} \subset R$  and  $\{\gamma_1, \gamma_2, \dots, \gamma_n\} \subset F^* \cup \{\infty\}$ , the intersections  $H_{\gamma_1}(a_1) \cap H_{\gamma_2}(a_2) \cap \dots \cap H_{\gamma_n}(a_n)$  construct an open basis of the space  $X(R, F)$ . Furthermore, for a subset  $H \subset F \cup \{\infty\}$  and  $a \in R$ , we have that  $\bigcup_{\alpha \in H} H_\alpha(a)$  is a closed subset of  $X(R, F)$  if and only if  $0 \in H$ .

In the following lemmata and corollary, we need not assume that  $F^*$  is a torsion group.

**Lemma 4.1.** *For a  $\sigma \in X(R, F)$  and an invertible element  $a$  in  $R$ , the following statements hold;*

- 1) *For any  $x, y \in R$ ,  $xR_\sigma y \subset p_0(\sigma)$  implies either  $x \in p_0(\sigma)$  or  $y \in p_0(\sigma)$ .*
- 2)  *$a \in R_\sigma \setminus p_0(\sigma)$  if and only if  $a^{-1} \in R_\sigma \setminus p_0(\sigma)$*
- 3) *Assume that either  $R$  is commutative or  $R_\sigma$  is a local ring with maximal*

compact.

**Proposition 4.4.** *Assume that  $R$  is a commutative ring and  $\sigma, \tau \in X(R, F)$ . If  $P(\sigma) \subset P(\tau)$  holds, then there are a subgroup  $H$  of  $G(\sigma)$  and a homomorphism  $\psi: H \rightarrow G(\tau)$  such that  $p_\beta(\tau) \cap R_\sigma \subset \bigcup_{\alpha \in \psi^{-1}(\beta)} p_\alpha(\sigma) \subset p_0(\tau) \cup p_\beta(\tau)$  holds for every  $\beta \in G(\tau)$ , and  $R_\sigma \subset R_\tau$  holds.*

Proof. Suppose that  $P(\sigma) \subset P(\tau)$ . Since  $G(\sigma)$  and  $G(\tau)$  are torsion groups, by Lemma 2.3, we get  $R_\sigma \subset R_\tau$ . We put  $H = \{\alpha \in G(\sigma) \mid p_\alpha(\sigma) \subset p_0(\tau)\}$ , then  $H$  is a subgroup of  $G(\sigma)$ . We can define a homomorphism  $\psi: H \rightarrow G(\tau)$  as follows; For any  $\alpha \in H$ , we can find an element  $a$  in  $p_\alpha(\sigma) \setminus p_0(\tau)$ , and  $\tau(a) = \tau(x)$  holds for every  $x \in p_\alpha(\sigma) \setminus p_0(\tau)$ . Because,  $\alpha^{-1}$  belongs to  $H$ , so we can find a  $b$  in  $p_{\alpha^{-1}}(\sigma) \setminus p_0(\tau)$ , which satisfies  $\sigma(ab) = \sigma(xb) = 1$  for every  $x \in p_\alpha(\sigma) \setminus p_0(\tau)$ . The condition  $P(\sigma) \subset P(\tau)$  means that for every  $x \in p_\alpha(\sigma) \setminus p_0(\tau)$ ,  $\tau(ab) = \tau(xb) = 1$  holds, so  $\tau(a) = \tau(x)$ . Therefore, we can define the image  $\psi(\alpha)$  of  $\alpha$  as  $\tau(a)$  for  $a \in p_\alpha(\sigma) \setminus p_0(\tau)$ . Then, it is easy to see that the map  $\psi: H \rightarrow G(\tau)$  is a group homomorphism. Further, for any  $\alpha \in H$  and  $\beta \in G(\tau)$  with  $\psi(\alpha) = \beta$ , from the definition of  $\psi$ ,  $p_\alpha(\sigma) \subset p_0(\tau) \cup p_\beta(\tau)$  follows. Hence, we get  $\bigcup_{\alpha \in \psi^{-1}(\beta)} p_\alpha(\sigma) \subset p_0(\tau) \cup p_\beta(\tau)$ . On the other hand, if  $\beta$  is an element in  $G(\tau)$  with  $p_\beta(\tau) \cap R_\sigma \neq \emptyset$ , then for each  $x \in p_\beta(\tau) \cap R_\sigma$ , there is an  $\alpha \in G(\sigma)$  with  $x \in p_\alpha(\sigma) \setminus p_0(\tau)$ , that is,  $\psi(\alpha) = \beta$  and  $x \in p_\alpha(\sigma)$ . Hence, we get  $p_\beta(\tau) \cap R_\sigma \subset \bigcup_{\alpha \in \psi^{-1}(\beta)} p_\alpha(\sigma)$  for every  $\beta \in G(\tau)$ .

**REMARK 4.5.** Let  $R$  be a commutative ring, and  $\sigma: R \rightarrow F$  a signature of  $R$ . By  $\sigma$ , a topology on affine  $n$ -space  $R^n$  is introduced as follows; For any  $\gamma_i \in G(\sigma) \cup \{\infty\}$  and  $f_i(X_1, X_2, \dots, X_n)$  in polynomial ring  $R[X_1, X_2, \dots, X_n]$ ,  $i = 1, 2, \dots, m$ , we put  $U(f_1, f_2, \dots, f_m, \gamma_1, \gamma_2, \dots, \gamma_m) = \{(a_1, a_2, \dots, a_n) \in R^n \mid \sigma(f_i(a_1, a_2, \dots, a_n)) = \gamma_i, i = 1, 2, \dots, n\}$ , where  $\sigma(f_i(a_1, a_2, \dots, a_n)) = \infty$  whenever  $f_i(a_1, a_2, \dots, a_n) \notin R_\sigma$ . Then, the sets  $U(f_1, f_2, \dots, f_m, \gamma_1, \gamma_2, \dots, \gamma_m)$  form an open basis on  $R^n$ . We can define a continuous map  $\psi_\sigma$  of the topological space  $R^n$  into  $X(R[X_1, X_2, \dots, X_n], F)$ ; Let  $(a_1, a_2, \dots, a_n)$  be any element in  $R^n$ , and let  $\psi_{(a_1, a_2, \dots, a_n)}: R[X_1, X_2, \dots, X_n] \rightarrow R; f(X_1, X_2, \dots, X_n) \rightsquigarrow f(a_1, a_2, \dots, a_n)$  a natural ring homomorphism. By Proposition 3.1, 1), there exists a signature  $\sigma_{(a_1, a_2, \dots, a_n)}: R[X_1, X_2, \dots, X_n] \rightarrow F$  with a morphism  $(\psi_{(a_1, a_2, \dots, a_n)}, I_F): \sigma_{(a_1, a_2, \dots, a_n)} \rightarrow \sigma$  in  $C_{\text{sig}}$ . Thus, we get a map  $\psi_\sigma: R^n \rightarrow X(R[X_1, X_2, \dots, X_n], F); (a_1, a_2, \dots, a_n) \rightsquigarrow \sigma_{(a_1, a_2, \dots, a_n)}$ , which is continuous, because of  $\psi_\sigma^{-1}(H_\gamma(f)) = U(f, \gamma)$  for  $f \in R[X_1, X_2, \dots, X_n]$  and  $\gamma \in G(\sigma) \cup \{\infty\}$ .

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### References

[1] E. Becker: *Partial orders on a field and valuation rings*, Comm. Algebra 7 (1979),

1933-1976.

- [2] E. Becker, J. Harman and A. Rosenberg: *Signatures of fields and extension theory*, J. Reine Angew. Math. **330** (1982), 53-75.
- [3] D.K. Harrison: Finite and infinite primes for rings and fields, Mem. Amer. Math. Soc. 68 (1966).
- [4] I. Kaplansky: Infinite abelian groups, University of Michigan Press 1954.
- [5] M. Nagata: Local rings, Huntington, New York 1975.
- [6] T. Tamura: Semigroup theory, (Japanese), Kyoritsu Koza, Gendaino Sugaku 8, Tokyo 1972.
- [7] D.K. Harrison and H.D. Warner: *Infinite primes of fields and completions*, Pacific J. Math. **45** (1973), 201-216.
- [8] M. Marshall: Abstract Witt rings, Queen's Papers in Pure and Applied Math. No. 57 (1980).
- [9] E. Becker: Hereditarily-Pythagorean fields and orderings of higher level, Monografias de Mat. no. 29, Rio de Janeiro (1978).

Osaka Women's University  
Daisen-cho, 2-1  
Sakai, Osaka 590  
Japan