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Osaka J. Math.
22 (1985), 327–338

NOTES ON SIGNATURES ON RINGS

Dedicated to Professor Hirosi Nagao on his 60th birthday

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(Received June 21, 1984)

0. Introduction

The notion of infinite prime introduced by Harrison [3] was investigated in [1], [2], [7] and [9] which were concerned with ordering on a field. In this note, we study about signatures on rings as some generalization of infinite primes and signatures of fields in [2]. In the section 1, we introduce notions of U-prime and signature of a ring which are generalizations of infinite prime and signature of field. In the section 2, we show that a U-prime of a commuative ring defines a signature on the ring. In the sections 3 and 4, we consider the category of signatures and a space of signatures on a ring which include notions of extension of signature and space of ordering on fields (cf. [2] and [8]), and investigate them. Throughout this paper, we assume that every ring has identity 1.

1. Preliminaries, definitions and notations

Let S be a multiplicative semigroup, and T a normal subsemigroup of S, (cf. [6], p. 195), denoted by $T\lhd S$, that is, T is a subsemigroup of S which satisfies 1) for $x, y\in S$, $xy\in T$ implies $yx\in T$, 2) if there is an $x\in T$ with $xy\in T$, then $y\in T$, and 3) for every $x\in S$, there exists an $x'\in S$ with $x'x\in T$. We can define a binary relation $\sim$ on S; for $x, y\in S$, $x\sim y$ if and only if there is a $z\in S$ such that both zx and zy are contained in T. Then, the relation $\sim$ is an equivalence relation on S, and is compatible with the multiplication of S, so the quotient set $S/\sim$, denoted by S/T, makes a group such that the canonical map $\psi: S\rightarrow S/T$; $x\mapsto [x]$ is a homomorphism with Ker $\psi=T$.

Let R be any ring with identity 1, and P a preprime of R ([3]), that is, P is closed under addition and multiplication of R and $-1\notin P$. We put $p(P)=P\cap -P$, $R_P=\{x\in R\,|\,xp(P)\cup p(P)x\subset p(P)\}$, $R_P^+=R_P\backslash p(P)$ $(:=\{x\in R_P\,|\,x\notin p(P)\})$, $P^+=P\backslash p(P)$ $(=P\backslash -P)$. We shall say a preprime P to be *complete quasi-prime*, if it satisfies the following conditions;

1) $p(P)$ is an ideal of R_P such that $R_P/p(P)$ is an integral domain,
2) $P^+\lhd R_P^+$ under the multiplication of R_P.

3) P is complete in R_P, that is, for $x \in R_P$, $x^2 \in P$ implies $x \in P \cup -P$.

A multiplicative semigroup F with unit element 1 and zero element 0 will be called a *f-semigroup*, if $F^* = F \backslash \{0\}$ makes a group with a unique element of order 2, denoted by -1, under the multplication of F. If P is a complete quasi-prime of R, then the quotient group $G(P) = R_P^+/P^+$ has a unique element $[-1]$ of order 2, and the formally composed semigroup $F(P) = G(P) \cup \{0\}$ makes an f-semigroup under the multiplication of $G(P)$ and $\alpha 0 = 0\alpha = 00 = 0$ for $\alpha \in G(P)$. Furthermore, we can define a map $\sigma: R_P \to F(P)$ by $\sigma(a) = 0$ or $[a]$ for $a \in p(P)$ or $a \in R_P^+$, respectively. Then, it can be verified that 1) $\sigma(-1) = [-1]$, 2) $\sigma(ab) = \sigma(a)\sigma(b)$ for every $a, b \in R_P$, and 3) for $a, b \in R_P$, either $\sigma(a) = 0$ or $\sigma(a) = \sigma(b)$ implies $\sigma(a+b) = \sigma(b)$.

Let π be a set of prime numbers, and suppose $2 \in \pi$. A complete quasi-prime P will be called a π-complete quasi-prime, if for each $q \in \pi$, there is a $\zeta_q \in R_P \backslash P$ such that $\zeta_q{}^q \in P$ and for any $x \in R_P$ with $x^q \in P$, $yx \in \bigcup_{1 \leq i \leq q} \zeta_q P^i$ for some $y \in P^+$.

REMARK 1.1. If R is a commutative ring and P is a π-complete quasi-prime, then for each $q \in \pi$, the q-torsion subgroup $G(P)_q = \{\alpha \in G(P) \mid {}^{\exists}n > 0; \alpha^{q^n} = [1]\}$ of $G(P)$ is isomorphic to a subgroup of $\boldsymbol{Z}(q^\infty)$. Because, since $G(P)_q$ has a unique minimal non trivial subgroup $\langle[\zeta_q]\rangle$, $G(P)_q$ is indecomposable, so by [4], p. 22, Theorem 10, $G(P)_q$ is isomorphic to $\boldsymbol{Z}(q^n)$ or $\boldsymbol{Z}(q^\infty)$.

Let R be a ring with identity 1, and F an abelian f-semigroup. A partial map $\sigma: R \rightharpoondown F$ will be called a *signature* of R with domain of definition R_σ, if σ is a map of a subset R_σ of R into F satisfying the following conditions;

(S 1) $-1 \in R_\sigma$ and $\sigma(-1) = -1$,

(S 2) $a, b \in R_\sigma$ implies $ab \in R_\sigma$ and $\sigma(ab) = \sigma(a)\sigma(b)$,

(S 3) for $a, b \in R_\sigma$, if $\sigma(a) = 0$ or $\sigma(a) = \sigma(b)$ then $a+b \in R_\sigma$ and $\sigma(a+b) = \sigma(b)$,

(S 4) for $a \in R$, if $a \notin R_\sigma$, then there exists a $b \in R_\sigma$ such that $\sigma(b) = 0$ and either $\sigma(ab) = 1$ or $\sigma(ba) = 1$.

Let $\sigma: R \rightharpoondown F$ be a signature. For $\alpha \in F$, we put $p_\alpha(\sigma) = \{x \in R_\sigma \mid \sigma(x) = \alpha\}$, $P(\sigma) = p_0(\sigma) \cup p_1(\sigma)$ and $G(\sigma) = \operatorname{Im} \sigma \cap F^*$.

Lemma 1.2. *Let $\sigma: R \rightharpoondown F$ be a signature of a ring R.*

1) *R_σ is a subring of R with prime ideal $p_0(\sigma)$ such that $R_\sigma/p_0(\sigma)$ is an integral domain.*

2) *$P(\sigma)$ is a preprime of R, and $R_\sigma = R_{P(\sigma)}$.*

3) *If $G(\sigma)$ is a subgroup of F^*, then $P(\sigma)$ is a complete quasi-prime of R, and $G(P(\sigma))$ and $G(\sigma)$ are group isomorphic.*

Proof. 1) If R_σ is closed under the addition of R, then it is easy to see that R_σ is a subring of R. Suppose $a+b\notin R_\sigma$ for some a and b in R_σ. There is a $c\in R_\sigma$ such that $\sigma(c)=0$, and $\sigma(c(a+b))=1$ or $\sigma((a+b)c)=1$. Since $\sigma(ca)=\sigma(ac)=\sigma(a)\sigma(c)=0$ and $\sigma(cb)=\sigma(bc)=0$, we get $\sigma(ca+cb)=\sigma(ac+bc)=0$ which is a contradiction. Hence, we get $R_\sigma+R_\sigma\subset R_\sigma$. It is easy to see that $p_0(\sigma)$ is an ideal of R_σ, and $R_\sigma/p_0(\sigma)$ is an integral domain. 2) From the definition of signature, it follows that $\mathrm{P}(\sigma)$ is a preprime of R and $p_0(\sigma)=\mathrm{P}(\sigma)\cap -\mathrm{P}(\sigma)$. We shall show $R_\sigma=R_{P(\sigma)}$. Since $R_\sigma\subset R_{P(\sigma)}$ is clear, it suffices to show $R_\sigma\supset R_{P(\sigma)}$. If $x\in R\backslash R_\sigma$, then there is a $y\in p_0(\sigma)$ with $xy\in p_1(\sigma)$ or $yx\in p_1(\sigma)$, so $xp_0(\sigma)\cup p_0(\sigma)x\not\subset p_0(\sigma)$, that is, $x\notin R_{P(\sigma)}$. 3) If $\mathrm{G}(\sigma)$ is a group, then it is easy to see that $\mathrm{P}(\sigma)^+=p_1(\sigma)$, $\mathrm{P}(\sigma)^+\triangleleft R_{P(\sigma)}^+$, $\sigma(R_{P(\sigma)}^+)=\mathrm{G}(\sigma)$, and $\mathrm{P}(\sigma)$ is complete. Furthermore, a map $\mathrm{G}(\mathrm{P}(\sigma))=R_{P(\sigma)}^+/\mathrm{P}(\sigma)^+\to \mathrm{G}(\sigma)$; $[x]\rightsquigarrow\sigma(x)$ is a group isomorphism.

REMARK. 1) If R is a field, then a signature $\sigma: R\to F$ with $p_0(\sigma)=\{0\}$ and $F=\mu\cup\{0\}$ coincides with the notion of signature defined by Becker, Harman and Rosenberg [2], where μ is the group of all roots of unity in the complices. 2) Let F be a finite field with characteristic $\neq 2$. The multiplicative semigroup F is an abelian f-semigroup. For a signature σ: $R\to F$, let π be the set of all prime factors of order $|G(\sigma)|$. Then, it is easy to see that $\mathrm{P}(\sigma)$ is a π-complete quasiprime of R.

Let R be a ring with identity 1, and U a non empty multiplicatively closed subset of R satisfying $U\cap -U=\phi$. A preprime P of R will be called a U-preprime of R, if $U\subset P$ and $P\cap -U=\phi$. A maximal U-preprime of R will be called a U-prime of R. Any Harrison's infinite prime is a $\{1\}$-prime.

Lemma 1.3. *Let U a non empty multiplicatively closed subset of R with $U\cap -U=\phi$, and P a U-prime of R. If either R is commutative or $Px=xP$ and $Ux=xU$ hold for every $x\in R_P^+$, then P is a complete quasi-prime of R.*

The proof of this lemma is obtained by checking the following facts;

(1.3.1) $U+P\subset P^+$.

(1.3.2) For $x\in R_P$ ($x\in R$, if R is commutative), if there are $u\in U$ and $y\in P$ with $(u+y)x\in P$, then $x\in P$. Hence $1\in P$.

(1.3.3) For $x\in R_P$ ($x\in R$, if R is commtative), if $x\notin p(P)$, then there is an $x'\in(\pm P)[x]$ with $x'x\in U+P$, where $(\pm P)[x]=\{\sum_i a_ix^i\in R\,|\,a_i\in P\cup -P\}$.

(1.3.4) $R_P/p(P)$ is an integral domain.

(1.3.5) For $x, y\in R_P$, $xy\in P^+$ implies $yx\in P^+$.

(1.3.6) P is complete in R_P.

(1.3.7) For any $x\in P^+$, there is an $x'\in P^+$ with $x'x\in U+P$.

(1.3.8) For $x\in R_P$ ($x\in R$, if R is commutative), if there is a $y\in P^+$ with $yx\in P^+$, then $x\in P^+$.

The proofs of these statements are obtained similarly to the case of Harrison's infinite prime; (1.3.1): Since $U \cap -P = \phi$, it follows that $U \subset P^+$ and $U+P \subset P^+$. (1.3.2): A subset $P' = \{x \in R_P \mid {}^{\exists}u \in U, {}^{\exists}y \in P; (u+y)x \in P\}$ of R is closed under addition and multiplication. Because, if $x_1, x_2 \in P'$, there are $u_i \in U$ and $y_i \in P$ with $(u_i+y_i)x_i \in P$, $i=1, 2$. If either x_1 or x_2 belongs to $p(P)$, then it is trivial that x_1+x_2 and x_1x_2 belong to P'. Otherwise, by assumption, there are $u_2' \in U$ and $y_2' \in P$ such that $x_1u_2 = u_2'x_1$ and $x_1y_2 = y_2'x_1$. Then $(u_1+y_1)(u_2+y_2)$ and $(u_1+y_1)(u_2'+y_2')$ belong to $U+P$, and $(u_1+y_1)(u_2+y_2)(x_1+x_2)$ and $(u_1+y_1)(u_2'+y_2')x_1x_2$ are in P. Furthermore, it is immeadiately seen that $P \subset P'$ and $P' \cap -U = \phi$, so we get $P = P'$. (1.3.3): For $x \in R_P$, if $x \notin p(P)$, then either $x \notin P$ or $-x \notin P$. By assumption, a subset $P[x] = P + Px + Px^2 + \cdots$, (resp. $P[-x] = P + P(-x) + P(-x)^2 + \cdots$) of R is closed under addition and multiplication. Since $P \subsetneqq P[x]$ or $P \subsetneqq P[-x]$, we get $P[x] \cap -U \neq \phi$ or $P[-x] \cap -U \neq \phi$, so we can find an element $y \in (\pm P)[x]$ such that $yx \in U+P$ holds. (1.3.4): For $x, y \in R_P$, suppose that $xy \in p(P)$ and $x \notin p(P)$. By (1.3.3), there is an $x' \in (\pm P)[x]$ $(\subset R_P)$ with $x'x \in U+P$, and (1.3.2) derives that $x'xy \in p(P)$ implies $y \in p(P)$. (1.3.5): For $x, y \in R_P$, suppose $xy \in P^+$. $(xy)x$ is in $Px = xP$, and for an element x' in $(\pm P)[x]$, also in R_P, with $x'x \in U+P$, we get $(x'x)yx \in x'xP \subset P$, so $yx \in P^+$ by (1.3.2) and (1.3.4). (1.3.6) is easy. (1.3.7): If $x \in P^+$, then $P[-x] = P - Px$ is closed under addition and multiplication, and $P \subsetneqq P[-x]$. Hence, there are $u \in U$ and $x', y \in P$ with $-u = y - x'x$, so we get $x'x = u + y \in U+P$ and $x' \in P^+$. (1.3.8) is immeadiately obtained from (1.3.2) and (1.3.7).

2. The connection between U-prime and signature

Theorem 2.1. *Let R be a commutative ring with identity 1, and U any non empty multiplicatively closed subset of R with $U \cap -U = \phi$. If P is a U-prime of R, then there exists a signature $\sigma: R \to F$ with $\mathrm{P}(\sigma) = P$ and group $\mathrm{G}(\sigma) = \mathrm{G}(P)$.*

Proof. By Lemma 1.3, U-prime P is a complete quasi-prime of R, so it defines a map $\sigma: R_P \to \mathrm{F}(P)$. Then, we put $R_\sigma = R_P$ and $F = \mathrm{F}(P)$. The conditions (S 1), (S 2) and (S 3) of signature were verified. (S 4) is proved in the following proposition. Then we have a signature $\sigma: R \to F$ with $P = \mathrm{P}(\sigma)$ and $\mathrm{G}(\sigma) = \mathrm{G}(P) = R_P^+/P^+$.

Proposition 2.2. *Let P be a U-prime of a commutative ring R, and let $A_P = \{a \in R \mid {}^{\exists}b_0 \in U+P, {}^{\exists}b_i \in P \cup -P, i=1, 2, \cdots, n; \sum_{i=0}^{n} b_i a^{n-i} = 0\}$.*

1) *$(R_P, p(P))$ is a valuation pair of R,* (cf. [3], Proposition. 2.5).
2) *If $x \in R \backslash p(P)$ then there is an $a \in A_P$ with $ax \in U+P$.*
3) *If x and y are elements of R with $xy \in U+P$, then $x \notin p(P)$ implies $y \in A_P$.*
4) *$R_P = A_P$.*

Proof. The proof of 1) is quite similar to [3], Proposition 2.5. 2) If $x \in R \backslash p(P)$, by (1.3.3) there is an $a \in (\pm P)[x]$ with $ax \in U+P$, then a can be represented as $-(b_1+b_2x+\cdots+b_nx^{n-1})$ for some $b_i\in P\cup -P$. If we put $ax=b_0$, then a satisfies an equation $b_0a^n+b_1b_0a^{n-1}+\cdots+b_nb_0^n=0$ with $b_0\in U+P$ and $b_ib_0^i\in P\cup -P$, $i=1, 2, \cdots, n$, so $a\in A_P$. 3) Suppose that x and y are in R and $xy\in U+P$. If $x\notin p(P)$, by 2), there is a $z\in A_P$ with $zx\in U+P$. Since $z\in A_P$, there are $a_0\in U+P$ and $a_i\in P\cup -P$, $i=1, 2, \cdots, m$, with $\sum_{i=0}^m a_iz^{m-i}=0$. Put $xy=b_0$ and $zx=c_0$, so we get that $\sum_{i=0}^m(a_ic_0^{m-i}b_0^i)y^{m-i}=(\sum_{i=0}^m a_iz^{m-i})b_0^m=0$, $a_0c_0^m\in U+P$ and $a_ic_0^{m-i}b_0^i\in P\cup -P$, hence $y\in A_P$. 4) In the first place, we show $A_P\supset R_P$: Let x be any element in R_P. If $x\in p(P)$, $x\in A_P$ is obvious. Otherwise, by (1.3.3) there is a $y\in(\pm P)[x]$ with $xy\in U+P$, so $y\notin p(P)$ and by 3) we get $x\in A_P$. Now, we show $A_P=R_P$: Let $(U+P)^{-1}R$ be the ring of quotients of R with respect to $U+P$, and ψ: $R\to(U+P)^{-1}R$ the canonical ring homomorphism. Then, $(U+P)^{-1}R_P$ may be regarded as a subring of $(U+P)^{-1}R$. By B', we denote the integral closure of $(U+P)^{-1}R_P$ in $(U+P)^{-1}R$. There is a prime ideal Q' of B' which lies over $(U+P)^{-1}R_Pp(P)$, (cf. [5], (10.8)). It follows that $B=\psi^{-1}(B')$ is a subring of R with $B\supset A_P\supset R_P$, and $Q=\psi^{-1}(Q')$ is a prime ideal of B with $Q\cap R_P=p(P)$. By 1), we get $B=A_P=R_P$.

Lemma 2.3. *Let R be a commutative ring, and σ: $R\to F$ a signature. If $\mathrm{G}(\sigma)$ is a torsion group, then $R_\sigma=\{a\in R\mid a^n\in\mathrm{P}(\sigma)$ for some integer $n>0\}$.*

Proof. Since $\mathrm{G}(\sigma)$ is a torsion group, it is clear that any element a in R_σ has a positive integer n with $a^n\in P(\sigma)$. Conversely, suppose that an element $a\in R$ does not belong to R_σ. There is a $b\in p_0(\sigma)$ with $ab\in p_1(\sigma)$. Then a^n is not contained in $P(\sigma)$ for every positive integer n. Because, if $a^n\in P(\sigma)$ for some $n>0$, it derives a contradiction $1=\sigma((ab)^n)=\sigma(a^n)\sigma(b^n)=0$.

Let R be a ring with identity 1. By [1], a preprime P is called a torsion preprime (resp. 2-torsion preprime) of R, if for each $a\in R$ there exists a positive integer n such that $a^n\in P$ (resp. $a^{2^n}\in P$) holds. From Theorem 2.1 and Lemma 2.3, the following corollaries immeadiately follow;

Corollary 2.4. *Let R be a commutative ring with 1 and U a non empty multiplicatively closed subset of R with $1\in U$ and $U\cap -U=\phi$.*

1) *If P is a torsion U-prime of R, then $p(P)$ is an ideal of R, i.e. $R_P=R$, so there is a signature σ: $R\to F$ such that $P=\mathrm{P}(\sigma)$, $R=R_\sigma$ and $\mathrm{G}(\sigma)$ is a torsion group.*

2) *If P is a 2-torsion U-prime of R, then there is a signature σ: $R\to F$ such that $P=\mathrm{P}(\sigma)$, $R=R_\sigma$ and $F^*\cong\boldsymbol{Z}(2^\infty)$.*

In particular, on a field, we have

Corollary 2.5. *Let K be a field.*

1) *For any signature σ: $K\to F$, K_σ is a valuation ring of K with maximal ideal $p_0(\sigma)$, and the residue field $k(\sigma)=K_\sigma/p_0(\sigma)$ has an induced signature $\bar{\sigma}$: $k(\sigma)$*

$\to F$ with $k(\sigma)_{\bar\sigma}=k(\sigma)$ and $\mathfrak{p}_0(\bar\sigma)=\{\bar 0\}$, and $\mathrm{P}(\bar\sigma)$ is a preordering on $k(\sigma)$.

2) *Let U be a non empty multiplicatively closed subset of K with $U\cap -U=\phi$. If P is a U-prime of K, K_P is a valuation ring of K with maximal ideal $p(P)$. If P is a torsion U-prime of K, then $K=K_P$, $p(P)=\{0\}$, and P is a preordering, i.e. $P^+=P\setminus\{0\}$ is a subgroup of $K^*=K\setminus\{0\}$,* (cf. [1], (3.3)).

3) *If O is a real valutation ring of K with maximal ideal p, i.e. the residue field O/p is a formally real field, then there is a signature $\sigma\colon K\to \mathrm{GF}(3)$ with $K_\sigma=O$ and $p_0(\sigma)=p$, where $\mathrm{GF}(3)=\{0, 1, -1\}$ is a multiplicative semigroup of prime field with charcteristic* 3.

Theorem 2.6. *Let R be a ring with identity* 1, *and $\sigma\colon R\to F$ a signature of R. Assume that $\mathrm{G}(\sigma)$ is a torsion group and $xp_\alpha(\sigma)=p_\alpha(\sigma)x$ holds for all $x\in R_\sigma\setminus p_0(\sigma)$ and $\alpha\in\mathrm{G}(\sigma)\cup\{0\}$. Then, there exists a signature $\tau\colon R\to F'$ of R satisfying the following conditions;*

1) *$\mathrm{P}(\tau)$ is a $p_1(\sigma)$-prime of R and $\mathrm{P}(\tau)\supset\mathrm{P}(\sigma)$,*

2) *$R_\tau=R_\sigma$ and $p_0(\tau)=p_0(\sigma)$,*

3) *there is a subgroup H of $\mathrm{G}(\sigma)$ such that $p_1(\tau)=\sigma^{-1}(H)$, $-1\notin H$ and $\mathrm{G}(\sigma)/H\cong\mathrm{G}(\tau)$ hold.*

Proof. Since $\mathrm{P}(\sigma)$ is a $p_1(\sigma)$-preprime of R, by Zorn's Lemma there exists a $p_1(\sigma)$-prime P of R containing $\mathrm{P}(\sigma)$. From the facts that $P\cap -p_1(\sigma)=\phi$ and $p_0(\sigma)\subset p(P)$, we can derive that $p_0(\sigma)=p(P)$ and $R_P=R_\sigma$; If there is an element $x\in R_P\setminus R_\sigma$, then there exists a $y\in p_0(\sigma)$ such that either xy or yx belongs to $p_1(\sigma)$. However, xy and yx are also contained in $p(P)$, so these are contrary to $p_1(\sigma)\cap p(P)=\phi$. Hence, we get $R_P\subset R_\sigma$. Furthermore, if there is an element $x\in p(P)\setminus p_0(\sigma)$, we have $x^n\in p_1(\sigma)\cap p(P)$ for some integer $n>0$, which is a contradiction. Therefore, we get $p_0(\sigma)=p(P)$ and $R_P=R_\sigma$. Now, we put $H=\sigma(P^+)$, so H is a subgroup of $\mathrm{G}(\sigma)$. We shall show $P^+=\sigma^{-1}(H)$; If $x\in\sigma^{-1}(H)$ then there is a $y\in P^+$ with $\sigma(x)=\sigma(y)$. Since $y^n\in p_1(\sigma)$ for some integer $n>0$, we have $xy^n=(xy^{n-1})y\in xp_1(\sigma)\cap P^+$. Hence, for any $x\in R$, it follows that $x\in\sigma^{-1}(H)$ if and only if $xp_1(\sigma)\cap P^+\neq\phi$. On the other hand, we can show that $P=\{x\in R_\sigma \mid xp_1(\sigma)\cap P\neq\phi\}$; The set $P'=\{x\in R_\sigma\mid xp_1(\sigma)\cap P\neq\phi\}$ is closed under addition and multiplication: Because, for $x, y\in P'$, there are $x_1, y_1\in p_1(\sigma)$ such that both xx_1 and yy_1 are in P. Since we may suppose that y is not in $p_0(\sigma)$, there is an $x_1'\in p_1(\sigma)$ with $x_1y=yx_1'$, and it follows that both $(x+y)(x_1y_1)$ and $(xy)(x_1'y_1)$ are contained in P. Hence, both $x+y$ and xy belong to P'. Furthermore, it is derived that $P\subset P'$ and $P'\cap -p_1(\sigma)=\phi$, because of $P\cap -p_1(\sigma)=\phi$. Hence, we get $P=P'$. Accordingly, we conclude that $\sigma^{-1}(H)=P^+=\bigcup_{\alpha\in H}p_\alpha(\sigma)$. From the assumption $xp_\alpha(\sigma)=p_\alpha(\sigma)x$ for $x\in R_\sigma\setminus p_0(\sigma)$ and $\alpha\in\mathrm{G}(\sigma)\cup\{0\}$, P is a complete quasi-prime of R. Therefore, we can define a signature $\tau\colon R\to \mathrm{F}(P)$ such that $R_\tau=R_P=R_\sigma$, $p_0(\tau)=p(P)=p_0(\sigma)$ and $\mathrm{G}(\tau)=\mathrm{G}(P)\cong\mathrm{G}(\sigma)/H$.

It is easy to check the conditions of signature for τ.

Corollary 2.7. *Let R be a commutative ring with identity* 1. *If $\sigma: R \rightharpoondown F$ is a signature of R such that* $\mathrm{G}(\sigma)$ *is a 2-torsion group, then* $\mathrm{P}(\sigma)$ *is a $p_1(\sigma)$-prime of R.*

Proof. Since $\mathrm{G}(\sigma)$ is a 2-torsion group, by Remark 1.1 every non-trivial subgroup H of $\mathrm{G}(\sigma)$ contains -1. By Theorem 1.7, $\mathrm{P}(\sigma)$ is a $p_1(\sigma)$-prime of R.

Corollary 2.8. *Let S be a commutative ring with identity* 1, *and R a subring of S containing* 1. *If $\sigma: R \rightharpoondown F$ a signature of R such that* $\mathrm{G}(\sigma)$ *is 2-torsion group, then σ can be extended to a signature $\tau: S \rightharpoondown F'$ of S, i.e. $S_\tau \cap R = R_\sigma$ and* $\mathrm{P}(\tau) \cap R = \mathrm{P}(\sigma)$ *hold.*

Proof. A signature $\tau: S \rightharpoondown F'$ is defined by a $p_1(\sigma)$-prime P of S containing $\mathrm{P}(\sigma)$. Then, τ is an extension of σ.

3. Category of signatures

Let $\sigma_1: R_1 \rightharpoondown F_1$ and $\sigma_2: R_2 \rightharpoondown F_2$ be signatures of rings R_1 and R_2. Suppose that $f: R_1 \to R_2$ is a ring homomorphism such that $f(1)=1$ and $f(R_{1\sigma_1}) \subset R_{2\sigma_2}$, and that $\xi: F_1 \rightharpoondown F_2$ is a partial homomorphism which is defined on $\mathrm{G}(\sigma_1)$ and satisfies $\xi(0)=0$, $\xi(-1)=-1$ and $\xi(\alpha\beta)=\xi(\alpha)\xi(\beta)$ if ξ is defined on α, β and $\alpha\beta$ for $\alpha, \beta \in F_1$. Then, the pair (f, ξ) will be called a morphism of signatures of σ_1 to σ_2, denoted by $(f, \xi): \sigma_1 \to \sigma_2$, if it satisfies $\xi(\sigma_1(x)) = \sigma_2(f(x))$ for all $x \in R_{1\sigma_1}$. Let $\sigma_i: R_i \rightharpoondown F_i$ and $\sigma'_i: R'_i \to F'_i$ be signatures of rings for $i=1, 2$, and (f, ξ): $\sigma_1 \to \sigma_2$ and (f', ξ'): $\sigma'_1 \to \sigma'_2$ morphisms of signatures. We define the equalitiy of morphisms that $(f, \xi) = (f', \xi')$ if and only if $\sigma_i = \sigma'_i$ (i.e. $R_i = R'_i$, $R_{i\sigma_i} = R'_{i\sigma'_i}$, $F_i = F'_i$ and $\sigma_i(x) = \sigma'_i(x)$ for all $x \in R_{i\sigma_i}$) for $i=1, 2, f=f'$ and for every $\alpha \in \mathrm{G}(\sigma_1) = \mathrm{G}(\sigma'_1)$, $\xi(\alpha) = \xi'(\alpha)$ hold. By $\boldsymbol{C}_{\mathrm{sig}}$, we denote the category of signatures in which objects are signatures of rings and morphisms are morphisms of signatures.

Proposition 3.1. *Let R and S be rings with identity* 1, *and $f: R \to S$ a ring homomorphism with $f(1)=1$.*

1) *If $\tau: S \rightharpoondown F$ is a signature of ring S with* $\mathrm{Im}\, f \supset p_0(\tau)$, *then there exists a signature $\sigma: R \rightharpoondown F$ of ring R with a morphism $(f, \mathrm{I}_F): \sigma \to \tau$ in $\boldsymbol{C}_{\mathrm{sig}}$.*

2) *If $f: R \to S$ is surjective, and if $\sigma: R \rightharpoondown F$ is a signature of ring R with* $\mathrm{Ker}\, f \subset p_0(\sigma)$, *then there exists a signature $\tau: S \rightharpoondown F$ of ring S with a morphism $(f, \mathrm{I}_F): \sigma \to \tau$ in $\boldsymbol{C}_{\mathrm{sig}}$.*

Proof. 1) Suppose that $\tau: S \to F$ is a signature of ring S and $f: R \to S$ is a ring homomorphism with $f(1)=1$ and $\mathrm{Im}\, f \supset p_0(\tau)$. On a subring $R_\sigma = \{x \in R \mid f(x) \in S_\tau\}$ of R, a map $\sigma: R_\sigma \to F$; $x \rightsquigarrow \tau(f(x))$ is defined. The condition

$\mathrm{Im} f \supset p_0(\tau)$ derives that a signature $\sigma: R \rightharpoondown F$ of ring R and a morphism (f, I_F): $\sigma \to \tau$ in $\boldsymbol{C}_{\mathrm{sig}}$ are defined. 2) Suppose that $f: R \to S$ is a surjective ring homomorphism, and $\sigma: R \rightharpoondown F$ is a signature of ring R with $\mathrm{Ker} f \subset p_0(\sigma)$. For a subring $S_\tau = f(R_\sigma)$, we can define a map $\tau: S_\tau \to F$ as follows: For any $a \in S_\tau$, there is a $b \in R_\sigma$ with $f(b) = a$, then we put $\tau(a) = \sigma(b)$. From the condition $\mathrm{Ker} f \subset p_0(\sigma)$, it is known that the map $\tau: S_\tau \to F$ is well defined. Then, it is easy to see that a signature $\tau: S \rightharpoondown F$ of ring S and a morphism (f, I_F): $\sigma \to \tau$ in $\boldsymbol{C}_{\mathrm{sig}}$ are defined.

Concerning commutative rings, the situation of Proposition 3.1, 2) is reformed as follows;

Theorem 3.2. *Let $f: R \to S$ be a ring homomorphism of a commutative ring R into a commutative ring S with $f(1)=1$. If $\sigma: R \rightharpoondown F$ is a signature of R such that $\mathrm{G}(\sigma)$ is a torsion group and $\mathrm{Ker} f \subset p_0(\sigma)$, then there exists a signature $\tau: S \rightharpoondown F'$ of ring S with a morphism (f, ξ): $\sigma \to \tau$ in $\boldsymbol{C}_{\mathrm{sig}}$.*

Proof. Suppose that $f: R \to S$ is a ring homomorphism with $f(1) = 1$, and $\sigma: R \rightharpoondown F$ is a signature of R with torsion group $\mathrm{G}(\sigma)$ and satisfying $\mathrm{Ker} f \subset p_0(\sigma)$. By Proposition 3.1, 2), for the surjective ring homomorphism $f: R \to \mathrm{Im} f$, there exists a signature σ': $\mathrm{Im} f \rightharpoondown F$ of the subring $\mathrm{Im} f$ of S with a morphism (f, I_P): $\sigma \to \sigma'$ in $\boldsymbol{C}_{\mathrm{sig}}$. Hence, we may assume that R is a subring of S with common identity, and it is sufficient to show that there exists a signature $\tau: S \rightharpoondown F'$ of S with a morphism (ι, ξ): $\sigma \to \tau$ in $\boldsymbol{C}_{\mathrm{sig}}$, where ι denotes the inclusion map $R \hookrightarrow S$. By Theorem 2.6, there exists a signature $\bar{\sigma}$: $R \rightharpoondown F''$ of R such that $R_{\bar{\sigma}} = R_\sigma$, $p_0(\bar{\sigma}) = p_0(\sigma)$ and $\mathrm{G}(\bar{\sigma}) \cong \mathrm{G}(\sigma)/H$ for some subgroup H of $\mathrm{G}(\sigma)$ hold, and $\mathrm{P}(\bar{\sigma})$ is a $p_1(\sigma)$-prime of R containing $\mathrm{P}(\sigma)$. Then, we can define a partial homomorphism ξ_1: $F \rightharpoondown F''$ such that ξ_1 induces a group homomorphism $\mathrm{G}(\sigma) \to \mathrm{G}(\bar{\sigma})$ and the pair (I_R, ξ_1) defines a morphism (I_R, ξ_1): $\sigma \to \bar{\sigma}$ in $\boldsymbol{C}_{\mathrm{sig}}$. On the other hand, by Zorn's Lemma, there exists a $p_1(\sigma)$-prime P of S containing $\mathrm{P}(\bar{\sigma})$, and by Theorem 2.1 the $p_1(\sigma)$-prime P defines a signature $\tau: S \rightharpoondown \mathrm{F}(P)$ of S such that $\mathrm{P}(\tau) = P$, $S_\tau = S_P$, $\mathrm{F}(P) = \mathrm{G}(P) \cup \{0\}$ and $\mathrm{G}(P) = S_P^+/P^+$ hold, and τ is induced from the canonical map $S_P^+ \to \mathrm{G}(P)$. From the fact that $\mathrm{P}(\bar{\sigma})$ is a $p_1(\sigma)$-prime of R, and $P \supset \mathrm{P}(\bar{\sigma})$, it follows that $P \cap R = \mathrm{P}(\bar{\sigma})$, $p(P) \cap R = p_0(\bar{\sigma})$ and $P^+ \cap R = \mathrm{P}(\bar{\sigma})^+ (= p_1(\bar{\sigma}))$ hold. Since $\mathrm{G}(\sigma)$ is a torsion group, so is also $\mathrm{G}(\bar{\sigma})$, and by Lemma 2.3 and Proposition 2.2, it is derived that $R_{P(\bar{\sigma})} (= R_{\bar{\sigma}}) = \{a \in R \mid a^n \in \mathrm{P}(\bar{\sigma})$ for some integer $n > 0\}$ is included in $S_P = \{a \in S \mid {}^{\exists} b_0 \in p_1(\sigma) + P,\ {}^{\exists} b_i \in P \cup -P$, $i = 1, 2, \cdots, n;\ \sum_i b_i a^{n-i} = 0$ for some $n > 0\}$. Hence we have that $R^+_{P(\bar{\sigma})} \subset S_P^+$, and the natural homomorphism $\mathrm{G}(\mathrm{P}(\bar{\sigma})) = R^+_{P(\bar{\sigma})}/\mathrm{P}(\sigma)^+ \to \mathrm{G}(P) = S_P^+/P^+$; $[a] \leadsto [a]$ defines a partial homomorphism ξ_2: $F'' \rightharpoondown \mathrm{F}(P)$ such that (ι, ξ_2): $\bar{\sigma} \to \tau$ is a morphism in $\boldsymbol{C}_{\mathrm{sig}}$. Thus, we obtain a signature $\tau: S \rightharpoondown F' = \mathrm{F}(P)$ of ring S and a morphism $(\iota, \xi_2 \circ \xi_1) = (\iota, \xi_2) \circ (\mathrm{I}_R, \xi_1)$: $\sigma \to \tau$ in $\boldsymbol{C}_{\mathrm{sig}}$.

ideal $p_0(\sigma)$, that is, every element in $R_\sigma \backslash p_0(\sigma)$ is invertible in R_σ. Then, $a \in p_0(\sigma)$ if and only if $a^{-1} \notin R_\sigma$.

Proof. 1) For elements $x, y \in R$, we suppose that $xR_\sigma y \subset p_0(\sigma)$ and $x \notin p_0(\sigma)$. If $x \notin R_\sigma$, then there is an $x' \in p_0(\sigma)$ with $x'x \in p_1(\sigma)$ or $xx' \in p_1(\sigma)$. Since both $x'xR_\sigma y$ and $xx'R_\sigma y$ are included in $p_0(\sigma)$, we may assume that $x \in R_\sigma$, and similary $y \in R_\sigma$. Then, $y \in p_0(\sigma)$ follows. 2) Suppose that $a \in R_\sigma \backslash p_0(\sigma)$. If $a^{-1} \notin R_\sigma$, then there is a $b \in p_0(\sigma)$ with $a^{-1}b \in p_1(\sigma)$ or $ba^{-1} \in p_1(\sigma)$, so it means either $a(a^{-1}b)$ or $(ba^{-1})a$ belongs to $p_0(\sigma)$, that is, $a \in p_0(\sigma)$, which is contrary to $a \notin p_0(\sigma)$. Hence, we get $a^{-1} \in R_\sigma \backslash p_0(\sigma)$. 3) First, we suppose that R is commutative. It is easy to see the "only if" part. If $a^{-1} \notin R_\sigma$, there is a $b \in p_0(\sigma)$ with $a^{-1}b \in p_1(\sigma)$, so by 1) $a(a^{-1}b) \in p_0(\sigma)$ implies $a \in p_0(\sigma)$. Next, we suppose that R_σ is a local ring with maximal ideal $p_0(\sigma)$. If $a^{-1} \notin R_\sigma$ then there is a $b \in p_0(\sigma)$ with $a^{-1}b \in p_1(\sigma)$ or $ba^{-1} \in p_1(\sigma)$, so either $a^{-1}b$ or ba^{-1} is invertible in R_σ. Hence, we get $a \in p_0(\sigma)$.

Lemma 4.2. *For a $\sigma \in \mathrm{X}(R, F)$, put $q(\sigma) = \{a \in R \mid RaR \subset p_0(\sigma)\}$. Then, the following properties hold;*

1) *$q(\sigma)$ is a prime ideal of R, and $q(\sigma) \subset p_0(\sigma)$.*

2) *If R is a local ring with maximal ideal $q(\sigma)$ then so is R_σ with maximal ideal $p_0(\sigma)$. If R is commutative, then the converse also holds.*

3) *If $p_0(\sigma) = \{0\}$, then $R = R_\sigma$, and $\mathrm{P}(\sigma)$ gives a partial ordering on the ring R.*

Proof. 1) It is easy to see that $q(\sigma)$ is an ideal of R, and $q(\sigma) \subset p_0(\sigma)$. For $x, y \in R$, we suppose that $xRy \subset q(\sigma)$ and $x \notin q(\sigma)$. We can find elements a and b in R with $axb \notin p_0(\sigma)$, so it follows that $axbR_\sigma(RyR) \subset p_0(\sigma)$ and $RyR \subset p_0(\sigma)$ by Lemma 4.1, 1), i.e. $y \in q(\sigma)$. 2) If R is a local ring with maximal ideal $q(\sigma)$, then every element in $R_\sigma \backslash p_0(\sigma)$ $(\subset R \backslash q(\sigma))$ is invertible in R, and by Lemma 4.1, 2), so is also in R_σ. Hence, R_σ is a local ring with maximal ideal $p_0(\sigma)$. If R is commutative and R_σ a local ring with maximal ideal $p_0(\sigma)$, then for any element $x \in R \backslash q(\sigma)$, we can find an element $a \in R$ such that $ax \in R_\sigma \backslash p_0(\sigma)$, that is, ax is invertible in R_σ, so x is invertible in R. 3) is easy.

Corollary 4.3. *Assume that R is a division ring, then the following hold.*

1) *For any $\sigma \in \mathrm{X}(R, F)$, R_σ is a local ring with maximal ideal $p_0(\sigma)$.*

2) *$\mathrm{X}(R, F)$ is a Hausdorff and totally disconnected space.*

3) *If F is a finite set, then $\mathrm{X}(R, F)$ is compact, that is, a Boolean space.*

Proof. 1) is obtained by Lemma 4.2, 2). 2) By Lemma 4.1, 3), it follows that $\mathrm{H}_0(a) = \mathrm{H}_\infty(a^{-1})$ is a clopen set of $\mathrm{X}(R, F)$ for any $a \neq 0$ in R, and so is also $\mathrm{H}_\gamma(a)$ for any $\gamma \in F \cup \{\infty\}$ and $a \in R$. Hence, $\mathrm{X}(R, F)$ is Hausdorff and totally disconnected. 3) Suppose that F is finite, then $(F \cup \{\infty\})^R$ is compact. Whenever $F \cup \{\infty\}$ is a discrete space, the subset $\mathrm{X}(R, F)$ becomes a closed subset of $(F \cup \{\infty\})^R$. Hence, under our topology on $F \cup \{\infty\}$, $\mathrm{X}(R, F)$ is also

REMARK 3.3. Let $\sigma: R \rightarrow F$ and $\tau: S \rightarrow F'$ be signatures of rings R and S. If (f, ξ): $\sigma \rightarrow \tau$ is a morphism in C_{sig}, then the following identities hold; 1) $R_\sigma = f^{-1}(S_\tau)$, 2) if $G(\sigma)$ is a group, then $p_0(\sigma) = f^{-1}(p_0(\tau))$ and $\bigcup_{\alpha \in \xi^{-1}(\beta)} p_\alpha(\sigma) = f^{-1}(p_\beta(\tau))$ for each $\beta \in G(\tau)$.

Proof. 1) It is easy that $R_\sigma \subset f^{-1}(S_\tau)$. To prove the opposite, we suppose that there is an $x \in R \backslash R_\sigma$ with $f(x) \in S_\tau$. Then, there is a $y \in p_0(\sigma)$ such that $xy \in p_1(\sigma)$ or $yx \in p_1(\sigma)$ hold. However, $xy \in p_1(\sigma)$ (resp. $yx \in p_1(\sigma)$) implies $\tau(f(xy)) = \xi(\sigma(xy)) = 1$ (resp. $\tau(f(yx)) = 1$) which is contrary to that $\tau(f(xy)) = \tau(f(x))\tau(f(y)) = \tau(f(x))\xi(\sigma(y)) = \tau(f(x))\xi(0) = \tau(f(x))0 = 0$ (resp. $\tau(f(yx)) = 0$). Hence, we get $R_\sigma = f^{-1}(S_\tau)$. 2) It is also easy that $p_0(\sigma) \subset f^{-1}(p_0(\tau))$. If $x \in f^{-1}(p_0(\tau))$, then we have $\xi(\sigma(x)) = \tau(f(x)) = 0$ and $\sigma(x) = 0$, i.e. $x \in p_0(\sigma)$, since $G(\sigma)$ is a group and $\xi(1) = 1$. Hence, we get $p_0(\sigma) = f^{-1}(p_0(\tau))$. Since $R_\sigma = f^{-1}(S_\tau)$ and $p_0(\sigma) = f^{-1}(p_0(\tau))$, it follows that $R_\sigma \backslash p_0(\sigma) = \bigcup_{\alpha \in G(\sigma)} p_\alpha(\sigma) = f^{-1}(S_\tau \backslash p_0(\tau)) = \bigcup_{\beta \in G(\tau)} f^{-1}(p_\beta(\tau))$. Since $\bigcup_{\alpha \in \xi^{-1}(\beta)} p_\alpha(\sigma) \subset f^{-1}(p_\beta(\tau))$ holds for every $\beta \in G(\tau)$, we get $\bigcup_{\alpha \in \xi^{-1}(\beta)} p_\alpha(\sigma) = f^{-1}(p_\beta(\tau))$ for evry $\beta \in G(\tau)$.

4. Space of signatures

In this secition, we assume that F is a f-semigroup with abelian torsion group F^*. Let R be any ring with identity 1, and $X(R, F)$ denote the set of signatures σ: $R \rightarrow F$ of the ring R over the f-semigroup F. We consider a set $F \cup \{\infty\}$ which is added a formal symbol ∞ to F. We make the set $F \cup \{\infty\}$ a topological space such that $\{\alpha\}$ and $\{\infty\}$ are open subsets for every $\alpha \in F^*$. Then, for any subset $H \subset F \cup \{\infty\}$, H is a closed subset if and only if $0 \in H$. Considering R as a descrete space, we make the power space $(F \cup \{\infty\})^R$ have a weak topology. We can introduce a topology on $X(R, F)$ as a subspace of $(F \cup \{\infty\})^R$. For any $\alpha \in F$ and $a \in R$, we put $H_\alpha(a) = \{\sigma \in X(R, F) \mid \sigma(a) = \alpha\}$ and $H_\infty(a) = \{\sigma \in X(R, F) \mid a \notin R_\sigma\}$. Then, for every finite subsets $\{a_1, a_2, \cdots, a_n\} \subset R$ and $\{\gamma_1, \gamma_2, \cdots, \gamma_n\} \subset F^* \cup \{\infty\}$, the intersections $H_{\gamma_1}(a_1) \cap H_{\gamma_2}(a_2) \cap \cdots \cap H_{\gamma_n}(a_n)$ construct an open basis of the space $X(R, F)$. Furthermore, for a subset $H \subset F \cup \{\infty\}$ and $a \in R$, we have that $\bigcup_{\alpha \in H} H_\alpha(a)$ is a closed subset of $X(R, F)$ if and only if $0 \in H$.

In the following lemmata and corollary, we need not assume that F^* is a torsion group.

Lemma 4.1. *For a $\sigma \in X(R, F)$ and an invertible element a in R, the following statements hold;*

1) *For any $x, y \in R$, $xR_\sigma y \subset p_0(\sigma)$ implies either $x \in p_0(\sigma)$ or $y \in p_0(\sigma)$.*
2) *$a \in R_\sigma \backslash p_0(\sigma)$ if and only if $a^{-1} \in R_\sigma \backslash p_0(\sigma)$*
3) *Assume that either R is commutative or R_σ is a local ring with maximal*

compact.

Proposition 4.4. *Assume that R is a commutative ring and σ, $\tau \in \mathrm{X}(R, F)$. If $\mathrm{P}(\sigma) \subset \mathrm{P}(\tau)$ holds, then there are a subgroup H of $\mathrm{G}(\sigma)$ and a homomorphism $\psi: H \to \mathrm{G}(\tau)$ such that $p_\beta(\tau) \cap R_\sigma \subset \bigcup_{\alpha \in \psi^{-1}(\beta)} p_\alpha(\sigma) \subset p_0(\tau) \cup p_\beta(\tau)$ holds for every $\beta \in \mathrm{G}(\tau)$, and $R_\sigma \subset R_\tau$ holds.*

Proof. Suppose that $\mathrm{P}(\sigma) \subset \mathrm{P}(\tau)$. Since $\mathrm{G}(\sigma)$ and $\mathrm{G}(\tau)$ are torsion groups, by Lemma 2.3, we get $R_\sigma \subset R_\tau$. We put $H = \{\alpha \in \mathrm{G}(\sigma) \mid p_\alpha(\sigma) \not\subset p_0(\tau)\}$, then H is a subgroup of $\mathrm{G}(\sigma)$. We can define a homomorphism $\psi: H \to \mathrm{G}(\tau)$ as follows; For any $\alpha \in H$, we can find an element a in $p_\alpha(\sigma) \backslash p_0(\tau)$, and $\tau(a) = \tau(x)$ holds for every $x \in p_\alpha(\sigma) \backslash p_0(\tau)$. Because, α^{-1} belongs to H, so we can find a b in $p_{\alpha^{-1}}(\sigma) \backslash p_0(\tau)$, which satisfies $\sigma(ab) = \sigma(xb) = 1$ for every $x \in p_\alpha(\sigma) \backslash p_0(\tau)$. The condition $\mathrm{P}(\sigma) \subset \mathrm{P}(\tau)$ means that for every $x \in p_\alpha(\sigma) \backslash p_0(\tau)$, $\tau(ab) = \tau(xb) = 1$ holds, so $\tau(a) = \tau(x)$. Therefore, we can define the image $\psi(\alpha)$ of α as $\tau(a)$ for $a \in p_\alpha(\sigma) \backslash p_0(\tau)$. Then, it is easy to see that the map $\psi: H \to \mathrm{G}(\tau)$ is a group homomorphism. Further, for any $\alpha \in H$ and $\beta \in \mathrm{G}(\tau)$ with $\psi(\alpha) = \beta$, from the definition of ψ, $p_\alpha(\sigma) \subset p_0(\tau) \cup p_\beta(\tau)$ follows. Hence, we get $\bigcup_{\alpha \in \psi^{-1}(\beta)} p_\alpha(\sigma) \subset p_0(\tau) \cup p_\beta(\tau)$. On the other hand, if β is an element in $\mathrm{G}(\tau)$ with $p_\beta(\tau) \cap R_\sigma \neq \phi$, then for each $x \in p_\beta(\tau) \cap R_\sigma$, there is an $\alpha \in \mathrm{G}(\sigma)$ with $x \in p_\alpha(\sigma) \backslash p_0(\tau)$, that is, $\psi(\alpha) = \beta$ and $x \in p_\alpha(\sigma)$. Hence, we get $p_\beta(\tau) \cap R_\sigma \subset \bigcup_{\alpha \in \psi^{-1}(\beta)} p_\alpha(\sigma)$ for every $\beta \in \mathrm{G}(\tau)$.

REMARK 4.5. Let R be a commutative ring, and $\sigma: R \to F$ a signature of R. By σ, a topology on affine n-space R^n is introduced as follows; For any $\gamma_i \in \mathrm{G}(\sigma) \cup \{\infty\}$ and $f_i(X_1, X_2, \cdots, X_n)$ in polynomial ring $R[X_1, X_2, \cdots, X_n]$, $i = 1, 2, \cdots, m$, we put $\mathrm{U}(f_1, f_2, \cdots, f_m, \gamma_1, \gamma_2, \cdots, \gamma_m) = \{(a_1, a_2, \cdots, a_n) \in R^n \mid \sigma(f_i(a_1, a_2, \cdots, a_n)) = \gamma_i, i = 1, 2, \cdots, n\}$, where $\sigma(f_i(a_1, a_2, \cdots, a_n)) = \infty$ whenever $f_i(a_1, a_2, \cdots, a_n) \notin R_\sigma$. Then, the sets $\mathrm{U}(f_1, f_2, \cdots, f_m, \gamma_1, \gamma_2, \cdots, \gamma_m)$ form an open basis on R^n. We can define a continuous map ψ_σ of the topologicl space R^n into $\mathrm{X}(R[X_1, X_2, \cdots, X_n], F)$; Let $(a_1, a_2, \cdots, a_n)$ be any element in R^n, and let $\psi_{(a_1, a_2, \cdots, a_n)}: R[X_1, X_2, \cdots, X_n] \to R$; $f(X_1, X_2, \cdots, X_n) \rightsquigarrow f(a_1, a_2, \cdots, a_n)$ a natural ring homomorphism. By Proposition 3.1, 1), there exists a signature $\sigma_{(a_1, a_2, \cdots, a_n)}: R[X_1, X_2, \cdots, X_n] \to F$ with a morphism $(\psi_{(a_1, a_2, \cdots, a_n)}, \mathrm{I}_F): \sigma_{(a_1, a_2, \cdots a_n)} \to \sigma$ in $\boldsymbol{C}_{\mathrm{sig}}$. Thus, we get a map $\psi_\sigma: R^n \to \mathrm{X}(R[X_1, X_2, \cdots, X_n], F)$; $(a_1, a_2, \cdots, a_n) \rightsquigarrow \sigma_{(a_1, a_2, \cdots, a_n)}$, which is continuous, because of $\psi_\sigma^{-1}(\mathrm{H}_\gamma(f)) = \mathrm{U}(f, \gamma)$ for $f \in R[X_1, X_2, \cdots, X_n]$ and $\gamma \in \mathrm{G}(\sigma) \cup \{\infty\}$.

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