



Title	Notes on signatures on rings
Author(s)	Kanzaki, Teruo
Citation	Osaka Journal of Mathematics. 1985, 22(2), p. 327-338
Version Type	VoR
URL	https://doi.org/10.18910/9811
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NOTES ON SIGNATURES ON RINGS

Dedicated to Professor Hiroshi Nagao on his 60th birthday

TERUO KANZAKI

(Received June 21, 1984)

0. Introduction

The notion of infinite prime introduced by Harrison [3] was investigated in [1], [2], [7] and [9] which were concerned with ordering on a field. In this note, we study about signatures on rings as some generalization of infinite primes and signatures of fields in [2]. In the section 1, we introduce notions of U -prime and signature of a ring which are generalizations of infinite prime and signature of field. In the section 2, we show that a U -prime of a commutative ring defines a signature on the ring. In the sections 3 and 4, we consider the category of signatures and a space of signatures on a ring which include notions of extension of signature and space of ordering on fields (cf. [2] and [8]), and investigate them. Throughout this paper, we assume that every ring has identity 1.

1. Preliminaries, definitions and notations

Let S be a multiplicative semigroup, and T a normal subsemigroup of S , (cf. [6], p. 195), denoted by $T \triangleleft S$, that is, T is a subsemigroup of S which satisfies 1) for $x, y \in S$, $xy \in T$ implies $yx \in T$, 2) if there is an $x \in T$ with $xy \in T$, then $y \in T$, and 3) for every $x \in S$, there exists an $x' \in S$ with $x'x \in T$. We can define a binary relation \sim on S ; for $x, y \in S$, $x \sim y$ if and only if there is a $z \in S$ such that both zx and zy are contained in T . Then, the relation \sim is an equivalence relation on S , and is compatible with the multiplication of S , so the quotient set S/\sim , denoted by S/T , makes a group such that the canonical map $\psi: S \rightarrow S/T$; $x \mapsto [x]$ is a homomorphism with $\text{Ker } \psi = T$.

Let R be any ring with identity 1, and P a preprime of R ([3]), that is, P is closed under addition and multiplication of R and $-1 \notin P$. We put $p(P) = P \cap -P$, $R_p = \{x \in R \mid xp(P) \cup p(P)x \subset p(P)\}$, $R_p^+ = R_p \setminus p(P)$ ($:= \{x \in R_p \mid x \notin p(P)\}$), $P^+ = P \setminus p(P)$ ($= P \setminus -P$). We shall say a preprime P to be *complete quasi-prime*, if it satisfies the following conditions;

- 1) $p(P)$ is an ideal of R_p such that $R_p/p(P)$ is an integral domain,
- 2) $P^+ \triangleleft R_p^+$ under the multiplication of R_p .

3) P is complete in R_P , that is, for $x \in R_P$, $x^2 \in P$ implies $x \in P \cup -P$.

A multiplicative semigroup F with unit element 1 and zero element 0 will be called a *f-semigroup*, if $F^* = F \setminus \{0\}$ makes a group with a unique element of order 2, denoted by -1 , under the multiplication of F . If P is a complete quasi-prime of R , then the quotient group $G(P) = R_P^+ / P^+$ has a unique element $[-1]$ of order 2, and the formally composed semigroup $F(P) = G(P) \cup \{0\}$ makes an f-semigroup under the multiplication of $G(P)$ and $\alpha 0 = 0\alpha = 00 = 0$ for $\alpha \in G(P)$. Furthermore, we can define a map $\sigma: R_P \rightarrow F(P)$ by $\sigma(a) = 0$ or $[a]$ for $a \in p(P)$ or $a \in R_P^+$, respectively. Then, it can be verified that 1) $\sigma(-1) = [-1]$, 2) $\sigma(ab) = \sigma(a)\sigma(b)$ for every $a, b \in R_P$, and 3) for $a, b \in R_P$, either $\sigma(a) = 0$ or $\sigma(a) = \sigma(b)$ implies $\sigma(a+b) = \sigma(b)$.

Let π be a set of prime numbers, and suppose $2 \in \pi$. A complete quasi-prime P will be called a π -complete quasi-prime, if for each $q \in \pi$, there is a $\zeta_q \in R_P \setminus P$ such that $\zeta_q^q \in P$ and for any $x \in R_P$ with $x^q \in P$, $yx \in \bigcup_{1 \leq i \leq q} \zeta_q P^i$ for some $y \in P^+$.

REMARK 1.1. If R is a commutative ring and P is a π -complete quasi-prime, then for each $q \in \pi$, the q -torsion subgroup $G(P)_q = \{\alpha \in G(P) \mid \alpha^n = 0; \alpha^{q^n} = [1]\}$ of $G(P)$ is isomorphic to a subgroup of $\mathbf{Z}(q^\infty)$. Because, since $G(P)_q$ has a unique minimal non trivial subgroup $\langle [\zeta_q] \rangle$, $G(P)_q$ is indecomposable, so by [4], p. 22, Theorem 10, $G(P)_q$ is isomorphic to $\mathbf{Z}(q^n)$ or $\mathbf{Z}(q^\infty)$.

Let R be a ring with identity 1, and F an abelian f-semigroup. A partial map $\sigma: R \rightarrow F$ will be called a *signature* of R with domain of definition R_σ , if σ is a map of a subset R_σ of R into F satisfying the following conditions;

- (S 1) $-1 \in R_\sigma$ and $\sigma(-1) = -1$,
- (S 2) $a, b \in R_\sigma$ implies $ab \in R_\sigma$ and $\sigma(ab) = \sigma(a)\sigma(b)$,
- (S 3) for $a, b \in R_\sigma$, if $\sigma(a) = 0$ or $\sigma(a) = \sigma(b)$ then $a+b \in R_\sigma$ and $\sigma(a+b) = \sigma(b)$,
- (S 4) for $a \in R$, if $a \notin R_\sigma$, then there exists a $b \in R_\sigma$ such that $\sigma(b) = 0$ and either $\sigma(ab) = 1$ or $\sigma(ba) = 1$.

Let $\sigma: R \rightarrow F$ be a signature. For $\alpha \in F$, we put $p_\alpha(\sigma) = \{x \in R_\sigma \mid \sigma(x) = \alpha\}$, $P(\sigma) = p_0(\sigma) \cup p_1(\sigma)$ and $G(\sigma) = \text{Im } \sigma \cap F^*$.

Lemma 1.2. Let $\sigma: R \rightarrow F$ be a signature of a ring R .

- 1) R_σ is a subring of R with prime ideal $p_0(\sigma)$ such that $R_\sigma/p_0(\sigma)$ is an integral domain.
- 2) $P(\sigma)$ is a preprime of R , and $R_\sigma = R_{P(\sigma)}$.
- 3) If $G(\sigma)$ is a subgroup of F^* , then $P(\sigma)$ is a complete quasi-prime of R , and $G(P(\sigma))$ and $G(\sigma)$ are group isomorphic.

Proof. 1) If R_σ is closed under the addition of R , then it is easy to see

that R_σ is a subring of R . Suppose $a+b \notin R_\sigma$ for some a and b in R_σ . There is a $c \in R_\sigma$ such that $\sigma(c)=0$, and $\sigma(c(a+b))=1$ or $\sigma((a+b)c)=1$. Since $\sigma(ca)=\sigma(ac)=\sigma(a)\sigma(c)=0$ and $\sigma(cb)=\sigma(bc)=0$, we get $\sigma(ca+cb)=\sigma(ac+bc)=0$ which is a contradiction. Hence, we get $R_\sigma+R_\sigma \subset R_\sigma$. It is easy to see that $p_0(\sigma)$ is an ideal of R_σ , and $R_\sigma/p_0(\sigma)$ is an integral domain. 2) From the definition of signature, it follows that $P(\sigma)$ is a preprime of R and $p_0(\sigma)=P(\sigma) \cap -P(\sigma)$. We shall show $R_\sigma=R_{P(\sigma)}$. Since $R_\sigma \subset R_{P(\sigma)}$ is clear, it suffices to show $R_\sigma \supset R_{P(\sigma)}$. If $x \in R \setminus R_\sigma$, then there is a $y \in p_0(\sigma)$ with $xy \in p_1(\sigma)$ or $yx \in p_1(\sigma)$, so $x p_0(\sigma) \cup p_0(\sigma) x \not\subset p_0(\sigma)$, that is, $x \notin R_{P(\sigma)}$. 3) If $G(\sigma)$ is a group, then it is easy to see that $P(\sigma)^+ = p_1(\sigma)$, $P(\sigma)^+ \triangleleft R_{P(\sigma)}^+$, $\sigma(R_{P(\sigma)}^+) = G(\sigma)$, and $P(\sigma)$ is complete. Furthermore, a map $G(P(\sigma)) = R_{P(\sigma)}^+/P(\sigma)^+ \rightarrow G(\sigma)$; $[x] \mapsto \sigma(x)$ is a group isomorphism.

REMARK. 1) If R is a field, then a signature $\sigma: R \rightarrow F$ with $p_0(\sigma) = \{0\}$ and $F = \mu \cup \{0\}$ coincides with the notion of signature defined by Becker, Harman and Rosenberg [2], where μ is the group of all roots of unity in the complices. 2) Let F be a finite field with characteristic $\neq 2$. The multiplicative semigroup F is an abelian f-semigroup. For a signature $\sigma: R \rightarrow F$, let π be the set of all prime factors of order $|G(\sigma)|$. Then, it is easy to see that $P(\sigma)$ is a π -complete quasiprime of R .

Let R be a ring with identity 1, and U a non empty multiplicatively closed subset of R satisfying $U \cap -U = \emptyset$. A preprime P of R will be called a U -preprime of R , if $U \subset P$ and $P \cap -U = \emptyset$. A maximal U -preprime of R will be called a U -prime of R . Any Harrison's infinite prime is a $\{1\}$ -prime.

Lemma 1.3. *Let U a non empty multiplicatively closed subset of R with $U \cap -U = \emptyset$, and P a U -prime of R . If either R is commutative or $Px = xP$ and $Ux = xU$ hold for every $x \in R_P^+$, then P is a complete quasi-prime of R .*

The proof of this lemma is obtained by checking the following facts;

(1.3.1) $U+P \subset P^+$.

(1.3.2) For $x \in R_P$ ($x \in R$, if R is commutative), if there are $u \in U$ and $y \in P$ with $(u+y)x \in P$, then $x \in P$. Hence $1 \in P$.

(1.3.3) For $x \in R_P$ ($x \in R$, if R is commutative), if $x \notin p(P)$, then there is an $x' \in (\pm P)[x]$ with $x'x \in U+P$, where $(\pm P)[x] = \{\sum_i a_i x^i \in R \mid a_i \in P \cup -P\}$.

(1.3.4) $R_P/p(P)$ is an integral domain.

(1.3.5) For $x, y \in R_P$, $xy \in P^+$ implies $yx \in P^+$.

(1.3.6) P is complete in R_P .

(1.3.7) For any $x \in P^+$, there is an $x' \in P^+$ with $x'x \in U+P$.

(1.3.8) For $x \in R_P$ ($x \in R$, if R is commutative), if there is a $y \in P^+$ with $yx \in P^+$, then $x \in P^+$.

The proofs of these statements are obtained similarly to the case of Harrison's

infinite prime; (1.3.1): Since $U \cap -P = \phi$, it follows that $U \subset P^+$ and $U+P \subset P^+$. (1.3.2): A subset $P' = \{x \in R_p \mid {}^a u \in U, {}^a y \in P; (u+y)x \in P\}$ of R is closed under addition and multiplication. Because, if $x_1, x_2 \in P'$, there are $u_i \in U$ and $y_i \in P$ with $(u_i + y_i)x_i \in P$, $i = 1, 2$. If either x_1 or x_2 belongs to $p(P)$, then it is trivial that $x_1 + x_2$ and $x_1 x_2$ belong to P' . Otherwise, by assumption, there are $u'_2 \in U$ and $y'_2 \in P$ such that $x_1 u_2 = u'_2 x_1$ and $x_1 y_2 = y'_2 x_1$. Then $(u_1 + y_1)(u_2 + y_2)$ and $(u_1 + y_1)(u'_2 + y'_2)$ belong to $U+P$, and $(u_1 + y_1)(u_2 + y_2)(x_1 + x_2)$ and $(u_1 + y_1)(u'_2 + y'_2)x_1 x_2$ are in P . Furthermore, it is immediately seen that $P \subset P'$ and $P' \cap -U = \phi$, so we get $P = P'$. (1.3.3): For $x \in R_p$, if $x \notin p(P)$, then either $x \notin P$ or $-x \notin P$. By assumption, a subset $P[x] = P + Px + Px^2 + \dots$, (resp. $P[-x] = P + P(-x) + P(-x)^2 + \dots$) of R is closed under addition and multiplication. Since $P \subsetneq P[x]$ or $P \subsetneq P[-x]$, we get $P[x] \cap -U \neq \phi$ or $P[-x] \cap -U \neq \phi$, so we can find an element $y \in (\pm P)[x]$ such that $yx \in U+P$ holds. (1.3.4): For $x, y \in R_p$, suppose that $xy \in p(P)$ and $x \notin p(P)$. By (1.3.3), there is an $x' \in (\pm P)[x] (\subset R_p)$ with $x'x \in U+P$, and (1.3.2) derives that $x'xy \in p(P)$ implies $y \in p(P)$. (1.3.5): For $x, y \in R_p$, suppose $xy \in P^+$. $(xy)x$ is in $Px = xP$, and for an element x' in $(\pm P)[x]$, also in R_p , with $x'x \in U+P$, we get $(x'x)yx \in x'xP \subset P$, so $yx \in P^+$ by (1.3.2) and (1.3.4). (1.3.6) is easy. (1.3.7): If $x \in P^+$, then $P[-x] = P - Px$ is closed under addition and multiplication, and $P \subsetneq P[-x]$. Hence, there are $u \in U$ and $x', y \in P$ with $-u = y - x'x$, so we get $x'x = u + y \in U+P$ and $x' \in P^+$. (1.3.8) is immediately obtained from (1.3.2) and (1.3.7).

2. The connection between U -prime and signature

Theorem 2.1. *Let R be a commutative ring with identity 1, and U any non empty multiplicatively closed subset of R with $U \cap -U = \phi$. If P is a U -prime of R , then there exists a signature $\sigma: R \rightarrow F$ with $P(\sigma) = P$ and group $G(\sigma) = G(P)$.*

Proof. By Lemma 1.3, U -prime P is a complete quasi-prime of R , so it defines a map $\sigma: R_p \rightarrow F(P)$. Then, we put $R_\sigma = R_p$ and $F = F(P)$. The conditions (S 1), (S 2) and (S 3) of signature were verified. (S 4) is proved in the following proposition. Then we have a signature $\sigma: R \rightarrow F$ with $P = P(\sigma)$ and $G(\sigma) = G(P) = R_p^+/P^+$.

Proposition 2.2. *Let P be a U -prime of a commutative ring R , and let $A_P = \{a \in R \mid {}^a b_0 \in U+P, {}^a b_i \in P \cup -P, i = 1, 2, \dots, n; \sum_{i=0}^n b_i a^{n-i} = 0\}$.*

- 1) $(R_p, p(P))$ is a valuation pair of R , (cf. [3], Proposition. 2.5).
- 2) If $x \in R \setminus p(P)$ then there is an $a \in A_P$ with $ax \in U+P$.
- 3) If x and y are elements of R with $xy \in U+P$, then $x \notin p(P)$ implies $y \in A_P$.
- 4) $R_P = A_P$.

Proof. The proof of 1) is quite similar to [3], Proposition 2.5. 2) If $x \in R \setminus p(P)$, by (1.3.3) there is an $a \in (\pm P)[x]$ with $ax \in U+P$, then a can be

represented as $-(b_1 + b_2x + \dots + b_nx^{n-1})$ for some $b_i \in P \cup -P$. If we put $ax = b_0$, then a satisfies an equation $b_0a^n + b_1b_0a^{n-1} + \dots + b_nb_0^n = 0$ with $b_0 \in U+P$ and $b_ib_0^i \in P \cup -P$, $i = 1, 2, \dots, n$, so $a \in A_P$. 3) Suppose that x and y are in R and $xy \in U+P$. If $x \notin p(P)$, by 2), there is a $z \in A_P$ with $zx \in U+P$. Since $z \in A_P$, there are $a_0 \in U+P$ and $a_i \in P \cup -P$, $i = 1, 2, \dots, m$, with $\sum_{i=0}^m a_iz^{m-i} = 0$. Put $xy = b_0$ and $zx = c_0$, so we get that $\sum_{i=0}^m (a_iz_0^{m-i}b_0^i)y^{m-i} = (\sum_{i=0}^m a_iz^{m-i})b_0^m = 0$, $a_0c_0^m \in U+P$ and $a_iz_0^{m-i}b_0^i \in P \cup -P$, hence $y \in A_P$. 4) In the first place, we show $A_P \supset R_P$: Let x be any element in R_P . If $x \in p(P)$, $x \in A_P$ is obvious. Otherwise, by (1.3.3) there is a $y \in (\pm P)[x]$ with $xy \in U+P$, so $y \notin p(P)$ and by 3) we get $x \in A_P$. Now, we show $A_P = R_P$: Let $(U+P)^{-1}R$ be the ring of quotients of R with respect to $U+P$, and $\psi: R \rightarrow (U+P)^{-1}R$ the canonical ring homomorphism. Then, $(U+P)^{-1}R_P$ may be regarded as a subring of $(U+P)^{-1}R$. By B' , we denote the integral closure of $(U+P)^{-1}R_P$ in $(U+P)^{-1}R$. There is a prime ideal Q' of B' which lies over $(U+P)^{-1}R_P p(P)$, (cf. [5], (10.8)). It follows that $B = \psi^{-1}(B')$ is a subring of R with $B \supset A_P \supset R_P$, and $Q = \psi^{-1}(Q')$ is a prime ideal of B with $Q \cap R_P = p(P)$. By 1), we get $B = A_P = R_P$.

Lemma 2.3. *Let R be a commutative ring, and $\sigma: R \rightarrow F$ a signature. If $G(\sigma)$ is a torsion group, then $R_\sigma = \{a \in R \mid a^n \in P(\sigma) \text{ for some integer } n > 0\}$.*

Proof. Since $G(\sigma)$ is a torsion group, it is clear that any element a in R_σ has a positive integer n with $a^n \in P(\sigma)$. Conversely, suppose that an element $a \in R$ does not belong to R_σ . There is a $b \in p_0(\sigma)$ with $ab \in p_1(\sigma)$. Then a^n is not contained in $P(\sigma)$ for every positive integer n . Because, if $a^n \in P(\sigma)$ for some $n > 0$, it derives a contradiction $1 = \sigma((ab)^n) = \sigma(a^n)\sigma(b^n) = 0$.

Let R be a ring with identity 1. By [1], a preprime P is called a torsion preprime (resp. 2-torsion preprime) of R , if for each $a \in R$ there exists a positive integer n such that $a^n \in P$ (resp. $a^{2^n} \in P$) holds. From Theorem 2.1 and Lemma 2.3, the following corollaries immediately follow;

Corollary 2.4. *Let R be a commutative ring with 1 and U a non empty multiplicatively closed subset of R with $1 \in U$ and $U \cap -U = \emptyset$.*

1) *If P is a torsion U -prime of R , then $p(P)$ is an ideal of R , i.e. $R_P = R$, so there is a signature $\sigma: R \rightarrow F$ such that $P = P(\sigma)$, $R = R_\sigma$ and $G(\sigma)$ is a torsion group.*

2) *If P is a 2-torsion U -prime of R , then there is a signature $\sigma: R \rightarrow F$ such that $P = P(\sigma)$, $R = R_\sigma$ and $F^* \cong \mathbb{Z}(2^\infty)$.*

In particular, on a field, we have

Corollary 2.5. *Let K be a field.*

1) *For any signature $\sigma: K \rightarrow F$, K_σ is a valuation ring of K with maximal ideal $p_0(\sigma)$, and the residue field $k(\sigma) = K_\sigma/p_0(\sigma)$ has an induced signature $\bar{\sigma}: k(\sigma)$*

$\neg F$ with $k(\sigma)_{\bar{\sigma}} = k(\sigma)$ and $\mathfrak{p}_0(\bar{\sigma}) = \{\bar{0}\}$, and $P(\bar{\sigma})$ is a preordering on $k(\sigma)$.

2) Let U be a non empty multiplicatively closed subset of K with $U \cap -U = \emptyset$. If P is a U -prime of K , K_P is a valuation ring of K with maximal ideal $\mathfrak{p}(P)$. If P is a torsion U -prime of K , then $K = K_P$, $\mathfrak{p}(P) = \{0\}$, and P is a preordering, i.e. $P^+ = P \setminus \{0\}$ is a subgroup of $K^* = K \setminus \{0\}$, (cf. [1], (3.3)).

3) If O is a real valuation ring of K with maximal ideal \mathfrak{p} , i.e. the residue field O/\mathfrak{p} is a formally real field, then there is a signature $\sigma: K \rightarrow \text{GF}(3)$ with $K_{\sigma} = O$ and $\mathfrak{p}_0(\sigma) = \mathfrak{p}$, where $\text{GF}(3) = \{0, 1, -1\}$ is a multiplicative semigroup of prime field with characteristic 3.

Theorem 2.6. Let R be a ring with identity 1, and $\sigma: R \rightarrow F$ a signature of R . Assume that $G(\sigma)$ is a torsion group and $x\mathfrak{p}_{\alpha}(\sigma) = \mathfrak{p}_{\alpha}(\sigma)x$ holds for all $x \in R_{\sigma} \setminus \mathfrak{p}_0(\sigma)$ and $\alpha \in G(\sigma) \cup \{0\}$. Then, there exists a signature $\tau: R \rightarrow F'$ of R satisfying the following conditions;

- 1) $P(\tau)$ is a $\mathfrak{p}_1(\sigma)$ -prime of R and $P(\tau) \supset P(\sigma)$,
- 2) $R_{\tau} = R_{\sigma}$ and $\mathfrak{p}_0(\tau) = \mathfrak{p}_0(\sigma)$,
- 3) there is a subgroup H of $G(\sigma)$ such that $\mathfrak{p}_1(\tau) = \sigma^{-1}(H)$, $-1 \notin H$ and $G(\sigma)/H \cong G(\tau)$ hold.

Proof. Since $P(\sigma)$ is a $\mathfrak{p}_1(\sigma)$ -preprime of R , by Zorn's Lemma there exists a $\mathfrak{p}_1(\sigma)$ -prime P of R containing $P(\sigma)$. From the facts that $P \cap -\mathfrak{p}_1(\sigma) = \emptyset$ and $\mathfrak{p}_0(\sigma) \subset \mathfrak{p}(P)$, we can derive that $\mathfrak{p}_0(\sigma) = \mathfrak{p}(P)$ and $R_P = R_{\sigma}$; If there is an element $x \in R_P \setminus R_{\sigma}$, then there exists a $y \in \mathfrak{p}_0(\sigma)$ such that either xy or yx belongs to $\mathfrak{p}_1(\sigma)$. However, xy and yx are also contained in $\mathfrak{p}(P)$, so these are contrary to $\mathfrak{p}_1(\sigma) \cap \mathfrak{p}(P) = \emptyset$. Hence, we get $R_P \subset R_{\sigma}$. Furthermore, if there is an element $x \in \mathfrak{p}(P) \setminus \mathfrak{p}_0(\sigma)$, we have $x^n \in \mathfrak{p}_1(\sigma) \cap \mathfrak{p}(P)$ for some integer $n > 0$, which is a contradiction. Therefore, we get $\mathfrak{p}_0(\sigma) = \mathfrak{p}(P)$ and $R_P = R_{\sigma}$. Now, we put $H = \sigma(P^+)$, so H is a subgroup of $G(\sigma)$. We shall show $P^+ = \sigma^{-1}(H)$; If $x \in \sigma^{-1}(H)$ then there is a $y \in P^+$ with $\sigma(x) = \sigma(y)$. Since $y^n \in \mathfrak{p}_1(\sigma)$ for some integer $n > 0$, we have $xy^n = (xy^{n-1})y \in x\mathfrak{p}_1(\sigma) \cap P^+$. Hence, for any $x \in R$, it follows that $x \in \sigma^{-1}(H)$ if and only if $x\mathfrak{p}_1(\sigma) \cap P^+ \neq \emptyset$. On the other hand, we can show that $P = \{x \in R_{\sigma} \mid x\mathfrak{p}_1(\sigma) \cap P \neq \emptyset\}$; The set $P' = \{x \in R_{\sigma} \mid x\mathfrak{p}_1(\sigma) \cap P \neq \emptyset\}$ is closed under addition and multiplication: Because, for $x, y \in P'$, there are $x_1, y_1 \in \mathfrak{p}_1(\sigma)$ such that both xx_1 and yy_1 are in P . Since we may suppose that y is not in $\mathfrak{p}_0(\sigma)$, there is an $x'_1 \in \mathfrak{p}_1(\sigma)$ with $x_1y = yx'_1$, and it follows that both $(x+y)(x_1y_1)$ and $(xy)(x'_1y_1)$ are contained in P . Hence, both $x+y$ and xy belong to P' . Furthermore, it is derived that $P \subset P'$ and $P' \cap -\mathfrak{p}_1(\sigma) = \emptyset$, because of $P \cap -\mathfrak{p}_1(\sigma) = \emptyset$. Hence, we get $P = P'$. Accordingly, we conclude that $\sigma^{-1}(H) = P^+ = \bigcup_{\alpha \in H} \mathfrak{p}_{\alpha}(\sigma)$.

From the assumption $x\mathfrak{p}_{\alpha}(\sigma) = \mathfrak{p}_{\alpha}(\sigma)x$ for $x \in R_{\sigma} \setminus \mathfrak{p}_0(\sigma)$ and $\alpha \in G(\sigma) \cup \{0\}$, P is a complete quasi-prime of R . Therefore, we can define a signature $\tau: R \rightarrow F(P)$ such that $R_{\tau} = R_P = R_{\sigma}$, $\mathfrak{p}_0(\tau) = \mathfrak{p}(P) = \mathfrak{p}_0(\sigma)$ and $G(\tau) = G(P) \cong G(\sigma)/H$.

It is easy to check the conditions of signature for τ .

Corollary 2.7. *Let R be a commutative ring with identity 1. If $\sigma: R \rightarrow F$ is a signature of R such that $G(\sigma)$ is a 2-torsion group, then $P(\sigma)$ is a $p_1(\sigma)$ -prime of R .*

Proof. Since $G(\sigma)$ is a 2-torsion group, by Remark 1.1 every non-trivial subgroup H of $G(\sigma)$ contains -1 . By Theorem 1.7, $P(\sigma)$ is a $p_1(\sigma)$ -prime of R .

Corollary 2.8. *Let S be a commutative ring with identity 1, and R a subring of S containing 1. If $\sigma: R \rightarrow F$ a signature of R such that $G(\sigma)$ is 2-torsion group, then σ can be extended to a signature $\tau: S \rightarrow F'$ of S , i.e. $S_\tau \cap R = R_\sigma$ and $P(\tau) \cap R = P(\sigma)$ hold.*

Proof. A signature $\tau: S \rightarrow F'$ is defined by a $p_1(\sigma)$ -prime P of S containing $P(\sigma)$. Then, τ is an extension of σ .

3. Category of signatures

Let $\sigma_1: R_1 \rightarrow F_1$ and $\sigma_2: R_2 \rightarrow F_2$ be signatures of rings R_1 and R_2 . Suppose that $f: R_1 \rightarrow R_2$ is a ring homomorphism such that $f(1)=1$ and $f(R_{1\sigma_1}) \subset R_{2\sigma_2}$, and that $\xi: F_1 \rightarrow F_2$ is a partial homomorphism which is defined on $G(\sigma_1)$ and satisfies $\xi(0)=0$, $\xi(-1)=-1$ and $\xi(\alpha\beta)=\xi(\alpha)\xi(\beta)$ if ξ is defined on α, β and $\alpha\beta$ for $\alpha, \beta \in F_1$. Then, the pair (f, ξ) will be called a morphism of signatures of σ_1 to σ_2 , denoted by $(f, \xi): \sigma_1 \rightarrow \sigma_2$, if it satisfies $\xi(\sigma_1(x)) = \sigma_2(f(x))$ for all $x \in R_{1\sigma_1}$. Let $\sigma_i: R_i \rightarrow F_i$ and $\sigma'_i: R'_i \rightarrow F'_i$ be signatures of rings for $i=1, 2$, and $(f, \xi): \sigma_1 \rightarrow \sigma_2$ and $(f', \xi'): \sigma'_1 \rightarrow \sigma'_2$ morphisms of signatures. We define the equality of morphisms that $(f, \xi) = (f', \xi')$ if and only if $\sigma_i = \sigma'_i$ (i.e. $R_i = R'_i$, $R_{i\sigma_i} = R'_{i\sigma'_i}$, $F_i = F'_i$ and $\sigma_i(x) = \sigma'_i(x)$ for all $x \in R_{i\sigma_i}$) for $i=1, 2$, $f=f'$ and for every $\alpha \in G(\sigma_1) = G(\sigma'_1)$, $\xi(\alpha) = \xi'(\alpha)$ hold. By C_{sig} , we denote the category of signatures in which objects are signatures of rings and morphisms are morphisms of signatures.

Proposition 3.1. *Let R and S be rings with identity 1, and $f: R \rightarrow S$ a ring homomorphism with $f(1)=1$.*

1) *If $\tau: S \rightarrow F$ is a signature of ring S with $\text{Im } f \supset p_0(\tau)$, then there exists a signature $\sigma: R \rightarrow F$ of ring R with a morphism $(f, I_F): \sigma \rightarrow \tau$ in C_{sig} .*

2) *If $f: R \rightarrow S$ is surjective, and if $\sigma: R \rightarrow F$ is a signature of ring R with $\text{Ker } f \subset p_0(\sigma)$, then there exists a signature $\tau: S \rightarrow F$ of ring S with a morphism $(f, I_F): \sigma \rightarrow \tau$ in C_{sig} .*

Proof. 1) Suppose that $\tau: S \rightarrow F$ is a signature of ring S and $f: R \rightarrow S$ is a ring homomorphism with $f(1)=1$ and $\text{Im } f \supset p_0(\tau)$. On a subring $R_\sigma = \{x \in R \mid f(x) \in S_\tau\}$ of R , a map $\sigma: R_\sigma \rightarrow F$; $x \mapsto \tau(f(x))$ is defined. The condition

$\text{Im } f \supset p_0(\tau)$ derives that a signature $\sigma: R \rightarrow F$ of ring R and a morphism $(f, I_F): \sigma \rightarrow \tau$ in \mathbf{C}_{sig} are defined. 2) Suppose that $f: R \rightarrow S$ is a surjective ring homomorphism, and $\sigma: R \rightarrow F$ is a signature of ring R with $\text{Ker } f \subset p_0(\sigma)$. For a subring $S_\tau = f(R_\sigma)$, we can define a map $\tau: S_\tau \rightarrow F$ as follows: For any $a \in S_\tau$, there is a $b \in R_\sigma$ with $f(b) = a$, then we put $\tau(a) = \sigma(b)$. From the condition $\text{Ker } f \subset p_0(\sigma)$, it is known that the map $\tau: S_\tau \rightarrow F$ is well defined. Then, it is easy to see that a signature $\tau: S \rightarrow F$ of ring S and a morphism $(f, I_F): \sigma \rightarrow \tau$ in \mathbf{C}_{sig} are defined.

Concerning commutative rings, the situation of Proposition 3.1, 2) is reformed as follows;

Theorem 3.2. *Let $f: R \rightarrow S$ be a ring homomorphism of a commutative ring R into a commutative ring S with $f(1)=1$. If $\sigma: R \rightarrow F$ is a signature of R such that $G(\sigma)$ is a torsion group and $\text{Ker } f \subset p_0(\sigma)$, then there exists a signature $\tau: S \rightarrow F'$ of ring S with a morphism $(f, \xi): \sigma \rightarrow \tau$ in \mathbf{C}_{sig} .*

Proof. Suppose that $f: R \rightarrow S$ is a ring homomorphism with $f(1)=1$, and $\sigma: R \rightarrow F$ is a signature of R with torsion group $G(\sigma)$ and satisfying $\text{Ker } f \subset p_0(\sigma)$. By Proposition 3.1, 2), for the surjective ring homomorphism $f: R \rightarrow \text{Im } f$, there exists a signature $\sigma': \text{Im } f \rightarrow F$ of the subring $\text{Im } f$ of S with a morphism $(f, I_F): \sigma \rightarrow \sigma'$ in \mathbf{C}_{sig} . Hence, we may assume that R is a subring of S with common identity, and it is sufficient to show that there exists a signature $\tau: S \rightarrow F'$ of S with a morphism $(\iota, \xi): \sigma \rightarrow \tau$ in \mathbf{C}_{sig} , where ι denotes the inclusion map $R \hookrightarrow S$. By Theorem 2.6, there exists a signature $\bar{\sigma}: R \rightarrow F''$ of R such that $R_{\bar{\sigma}} = R_\sigma$, $p_0(\bar{\sigma}) = p_0(\sigma)$ and $G(\bar{\sigma}) \cong G(\sigma)/H$ for some subgroup H of $G(\sigma)$ hold, and $P(\bar{\sigma})$ is a $p_1(\sigma)$ -prime of R containing $P(\sigma)$. Then, we can define a partial homomorphism $\xi_1: F \rightarrow F''$ such that ξ_1 induces a group homomorphism $G(\sigma) \rightarrow G(\bar{\sigma})$ and the pair (I_R, ξ_1) defines a morphism $(I_R, \xi_1): \sigma \rightarrow \bar{\sigma}$ in \mathbf{C}_{sig} . On the other hand, by Zorn's Lemma, there exists a $p_1(\sigma)$ -prime P of S containing $P(\bar{\sigma})$, and by Theorem 2.1 the $p_1(\sigma)$ -prime P defines a signature $\tau: S \rightarrow F(P)$ of S such that $P(\tau) = P$, $S_\tau = S_P$, $F(P) = G(P) \cup \{0\}$ and $G(P) = S_P^+ / P^+$ hold, and τ is induced from the canonical map $S_P^+ \rightarrow G(P)$. From the fact that $P(\bar{\sigma})$ is a $p_1(\sigma)$ -prime of R , and $P \supset P(\bar{\sigma})$, it follows that $P \cap R = P(\bar{\sigma})$, $p(P) \cap R = p_0(\bar{\sigma})$ and $P^+ \cap R = P(\bar{\sigma})^+ (= p_1(\bar{\sigma}))$ hold. Since $G(\sigma)$ is a torsion group, so is also $G(\bar{\sigma})$, and by Lemma 2.3 and Proposition 2.2, it is derived that $R_{P(\bar{\sigma})} (= R_{\bar{\sigma}}) = \{a \in R \mid a^n \in P(\bar{\sigma}) \text{ for some integer } n > 0\}$ is included in $S_P = \{a \in S \mid {}^a b_0 \in p_1(\sigma) + P, {}^a b_i \in P \cup -P, i=1, 2, \dots, n; \sum_i b_i a^{n-i} = 0 \text{ for some } n > 0\}$. Hence we have that $R_{P(\bar{\sigma})}^+ \subset S_P^+$, and the natural homomorphism $G(P(\bar{\sigma})) = R_{P(\bar{\sigma})}^+ / P(\sigma)^+ \rightarrow G(P) = S_P^+ / P^+$; $[a] \mapsto [a]$ defines a partial homomorphism $\xi_2: F'' \rightarrow F(P)$ such that $(\iota, \xi_2): \bar{\sigma} \rightarrow \tau$ is a morphism in \mathbf{C}_{sig} . Thus, we obtain a signature $\tau: S \rightarrow F' = F(P)$ of ring S and a morphism $(\iota, \xi_2 \circ \xi_1) = (\iota, \xi_2) \circ (I_R, \xi_1): \sigma \rightarrow \tau$ in \mathbf{C}_{sig} .

ideal $p_0(\sigma)$, that is, every element in $R_\sigma \setminus p_0(\sigma)$ is invertible in R_σ . Then, $a \in p_0(\sigma)$ if and only if $a^{-1} \notin R_\sigma$.

Proof. 1) For elements $x, y \in R$, we suppose that $xR_\sigma y \subset p_0(\sigma)$ and $x \notin p_0(\sigma)$. If $x \notin R_\sigma$, then there is an $x' \in p_0(\sigma)$ with $x'x \in p_1(\sigma)$ or $xx' \in p_1(\sigma)$. Since both $x'R_\sigma y$ and $xx'R_\sigma y$ are included in $p_0(\sigma)$, we may assume that $x \in R_\sigma$, and similarly $y \in R_\sigma$. Then, $y \in p_0(\sigma)$ follows. 2) Suppose that $a \in R_\sigma \setminus p_0(\sigma)$. If $a^{-1} \notin R_\sigma$, then there is a $b \in p_0(\sigma)$ with $a^{-1}b \in p_1(\sigma)$ or $ba^{-1} \in p_1(\sigma)$, so it means either $a(a^{-1}b)$ or $(ba^{-1})a$ belongs to $p_0(\sigma)$, that is, $a \in p_0(\sigma)$, which is contrary to $a \notin p_0(\sigma)$. Hence, we get $a^{-1} \in R_\sigma \setminus p_0(\sigma)$. 3) First, we suppose that R is commutative. It is easy to see the "only if" part. If $a^{-1} \notin R_\sigma$, there is a $b \in p_0(\sigma)$ with $a^{-1}b \in p_1(\sigma)$, so by 1) $a(a^{-1}b) \in p_0(\sigma)$ implies $a \in p_0(\sigma)$. Next, we suppose that R_σ is a local ring with maximal ideal $p_0(\sigma)$. If $a^{-1} \notin R_\sigma$ then there is a $b \in p_0(\sigma)$ with $a^{-1}b \in p_1(\sigma)$ or $ba^{-1} \in p_1(\sigma)$, so either $a^{-1}b$ or ba^{-1} is invertible in R_σ . Hence, we get $a \in p_0(\sigma)$.

Lemma 4.2. For a $\sigma \in X(R, F)$, put $q(\sigma) = \{a \in R \mid RaR \subset p_0(\sigma)\}$. Then, the following properties hold;

- 1) $q(\sigma)$ is a prime ideal of R , and $q(\sigma) \subset p_0(\sigma)$.
- 2) If R is a local ring with maximal ideal $q(\sigma)$ then so is R_σ with maximal ideal $p_0(\sigma)$. If R is commutative, then the converse also holds.
- 3) If $p_0(\sigma) = \{0\}$, then $R = R_\sigma$, and $P(\sigma)$ gives a partial ordering on the ring R .

Proof. 1) It is easy to see that $q(\sigma)$ is an ideal of R , and $q(\sigma) \subset p_0(\sigma)$. For $x, y \in R$, we suppose that $xRy \subset q(\sigma)$ and $x \notin q(\sigma)$. We can find elements a and b in R with $axb \notin p_0(\sigma)$, so it follows that $axbR_\sigma(RyR) \subset p_0(\sigma)$ and $RyR \subset p_0(\sigma)$ by Lemma 4.1, 1), i.e. $y \in q(\sigma)$. 2) If R is a local ring with maximal ideal $q(\sigma)$, then every element in $R_\sigma \setminus p_0(\sigma) (\subset R \setminus q(\sigma))$ is invertible in R , and by Lemma 4.1, 2), so is also in R_σ . Hence, R_σ is a local ring with maximal ideal $p_0(\sigma)$. If R is commutative and R_σ a local ring with maximal ideal $p_0(\sigma)$, then for any element $x \in R \setminus q(\sigma)$, we can find an element $a \in R$ such that $ax \in R_\sigma \setminus p_0(\sigma)$, that is, ax is invertible in R_σ , so x is invertible in R . 3) is easy.

Corollary 4.3. Assume that R is a division ring, then the following hold.

- 1) For any $\sigma \in X(R, F)$, R_σ is a local ring with maximal ideal $p_0(\sigma)$.
- 2) $X(R, F)$ is a Hausdorff and totally disconnected space.
- 3) If F is a finite set, then $X(R, F)$ is compact, that is, a Boolean space.

Proof. 1) is obtained by Lemma 4.2, 2). 2) By Lemma 4.1, 3), it follows that $H_0(a) = H_\infty(a^{-1})$ is a clopen set of $X(R, F)$ for any $a \neq 0$ in R , and so is also $H_\gamma(a)$ for any $\gamma \in F \cup \{\infty\}$ and $a \in R$. Hence, $X(R, F)$ is Hausdorff and totally disconnected. 3) Suppose that F is finite, then $(F \cup \{\infty\})^R$ is compact. Whenever $F \cup \{\infty\}$ is a discrete space, the subset $X(R, F)$ becomes a closed subset of $(F \cup \{\infty\})^R$. Hence, under our topology on $F \cup \{\infty\}$, $X(R, F)$ is also

REMARK 3.3. Let $\sigma: R \rightarrow F$ and $\tau: S \rightarrow F'$ be signatures of rings R and S . If $(f, \xi): \sigma \rightarrow \tau$ is a morphism in \mathcal{C}_{sig} , then the following identities hold; 1) $R_\sigma = f^{-1}(S_\tau)$, 2) if $G(\sigma)$ is a group, then $p_0(\sigma) = f^{-1}(p_0(\tau))$ and $\bigcup_{\alpha \in \xi^{-1}(\beta)} p_\alpha(\sigma) = f^{-1}(p_\beta(\tau))$ for each $\beta \in G(\tau)$.

Proof. 1) It is easy that $R_\sigma \subset f^{-1}(S_\tau)$. To prove the opposite, we suppose that there is an $x \in R \setminus R_\sigma$ with $f(x) \in S_\tau$. Then, there is a $y \in p_0(\sigma)$ such that $xy \in p_1(\sigma)$ or $yx \in p_1(\sigma)$ hold. However, $xy \in p_1(\sigma)$ (resp. $yx \in p_1(\sigma)$) implies $\tau(f(xy)) = \xi(\sigma(xy)) = 1$ (resp. $\tau(f(yx)) = 1$) which is contrary to that $\tau(f(xy)) = \tau(f(x))\tau(f(y)) = \tau(f(x))\xi(\sigma(y)) = \tau(f(x))\xi(0) = \tau(f(x))0 = 0$ (resp. $\tau(f(yx)) = 0$). Hence, we get $R_\sigma = f^{-1}(S_\tau)$. 2) It is also easy that $p_0(\sigma) \subset f^{-1}(p_0(\tau))$. If $x \in f^{-1}(p_0(\tau))$, then we have $\xi(\sigma(x)) = \tau(f(x)) = 0$ and $\sigma(x) = 0$, i.e. $x \in p_0(\sigma)$, since $G(\sigma)$ is a group and $\xi(1) = 1$. Hence, we get $p_0(\sigma) = f^{-1}(p_0(\tau))$. Since $R_\sigma = f^{-1}(S_\tau)$ and $p_0(\sigma) = f^{-1}(p_0(\tau))$, it follows that $R_\sigma \setminus p_0(\sigma) = \bigcup_{\alpha \in G(\sigma)} p_\alpha(\sigma) = f^{-1}(S_\tau \setminus p_0(\tau)) = \bigcup_{\beta \in G(\tau)} f^{-1}(p_\beta(\tau))$. Since $\bigcup_{\alpha \in \xi^{-1}(\beta)} p_\alpha(\sigma) \subset f^{-1}(p_\beta(\tau))$ holds for every $\beta \in G(\tau)$, we get $\bigcup_{\alpha \in \xi^{-1}(\beta)} p_\alpha(\sigma) = f^{-1}(p_\beta(\tau))$ for every $\beta \in G(\tau)$.

4. Space of signatures

In this section, we assume that F is a f -semigroup with abelian torsion group F^* . Let R be any ring with identity 1, and $X(R, F)$ denote the set of signatures $\sigma: R \rightarrow F$ of the ring R over the f -semigroup F . We consider a set $F \cup \{\infty\}$ which is added a formal symbol ∞ to F . We make the set $F \cup \{\infty\}$ a topological space such that $\{\alpha\}$ and $\{\infty\}$ are open subsets for every $\alpha \in F^*$. Then, for any subset $H \subset F \cup \{\infty\}$, H is a closed subset if and only if $0 \in H$. Considering R as a discrete space, we make the power space $(F \cup \{\infty\})^R$ have a weak topology. We can introduce a topology on $X(R, F)$ as a subspace of $(F \cup \{\infty\})^R$. For any $\alpha \in F$ and $a \in R$, we put $H_\alpha(a) = \{\sigma \in X(R, F) \mid \sigma(a) = \alpha\}$ and $H_\infty(a) = \{\sigma \in X(R, F) \mid a \notin R_\sigma\}$. Then, for every finite subsets $\{a_1, a_2, \dots, a_n\} \subset R$ and $\{\gamma_1, \gamma_2, \dots, \gamma_n\} \subset F^* \cup \{\infty\}$, the intersections $H_{\gamma_1}(a_1) \cap H_{\gamma_2}(a_2) \cap \dots \cap H_{\gamma_n}(a_n)$ construct an open basis of the space $X(R, F)$. Furthermore, for a subset $H \subset F \cup \{\infty\}$ and $a \in R$, we have that $\bigcup_{\alpha \in H} H_\alpha(a)$ is a closed subset of $X(R, F)$ if and only if $0 \in H$.

In the following lemmata and corollary, we need not assume that F^* is a torsion group.

Lemma 4.1. *For a $\sigma \in X(R, F)$ and an invertible element a in R , the following statements hold;*

- 1) *For any $x, y \in R$, $xR_\sigma y \subset p_0(\sigma)$ implies either $x \in p_0(\sigma)$ or $y \in p_0(\sigma)$.*
- 2) *$a \in R_\sigma \setminus p_0(\sigma)$ if and only if $a^{-1} \in R_\sigma \setminus p_0(\sigma)$*
- 3) *Assume that either R is commutative or R_σ is a local ring with maximal*

compact.

Proposition 4.4. *Assume that R is a commutative ring and $\sigma, \tau \in X(R, F)$. If $P(\sigma) \subset P(\tau)$ holds, then there are a subgroup H of $G(\sigma)$ and a homomorphism $\psi: H \rightarrow G(\tau)$ such that $p_\beta(\tau) \cap R_\sigma \subset \bigcup_{\alpha \in \psi^{-1}(\beta)} p_\alpha(\sigma) \subset p_0(\tau) \cup p_\beta(\tau)$ holds for every $\beta \in G(\tau)$, and $R_\sigma \subset R_\tau$ holds.*

Proof. Suppose that $P(\sigma) \subset P(\tau)$. Since $G(\sigma)$ and $G(\tau)$ are torsion groups, by Lemma 2.3, we get $R_\sigma \subset R_\tau$. We put $H = \{\alpha \in G(\sigma) \mid p_\alpha(\sigma) \not\subset p_0(\tau)\}$, then H is a subgroup of $G(\sigma)$. We can define a homomorphism $\psi: H \rightarrow G(\tau)$ as follows; For any $\alpha \in H$, we can find an element a in $p_\alpha(\sigma) \setminus p_0(\tau)$, and $\tau(a) = \tau(x)$ holds for every $x \in p_\alpha(\sigma) \setminus p_0(\tau)$. Because, α^{-1} belongs to H , so we can find a b in $p_{\alpha^{-1}}(\sigma) \setminus p_0(\tau)$, which satisfies $\sigma(ab) = \sigma(xb) = 1$ for every $x \in p_\alpha(\sigma) \setminus p_0(\tau)$. The condition $P(\sigma) \subset P(\tau)$ means that for every $x \in p_\alpha(\sigma) \setminus p_0(\tau)$, $\tau(ab) = \tau(xb) = 1$ holds, so $\tau(a) = \tau(x)$. Therefore, we can define the image $\psi(\alpha)$ of α as $\tau(a)$ for $a \in p_\alpha(\sigma) \setminus p_0(\tau)$. Then, it is easy to see that the map $\psi: H \rightarrow G(\tau)$ is a group homomorphism. Further, for any $\alpha \in H$ and $\beta \in G(\tau)$ with $\psi(\alpha) = \beta$, from the definition of ψ , $p_\alpha(\sigma) \subset p_0(\tau) \cup p_\beta(\tau)$ follows. Hence, we get $\bigcup_{\alpha \in \psi^{-1}(\beta)} p_\alpha(\sigma) \subset p_0(\tau) \cup p_\beta(\tau)$. On the other hand, if β is an element in $G(\tau)$ with $p_\beta(\tau) \cap R_\sigma \neq \emptyset$, then for each $x \in p_\beta(\tau) \cap R_\sigma$, there is an $\alpha \in G(\sigma)$ with $x \in p_\alpha(\sigma) \setminus p_0(\tau)$, that is, $\psi(\alpha) = \beta$ and $x \in p_\alpha(\sigma)$. Hence, we get $p_\beta(\tau) \cap R_\sigma \subset \bigcup_{\alpha \in \psi^{-1}(\beta)} p_\alpha(\sigma)$ for every $\beta \in G(\tau)$.

REMARK 4.5. Let R be a commutative ring, and $\sigma: R \rightarrow F$ a signature of R . By σ , a topology on affine n -space R^n is introduced as follows; For any $\gamma_i \in G(\sigma) \cup \{\infty\}$ and $f_i(X_1, X_2, \dots, X_n)$ in polynomial ring $R[X_1, X_2, \dots, X_n]$, $i=1, 2, \dots, m$, we put $U(f_1, f_2, \dots, f_m, \gamma_1, \gamma_2, \dots, \gamma_m) = \{(a_1, a_2, \dots, a_n) \in R^n \mid \sigma(f_i(a_1, a_2, \dots, a_n)) = \gamma_i, i=1, 2, \dots, m\}$, where $\sigma(f_i(a_1, a_2, \dots, a_n)) = \infty$ whenever $f_i(a_1, a_2, \dots, a_n) \notin R_\sigma$. Then, the sets $U(f_1, f_2, \dots, f_m, \gamma_1, \gamma_2, \dots, \gamma_m)$ form an open basis on R^n . We can define a continuous map ψ_σ of the topological space R^n into $X(R[X_1, X_2, \dots, X_n], F)$; Let (a_1, a_2, \dots, a_n) be any element in R^n , and let $\psi_{(a_1, a_2, \dots, a_n)}: R[X_1, X_2, \dots, X_n] \rightarrow R$; $f(X_1, X_2, \dots, X_n) \mapsto f(a_1, a_2, \dots, a_n)$ a natural ring homomorphism. By Proposition 3.1, 1), there exists a signature $\sigma_{(a_1, a_2, \dots, a_n)}: R[X_1, X_2, \dots, X_n] \rightarrow F$ with a morphism $(\psi_{(a_1, a_2, \dots, a_n)}, I_F): \sigma_{(a_1, a_2, \dots, a_n)} \rightarrow \sigma$ in \mathcal{C}_{sig} . Thus, we get a map $\psi_\sigma: R^n \rightarrow X(R[X_1, X_2, \dots, X_n], F)$; $(a_1, a_2, \dots, a_n) \mapsto \sigma_{(a_1, a_2, \dots, a_n)}$, which is continuous, because of $\psi_\sigma^{-1}(H_\gamma(f)) = U(f, \gamma)$ for $f \in R[X_1, X_2, \dots, X_n]$ and $\gamma \in G(\sigma) \cup \{\infty\}$.

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Osaka Women's University
Daisen-cho, 2-1
Sakai, Osaka 590
Japan