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Asano, K.
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# ON ONE-SIDED HEEGAARD SPLITTINGS AND INVOLUTIONS ON A CLASS OF LENS SPACES

KOUHEI ASANO

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**1. Introduction.** Let $F$ be a closed non-orientable surface in the 3-manifold $M$ such that the exterior of a regular neighbourhood of $F$ is homeomorphic to a solid torus. Then the pair $(M,F)$ is called a one-sided Heegaard splitting of $M$ [13]. This technique is useful for studying 3-manifolds which are not sufficiently large, for example [1], [7], [12], [13] and [14]. In this paper, we will give the minimum one-sided Heegaard splitting of lens spaces [Theorem 2.1].

An *involution* $\varphi$ *on a space* $X$ is a homeomorphism from $X$ onto itself such that $\varphi^2$ is the identity on $X$. Two involutions $\varphi$ and $\varphi'$ are said to be *equivalent* to each other, if there exists an autohomeomorphism $\psi$ of $X$ such that $\varphi=\psi\varphi'\psi^{-1}$. By [9], [10], [11] and [12], we can classify the fixed point free involutions on lens spaces $L(1, 0)$, $L(2, 1)$ and $L(4\alpha, 2\alpha-1)$ up to the equivalence. As an application of Theorem 2.1, we consider the fixed point free involutions on a certain family of lens spaces and will obtain

**Theorem 5.1.** *Let* $\mu_1$ *and* $\mu_2$ *be integers such that* $\mu_1\mu_2\neq 0$ *and* $\mu_1\mu_2\neq -2$. *Then the orbit space of a fixed point free involution on* $L(8\mu_1\mu_2-2, 4\mu_1\mu_2-2\mu_1-1)$ *is homeomorphic to a Seifert fiber space.*

In §2, we will give the minimum one-sided Heegaard splitting of $L(2\alpha, \beta)$. Using the lemmas proved in §3, we will find and invariant subspace under an involution on $L(8\mu_1\mu_2-2, 4\mu_1\mu_2-2\mu_1-1)$ [Lemma 4.1]. Finally the proof of Theorem 5.1 will be completed in §5.

Throughout this paper we work in the piecewise linear category. For a subcomplex $X$ of a complex $Y$, the regular neighbourhood of $X$ in $Y$ will denoted by $N(X)$. The boundary, the interior and the closure of a manifold $Q$ will be denoted by $\partial Q$, $\mathring{Q}$ and $\bar{Q}$, respectively.

Two submanifolds $X$ and $Y$ of $Q$ are said to be *parallel,* if there exists an embedding $\psi$: $X\times I\to Q$ such that $\psi(X\times \{0\})=X$ and $\psi^{-}(\partial(X\times I)-X\times \{0\})=Y$, where $I$ denotes the unit interval [1, 0].

A surface $F$ properly embedded in a 3-manifold $Q$ is said to be *compressible* in $Q$, if

1) there exists a disk $D$ such that $D\cap F=\partial D$ and $\partial D$ is essential on $F$, or

2) there exists a 3-ball $E$ in $Q$ such that $\partial E=F$.

We say that $F$ is incompressible in $Q$, if $F$ is not compressible.

Let $V$ and $V'$ be a solid torus of genus 1. Let $m$ and $m'$ be a meridian of $V$ and $V'$. Then a lens space $L(\alpha, \beta)$ of type $(\alpha, \beta)$ is the 3-manifold obtained by gluing $V'$ and $V$ via a homeomorphism $\psi$ from $\partial V'$ onto $\partial V$ such that $\psi m'\sim\alpha l+\beta m$ on $\partial V$.

We call the connected sum of $\lambda$-copies of a projective plane a non-orientable surface of genus $\lambda$.

R. Myers [Notices, vol. 25, 1978, A-607] and B.D. Evans [Notices, vol. 26, 1979, A-308] announced that they classifyed the fixed point free involutions on Seifert fiber spaces which have finite fundamental group. The author wish to thank the refree for bringing this to his attention.

The author would like to express his gratitude to Prof. J.S. Birman for helpful suggestions, and to Prof. F. Hosokawa and Prof. S. Suzuki for valuable discussions during the revision.

**2. One-sided Heegaard splitting of $L(2\alpha, \beta)$.** Let $(2\alpha, \beta)$ be a pair of integers such that $\alpha\beta$ is positive and $|\beta|<2|\alpha|$. According to [3], each $L(2\alpha, \beta)$ contains a non-orientable surface. Let $\lambda$ be the minimum number of genus of non-orientable surfaces which can be embedded in $L(2\alpha, \beta)$. By $F_\lambda$ we denote a non-orientable surface of genus $\lambda$ embedded in $L(2\alpha, \beta)$. If $\lambda>2$ and $F_\lambda$ is compressible, there exists a non-orientable surface of genus smaller than $\lambda$. If $\lambda=2$, $F_\lambda$ is incompressible by [1], [12] and [7]. Hence $F_\lambda$ is incompressible in $L(2\alpha, \beta)$. It follows from [4] that $L(2\alpha, \beta)$-$\mathring{N}(F_\lambda)$ is homeomorphic to a solid torus of genus $\lambda-1$. Thus we can construct $L(2\alpha, \beta)$ by gluing a regular neighbourhood $N(F_\lambda)$ of $F_\lambda$ and a solid torus $V_{\lambda-1}$ of genus $\lambda-1$.

Let $\pi$: $T_{\lambda-1}\rightarrow F_\lambda$ be an orientable double covering of $F_\lambda$. We will consider $N(F_\lambda)$ as the mapping cylinder of $\pi$. For a subcomplex $X$ of $F_\lambda$, we denote the mapping cylinder of $\pi|\pi^{-1}X$ by $M(X)$.

First we will give a description of $F_\lambda$, $T_{\lambda-1}$ and $\pi$. Let $T_{\lambda-1}$ be a closed orientable surface of genus $\lambda-1$ represented in $R^3$ in such a way that it is invariant under the reflection about the $xy$ plane as illustrated in Fig. 2.1. By $\tilde{p}$, $a_1,\cdots,a_{\lambda-1},b_1,\cdots,b_{\lambda-1},\tilde{c}_1,\cdots,\tilde{c}_\lambda$ and $d_1,\cdots,d_{\lambda-1}$, we denote a base point, oriented simple closed curves and arcs, as in Fig. 2.1.

We define a homeomorphism $\iota$: $T_{\lambda-1}\rightarrow T_{\lambda-1}$ by $\iota(x, y, z)=(x, y, -z)$. Suppose that each $N(\tilde{c}_\mu)$ is of the form $S^1\times[-1, 1]$ such that $\iota(x, t)=(x, -t)$, where $x\in S^1$ and $t\in[-1, 1]$. The homeomorphism of $S^1\times[-1, 1]$ onto itself given by $(\exp i\Theta, t)\to(\exp i(\Theta+\pi(t+1)), t)$ induces a homeomorphism $\tau_\mu$ of $N(\tilde{c}_\mu)$, fixed on its boundary. Then $\tau_1\cup\cdots\cup\tau_\lambda$ can be extended to a homeomorphism $\tau$ of $T_{\lambda-1}$ so that $\tau\,|\,T_{\lambda-1}-\mathring{N}(\tilde{c}\cup\cdots\cup\tilde{c}_\lambda)$ is the identity. We choose orientations so that $\tau(a_\mu)\sim a_\mu+b_\mu$ on $T_{\lambda-1}$. Clearly $\tau\cdot\iota$ is an orientation reversing, fixed point free involution on $T_{\lambda-1}$. If we denote the orbit space and the projection of $\tau\cdot\iota$ by $F_\lambda$ and $\pi$, respectively, then $\pi\colon T_{\lambda-1}\to F_\lambda$ is an orientable double cover of a non-orientable surface of genus $\lambda$. Let $p=\pi\tilde{p}$ and $c_\mu=\pi\tilde{c}_\mu$. We take oriented arcs $e_1,\cdots,e_\lambda$ from $p$ to a point in $c_\mu$ on $F_\lambda$, as in Fig. 2.1.

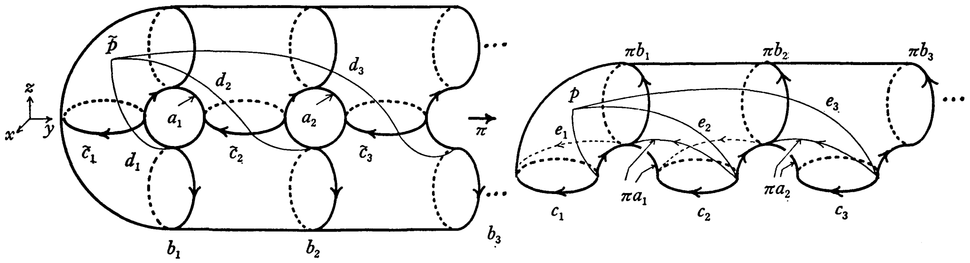


Fig. 2.1

Let $z_\mu$, $\mu=1,\cdots,\lambda$, be the element of $\pi_1(F_\lambda, p)$ represented by $e_\mu c_\mu e_\mu^{-1}$. By $x_\mu$ and $y_\mu$, $\mu=1,\cdots,\lambda-1$, we denote the element of $\pi_1(T_{\lambda-1},\tilde{p})$ represented by $d_\mu a_\mu d_\mu^{-1}$ and $d_\mu b_\mu d_\mu^{-1}$, respectively. Then we can show that $\pi^\sharp(x_\mu)=z_{\mu+1}z_\mu^{-1}$ and $\pi^\sharp(y_\mu)=z_\mu z_1^2\cdots z_{\mu-1}^2 z_\mu$.

Let $(2\alpha, \beta)$ be a pair of relatively prime integers such that $\alpha\beta$ is positive and $|\beta|<2|\alpha|$. For each pair $(2\alpha, \beta)$, we define the function $N(2\alpha, \beta)$ recursively by

$N(2, 1)=N(-2, -1)=1$ and $N(2\alpha, \beta)=N(2\alpha', \beta')+1$, where $\alpha'=\alpha-\beta$, $\beta'\equiv\beta \pmod{2|\alpha'|}$, $\alpha'\beta'$ is positive and $|\beta'|<2|\alpha'|$.

By [3], we can show that $N(2\alpha, \beta)$ is the minimum number of non-orientable surfaces which can be embedded in $L(2\alpha, \beta)$. Furthermore we will define the sequence $\{I_\mu(2\alpha, \beta),\ 1\le\mu\le N(2\alpha, \beta)-1\}$ of integers. Since $N(2, 1)=N(-2, -1)=1$, $\{I_\mu(2, 1)\}$ and $\{I_\mu(-2, -1)\}$ are defined to be $\emptyset$. Assume that we have defined the sequence $\{I_\mu(2\alpha', \beta')\}$. We define $\{I_\mu(2\alpha, \beta)\}$ as follows:

$$I_\mu(2\alpha, \beta)=\begin{cases} I_\mu(2\alpha', \beta') & \text{if } 1\le\mu\le N(2\alpha, \beta)-2\,, \\ I & \text{if } \mu=N(2\alpha, \beta)-1\,, \end{cases}$$

where $I$ denotes the integer such that $\beta=\beta'+2\alpha' I$.

Note that, if we make use of the fact that $|\beta'|<2|\alpha'|$, it follows that $I_\mu(2\alpha, \beta)\neq-1$ for each $\mu$.

**Theorem 2.1.**[†] *Let $\lambda=N(2\alpha, \beta)$ and let $a'_1, \cdots, a'_{\lambda-1}$ be mutually disjoint simple closed curves on $T_{\lambda-1}$ with the following properties:*

1) $a'_\mu \cap (\bigcup_{\nu=1}^{\lambda-1} b_\nu \cup d_\nu) = a'_\mu \cap b_\mu = a'_\mu \cap d_\mu$ .

2) *If* $x'_\mu = d_\mu a'_\mu d_\mu^{-1}$, *then* $x'_\mu = x_\mu y_\mu^{-l_\mu(2\alpha,\beta)}$ .

*Let $V_{\lambda-1}$ be a solid torus of genus $\lambda-1$ with meridian disks $D_1, \cdots, D_{\lambda-1}$. Then the union of $M(F_\lambda)$ and $V_{\lambda-1}$ such that $M(F_\lambda)\cap V_{\lambda-1}=T_{\lambda-1}=\partial V_{\lambda-1}$ and $\partial D_\mu = a'_\mu$, $1\leq\mu\leq\lambda-1$, is homeomorphic to $L(2\alpha, \beta)$.*

Before we state the proof, we summarize notations about a surgery on links in the 3-sphere $S^3$[15]. A *link L with surgery coefficients* is a finite, disjoint collection of oriented simple closed curves $k_1, \cdots, k_\nu$ in $S^3$ with ratio $\gamma_\mu/\delta_\mu$ associated with each component $k_\mu$. Let $l_\mu$ and $m_\mu$ be a longitude and a meridian of $N(k_\mu)$; that is, $l_\mu\sim k_\mu$ in $N(k_\mu)$, $l_\mu\sim 0$ in $S^3-\mathring{N}(k_\mu)$ and the linking number of $m_\mu$ with $k_\mu$ is 1. Let $Q$ be the 3-manifold obtained by replacing each $N(k_\mu)$ by a solid torus $N_\mu$ with a meridian $m'_\mu$, so that $m'_\mu\sim\gamma_\mu m_\mu+\delta_\mu l_\mu$ on $\partial N(k_\mu)$. Then we call $Q$ the *result of a Dehn surgery on L.*

The following lemma is proved in [6].

**Lemma 2.2.** *Let $\gamma_1, \cdots, \gamma_\nu$ be integers and let $L_0$ be a link with surgery coefficients as shown in Fig. 2.2. Then the result of a Dehn surgery on $L_0$ is homeomorphic to $L(\gamma, \delta)$, where*

$$\frac{\gamma}{\delta}=\gamma_\nu-\cfrac{1}{\gamma_{\nu-1}-\cfrac{1}{\ddots\; \gamma_2-\cfrac{1}{\gamma_1}}}\,.$$

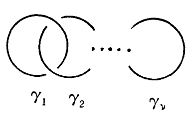


Fig. 2.2

Proof of Theorem 2.1. Let $L_1$ be a trivial link with the components $k_1, \cdots, k_\lambda$ such that the coefficient associated with each $k_\mu$ is 2. Then, if we perform a Dehn surgery on $L_1$, a longitude $l_\mu$ of each $N(k_\mu)$ bounds a Möbius band $M_\mu$ in a solid torus $N_\mu$ by which we have replaced $N(k_\mu)$. In $S^3-\mathring{N}(k_1\cup\cdots\cup k_\lambda)$, there exists a $\lambda$-punctured sphere $S$ such that $\partial S=l_1\cup\cdots\cup l_\lambda$.

By $Q_1$ we denote the result of a Dehn surgery on $L_1$. Assume that $M(F_\lambda)$ is embedded in $Q_1$ so that $F_\lambda=S\cup M_1\cup\cdots\cup M_\lambda$, $M(F_\lambda)=N(S)\cup N_1\cup\cdots\cup N_\lambda$, $c_\mu$ is a centerline of $M_\mu$ and $2c_\mu\sim l_\mu$ in $N_\mu$, $1\leq\mu\leq\lambda$. Then $V=Q_1-\mathring{M}(F_\lambda)$ is a solid torus of genus $\lambda-1$.

For $1\leq\mu\leq\lambda-1$, we take oriented simple closed curves $\hat{a}_1, \cdots, \hat{a}_{\lambda-1}$ in $\mathring{V}$ which is parallel to $a_\mu$. Let $L_2$ be a link obtained from $L_2$ by adding $\hat{a}_1, \cdots,$

† J.S. Birman and J.H, Rubinstein have obtained independently the essentially same result as Theorem 2.1, using a different method.

$\hat{a}_{\lambda-1}$ with coefficients $-I_1(2\alpha, \beta), \cdots, -I_{\lambda-1}(2\alpha, \beta)$. Then we can show that each $a'_\mu$, $1 \leq \mu \leq \lambda-1$, bounds a disk in the result $Q_2$ of a Dehn surgery on $L_2$.

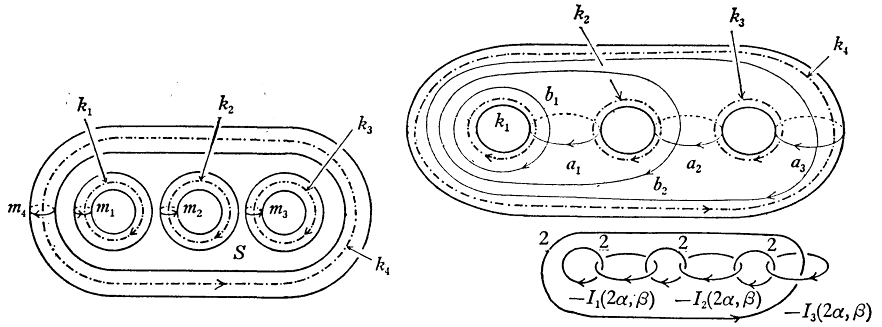


Fig. 2.3

Since $Q_2-M(F_\lambda)$ is a solid torus of genus $\lambda-1$, $Q_2$ is homeomorphic to the union of $M(F_\lambda)$ and $V_{\lambda-1}$ such that $\partial D_\mu = a'_\mu$, $1 \leq \mu \leq \lambda-1$. From the definition of the sequence $\{I_\mu(2\alpha, \beta)\}$ and Lemma 2.2, it follows that $Q_2$ is homeomorphic to $L(2\alpha, \beta)$. The proof is completed.

**Corollary 2.3.** *If $\lambda=3$, there exists a homeomorphism $\psi$ from $L(2\alpha, \beta)$ onto a Seifert fiber space such that each $\psi c_1$, $\psi c_2$ and $\psi c_3$ is a fiber.*

Proof. Let $L$, $L'$ and $L''$ be links with coefficients in $S^3$, as shown in Fig. 2.4.

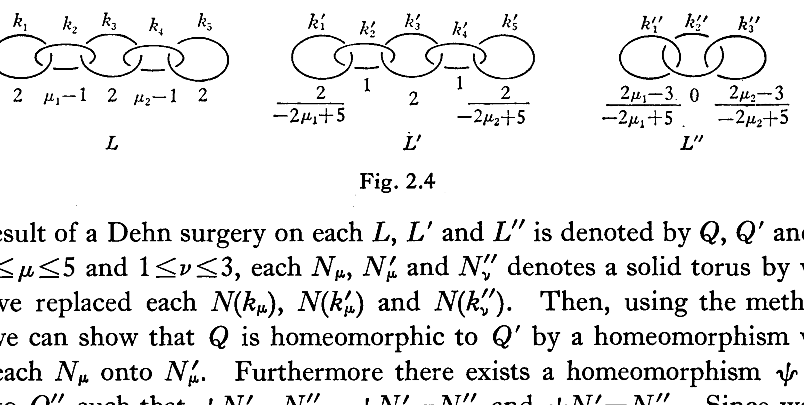


Fig. 2.4

The result of a Dehn surgery on each $L$, $L'$ and $L''$ is denoted by $Q$, $Q'$ and $Q''$. For $1 \leq \mu \leq 5$ and $1 \leq \nu \leq 3$, each $N_\mu$, $N'_\mu$ and $N''_\nu$ denotes a solid torus by which we have replaced each $N(k_\mu)$, $N(k'_\mu)$ and $N(k''_\nu)$. Then, using the method in [15], we can show that $Q$ is homeomorphic to $Q'$ by a homeomorphism which takes each $N_\mu$ onto $N'_\mu$. Furthermore there exists a homeomorphism $\psi$ from $Q'$ onto $Q''$ such that $\psi N'_1 = N''_1$. $\psi N'_3 = N''_2$ and $\psi N'_5 = N''_3$. Since we may consider $Q''$ as a Seifert fiber space having a core of each $N''_\mu$ as a fiber, the proof is completed.

Let $\mu_1 = I_1(2\alpha, \beta)+1$ and $\mu_2 = I_2(2\alpha, \beta)+1$. Then, since $N(2\alpha, \beta)$ is the minimum number of genus of non-orientable surfaces which can be embedded in $L(2\alpha, \beta)$, by [3], we have

**Proposition 2.4.** *The minimum number $\lambda$ of the genus of non-orientable* *surfaces which can be embedded in* $L(2\alpha, \beta)$, *is* 3, *if and only if* $\alpha=4\mu_1\mu_2-1$ *and* $\beta=4\mu_1\mu_2-2\mu_1-1$.

**3. System of curves on $F_3$.** From now on we restrict ourselves to the case that $\lambda=3$. Let $a^*$, $b^*$, $c^*$ and $e_{c^*}$ be oriented simple closed curves and an arc on $F_3$, as shown in Fig. 3.1. Then we have $\{e_1a^*e_1^{-1}\}=z_1z_2$, $\{e_2b^*e_2^{-1}\}=z_2z_3$ and $\{e_{c^*}c^*e_{c^*}^{-1}\}=z_1z_2z_3$ in $\pi_1(F_3, p)$, where $\{c\}$ denotes the element of $\pi_1(F_3, p)$ represented by a $p$-based loop $c$. Note that $N(a^* \cup b^*)$ is an orientable surface of genus 1 and $F-\mathring{N}(a^* \cup b^*)$ is a Möbius band having $c^*$ as a center-line.

For every essential simple closed curve $c$ on $F_3$, there exists a homeomorphism $\rho$ from $F_3$ onto itself which takes $c$ onto either $c^*$, $c_1$, $a^*$, $\partial N(c^*)$ or $\partial N(c_1)$. We say that $c$ is of type I, II, III, IV or V, according as $\rho(c)$ coincides with $c^*$, $c_1$, $a^*$, $\partial N(c^*)$ or $\partial N(c_1)$.

Since an autohomeomorphism of $N(a^* \cup b^*)$ can be extended to $F_3$, there exists a homomorphism from the homeotopy group $\mathcal{H}(N(a^* \cup b^*))$ of $N(a^* \cup b^*)$ into $\mathcal{H}(F_3)$. According to [2], the homomorphism is an isomorphism. More presisely,

**Proposition 3.1.** *Let* $GL(2, Z)$ *be the group of all invertible matices over* $Z$. *Then* $GL(2, Z)$ *is isomorphic to* $\mathcal{H}(F_3)$ *by an isomorphism which maps each matrix* $\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$ *to an isotopy class of an autohomeomorphism* $\rho$ *such that* $\rho a^* \sim \alpha_{11}a^* + \alpha_{12}b^*$ *and* $\rho b^* \sim \alpha_{21}a^* + \alpha_{22}b^*$ *on* $F_3$.

It follows from the above proposition that every simple closed curve of type *I* on $F_3$ is ambient isotopic to $c^*$ or $-c^*$ on $F_3$.

By $a_1^*$, $b_1^*$, $a_2^*$ and $b_2^*$, we denote simple closed curves on $T_2$ such that $\pi^{-1}(a^*)=a_1^* \cup b_2^*$, $\pi^{-1}(b^*)=b_1^* \cup a_2^*$, $\pi^{\#}(\{\tilde{e}_1a_1^*\tilde{e}_1^{-1}\})=z_1z_2$ and $\pi^{\#}(\{\tilde{e}_2b_1^*\tilde{e}_2^{-1}\})=z_2z_3$ [Fig. 3.1].

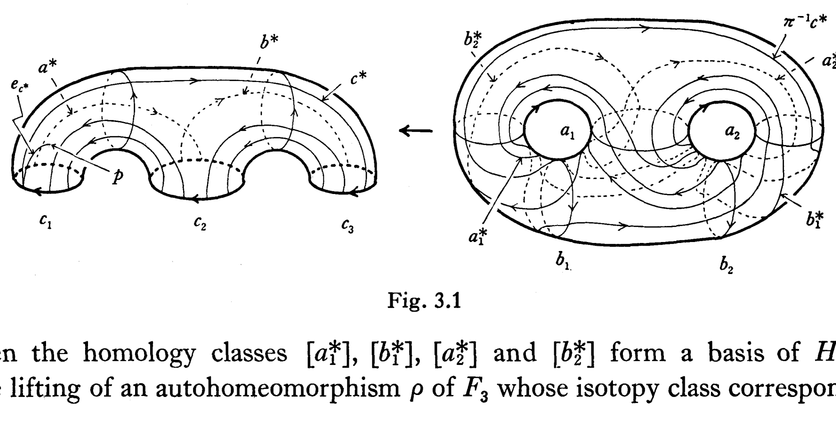


Fig. 3.1

Then the homology classes $[a_1^*]$, $[b_1^*]$, $[a_2^*]$ and $[b_2^*]$ form a basis of $H_1(T_2)$. The lifting of an autohomeomorphism $\rho$ of $F_3$ whose isotopy class corresponds to

$\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$ induces an automorphism $\tilde{\rho}^{\#}$ such that

$$\begin{pmatrix}\tilde{\rho}^{\#}[a_1^*]\\ \tilde{\rho}^{\#}[b_1^*]\\ \tilde{\rho}^{\#}[a_2^*]\\ \tilde{\rho}^{\#}[b_2^*]\end{pmatrix}=\begin{pmatrix}\alpha_{11} & \alpha_{12} & 0 & 0\\ \alpha_{21} & \alpha_{22} & 0 & 0\\ 0 & 0 & \alpha_{22} & \alpha_{21}\\ 0 & 0 & \alpha_{12} & \alpha_{11}\end{pmatrix}\begin{pmatrix}[a_1^*]\\ [b_1^*]\\ [a_2^*]\\ [b_2^*]\end{pmatrix}$$

or

$$\begin{pmatrix}\tilde{\rho}^{\#}[a_1^*]\\ \tilde{\rho}^{\#}[b_1^*]\\ \tilde{\rho}^{\#}[a_2^*]\\ \tilde{\rho}^{\#}[b_2^*]\end{pmatrix}=\begin{pmatrix}0 & 0 & \alpha_{12} & \alpha_{11}\\ 0 & 0 & \alpha_{22} & \alpha_{21}\\ \alpha_{21} & \alpha_{22} & 0 & 0\\ \alpha_{11} & \alpha_{12} & 0 & 0\end{pmatrix}\begin{pmatrix}[a_1^*]\\ [b_1^*]\\ [a_2^*]\\ [b_2^*]\end{pmatrix} \text{ in } H_1(T_2).$$

We have an another basis $\{[a_1'], [b_1], [a_2'], [b_2]\}$ of $H_1(T_2)$ defined in §2. In this paper it is convenient to use the basis $\{[a_1'], [b_1], [a_2'], [b_2]\}$. We now find the matrix associated with $\tilde{\rho}^{\#}$ with respect to $\{[a_1'], [b_1], [a_2'], [b_2]\}$.

**Lemma 3.2.** *Let $\rho$ be a homeomorphism from $F_3$ onto itself whose isotopy clas corresponds to* $\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$. *Then*

$$\begin{pmatrix}\tilde{\rho}^{\#}[a_1']\\ \tilde{\rho}^{\#}[b_1]\\ \tilde{\rho}^{\#}[a_2']\\ \tilde{\rho}^{\#}[b_2]\end{pmatrix}=\begin{pmatrix}\alpha_{11} & \mu_1\alpha_{11}-\alpha_{12}-\mu_1\alpha_{22} & \alpha_{12} & \mu_2\alpha_{12}+\mu_1\alpha_{21}\\ 0 & \alpha_{22} & 0 & -\alpha_{21}\\ \alpha_{21} & \mu_2\alpha_{12}+\mu_1\alpha_{21} & \alpha_{22} & -\mu_2\alpha_{11}-\alpha_{21}+\mu_2\alpha_{22}\\ 0 & -\alpha_{12} & 0 & \alpha_{11}\end{pmatrix}\begin{pmatrix}[a_1']\\ [b_1]\\ [a_2']\\ [b_2]\end{pmatrix}$$

or

$$\begin{pmatrix}\tilde{\rho}^{\#}[a_1']\\ \tilde{\rho}^{\#}[b_1]\\ \tilde{\rho}^{\#}[a_2']\\ \tilde{\rho}^{\#}[b_2]\end{pmatrix}=\begin{pmatrix}-\alpha_{11} & -\mu_1\alpha_{11}-\mu_1\alpha_{22} & -\alpha_{12} & \alpha_{11}-\mu_2\alpha_{12}+\mu_1\alpha_{21}\\ 0 & \alpha_{22} & 0 & -\alpha_{21}\\ -\alpha_{21} & \mu_2\alpha_{12}-\mu_1\alpha_{21}+\alpha_{22} & -\alpha_{22} & -\mu_2\alpha_{11}-\mu_2\alpha_{22}\\ 0 & -\alpha_{12} & 0 & \alpha_{11}\end{pmatrix}\begin{pmatrix}[a_1']\\ [b_1]\\ [a_2']\\ [b_2]\end{pmatrix},$$

*where $\mu_1=I_1(2\alpha, \beta)+1$ and $\mu_2=I_2(2\alpha, \beta)+1$.*

Proof. First we will find the matrix associated with the change of bases. Since $\pi^{\#}(x_1)=z_2z_1^{-1}$, $\pi^{\#}(y_1)=z_1^2$, $\pi^{\#}(x_2)=z_3z_2^{-1}$ and $\pi^{\#}(y_2)=z_2z_1^2z_2$, we can show that $z_1z_2=\pi^{\#}(y_1^{-1}x_1^{-1}y_2)$ and $z_2z_3=\pi^{\#}(x_2^{-1}y_2^{-1}x_1y_1x_1^{-1}y_1^{-1})$. Hence we have $a_1^*\sim a_1-b_1+b_2$ and $b_1^*\sim -a_2-b_2$. The covering transformation of $\pi$ takes $a_1^*$ and $b_1^*$ onto $b_2^*$ and $a_2^*$, respectively. Thus, by using the fact that $z_1^{-1}z_1z_2z_1=\pi^{\#}(x_1y_1)$ and $z_1^{-1}z_2z_3z_1=\pi^{\#}(y_1^{-2}x_1^{-1}y_2x_2x_1y_1)$, we obtain $a_2^*\sim -b_1+a_2+b_2$ and $b_2^*\sim a_1+b_1$. Since $a_1'\sim a_1-(\mu_1-1)b_1$ and $a_2'\sim a_2-(\mu_2-1)b_2$,

$$\begin{pmatrix}[a_1^\sharp]\\ [b_1^\sharp]\\ [a_2^\sharp]\\ [b_2^\sharp]\end{pmatrix}=\begin{pmatrix}-1 & -\mu_1 & 0 & 1\\ 0 & 0 & -1 & -\mu_2\\ 0 & -1 & 1 & \mu_2\\ 1 & \mu_1 & 0 & 0\end{pmatrix}\begin{pmatrix}[a_1']\\ [b_1]\\ [a_2']\\ [b_2]\end{pmatrix},$$

Using the above equation, we can compute the matrix associated with $\tilde{\rho}^\sharp$ with respect to $\{[a_1'], [b_1], [a_2'], [b_2]\}$.

**4. Invariant subspace.** The purpose of this section is to prove

**Lemma 4.1.** *Every involution of $L(2\alpha, \beta)$ is equivalent to $\varphi$ which has one of the following properties*:

(1) *$\varphi F_3 \cap F_3$ consists of three curves of type* II.

(2) *$\varphi F_3 \cap F_3$ consists of a curve of type* I.

**Assertion A.** *Let $F$ be an incompressible surface in $L(2\alpha, \beta)$ such that $F \cap F_3$ consists of simple closed curves. Then each component of $F \cap V_2$ is orientable.*

Proof. Suppose that $F \cap V_2$ is non-orientable. Let $\tilde{L}(2\alpha, \beta)$ denote the orientable double covering of $L(2\alpha, \beta)$. Then $\tilde{L}(2\alpha, \beta)$ can be considered as the union of two copies of $V_2$ and the double covering of $M(F_3)$. Hence the lifting $\tilde{F}_3$ of $F_3$ is orientable, but the lifting $\tilde{F}$ of $F$ is non-orientable. Since $F$ is isotptic to $F_3$ in $L(2\alpha, \beta)$ by [13], $\tilde{F}$ is isotopic to $\tilde{F}_3$ in $\tilde{L}(2\alpha, \beta)$. This contradicts the fact that $\tilde{F}_3$ is orientable.

Let $\varphi_0$ be an involution of $L(2\alpha, \beta)$. Then, by [10], we may suppose that $\varphi_0 F_3$ is transverse with respect to $F_3$, i.e., $M(c) \subset \varphi_0 F_3$ for each curve $c$ in $\varphi_0 F_3 \cap F_3$. It follows from [12] that $\varphi_0$ is equivalent to $\varphi_1$ such that $\varphi_1 F_3 \cap F_3$ consists of essential simple closed curves on $\varphi_1 F_3$ and $F_3$.

Using Assertion A, we can divide our consideration into the following three cases:

Case 1: $\varphi_1 F_3 \cap F_3$ contains three curves of type II on $\varphi_1 F_3$.

Case 2: $\varphi_1 F_3 \cap F_3$ contains a curve of type I on $\varphi_1 F_3$.

Case 3: $\varphi_1 F_3 \cap F_3$ contains precisely one curve of type II on $\varphi_1 F_3$.

In the rest of this section we will give the proof of Lemma 4.1 for each case.

*Case* 1. In this case each curve of $\varphi_1 F_3 \cap F_3$ is of either type II or type V. Suppose that $\varphi_1 F_3 \cap F_3$ contains a curve of type V on $\varphi_1 F_3$. Let $c$ be a simple closed curve of type V on $\varphi_1 F_3$ which bounds a Möbius band $B$ on $\varphi_1 F_3$ such that $B \cap F_3$ consists of $c$ and a centerline $c'$ of $B$. Then $c'$ is of type II on $\varphi_1 F_3$. On $F_3$, $c$ is two-sided, so $c$ is of type V. Hence $c$ also bounds a Möbius band $B'$ on $F_3$.

We now show that $B'$ contains $c'$. Since $c'$ and $c$ are of type II and V on $F_3$, respectively, there exists an autohomeomorphism $\rho$ of $F_3$ such that $\rho c' = c_1$

and $\rho c=\partial N(c_\mu)$, $\mu=1$, 2 or 3. An annulus $B\cap V_2$ has the boundaries $k_1$ and $k_2$ such that $\pi k_1=c'$ and $\pi k=c$. If we suppose that $B'\not\supset c'$, then $\rho c=\partial N(c_2)$ or $\partial N(c_3)$. Hence, in order to show that $B'\supset c'$, it suffices to prove that $k_1\not\sim \tilde{\rho}^{-1}c_2$, $-\tilde{\rho}^{-1}c_3$, $\tilde{\rho}^{-1}c_3$ and $-\tilde{\rho}^{-1}c_3$ in $V_2$, where $\tilde{\rho}$ denotes the lifting of $\rho$. Let $\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$ be a matrix in $GL(2, Z)$ corresponding to the isotopy class of $\rho^{-1}$. Then it follows from Lemma 3.2 that

$$\tilde{\rho}^{-1}\tilde{c}_1\sim\alpha_{22}b_1-\alpha_{21}b_2,\ \tilde{\rho}^{-1}\tilde{c}_2\sim-(\alpha_{22}+\alpha_{21})b_1+(\alpha_{21}+\alpha_{11})b_2$$

and

$$\tilde{\rho}^{-1}\tilde{c}_3\sim-\alpha_{12}b_1+\alpha_{11}b_2 \text{ in } T_2\,.$$

Since $\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}=\pm1$, it can be easily shown that $k_1=\tilde{\rho}^{-1}\tilde{c}_1\not\sim\tilde{\rho}^{-1}\tilde{c}_2$, $-\tilde{\rho}^{-1}\tilde{c}_2$, $\tilde{\rho}^{-1}\tilde{c}_3$ and $-\tilde{\rho}^{-1}\tilde{c}_3$ in $V_2$. Let $F_3'$ be the surface obtained by deforming $F_3-B'\cup B$ slightly keeping the exterior of $N(B)$ fixed until it intersects $B$ in $c'$ [Fig. 4.1].

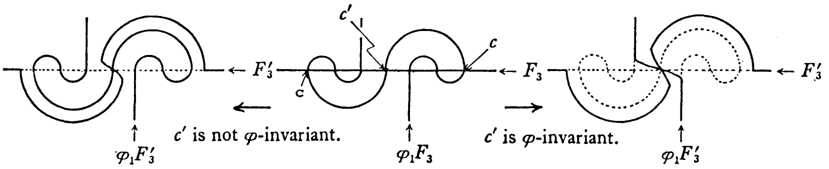


Fig. 4.1

Then $\varphi_1F_3'\cap F_3'$ has fewer components that $\varphi_1F_3\cap F_3$. Since we can deform $F_3'$ onto $F_3$ by an ambient isotopy, we obtain an involution $\varphi_2$ such that $\varphi_2F_3\cup F_3$ is isotopic to $\varphi_1F_3'\cup F_3'$ in $L(2\alpha, \beta)$. Repeating these procedures, we can show that $\varphi_1$ is equivalent to $\varphi$ such that $\varphi F_3\cap F_3$ consists of three curves of type II on $\varphi F_3$ and $F_3$.

*Case* 2. In this case each curve of $\varphi_1F_3\cap F_3$ is of either type I, type III or type IV.

**Assertion B.** *Let $k_1$ and $k_2$ be simple closed curves on $T_2$ such that $\pi k_1$ is of type* I *or* IV, *and $\pi k_2$ is of type* III. *Then $k_1$ is not homologous to $\varepsilon k_2$, for $\varepsilon=1$ and $-1$, in $V_2$.*

Proof. Let $\rho$ be an autohomeomorphism of $F_3$ such that $\rho\pi k_1$ coincides with $c^*$ or $\partial N(c^*)$, and $\rho\pi k_2=a^*$. By $\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$ we denote a matix in $GL(2, Z)$ corresponding to the isotopy class of $\rho^{-1}$. Then, by using Lemma 3.4, $k_2$ or $-k_2$ is homologous to either

$$-\alpha_{11}a_1'-\mu_1\alpha_{11}b_1-\alpha_{12}a_2'+(\alpha_{11}-\mu_2\alpha_{12})b_2$$

or

$$\alpha_{11}a_1'+(\mu_1\alpha_{11}-\alpha_{12})b_1+\alpha_{12}a_2'+\mu_2\alpha_{12}b_2 \text{ on } T_2.$$

Suppose that $k_1\sim\varepsilon k_2$ in $V_2$ for $\varepsilon=1$ or $-1$. Since $k_1\sim 0$ on $T_2$, one of the following systems of equations holds.

$$\begin{cases}\mu_1\alpha_{11}=0,\\ \alpha_{11}-\mu_2\alpha_{12}=0.\end{cases}\qquad \begin{cases}\mu_1\alpha_{11}-\alpha_{12}=0,\\ \mu_2\alpha_{12}=0.\end{cases}$$

Using $\mu_1\mu_2\neq 0$, we can show that it is impossible. This completes the proof.

Now we return to the proof for Case 2. Without loss of generality, we may assume that $\varphi_1F_3\cap F_3$ contains $c^*$. Suppose that there exists a curve of type IV on $\varphi_1F_3$ in $\varphi_1F_3\cap F_3$. Let $c$ be a simple closed curve of type IV on $\varphi_1F_3$ which bounds a Möbius band $B$ such that $B\cap F_3=c\cup c^*$. Since $B\cap V_2$ is an annulus, it follows from Assertion B that $c$ is of type IV on $F_3$. Hence $c$ bounds a Möbius band $B'$ on $F_3$. Let $F_3'$ denote the surface obtained by deforming $F_3-B'\cup B$ slightly so that it is disjoint from $B$. Then, as is similar to Case 1 [Fig. 4.1], $\varphi_1F_3'\cap F_3'$ contains fewer curves of type IV on $\varphi_1F_3'$ than $\varphi_1F_3\cap F_3$. Repeating these procedures, we can show that $\varphi_1$ is equivalent to $\varphi_2$ such that $\varphi_2F_3\cap F_3$ does not contain a curve of type IV on $\varphi_2F_3$ and $F_3$.

Suppose that $\varphi_2F_3\cap F_3$ contains at least two curves of type III on $\varphi_2F_3$. Then there exists an annulus $A$ on $\varphi_2F_3$ such that $A\cap F_3=\partial A$. Let $A'$ be an annulus on $F_3$ which bounds $\partial A$. Deforming $F_3-A'\cup A$ slightly until it is disjoint from $A$, we obtain $A'$ such that $\varphi_2F_3'\cap F_3'$ has fewer components than $\varphi_2F_3\cap F_3$. Hence we can find an involution $\varphi$ which is equivalent to $\varphi_2$ such that $\varphi F_3\cap F_3$ consists of $c^*$ and at most one curve of type III on $\varphi F_3$. If $\varphi F_3\cap F_3$ contains a curve $c$ of type III, then $c$ is $\varphi$-invariant. Since any two-sided curve in $\varphi F_3\cap F_3$ is not $\varphi$-invariant [12], the proof is completed.

*Case* 3. We will show that this case can not occur except for $\mu_1\mu_2=-2$.

**Assertion C.** *Suppose that $\mu_1\mu_2\neq-2$. Let $l_1$ and $l_2$ be disjoint simple closed curves on $T_2$ such that $\pi l_1$ is of type* II *or* V, *and $\pi l_2$ is of type* III *on $F_3$. Then $l_1$ is not homologous to $\varepsilon l_2$, for $\varepsilon=1$ and $-1$, in $V_2$.*

Proof. Let $\rho$ be an autohomeomorphism of $F_3$ such that $\rho\pi l_1$ coincides with $\partial N(c_1)$ or $c_1$ and $\rho\pi l_2=b^*$. By $\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$ we denote a matrix corresponding to $\rho^{-1}$. Then, by Lemma 3.2, $l_1$ is homologous to $\alpha_{22}b_1-\alpha_{21}b_2$ and $l_2$ is homologous to either

$$\varepsilon(-\alpha_{21}a_1'-\mu_1\alpha_{21}b_1-\alpha_{22}a_2'+(\alpha_{21}-\mu_2\alpha_{22})b_2)$$

or

$$\varepsilon(\alpha_{21}a_1'+(\mu_1\alpha_{21}-\alpha_{22})b_1+\alpha_{22}a_2'+\mu_2\alpha_{22}b_2), \text{ for } \varepsilon=1 \text{ or } -1, \text{ on } T_2.$$

If we assume that $l_1 \sim \varepsilon l_2$ in $V_2$, for $\varepsilon=1$ or $-1$, one of the following systems of equations holds.

$$\begin{cases}\alpha_{22}-\varepsilon\mu_1\alpha_{21}=0\,, \\ -\alpha_{21}-\varepsilon(\alpha_{21}-\mu_2\alpha_{22})=0\,.\end{cases} \qquad \begin{cases}\alpha_{22}+\varepsilon(\mu_1\alpha_{21}-\alpha_{22})=0\,, \\ -\alpha_{21}+\varepsilon\mu_2\alpha_{22}=0\,.\end{cases}$$

Each of the above systems does not hold except for $\varepsilon=1$ and $\mu_1\mu_2=-2$. Hence the proof is completed.

By making use of Assertion C and the same method as in the proof for Case 1, we can show that $\varphi_1$ is equivalent to $\varphi_2$ such that $\varphi_2 F_3 \cap F_3$ does not contain a curve of type $V$. Let $B$ be a Möbius band in $\varphi_2 F_3$ such that $B$ intersects $F_3$ in $\partial B$ and a centerline of $B$. Then $\partial B$ is of type III on $F_3$. If $\mu_1\mu_2 \neq -2$, this contradicts Assertion C.

**5. Orbit space.** In this section we will complete the proof of the following main theorem.

**Theorem 5.1.** *Let $\mu_1$ and $\mu_2$ be integers such that $\mu_1\mu_2 \neq 0$ and $\mu_1\mu_2 \neq -2$. Then the orbit space of a fixed point free involution on $L(8\mu_1\mu_2-2,\ 4\mu_1\mu_2-2\mu_1-1)$ is homeomorphic to a Seifert fiber space.*

By Lemma 4.1, we can divide the proof into the following two cases:

Case 1: $\varphi F_3 \cap F_3$ consists of three curves of type II on $\varphi F_3$ and $F_3$.
Case 2: $\varphi F_3 \cap F_3 = c^*$.

*Case* 1. To prove Theorem 5.1 for Case 1, we need the following lemma which can be shown easily.

**Lemma 5.2.** *Let $A$ be an annulus properly embedded in $V_2$ such that $A$ is incompressible and $\partial A$ bounds an annulus $A'$ on $T_2$. Then $A \cup A'$ bounds a solid torus $U$ in $V_2$. Furthermore $A$ is parallel to $A'$, if and only if the inclusion from $A'$ into $U$ induces an isomorphism from $H_1(A')$ onto $H_1(U)$.*

Let $G_1$ and $G_2$ be 2-punctured disks obtained by cutting $T_2$ along $\varphi F_3 \cap T_2$. First we will show that $G=\varphi F_3 \cap V_2$ is parallel to $G_1$ and $G_2$. Each $G$, $G_1$ and $G_2$ is incompressible in $V_2$. Hence we can deform $D_1$ and $D_2$ so that $G \cap (D_1 \cup D_2)$ consists of arcs, where $D_1$ and $D_2$ are meridian disks of $V_2$, as in §2. Since $V_2$ is irreducible, we can construct a system $\{D'_1, D'_2\}$ of meridian disks of $V_2$ such that each curve $G \cap (D'_1 \cup D'_2)$ is not parallel to $\partial G$ in $G$.

From the fact that $G$ is incompressible, it follows that $G \cap D'_\mu \neq \emptyset$, for $\mu=1, 2$. Suppose that each of the innermost curves of $G \cap (D'_1 \cup D'_2)$ on $D'_1 \cup D'_2$ connects two points in the same component of $\partial G$. Then there exists a disk $\Delta$ on $G$ such that each $l_1=\Delta \cap D'_1$ and $l_2=\Delta \cap D'_2$ is an arc in $\partial\Delta$ and $\partial\Delta - {}^\circ(\Delta \cup (D'_1 \cup D'_2)) \subset \partial G$.

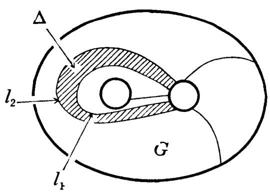


Fig. 5.1

For $\mu=1, 2$, $l_\mu$ separates $D'_\mu$ into two disks $\nabla_{\mu1}$ and $\nabla_{\mu2}$. Among $\nabla_{1\mu}\cup\Delta\cup\nabla_{2\nu}$, $\mu$, $\nu=1, 2$, there exists at least one disk $D_1'^*$ such that $\partial D_1'^*$ is not homologous to zero and $\partial D_2$ in $T_2$. Deforming $D_1'^*$ slightly, we obtain a system $\{D_1'^{**}, D_2'\}$ of $V_2$ such that $G\cap(D_1'^{**}\cup D_2')$ has fewer components than $G\cap(D_1'\cup D_2')$. Hence we can construct a system $\{D_1'', D_2''\}$ of meridian disks of $V_2$ such that at least one innermost curve $b$ of $G\cap(D_1''\cup D_2'')$ in $D_1''\cup D_2''$ connects two points in distinct components of $\partial G$.

Assume that $b\subset D_1''$. Let $\Delta_1$ be a disks in $D_1''$ such that $\Delta_1\cap G=b$ and $c=\partial\Delta_1-\mathring{b}$ is contained in $\partial D_1''$. Furthermore we assume that $c\subset G_1$. Cutting $G$ and $G_1$ along $b$ and $c$, we obtain annuli $A$ and $A'$. It can be shown easily that $A$ is incompressible. Applying Lemma 5.2 to $A$, we can show that the union of $A$, $A'$ and two copies $\Delta_1'$ and $\Delta_1''$ of $\Delta_1$ bounds a solid torus $U$ of genus 1. Thus $G\cup G_1$ bounds a solid torus $U'$ of genus 2.

Let $\Delta_2$ be a meridian disk of $U$ such that $\partial\Delta_2\subset A\cup A'$. Then $\Delta_1$ and $\Delta_2$ form a system of meridian disks of $U'$. Note that each $\Delta_1\cap G$ and $\Delta_1\cap G_1$ is a single arc connecting distinct components $k_1$ and $k_2$ of $\partial G=\partial G_1$.

Suppose that $G$ is not parallel to $G_1$. Then $A\cup\Delta_1'\cup\Delta_1''$ is not parallel to $A'$. Thus, by Lemma 5.2, we have $|Sc(k_3, \partial\Delta_2)|>1$, where $k_3=\partial G-(k_1\cup k_2)$ and $Sc(k_3, \partial\Delta_2)$ denotes the intersection number of $k_3$ with $\partial\Delta_2$ in $G\cup G_1$. Since $k_1$ and $k_3$ generate $H_1(G_1)$, the homomorphism $\Psi$ from $H_1(G_1)$ into $H_1(U')$ induced by the inclusion is not onto.

From Lemma 3.1 and the fact that $\tilde{c}_1\sim b_1$ and $\tilde{c}_3\sim -b_2$, it follows that there exists a matrix $\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$ in $GL(2, Z)$ such that $k_1\sim\alpha_{22}b_1-\alpha_{21}b_2$ and $k_2\sim\alpha_{12}b_1-\alpha_{11}b_2$ on $T_2$. Since $\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}=\pm1$, the inclusion from $G_1$ into $V_2$ induces an isomorphism from $H_1(G)$ onto $H_1(V_2)$. This contradicts the fact that $\Psi$ is not onto. Thus $G$ is parallel to $G_1$.

Since $\varphi M(F_3)\cup M(F_3)$ is $\varphi$-invariant, $\varphi$ takes $U''={}^-(U'-\varphi M(F_3))$ onto $U''$ or ${}^-(V_2-(\varphi M(F_3)\cup U''))$. Using the fact that a solid torus of genus 2 does not admit a free involution, we have $\varphi U''={}^-(V_2-(\varphi M(F_3)\cup U''))$. Then $G$ is parallel to $G_2$.

From this, it follows that $L(2\alpha, \beta)-\mathring{N}(\varphi F_3\cap F_3)$ is homeomorphic to the product of a 2-punctured disk and a circle. Hence we may consider $L(2\alpha, \beta)$ as a Seifert fiber space having each curve of $\varphi F_3\cap F_3$ as a fiber. By [5], we can show that $\varphi$ maps each fiber onto a fiber. Therefore the orbit space of $\varphi$ is also a Seifert fiber space.

*Case* 2. By Corollary 2.3, we may consider $L(2\alpha, \beta)-\mathring{N}(c_1 \cup c_3)$ as a Seifert fiber space. Hence, in order to complete the proof, it suffices to show that there exists a $\varphi$-invariant link $k_1 \cup k_2$ in $L(2\alpha, \beta)$ which is ambient isotopic to $c_1 \cup c_3$.

Let $G=V_2 \cap \varphi F_3$. Then we have $\partial G=\pi^{-1}c^*$. Cutting $T_2$ along $\partial G$, we obtain $G_1$ and $G_2$ where $G_\mu$ contains $a_\mu^*$ and $b_\mu^*$, $\mu=1, 2$. Furthermore $G$ separates $V_2$ into $U_1$ and $U_2$ such that $\partial U_\mu=G_\mu \cup G$, $\mu=1, 2$. Let $c$ be a simple closed curve on $T_2$ as shown in Fig. 5.2. Then $c$ bound a disk $D$ in $V_2$. Since $\pi^{-1}c^* \cap \partial D$ consists of four points, we can deform $D$ so that it intersects $G$ in two arcs. Each $D \cap G_\mu$ separates $G_\mu$ into a disk $G'_\mu$ and an annulus $G''_\mu$. The union $G'_1 \cup D \cup G'_2$ is an orientable surface of genus 1. See Fig. 5.2. Let $G_3$ be a surface obtained by pushing $G'_1 \cup D \cup G'_2$ into $V_2$ so that $G_3 \cap T_2=\partial G_3$.

First we will show that $G$ is ambient isotopic to $G_3$ in $V_2$. Let $V'$ and $V''$ be solid tori in $V_2$ such that $\partial V'=G'_1 \cup D \cup G''_2$ and $\partial V''=G''_1 \cup D \cup G'_2$. Suppose that $U_1 \cap D$ consists of two disks $\Delta_1$ and $\Delta_2$. Then $G'=G \cap V'$ is a disk and $G''=G \cap V''$ is an annulus. Obviously, $G'$ is parallel to $G'_1$. According to [16], $G''$ is parrallel to $G''_1 \cup \Delta_1 \cup \Delta_2$ or $G'_2, \cup \Delta_0$, where $\Delta_0={}^-((D-(\Delta_1 \cup \Delta_2))$. If $G''$ is parallel to $G''_1 \cup \Delta_1 \cup \Delta_2$, $G$ is parallel to $G_1$. If $G'$ is parallel to $G'_2 \cup \Delta_0$, $G$ is ambient isotopic to $G_3$. Similarly, we can show that $G$ is parallel to $G_2$ or is ambient isotopic to $G_3$, for the case that $U_2 \cup D$ is disconnected.

Suppose that $G$ is parallel to $G_1$. Then $G$ is parallel to $G_2$. Since each $\partial D_\nu \cap \partial G$, $\nu=1, 2$, consists of $|\mu_\nu|$ points, $G_1$ is parrallel to $G_2$, if and only if $|\mu_1\mu_2|=1$. Thus $G$ is ambient isotopic to $G_3$.

We take oriented simple closed curves $f$ and $g$ on $G'_1 \cup D \cup G'_2$ so that each $f$ and $g$ is a centerline of an annulus $G'_\mu \cup D$, $\mu=1, 2$, and $f \cap g$ is a single point, as shown in Fig. 5.2.

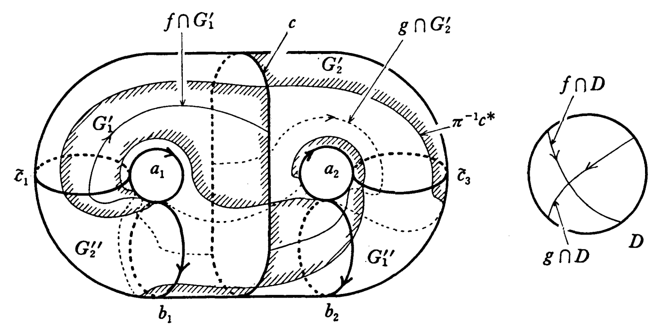


Fig. 5.2

Let $f^*$ and $g^*$ denote simple closed curves on $G$ which is parallel to $f$ and $g$, respectively. Since each $f^*$ and $g^*$ is of type III on $G\subset\varphi F_3$, there exists a matrix $\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}\in GL(2, Z)$ such that

(1) $$\varphi a^*\sim\alpha_{11}f^*+\alpha_{12}g^* \text{ and } \varphi b^*\sim\alpha_{21}f^*+\alpha_{22}g^* \text{ in } \varphi F_3\,.$$

The union $\varphi F_3\cup F_3$ separates $L(2\alpha, \beta)$ into solid tori $U_1'$ and $U_2'$ such that $U_\mu'\cap V_2=U_\mu$, $\mu=1, 2$. Let $\hat{b}_1$ and $\hat{b}_2$ be simple closed curves obtained by pushing $b_1$ int $U_2$ and $b_2$ into $U_1$, respectively. Since $G$ is ambient isotopic to $G_3$, $\{[f^*], [\hat{b}_2]\}$ is a basis of $H_1(U_1')$ and $\{[g^*], [\hat{b}_1]\}$ is a basis of $H_1(U_2')$. It can be shown that

(2) $$f^*\sim\mu_1\hat{b}_1 \text{ in } U_2' \text{ and } g^*\sim\mu_2\hat{b}_2 \text{ in } U_1'\,.$$

By the argument in the proof of Lemma 3.2, we have

$$Sc(a_1^*, b_1) = -1,\ Sc(a_1^*, a_2') = 1,\ Sc(b_1^*, b_1) = 0,\ Sc(b_1^*, a_2') = \mu_2\,,$$
$$Sc(a_2^*, a_1') = -1,\ Sc(a_2^*, b_2) = 1,\ Sc(b_2^*, a_1') = \mu_1 \text{ and } Sc(b_2^*, b_2) = 0\,.$$

From this and the fact that $\pi^{-1}a^*=a_1^*\cup b_1^*$ and $\pi^{-1}b^*=b_1^*\cup a_2^*$, it follows that

(3) $$a^*\sim-f^*+\hat{b}_2 \text{ and } b^*\sim-\mu_2\hat{b}_2 \text{ in } U_1'\,,$$

(4) $$a^*\sim g^*-\hat{b}_1 \text{ and } b^*\sim-\mu_1\hat{b}_1 \text{ in } U_2'\,.$$

In $U_1'$, $b^*+b^*\sim0$ and $\mu_2(a^*+f^*)+b^*\sim0$, by (2) and (3). Using (1), (3), (4) and the fact that $\varphi U_1'=U_2'$, we can show that

(5) $$\begin{aligned}&\varphi b^*+\varphi g^*\sim\alpha_{21}f^*+\alpha_{22}g^*-\varepsilon\alpha_{21}a^*+\varepsilon\alpha_{11}b^*\\ &\quad\sim(\alpha_{22}+\varepsilon\alpha_{11})g^*+(\mu_1\alpha_{21}-\varepsilon\alpha_{11}-\varepsilon\mu_1\alpha_{21})b^*\sim0\end{aligned}$$

and

(6) $$\begin{aligned}&\mu_2(\varphi a^*+\varphi f^*)+\varphi b^*\\ &\sim\mu_2(\alpha_{11}f^*+\alpha_{12}g^*+\varepsilon\alpha_{22}a^*-\varepsilon\alpha_{12}b^*)+\alpha_{21}f^*+\alpha_{22}g^*\\ &\sim(\varepsilon\mu_2\alpha_{12}+\varepsilon\mu_1\mu_2\alpha_{22}+\mu_1\mu_2\alpha_{21}+\mu_1\alpha_{21})b^*+(-\varepsilon\mu_2\alpha_{12}+\mu_2\alpha_{12}+\alpha_{22})g^*\\ &\sim0, \text{ in } U_2'\,,\end{aligned}$$

where $\varepsilon=\alpha_{11}\alpha_{23}-\alpha_{12}\alpha_{21}$.

It is not difficult to show that (5) and (6) does not hold at the same time except for the case that $\varepsilon=1$, $\alpha_{11}=\alpha_{22}=0$, $\alpha_{12}\alpha_{21}=-1$ and $\mu_1=\mu_2$. Thus we obtain $\varphi f^*\sim\pm b^*$ and $\varphi g^*\sim\mp a^*$ on $F_3$.

Let $k=G_2'\cap c_3$ [Fig. 5.2]. Then $g$ intersects $k$ in a single point and $k\cap f=\emptyset$. Let $k^*$ be an arc on $G$ which is parallel to $k$. Joining the end points of $k^*$ by a vertical line in $\varphi F_3\cap M(F_3)$, we obtain a one-sided simple closed curve $k_1$ on $\varphi F_3$, which is ambient isotopic to $c_3$ in $L(2\alpha, \beta)$. Each $k_1 \cap c^*$ and $k_1 \cap g^*$ consists of a point and $k_1 \cap f^* = \emptyset$. Thus, since $c^*$ is $\varphi$-invariant, $\varphi f^* \sim \pm b^*$ and $\varphi g^* \sim \mp a^*$ in $F_3$, we can show that $k_2 = \varphi k_1$ is homotopic to $c_1$ in $F_3$. Therefore a link $k_1 \cup k_2$ is ambient isotopic to $c_1 \cup c_3$ in $L(2\alpha, \beta)$, and we have proved Theorem 5.1.

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Faculty of General Education
Kinki University
Higashi-Osaka 577, Japan