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ALGEBRAIC STEENROD OPERATIONS IN THE SPECTRAL SEQUENCE ASSOCIATED WITH A PAIR OF HOPF ALGEBRAS

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Araki [3], [4] and Vazquez [10] investigated behaviors of Steenrod reduced powers in the spectral sequence associated with a fibre space in the sense of Serre. The main purpose of this paper is to establish an algebraic analogy to their works. For example, works of Adams [1], [2] and others, [6], [11], [12], implicitly contain a useful, direct application of our results.

1. Steenrod operations in the spectral sequence associated with an algebraic system $\mathfrak{S}$

DEFINITION 1. By a graded differential algebra $\mathfrak{S}=\{C, \delta, F, \underset{i}{\cup}\}$ with a decreasing filtration F and with cup-i-products $\underset{i}{\cup}$, we mean

1) a graded cochain complex C over the field Z_2:

$$C\colon\ C^0\to C^1\to\cdots\to C^n\xrightarrow{\delta^n} C^{n+1}\to\cdots,\ \text{where } \delta^n\colon\ C^n\to C^{n+1}$$

is a morphism of graded vector spaces over Z_2,

2) for each integer p, F^pC is a subcomplex of C such that
   i) $F^{p+1}C$ is a subcomplex of F^pC (in notation: $F^pC\supset F^{p+1}C$)
   ii) $F^pC=C$ if $p\leq 0$, and iii) $F^pC^n=0$ if $p>n$,
3) for each integer i there exists a Z_2-linear map $\underset{i}{\cup}\colon C\otimes C\to C$ such that if $x\in F^pC^{m,s}$ and $y\in F^qC^{n,t}$, then $x\underset{i}{\cup}y\in F^{\alpha}C^{m+n-i,s+t}$ for $\alpha=\mathrm{Max}\{p+q-i, p, q\}$, where $x\underset{i}{\cup}y=\underset{i}{\cup}(x\otimes y)$, $x\cup y=x\underset{0}{\cup}y$ in notations, and s,t stand for gradings. $\underset{i}{\cup}$ satisfies the following conditions:
   i) $\underset{i}{\cup}$ is trivial if $i<0$, ii) For $x\in F^qC^m$ and $y\in F^qC^n$, $x\underset{i}{\cup}y=0$ if $i>m$ or n,

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iii) $x\cup(y\cup z)=(x\cup y)\cup z$, iv), $1\cup x=x\cup 1=x$ for some $1\in C^{0,0}$, and
v) $\delta(x\underset{i}{\cup}y)=x\underset{i-1}{\cup}y+y\underset{i-1}{\cup}x+\delta x\underset{i}{\cup}y+x\underset{i}{\cup}\delta y$.

Associating $\mathfrak{S}$ with an exact couple $\langle D, E, i, j, k\rangle$ by defining $D_1^{p,q}=H^{p+q}(F^pC)$, $E_1^{p,q}=H^{p+q}(F^pC/F^{p+1}C)$, and i, j, k as usual, we have a spectral sequence $\{E_\gamma, d_\gamma|\gamma\geq 1\}$. Let us define Steenrod operations in the spectral sequence as Araki [4] and Vazquez [10] did. Define a map $\theta_i\colon C\to C$ by $\theta_i(x)=x\underset{i}{\cup}x+x\underset{i+1}{\cup}\delta x$, then we have

Proposition 1. *θ_i induces Steenrod operations ${}_BSt_i$, ${}_FSt_i$ in the spectral sequence associated with the algebraic system $\mathfrak{S}$ such that*

$$ {}_BSt_i\colon E_\gamma^{p,q}\to E_{2\gamma-2}^{2p-i,2q} \text{ for } \infty\geq\gamma\geq 2\,, $$

and

$$ {}_FSt_i\colon E_\gamma^{p,q}\to E_\gamma^{p,2q+p-i} \text{ for } \infty\geq\gamma\geq 1\,. $$

They are all Z_2-homomorphisms.

Proof. It is straightforward by definition that if we denote, as usual, $Z_\gamma^{p,q}=\{x\in F^pC^{p+q}|\delta x\in F^{p+\gamma}C^{p+q+1}\}$, $B_\gamma^{p,q}=\{x\in F^pC^{p+q}|^\exists y\in F^{p-\gamma}C^{p+q-1},\ \delta y=x\}$, $Z_\infty^{p,q}=\{x\in F^pC^{p+q}|\delta x=0\}$, and $B_\infty^{p,q}=\{x\in F^pC^{p+q}|^\exists y\in C^{p+q-1},\ \delta y=x\}$, then $\theta_i(Z_\gamma^{p,q})\subset Z_{2\gamma-1}^{2p-i,2q}\cap Z_\gamma^{p,2q+p-i}\subset Z_{2\gamma-2}^{2p-i,2q}\cap Z_\gamma^{p,2q+p-i}$, $\theta_i(B_{\gamma-1}^{p,q})\subset B_{2\gamma-3}^{2p-i,2q}\cap B_{\gamma-1}^{p,2q+p-i}$, $\theta_i(Z_\infty^{p,q})\subset Z_\infty^{2p-i,2q}\cap Z_\infty^{p,2q+p-i}$, and $\theta_i(B_\infty^{p,q})\subset B_\infty^{2p-i,2q}\cap B_\infty^{p,2q+p-i}$. Note that the restriction on $\gamma\geq 2$ comes from the following observation. If $x\in B_{\gamma-1}^{p,q}$, then

$$ \theta_i(x)=\delta y\underset{i}{\cup}\delta y=\delta(y\underset{i}{\cup}x+y\underset{i-1}{\cup}y)\,, $$

where $\delta y=x$ with $y\in F^{p-\gamma+1}C^{p+q-1}$. Since $y\underset{i}{\cup}x+y\underset{i-1}{\cup}y\in F^{2p-i-(2\gamma-3)}C^{2p+2q-i-1}$ if $\gamma\geq 2$, $\theta_i(x)\in B_{2\gamma-3}^{2p-i,2q}$ for $\gamma\geq 2$. Hence, θ_i induces ${}_BSt_i$ and ${}_FSt_i$ as stated in Proposition 1. For $x_1, x_2\in Z_\gamma^{p,q}$

$$ \theta_i(x_1+x_2)=\theta_i(x_1)+\theta_i(x_2)+\delta(x_1\underset{i+1}{\cup}x_2)+x_2\underset{i+1}{\cup}\delta x_1+\delta x_1\underset{i+1}{\cup}x_2 $$

from the bilinearity of $\underset{i}{\cup}$. Since

$$ \delta(x_1\underset{i+1}{\cup}x_2)\in B_1^{2p-i,2q}\cap B_0^{p,2q+p-i}\subset B_{2\gamma-3}^{2p-i,2q}\cap B_{\gamma-1}^{p,2q+p-i}\,, $$

and

$$ x_2\underset{i+1}{\cup}\delta x_1+\delta x_1\underset{i+1}{\cup}x_2\in Z_{2\gamma-3}^{2p-i+1,2q-1}\cap Z_{\gamma-1}^{p+1,2q+p-i-1}\,, $$

${}_BSt_i$ and ${}_FSt_i$ are Z_2-homomorphisms.

For completeness sake let us show some properties of Steenrod operations which are useful for their applications. (For example, for computation of cohomology of the Steenrod algebra.) Let $E^{a,b}_{\gamma,s}$ be the subvector space of $E^{a,b}_{\gamma}$ spanned by $(d_s, \cdots, d_{\gamma+1}, d_\gamma)$-cocycles and let $\kappa^{\gamma}_{s+1}\colon E^{a,b}_{\gamma,s} \to E^{a,b}_{s+1}$ be the natural epimorphism. An element in $E^{a,b}_{2,b}$ will be said to be g-transgressive.

Proposition 2. *${}_FSt_i\colon E^{p,q}_\gamma \to E^{p,2q+p-i}_\gamma$ is trivial if $i<p$ or $i>p+q$,*

$$ {}_BSt_i\colon E^{p,q}_\gamma \to E^{2p-i,2q}_{2\gamma-2} \text{ is trivial if } i>p \text{ or } i<0, $$

and

$$ {}_BSt_p = \kappa^{\gamma}_{2\gamma-2}\ {}_FSt_p . $$

Proof. If $p>i$ and $x \in Z^{p,q}_\gamma$, then $\theta_i(x) \in F^{2p-i}C^{2p+2q-i} \subset F^{p+1}C^{2p+2q-i}$ and $\delta(\theta_i(x)) = \delta x \underset{i+1}{\cup} \delta x \in F^{p+\gamma}C^{2p+2q-i+1}$. Hence, $\theta_i(x) \in Z^{p+1,2q+p-i-1}_{\gamma-1}$, so that by definition the triviality of ${}_FSt_i$ is proved if $p>i$. The rest of the proof is immediate, and hence, is omitted.

Proposition 3. *If $\alpha \in E^{p,q}_{\gamma,c}$, then ${}_FSt_i(\alpha) \in E^{p,2q+p-i}_{\gamma,d}$, where*

$$ d = Max\{p+2c-i, c\} $$

and ${}_BSt_i(\alpha) \in E^{2p-i,2q}_{\gamma,2c}$.

Proof. Recall that

$$ E^{p,q}_{\gamma,c} = Z^{p,q}_{c+1} + Z^{p+1,q-1}_{\gamma-1} / Z^{p+1,q-1}_{\gamma-1} + B^{p,q}_{\gamma-1} \subset E^{p,q}_{\gamma}. $$

If x is a representative of α, then $\theta_i(x) \in F^pC^{2p+2q-i} \cap F^{2p-i}C^{2p+2q-i}$ and $\delta(\theta_i(x)) \in F^{2p+2c-i+1}C^{2p+2q-i+1} \cap F^{p+c+1}C^{2p+2q-i+1}$. If $i \geq p$, then $\theta_i(x) \in Z^{p,2q+p-i}_{d+1}$ where $d = \text{Max}\{p+2c-i, c\}$, while if $p \geq i$, then $\theta_i(x) \in Z^{2p-i,2q}_{2c+1}$. Hence, the proof is completed.

Proposition 4. *If $\alpha \in E^{p,q}_2$ is g-transgressive, then ${}_FSt_i(\alpha) \in E^{p,2q+p-i}_2$ is also g-transgressive. Moreover we have*

$$ (1) \qquad \kappa^{\lambda}_{2q}\ {}_BSt_{i+1}\, d_{q+1}\, \kappa^{2}_{q+1}(\alpha) = d_\lambda \kappa^{2}_{\lambda}\ {}_FSt_i(\alpha) , $$

where $\lambda = 2q+(p-i)+1$ and $\kappa^{\lambda}_{2q} = \kappa^{2q}_{\lambda}$ if $\lambda > 2q$.

Proof. It is obvious from Proposition 3 that ${}_FSt_i(\alpha)$ is g-transgressive. If x is a representative of α, then both sides of (1) is represented by $\delta x \underset{i+1}{\cup} \delta x$. Hence, the proof is completed.

2. Comparison theorem in homological algebra

To prepare for later sections the algebraic Steenrod operations are introduced by the iterated use of a comparison theorem in relative homological algebra [5] (For the theorem in a more general and rigorous setting, see [8]), and the explicit formulas of chain homotopies [1], [11] involved in the theorem are presented in this section.

Let α: $A \to B$ be a morphism of graded augmented algebras A and B over a commutative ring R with unity, and let M and N be left graded modules over algebras A and B respectively. A morphism of graded R-modules f: $M \to N$ is called a α-homomorphism iff $f(ax) = \alpha(a)f(x)$ for $a \in A$ and $x \in M$.

Proposition 5. *Let ε: $\mathfrak{X} \to M$ be a R-split exact resolution of M in the category ${}_A\mathfrak{M}$ of left A-modules and let η: $\mathfrak{Y} \to N$ be a R-split exact resolution of N in the category ${}_B\mathfrak{M}$. Then, for any α-homomorphism f: $M \to N$ there exists a α-chain map extension F: $\mathfrak{X} \to \mathfrak{Y}$ of f in the sense that*
1) *for each $n \geq 0$, F_n: $X_n \to Y_n$ is a α-homomorphism, and*
2) *$d_n F_n = F_{n-1} \partial_n$ for $n \geq 1$ and $f\varepsilon = \eta F_0$, where*

$$
\begin{tikzcd}
M \arrow[r, shift left, "s_{-1}"'] & X_0 \arrow[l, shift left, "\varepsilon"'] \arrow[r, shift left, "s_0"'] \arrow[d, "F_0"] & X_1 \arrow[l, shift left, "\partial_1"'] \arrow[r, shift left] \arrow[d, "F_1"] & \cdots \arrow[l, shift left] & \cdots \arrow[r, shift left] & X_n \arrow[l, shift left, "\partial_n"'] \arrow[r, shift left, "s_n"'] \arrow[d, "F_n"] & \cdots \arrow[l, shift left] & : \mathfrak{X} \arrow[d, "F"] \\
N \arrow[r, shift left, "t_{-1}"'] \arrow[u, leftarrow, "f"] & Y_0 \arrow[l, shift left, "\eta"'] \arrow[r, shift left, "t_0"'] & Y_1 \arrow[l, shift left, "d_1"'] \arrow[r, shift left] & \cdots \arrow[l, shift left] & \cdots \arrow[r, shift left] & Y_n \arrow[l, shift left, "d_n"'] \arrow[r, shift left, "t_n"'] & \cdots \arrow[l, shift left] & : \mathfrak{Y}
\end{tikzcd}
$$

3) *If F, F' are α-chain map extensions of f, then there exists a α-chain homotopy h: $\mathfrak{X} \to \mathfrak{Y}$ connecting F with F'.*

Proof. First let us observe that the proposition is the usual comparison theorem in case when $A = B$ and α is the identity map. The following remarks enable us to reduce the proposition to the classical theorem; 1) any B-module Z can be considered as an A-module by definition $az = \alpha(a)z$ for $a \in A$ and $z \in Z$, 2) any morphism g: $Z \to Z'$ in ${}_B\mathfrak{M}$ can be regarded as a morphism in ${}_A\mathfrak{M}$ by considering Z, Z' as A-modules because

$$g(az) = g(\alpha(a)z) = \alpha(a)g(z) = ag(z),$$

3) a R-homomorphism k: $X \to Y$ is a α-homomorphism iff k is a morphism in ${}_A\mathfrak{M}$ considering Y as an A-module. For $k(ax) = \alpha(a)k(x) = ak(x)$. From 1) and 2), η: $\mathfrak{Y} \to N$ can be considered as a R-split exact complex of N in ${}_A\mathfrak{M}$, and from 3) f: $M \to N$ is a morphism in ${}_A\mathfrak{M}$. It follows from the usual comparison theorem that there exists a chain map extension F of f in ${}_A\mathfrak{M}$. From 3) F is a α-homomorphism. It is immediate to see the rest of the proof. This proves the proposition.

Let us apply the proposition to the following case. Let A be a cocommutative Hopf algebra over Z_2 and let α: $A \to A \otimes A$ be the cocommutative comultiplication Δ. Since $M=Z_2$ and $N=Z_2\otimes Z_2\cong Z_2$ can be considered by augmentations as a left A-module and a left $A\otimes A$-module respectively, the α-map $f\colon Z_2\to Z_2\otimes Z_2$ defined by $f(1)=1\otimes 1$, can be extended to a Δ-chain map $h^0\colon \mathfrak{X}\to\mathfrak{X}\otimes\mathfrak{X}$ by the direct application of the proposition, where $\mathfrak{X}$ is a Z_2-split exact resolution of Z_2. If $\rho\colon \mathfrak{X}\otimes\mathfrak{X}\to\mathfrak{X}\otimes\mathfrak{X}$ is the twisting chain map, then ρh^0 is again a Δ-chain map extension of f, because Δ is cocommutative. Hence, there exists a Δ-chain homotopy h^1 connecting h^0 with ρh^0. Since ρh^1 is a Δ-chain homotopy and since $h^1+\rho h^1$ is a Δ-chain map extension of the trivial Δ-homomorphism $0\colon Z_2\to\mathscr{I}md_1$, there exists a Δ-chain homotopy h^2 connecting h^1 and ρh^1. By the iterated use of the same arguments we have a sequence of Δ-chain homotopies $\{h^0, h^1, \cdots, h^i, \cdots\}$. Hence, we have

Proposition 6. *Let A be a cocommutative Hopf algebra over Z_2 and let $\Delta\colon A\to A\otimes A$ be the comultiplication. If $\varepsilon\colon \mathfrak{X}\to Z_2$ is a Z_2-split exact resolution of the A-module Z_2, then there exists a sequence of Δ-homomorphisms $h^i\colon \mathfrak{X}\to\mathfrak{X}\otimes\mathfrak{X}$ for $i=0, 1,\cdots, n, \cdots$ such that* 1) *h^0 is a grade preserving Δ-chain map and* 2) *for $i>0$ h^i is a Δ-chain homotopy connecting h^{i-1} with ρh^{i-1} which raises the homological dimensions by i and preserves the grading, where $\rho\colon \mathfrak{X}\otimes\mathfrak{X}\to\mathfrak{X}\otimes\mathfrak{X}$ is the twisting chain map.*

Consider a diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(\mathfrak{X}, Z_2)\otimes\mathrm{Hom}_A(\mathfrak{X}, Z_2) \arrow[rr, dotted, "\underset{i}{\cup}"] \arrow[dr, "\chi"'] & & \mathrm{Hom}_A(\mathfrak{X}, Z_2) \\
& \mathrm{Hom}_{A\otimes A}(\mathfrak{X}\otimes\mathfrak{X}, Z_2) \arrow[ur, "h^{i\#}"'] &
\end{tikzcd}$$

where χ is the Z_2-chain map defined by $\chi(f\otimes g)(x\otimes y)=f(x)g(y)$ for $f, g\in \mathrm{Hom}_A(\mathfrak{X}, Z_2)$ and for $x, y\in\mathfrak{X}$.

DEFINITION 2. The cup-i-product $\underset{i}{\cup}$ in the cochain complex $C=\mathrm{Hom}_A(\mathfrak{X}, Z_2)$ is defined by $h^{i\#}\cdot\chi$.

Denoting $\mathrm{Hom}_A^s(X_p, Z_2)$ by $C^{p,s}$ for each homological dimension $p\geq 0$ and the grading $s\geq 0$, we have the cochain complex

$$C^{*s}=\{C^{p,s} \text{ for } p=0, 1, \cdots, n, \cdots\}$$

such that $C=\{C^{*s}\mid s=0, 1, \cdots\}$. Then $f\underset{i}{\cup}g=\underset{i}{\cup}(f\otimes g)\in C^{p+q-i,s+t}$ for $f\in C^{p,s}$ and $g\in C^{q,t}$. It is immediate to see by definition the coboundary formula

$$\delta(f \underset{i}{\cup} g) = f \underset{i-1}{\cup} g + g \underset{i-1}{\cup} f + \delta f \underset{i}{\cup} g + f \underset{i}{\cup} \delta g .$$

For

$$\begin{aligned}\delta(f \underset{i}{\cup} g) &= \partial^{\#} h^{i\#} \chi(f \otimes g) = \chi(f \otimes g) h^{i} \partial \\ &= \chi(f \otimes g)(h^{i-1} + \rho h^{i-1} + d h^{i}) \\ &= h^{i-1\#}\chi(f \otimes g) + h^{i-1\#}\rho^{\#}\chi(f \otimes g) + h^{i\#} d^{\#} \chi(f \otimes g) \\ &= h^{i-1\#}\chi(f \otimes g) + h^{i-1\#}\chi(g \otimes f) + h^{i\#}\chi(\delta \otimes 1 + 1 \otimes \delta)(f \otimes g) \\ &= f \underset{i-1}{\cup} g + g \underset{i-1}{\cup} f + \delta f \underset{i}{\cup} g + f \underset{i}{\cup} \delta g .\end{aligned}$$

DEFINITION 3. Algebraic Steenrod operation ${}_{A}Sq_{i}$: $H^{p,s}(A) \to H^{2p-i,2s}(A)$ is defined by ${}_{A}Sq_{i}(\xi) = \overline{f \underset{i}{\cup} f}$, where $\xi \in H^{p,s}(A)$ is represented by $f \in C^{p,s}$ with $\delta f = 0$, and the bar over $f \underset{i}{\cup} f$ stands for the cohomology class.

Adams [1] and others (for example, see [11]) computed explicitely a Δ-homomorphism h^{i} in case when $\mathfrak{X}$ is the bar resolution $B(A)$. If

$$\Delta(a) = \sum a' \otimes a''$$

for $a \in A$, then we have

$$\begin{aligned}&h_{n}^{0}([a_{1} | a_{2} | \cdots | a_{n}]) \\ &\quad = 1 \otimes [a_{1} | \cdots | a_{n}] + \sum_{1 \leq \rho \leq n} [a_{1}' | \cdots | a_{\rho}'] \otimes a_{1}'' \cdots a_{\rho}'' [a_{\rho+1} | \cdots | a_{n}] ,\end{aligned}$$

for odd i,

$$\begin{aligned}&h_{n}^{i}([a_{1} | \cdots | a_{n}]) \\ &= \sum_{0 \leq \rho_{0} < \rho_{1} < \cdots \rho_{i} \leq n} [a_{1}' | \cdots | a_{\rho_{0}}' | a_{\rho_{0}+1}' \cdots a_{\rho_{1}}' | a_{\rho_{1}+1}' | \cdots | a_{\rho_{2}}' | \cdots | a_{\rho_{i-1}+1}' \cdots a_{\rho_{i}}' | a_{\rho_{i}+1} | \cdots | a_{n}] \\ &\quad \otimes a_{1}'' \cdots a_{\rho_{0}}'' [a_{\rho_{0}+1}'' | \cdots | a_{\rho_{1}}'' | a_{\rho_{1}+1}'' \cdots a_{\rho_{2}}'' | \cdots | a_{\rho_{i-1}+1}'' | \cdots | a_{\rho_{i}}''] ,\end{aligned}$$

for even i,

$$\begin{aligned}h_{n}^{i}([a_{1} | \cdots | a_{n}]) &= \sum_{0 \leq \rho_{0} < \rho_{1} < \cdots < \rho_{i} \leq n} [a_{1}' | \cdots | a_{\rho_{0}}' | a_{\rho_{0}+1}' \cdots a_{\rho_{1}}' | \cdots | a_{\rho_{i-1}+1}' | \cdots | a_{\rho_{i}}'] \\ &\quad \otimes a_{1}'' \cdots a_{\rho_{0}}'' [a_{\rho_{0}+1}'' | \cdots | a_{\rho_{1}}'' | \cdots | a_{\rho_{i-1}+1}'' \cdots a_{\rho_{i}}'' | a_{\rho_{i}+1} | \cdots | a_{n}] .\end{aligned}$$

Let us sketch the method of computation for completeness sake. Let S be the contracting homotopy for $B(A)$, then $t = S \otimes 1 + \varepsilon \otimes S$ is a contracting homotopy for $B(A) \otimes B(A)$. Define $h_{0}^{0} = \Delta$, then $h_{1}^{0} S_{0} = t_{0} h_{0}^{0}$ determine h_{1}^{0} by $h_{1}^{0}(ax) = \Delta(a) h_{1}^{0}(x)$. Inductively h^{0} is obtained easily. Define $h_{0}^{1} = t_{0}(h_{0}^{0} + \rho h_{0}^{0})$, then h_{1}^{1} is calculated by $h_{1}^{1} S_{0} = t_{1}(h_{1}^{0} + \rho h_{1}^{0}) S_{0} + t_{1} h_{0}^{1}$ and $h_{1}^{1}(ax) = \Delta(a) h_{1}^{1}(x)$. Repeat this process, we get the above formula.

3. $\mathfrak{S}(\Gamma, \Lambda)$ associated with a pair of Hopf algebras (Γ, Λ)

Let (Γ, Λ) be a pair of connected locally finite cocommutative Hopf algebras over Z_2 such that the subhopf algebra Λ is central in Γ. Then we have a sequence of Hopf algebras

$$\Lambda \xrightarrow{i} \Gamma \xrightarrow{\pi} \Omega = \Gamma / I(\Lambda)\cdot\Gamma$$

where the inclusion i and the projection π are morphisms of Hopf algebras (see [1]). In this setting we are going to associate with a pair of Hopf algebras (Γ, Λ) a graded differential algebra $\mathfrak{S}(\Gamma, \Lambda)=\{C, \delta, F, \underset{i}{\cup}\}$ with a decreasing filtration F and with cup-i-products $\underset{i}{\cup}$ so that behaviors of algebraic Steenrod operations can be discussed in the spectral sequence $\{E_\gamma, d_\gamma\}$ associated with $\mathfrak{S}(\Gamma, \Lambda)$.

Recall the filtration in the bar construction $B(\Gamma)$ which Adams introduced in [1]. For each integer p define a subcomplex $F_pB(\Gamma)$ of $B(\Gamma)$ such that $F_pB(\Gamma)_n$ is the Γ-submodule of $B(\Gamma)_n=\Gamma\otimes I(\Gamma)^n$ generated by elements of the form $\gamma[\gamma_1|\cdots|\gamma_n]$ with the property that $\gamma_s\in I(\Lambda)$ for at least $(n-p)$ values of s. Then it is immediate to see that F is the canonical increasing filtration in $B(\Gamma)$. Define the product filtration $\overset{\times}{F}$ in $B(\Gamma)\otimes B(\Gamma)$ by

$$\overset{\times}{F}_p(B(\Gamma)\otimes B(\Gamma)) = \bigcup_{p\geq s\geq 0} F_{p-s}B(\Gamma)\otimes F_sB(\Gamma)\ .$$

Then $(B(\Gamma)\otimes B(\Gamma), \overset{\times}{F})$ is a resolution of $\Gamma\otimes\Gamma$-module Z_2 with the increasing filtration $\overset{\times}{F}$. Let $\Delta\colon \Gamma\to\Gamma\otimes\Gamma$ be the cocommutative diagonal and let ρ be the twisting chain map of $B(\Gamma)\otimes B(\Gamma)$. Then we have

Theorem 1. *There exists a sequence of Δ-homomorphisms*

$$h^i\colon B(\Gamma)\to B(\Gamma)\otimes B(\Gamma)$$

for $i=0, 1, \cdots, n, \cdots$ such that 1) *h^0 is a Δ-chain map which preserves grading and filtration,* 2) *h^i is a Δ-chain homotopy connecting h^{i-1} and ρh^{i-1} which preserves grading, raises homological dimension by i, and satisfies the filtration condition*

$$h^i(F_pB(\Gamma))\subset \overset{\times}{F}_\alpha(B(\Gamma)\otimes B(\Gamma))$$

for $\alpha=Min\{2p, p+i\}$.

Proof. In virtue of Proposition 6 it remains only to prove that h_n^i shown in §2 satisfies the filtration condition. By denoting

$$\Delta(\gamma) = \sum \gamma'\otimes\gamma''$$

the three formulas $h_n^i([\gamma_1|\cdots|\gamma_n])$ show that for each j with $n\geq j\geq 1$ exactly one of the three elements γ_j, γ_j', and γ_j'' appears solely between bars. For example, if i is odd, each of the elements $\gamma'_1, \cdots, \gamma'_{\rho_0}, \gamma''_{\rho_0+1}, \cdots, \gamma''_{\rho_1}, \gamma'_{\rho_1+1} \cdots, \gamma'_{\rho_2}, \cdots, \gamma''_{\rho_{i-1}+1}, \cdots, \gamma''_{\rho_i}, \gamma_{\rho_i+1}, \cdots, \gamma_n$ appears solely in $|\quad|$. It follows that if $[\gamma_1|\cdots|\gamma_n] \in F_p B(\Gamma)_n$, each term of the sum on the right hand sides of the formulas contains at least $(n-p)$ elements in $I(\Lambda)$. By definition of the product filtration $\overset{\times}{F}$ we obtain

$$h^i(F_p B(\Gamma)) \subset \overset{\times}{F}_{p+i}(B(\Gamma)\otimes B(\Gamma)) .$$

If $p \geq i$, then the proof is complete, because Min $\{p+i, 2p\} = p+i$. If $i \geq p$, it is seen that among i products

$$\gamma'_{\rho_0+1}\cdots\gamma'_{\rho_1}, \gamma''_{\rho_1+1}\cdots\gamma''_{\rho_2}, \cdots, \gamma'_{\rho_{i-1}+1}\cdots\gamma'_{\rho_i}$$

(or $\gamma''_{\rho_{i-1}+1}\cdots\gamma''_{\rho_i}$ if i is even) there exist at least $(i-p)$ products contained in $I(\Lambda)$. Otherwise, at least $(p+1)$ products are not contained in $I(\Lambda)$. Then γ_s are not in $I(\Lambda)$ for at least $(p+1)$ values of s. This is a contradiction. It follows that each term of the sum for $h^i_n([\gamma_1|\cdots|\gamma_n])$ has at least

$$(n-p)+(i-p) = n+i-2p$$

elements in $I(\Lambda)$. Therefore,

$$h^i(F_p B(\Gamma)) \subset \overset{\times}{F}_{2p}(B(\Gamma)\otimes B(\Gamma))$$

if $i \geq p$, where Min $\{p+i, 2p\} = 2p$. This completes the proof.

Now let us dualize what we have obtained in this section. Let (C, δ) be the cochain complex $\mathrm{Hom}_\Gamma(B(\Gamma), Z_2)$ over Z_2. For each integer p define a subcomplex $F^p(C)$ by the image of

$$\mathrm{Hom}_\Gamma (B(\Gamma)/F_{p-1}B(\Gamma), Z_2)$$

under the dual of the projection

$$p\colon B(\Gamma) \to B(\Gamma)/F_{p-1}B(\Gamma) .$$

Then it is seen that (C, δ, F) is a cochain complex with a decreasing filtration. Let us call it Adams filtered complex associated with (Γ, Λ).

Theorem 2. *Let (C, δ, F) be Adams filtered complex associated with a pair of Hopf algebras over Z_2. Then there exist a Z_2-linear map $\underset{i}{\cup}: C\otimes C \to C$ such that $\mathfrak{S}(\Gamma, \Lambda) = \{C, \delta, F, \underset{i}{\cup}\}$ is a graded differential algebra with a decreasing filtration F and with cup-i-products in the sense of Definition* 1.

Proof. Let $h^i\colon B(\Gamma) \to B(\Gamma)\otimes B(\Gamma)$ be the Δ-homomorphism in Theorem 1 and define $\underset{i}{\cup}: C\otimes C \to C$ by $h^{i\#}\chi$ as was considered in Definition 2. Since $\underset{i}{\cup}$

is the cup-i-product in $C=\mathrm{Hom}_\Gamma(B(\Gamma), Z_2)$, it is easy to see that $\underset{i}{\cup}$ satisfies all the necessary conditions except the filtration condition. Consequently, it is sufficient to show that if $f\in F^pC^{m,s}$ and $g\in F^qC^{n,t}$, then $f\underset{i}{\cup}g\in F^\alpha C^{m+n-i,s+t}$ for $\alpha=\mathrm{Max}\{p+q-i, p, q\}$. Consider first the case when

$$\alpha=\mathrm{Max}\{p+q-i, p, q\}=p+q-i\,,$$

then

$$\mathrm{Min}\{(\alpha-1)+i, 2(\alpha-1)\}=(\alpha-1)+i=p+q-1$$

except the case when $p=q=i$. By Theorem 1

$$h^i(F_{\alpha-1}B(\Gamma))\subset\overset{\times}{F}_{p+q-1}(B(\Gamma)\otimes B(\Gamma))\,.$$

If

$$h^i(x)=\sum x'\otimes x''$$

for $x\in F_{\alpha-1}B(\Gamma)_{m+n-i,s+t}$, then $x'\in F_\xi B(\Gamma)_{\rho,\theta}$ and $x''\in F_\eta B(\Gamma)_{\sigma,\nu}$ with the property that $\xi+\eta=p+q-1$, $\rho+\sigma=m+n$, and $\theta+\nu=s+t$. Then

$$(f\underset{i}{\cup}g)(x)=\sum f(x')\cdot g(x'')=0\,,$$

because $\xi<p$ or $\eta<q$. Therefore, $f\underset{i}{\cup}g\in F^{p+q-i}C^{m+n-i,s+t}$. If $\alpha=p$, then $p\geq q$ and $i\geq q$. In this case also,

$$(f\underset{i}{\cup}g)(F_{p-1}B(\Gamma))=0$$

can be shown because

$$h^i(F_{p-1}B(\Gamma))\subset\overset{\times}{F}_{p-1+i}(B(\Gamma)\otimes B(\Gamma))\cap\overset{\times}{F}_{2p-2}(B(\Gamma)\otimes B(\Gamma))\,.$$

Hence, the proof is completed.

From Theorem 2 and Proposition 1 we obtain

Theorem 3. *Let* (Γ, Λ) *be a pair of connected locally finite cocommutative Hopf algebras over* Z_2 *such that* Λ *is central in* Γ, *and let* $\{E_\gamma, d_\gamma\}$ *be Adams spectral sequence associated with the system* $\mathfrak{S}(\Gamma, \Lambda)$. *Then there exist algebraic Steenrod operations* ${}_BSt_i$: $E_\gamma^{p,q}\to E_{2\gamma-2}^{2p-i,2q}$ *for* $\infty\geq\gamma\geq2$ *and* ${}_FSt_i$: $E_\gamma^{p,2q+p-i}$ *for* $\infty\geq\gamma\geq1$.

4. Some properties of algebraic Steenrod operations

Theorem 4. ${}_BSt_i$ *and* ${}_FSt_i$ *defined in Adams spectral sequence satisfy Propositions* 2, 3, *and* 4.

Theorem 5. *Let* (Γ, Λ) *and* (Γ', Λ') *be pairs of Hopf algebras over* Z_2 *both of which satisfy the conditions stated before, and let* E_γ *and* E'_γ *be Adams spectral sequences associated with* $\mathfrak{S}(\Gamma, \Lambda)$ *and* $\mathfrak{S}(\Gamma', \Lambda')$ *respectively. If* f: $(\Gamma, \Lambda)\to(\Gamma', \Lambda')$ *be a morphism of pairs of Hopf algebras, then* f *induces a sequence of homomorphisms* ϕ_γ: $E'_\gamma\to E_\gamma$ *for* $\gamma\geq 1$ *such that*

$$\phi_\gamma\ {}_FSt_i = {}_FSt_i\ \phi_\gamma \quad and \quad \phi_{2\gamma-2}\ {}_BSt_i = {}_BSt_i\ \phi_\gamma$$

for $\gamma\geq 2$.

Proof. It is obvious that f induces a chain map $B(f)$: $B(\Gamma)\to B(\Gamma')$ preserving filtrations and gradings. If h^i and h'^i are Δ-homomorphisms in Theorem 1, then we have $h'^i B(f)=(B(f)\otimes B(f))h^i$. Consequently, $B(f)$ induces a morphism $\mathfrak{S}(f)$: $\mathfrak{S}(\Gamma', \Lambda')\to\mathfrak{S}(\Gamma, \Lambda)$. By a morphism $\mathfrak{S}(f)$ of the system $\mathfrak{S}$ we mean that $\mathfrak{S}(f)$ is a chain map compatible with gradings, filtrations, and cup-i-products. Therefore, it is straightforward to verify the theorem.

Theorem 6. *Let* $\Lambda\xrightarrow{i}\Gamma\xrightarrow{\pi}\Omega$ *be a sequence of Hopf algebras as stated before, and let* $\{E_\gamma\}$ *be Adams spectral sequence associated with* (Γ, Λ). *Then the natural maps* $B(\pi)$: $B(\Gamma)\to B(\Omega)$ *and* $B(i)$: $B(\Lambda)\to B(\Gamma)$ *induce isomorphisms* $B(\pi)^*$: $H^p(\Omega)\to E_2^{p,0}$ *and* $B(i)^*$: $E_2^{0,q}\to H^q(\Lambda)$ *respectively. If* $E_2^{p,0}$ *and* $E_2^{0,q}$ *are identified with* $H^p(\Omega)$ *and* $H^q(\Lambda)$ *respectively, then* ${}_BSt_i$: $E_2^{p,0}\to E_2^{2p-i,0}$ *coincides with* ${}_\Omega Sq_i$: $H^p(\Omega)\to H^{2p-i}(\Omega)$, *and* ${}_FSt_i$: $E_2^{0,q}\to E_2^{0,2q-i}$ *coincides with* ${}_\Lambda Sq_i$: $H^q(\Lambda)\to H^{2q-i}(\Lambda)$. *Moreover,* ${}_BSt_i$: $E_\infty^{p,q}\to E_\infty^{2p-i,2q}$ *for* $i\leq p$ *and* ${}_FSt_i$: $E_\infty^{p,q}\to E_\infty^{p,2q+p-i}$ *for* $i\geq p$ *are induced by* ${}_\Gamma Sq_i$: $H^{p+q}(\Gamma)\to H^{2p+2q-i}(\Gamma)$.

Proof. Adams has shown in [1] that $B(\pi)^*$ and $B(i)^*$ are isomorphisms. Hence, a morphism of pairs of Hopf algebras π: $(\Gamma, \Lambda)\to(\Omega, Z_2)$ induces the isomorphism ϕ_2: $E_2'^{p,0}\to E_2^{p,0}$ for each p, because

$$E_2'^{p,0} = E_\infty'^{p,0} = H^p(\Omega)\,.$$

Since ${}_BSt_i$: $E_2'^{p,0}\to E_2'^{2p-i,0}$ is exactly ${}_\Omega Sq_i$: $H^p(\Omega)\to H^{2p-i}(\Omega)$, we obtain ${}_BSt_i\ \phi_2=\phi_2\ {}_\Omega Sq_i$. Similarly, $\phi_2\ {}_FSt_i={}_\Lambda Sq_i\ \phi_2$. From the facts that $H^{p+q}(\Gamma)$ is filtered by $F^pH^{p+q}(\Gamma)=Z_\infty^{p,q}/B_\infty^{p,q}$ with the property that $E_\infty^{p,q}=F^pH^{p+q}(\Omega)/F^{p+1}H^{p+q}(\Omega)$ and that ${}_\Gamma Sq_i$ maps $F^pH^{p+q}(\Gamma)$ into $F^{2p-i}H^{2p+2q-i}(\Gamma)\subset H^{2p+2q-i}(\Gamma)$, it is immediate to see that ${}_BSt_i$: $E_\infty^{p,q}\to E_\infty^{2p-i,2q}$ is induced by ${}_\Gamma Sq_i$. The rest of the proof is obvious. Hence, the proof is complete.

In a subsequent paper the author wishes to discuss higher cohomology operations involved in the Cartan formula and Massey-Uehara products.

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Bibliography

[1] J.F. Adams: *On the non-existence of elements of Hopf invariant one*, Ann. of Math. **72** (1960), 20–104.

[2] ————: Stable Homotopy Theory, Lecture Notes in Math. 3, Springer, 1969.

[3] S. Araki: *Steenrod reduced powers in the spectral sequences associated with a fibering*, Mem. Fac. Sci. Kyushu Univ. Ser. A. **11** (1957), 15–64.

[4] ————: *Steenrod reduced powers in the spectral sequences associated with a fibering* II, Mem. Fac. Sci. Kyushu Univ. Ser. A. **11** (1957), 81–97.

[5] S. Eilenberg and J.C. Moore: Foundation of Relative Homological Algebra, Mem. Amer. Math. Soc. 55, 1965.

[6] A. Liulevicius: The Factorization of Cyclic Reduced Powers by Secondary Cohomology Operations, Mem. Amer. Math. Soc. 42, 1962.

[7] J.W. Milnor and J.C. Moore: *On the structure of Hopf algebras*, Ann. of Math. **81** (1965), 211–264.

[8] H. Uehara, F.S. Brenneman, L.D. Olson: *A comparison theorem in fibred categories*, Abstract ♯71T–A27, Notices Amer. Math. Soc. **18** (1971), 396.

[9] H. Uehara: On Cohomology of Hopf Algebras, Lecture notes, Universität des Saarlandes, 1970.

[10] R. Vazquez: *Note on Steenrod squares in the spectral sequence of a fibre space*, Bol. Soc. Mat. Mexicana **2** (1957), 1–8.

[11] A. Zachariou: On Cup-i-products in the Cobar Construction F(A*), Univ. of Manchester, 1966.

[12] ————: *A subalgebra of* $Ext_A^{**}(Z_2, Z_2)$, Bull. Amer. Math. Soc. **73** (1967), 647–648.