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ALGEBRAIC STEENROD OPERATIONS IN THE SPECTRAL SEQUENCE ASSOCIATED WITH A PAIR OF HOPF ALGEBRAS

HIROSHI UEHARA*

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Araki [3], [4] and Vazquez [10] investigated behaviors of Steenrod reduced powers in the spectral sequence associated with a fibre space in the sense of Serre. The main purpose of this paper is to establish an algebraic analogy to their works. For example, works of Adams [1], [2] and others, [6], [11], [12], implicitly contain a useful, direct application of our results.

1. Steenrod operations in the spectral sequence associated with an algebraic system \mathfrak{S}

DEFINITION 1. By a graded differential algebra $\mathfrak{S} = \{C, \delta, F, \cup_i\}$ with a decreasing filtration F and with cup-i-products \cup_i , we mean

1) a graded cochain complex C over the field Z_2 :

$$C: C^0 \rightarrow C^1 \rightarrow \cdots \rightarrow C^n \xrightarrow{\delta^n} C^{n+1} \rightarrow \cdots, \text{ where } \delta^n: C^n \rightarrow C^{n+1}$$

is a morphism of graded vector spaces over Z_2 ,

2) for each integer p , $F^p C$ is a subcomplex of C such that

- i) $F^{p+1} C$ is a subcomplex of $F^p C$ (in notation: $F^p C \supset F^{p+1} C$)
- ii) $F^p C = C$ if $p \leq 0$, and iii) $F^p C^n = 0$ if $p > n$,

3) for each integer i there exists a Z_2 -linear map $\cup_i: C \otimes C \rightarrow C$ such that if $x \in F^p C^{m,s}$ and $y \in F^q C^{n,t}$, then $x \cup_i y \in F^{\alpha} C^{m+n-i, s+t}$ for $\alpha = \text{Max}\{p+q-i, p, q\}$, where $x \cup_i y = \cup_i(x \otimes y)$, $x \cup_0 y = x \cup_0 y$ in notations, and s, t stand for gradings. \cup_i satisfies the following conditions:

- i) \cup_i is trivial if $i < 0$, ii) For $x \in F^q C^m$ and $y \in F^q C^n$, $x \cup_i y = 0$ if $i > m$ or n ,

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- iii) $x \cup (y \cup z) = (x \cup y) \cup z$, iv), $1 \cup x = x \cup 1 = x$ for some $1 \in C^{0,0}$, and
- v) $\delta(x \cup y) = x \cup_{i-1} y + y \cup_{i-1} x + \delta x \cup_i y + x \cup_i \delta y$.

Associating \mathfrak{S} with an exact couple $\langle D, E, i, j, k \rangle$ by defining $D_1^{p,q} = H^{p+q}(F^p C)$, $E_1^{p,q} = H^{p+q}(F^p C / F^{p+1} C)$, and i, j, k as usual, we have a spectral sequence $\{E_\gamma, d_\gamma \mid \gamma \geq 1\}$. Let us define Steenrod operations in the spectral sequence as Araki [4] and Vazquez [10] did. Define a map $\theta_i: C \rightarrow C$ by $\theta_i(x) = x \cup_i x + x \cup_{i+1} \delta x$, then we have

Proposition 1. θ_i induces Steenrod operations ${}_B St_i$, ${}_F St_i$ in the spectral sequence associated with the algebraic system \mathfrak{S} such that

$${}_B St_i: E_\gamma^{p,q} \rightarrow E_{2\gamma-2}^{2p-i, 2q} \text{ for } \infty \geq \gamma \geq 2,$$

and

$${}_F St_i: E_\gamma^{p,q} \rightarrow E_\gamma^{p, 2q+p-i} \text{ for } \infty \geq \gamma \geq 1.$$

They are all Z_2 -homomorphisms.

Proof. It is straightforward by definition that if we denote, as usual, $Z_\gamma^{p,q} = \{x \in F^p C^{p+q} \mid \delta x \in F^{p+\gamma} C^{p+q+1}\}$, $B_\gamma^{p,q} = \{x \in F^p C^{p+q} \mid y \in F^{p-\gamma} C^{p+q-1}, \delta y = x\}$, $Z_\infty^{p,q} = \{x \in F^p C^{p+q} \mid \delta x = 0\}$, and $B_\infty^{p,q} = \{x \in F^p C^{p+q} \mid y \in C^{p+q-1}, \delta y = x\}$, then $\theta_i(Z_\gamma^{p,q}) \subset Z_{2\gamma-1}^{2p-i, 2q} \cap Z_\gamma^{2q+p-i} \subset Z_{2\gamma-2}^{2p-i, 2q} \cap Z_\gamma^{2q+p-i}$, $\theta_i(B_\gamma^{p,q}) \subset B_{2\gamma-3}^{2p-i, 2q} \cap B_{\gamma-1}^{2q+p-i}$, $\theta_i(Z_\infty^{p,q}) \subset Z_\infty^{2p-i, 2q} \cap Z_\infty^{2q+p-i}$, and $\theta_i(B_\infty^{p,q}) \subset B_\infty^{2p-i, 2q} \cap B_\infty^{2q+p-i}$. Note that the restriction on $\gamma \geq 2$ comes from the following observation. If $x \in B_{\gamma-1}^{p,q}$, then

$$\theta_i(x) = \delta y \cup_i \delta y = \delta(y \cup_i x + y \cup_{i-1} y),$$

where $\delta y = x$ with $y \in F^{p-\gamma+1} C^{p+q-1}$. Since $y \cup_i x + y \cup_{i-1} y \in F^{2p-i-(2\gamma-3)} C^{2p+2q-i-1}$ if $\gamma \geq 2$, $\theta_i(x) \in B_{2\gamma-3}^{2p-i, 2q}$ for $\gamma \geq 2$. Hence, θ_i induces ${}_B St_i$ and ${}_F St_i$ as stated in Proposition 1. For $x_1, x_2 \in Z_\gamma^{p,q}$

$$\theta_i(x_1 + x_2) = \theta_i(x_1) + \theta_i(x_2) + \delta(x_1 \cup_{i+1} x_2) + x_2 \cup_{i+1} \delta x_1 + \delta x_1 \cup_{i+1} x_2$$

from the bilinearity of \cup_i . Since

$$\delta(x_1 \cup x_2) \in B_1^{2p-i, 2q} \cap B_0^{p, 2q+p-i} \subset B_{2\gamma-3}^{2p-i, 2q} \cap B_{\gamma-1}^{2q+p-i},$$

and

$$x_2 \cup_{i+1} \delta x_1 + \delta x_1 \cup_{i+1} x_2 \in Z_{2\gamma-3}^{2p-i+1, 2q-1} \cap Z_{\gamma-1}^{p+1, 2q+p-i-1},$$

${}_B St_i$ and ${}_F St_i$ are Z_2 -homomorphisms.

For completeness sake let us show some properties of Steenrod operations

which are useful for their applications. (For example, for computation of cohomology of the Steenrod algebra.) Let $E_{\gamma, s}^{a, b}$ be the subvector space of $E_{\gamma}^{a, b}$ spanned by $(d_s, \dots, d_{\gamma+1}, d_{\gamma})$ -cocycles and let $\kappa_{s+1}^{\gamma}: E_{\gamma, s}^{a, b} \rightarrow E_{s+1}^{a, b}$ be the natural epimorphism. An element in $E_{2, b}^{a, b}$ will be said to be g -transgressive.

Proposition 2. ${}_F St_i: E_{\gamma}^{p, q} \rightarrow E_{\gamma}^{p, 2q+p-i}$ is trivial if $i < p$ or $i > p+q$,

${}_B St_i: E_{\gamma}^{p, q} \rightarrow E_{2\gamma-2}^{2p-i, 2q}$ is trivial if $i > p$ or $i < 0$,

and

$${}_B St_p = \kappa_{2\gamma-2}^{\gamma} {}_F St_p.$$

Proof. If $p > i$ and $x \in Z_{\gamma}^{p, q}$, then $\theta_i(x) \in F^{2p-i} C^{2p+2q-i} \subset F^{p+1} C^{2p+2q-i}$ and $\delta(\theta_i(x)) = \delta x \bigcup_{i+1} \delta x \in F^{p+q} C^{2p+2q-i+1}$. Hence, $\theta_i(x) \in Z_{\gamma-1}^{p+1, 2q+p-i-1}$, so that by definition the triviality of ${}_F St_i$ is proved if $p > i$. The rest of the proof is immediate, and hence, is omitted.

Proposition 3. If $\alpha \in E_{\gamma, c}^{p, q}$, then ${}_F St_i(\alpha) \in E_{\gamma, d}^{p, 2q+p-i}$, where

$$d = \text{Max}\{p+2c-i, c\}$$

and ${}_B St_i(\alpha) \in E_{\gamma, 2c}^{2p-i, 2q}$.

Proof. Recall that

$$E_{\gamma, c}^{p, q} = Z_{c+1}^{p, q} + Z_{\gamma-1}^{p+1, q-1} / Z_{\gamma-1}^{p+1, q-1} + B_{\gamma-1}^{p, q} \subset E_{\gamma}^{p, q}.$$

If x is a representative of α , then $\theta_i(x) \in F^p C^{2p+2q-i} \cap F^{2p-i} C^{2p+2q-i}$ and $\delta(\theta_i(x)) \in F^{2p+2c-i+1} C^{2p+2q-i+1} \cap F^{p+c+1} C^{2p+2q-i+1}$. If $i \geq p$, then $\theta_i(x) \in Z_{\alpha+1}^{p, 2q+p-i}$ where $d = \text{Max}\{p+2c-i, c\}$, while if $p \geq i$, then $\theta_i(x) \in Z_{2c+1}^{2p-i, 2q}$. Hence, the proof is completed.

Proposition 4. If $\alpha \in E_2^{p, q}$ is g -transgressive, then ${}_F St_i(\alpha) \in E_2^{p, 2q+p-i}$ is also g -transgressive. Moreover we have

$$(1) \quad \kappa_{2q}^{\lambda} {}_B St_{i+1} d_{q+1} \kappa_{q+1}^2(\alpha) = d_{\lambda} \kappa_{\lambda}^2 {}_F St_i(\alpha),$$

where $\lambda = 2q + (p-i) + 1$ and $\kappa_{2q}^{\lambda} = \kappa_{\lambda}^{2q}$ if $\lambda > 2q$.

Proof. It is obvious from Proposition 3 that ${}_F St_i(\alpha)$ is g -transgressive. If x is a representative of α , then both sides of (1) is represented by $\delta x \bigcup_{i+1} \delta x$. Hence, the proof is completed.

2. Comparison theorem in homological algebra

To prepare for later sections the algebraic Steenrod operations are introduced by the iterated use of a comparison theorem in relative homological

algebra [5] (For the theorem in a more general and rigorous setting, see [8]), and the explicit formulas of chain homotopies [1], [11] involved in the theorem are presented in this section.

Let $\alpha: A \rightarrow B$ be a morphism of graded augmented algebras A and B over a commutative ring R with unity, and let M and N be left graded modules over algebras A and B respectively. A morphism of graded R -modules $f: M \rightarrow N$ is called a α -homomorphism iff $f(ax) = \alpha(a)f(x)$ for $a \in A$ and $x \in M$.

Proposition 5. *Let $\varepsilon: \mathfrak{X} \rightarrow M$ be a R -split exact resolution of M in the category ${}_A\mathfrak{M}$ of left A -modules and let $\eta: \mathfrak{Y} \rightarrow N$ be a R -split exact resolution of N in the category ${}_B\mathfrak{M}$. Then, for any α -homomorphism $f: M \rightarrow N$ there exists a α -chain map extension $F: \mathfrak{X} \rightarrow \mathfrak{Y}$ of f in the sense that*

- 1) for each $n \geq 0$, $F_n: X_n \rightarrow Y_n$ is a α -homomorphism, and
- 2) $d_n F_n = F_{n-1} \partial_n$ for $n \geq 1$ and $f\varepsilon = \eta F_0$, where

$$\begin{array}{ccccccc}
 M & \xleftarrow{\varepsilon} & X_0 & \xleftarrow{\partial_1} & X_1 & \xleftarrow{\partial_n} & \cdots & \xleftarrow{s_n} & X_n & \xleftarrow{\partial_n} & \cdots & : \mathfrak{X} \\
 f \downarrow & & \downarrow s_{-1} & & \downarrow s_0 & & & & \downarrow F_n & & & \downarrow F \\
 N & \xleftarrow{\eta} & Y_0 & \xleftarrow{d_1} & Y_1 & \xleftarrow{d_n} & \cdots & \xleftarrow{t_{-1}} & Y_n & \xleftarrow{t_0} & \cdots & : \mathfrak{Y}
 \end{array}$$

- 3) If F, F' are α -chain map extensions of f , then there exists a α -chain homotopy $h: \mathfrak{X} \rightarrow \mathfrak{Y}$ connecting F with F' .

Proof. First let us observe that the proposition is the usual comparison theorem in case when $A=B$ and α is the identity map. The following remarks enable us to reduce the proposition to the classical theorem; 1) any B -module Z can be considered as an A -module by definition $az = \alpha(a)z$ for $a \in A$ and $z \in Z$, 2) any morphism $g: Z \rightarrow Z'$ in ${}_B\mathfrak{M}$ can be regarded as a morphism in ${}_A\mathfrak{M}$ by considering Z, Z' as A -modules because

$$g(az) = g(\alpha(a)z) = \alpha(a)g(z) = ag(z),$$

- 3) a R -homomorphism $k: X \rightarrow Y$ is a α -homomorphism iff k is a morphism in ${}_A\mathfrak{M}$ considering Y as an A -module. For $k(ax) = \alpha(a)k(x) = ak(x)$. From 1) and 2), $\eta: \mathfrak{Y} \rightarrow N$ can be considered as a R -split exact complex of N in ${}_A\mathfrak{M}$, and from 3) $f: M \rightarrow N$ is a morphism in ${}_A\mathfrak{M}$. It follows from the usual comparison theorem that there exists a chain map extension F of f in ${}_A\mathfrak{M}$. From 3) F is a α -homomorphism. It is immediate to see the rest of the proof. This proves the proposition.

Let us apply the proposition to the following case. Let A be a cocommutative Hopf algebra over Z_2 and let $\alpha: A \rightarrow A \otimes A$ be the cocommutative

comultiplication Δ . Since $M=Z_2$ and $N=Z_2 \otimes Z_2 \cong Z_2$ can be considered by augmentations as a left A -module and a left $A \otimes A$ -module respectively, the α -map $f: Z_2 \rightarrow Z_2 \otimes Z_2$ defined by $f(1)=1 \otimes 1$, can be extended to a Δ -chain map $h^0: \mathfrak{X} \rightarrow \mathfrak{X} \otimes \mathfrak{X}$ by the direct application of the proposition, where \mathfrak{X} is a Z_2 -split exact resolution of Z_2 . If $\rho: \mathfrak{X} \otimes \mathfrak{X} \rightarrow \mathfrak{X} \otimes \mathfrak{X}$ is the twisting chain map, then ρh^0 is again a Δ -chain map extension of f , because Δ is cocommutative. Hence, there exists a Δ -chain homotopy h^1 connecting h^0 with ρh^0 . Since ρh^1 is a Δ -chain homotopy and since $h^1 + \rho h^1$ is a Δ -chain map extension of the trivial Δ -homomorphism $0: Z_2 \rightarrow \mathcal{I}md_1$, there exists a Δ -chain homotopy h^2 connecting h^1 and ρh^1 . By the iterated use of the same arguments we have a sequence of Δ -chain homotopies $\{h^0, h^1, \dots, h^i, \dots\}$. Hence, we have

Proposition 6. *Let A be a cocommutative Hopf algebra over Z_2 and let $\Delta: A \rightarrow A \otimes A$ be the comultiplication. If $\varepsilon: \mathfrak{X} \rightarrow Z_2$ is a Z_2 -split exact resolution of the A -module Z_2 , then there exists a sequence of Δ -homomorphisms $h^i: \mathfrak{X} \rightarrow \mathfrak{X} \otimes \mathfrak{X}$ for $i=0, 1, \dots, n, \dots$ such that 1) h^0 is a grade preserving Δ -chain map and 2) for $i>0$ h^i is a Δ -chain homotopy connecting h^{i-1} with ρh^{i-1} which raises the homological dimensions by i and preserves the grading, where $\rho: \mathfrak{X} \otimes \mathfrak{X} \rightarrow \mathfrak{X} \otimes \mathfrak{X}$ is the twisting chain map.*

Consider a diagram

$$\begin{array}{ccc}
 \text{Hom}_A(\mathfrak{X}, Z_2) \otimes \text{Hom}_A(\mathfrak{X}, Z_2) & \xrightarrow{\cup} & \text{Hom}_A(\mathfrak{X}, Z_2) \\
 \chi \searrow & & \nearrow h^{i*} \\
 \text{Hom}_{A \otimes A}(\mathfrak{X} \otimes \mathfrak{X}, Z_2) & &
 \end{array}$$

where χ is the Z_2 -chain map defined by $\chi(f \otimes g)(x \otimes y) = f(x)g(y)$ for $f, g \in \text{Hom}_A(\mathfrak{X}, Z_2)$ and for $x, y \in \mathfrak{X}$.

DEFINITION 2. The cup- i -product \cup_i in the cochain complex $C = \text{Hom}_A(\mathfrak{X}, Z_2)$ is defined by $h^{i*} \cdot \chi$.

Denoting $\text{Hom}_A^s(X_p, Z_2)$ by $C^{p,s}$ for each homological dimension $p \geq 0$ and the grading $s \geq 0$, we have the cochain complex

$$C^{*s} = \{C^{p,s} \text{ for } p = 0, 1, \dots, n, \dots\}$$

such that $C = \{C^{*s} \mid s = 0, 1, \dots\}$. Then $f \cup_i g = \cup_i (f \otimes g) \in C^{p+q-i, s+t}$ for $f \in C^{p,s}$ and $g \in C^{q,t}$. It is immediate to see by definition the coboundary formula

$$\delta(f \cup_i g) = f \cup_{i-1} g + g \cup_{i-1} f + \delta f \cup_i g + f \cup_i \delta g.$$

For

$$\begin{aligned} \delta(f \cup_i g) &= \partial^* h^i \chi(f \otimes g) = \chi(f \otimes g) h^i \partial \\ &= \chi(f \otimes g) (h^{i-1} + \rho h^{i-1} + dh^i) \\ &= h^{i-1} \chi(f \otimes g) + h^{i-1} \rho \chi(f \otimes g) + h^i d \chi(f \otimes g) \\ &= h^{i-1} \chi(f \otimes g) + h^{i-1} \chi(g \otimes f) + h^i \chi(\delta \otimes 1 + 1 \otimes \delta)(f \otimes g) \\ &= f \cup_{i-1} g + g \cup_{i-1} f + \delta f \cup_i g + f \cup_i \delta g. \end{aligned}$$

DEFINITION 3. Algebraic Steenrod operation ${}_A Sq_i: H^{p,s}(A) \rightarrow H^{2p-i, 2s}(A)$ is defined by ${}_A Sq_i(\xi) = \overline{f \cup_i f}$, where $\xi \in H^{p,s}(A)$ is represented by $f \in C^{p,s}$ with $\delta f = 0$, and the bar over $f \cup_i f$ stands for the cohomology class.

Adams [1] and others (for example, see [11]) computed explicitly a Δ -homomorphism h^i in case when \mathfrak{X} is the bar resolution $B(A)$. If

$$\Delta(a) = \sum a' \otimes a''$$

for $a \in A$, then we have

$$\begin{aligned} h_n^0([a_1 | a_2 | \cdots | a_n]) \\ = 1 \otimes [a_1 | \cdots | a_n] + \sum_{1 \leq \rho \leq n} [a'_1 | \cdots | a'_\rho] \otimes a''_1 \cdots a''_\rho [a_{\rho+1} | \cdots | a_n], \end{aligned}$$

for odd i ,

$$\begin{aligned} h_n^t([a_1 | \cdots | a_n]) \\ = \sum_{0 \leq \rho_0 < \rho_1 < \cdots < \rho_i \leq n} [a'_1 | \cdots | a'_{\rho_0} | a'_{\rho_0+1} \cdots a'_{\rho_1} | a'_{\rho_1+1} | \cdots | a'_{\rho_2} | \cdots | a'_{\rho_{i-1}+1} \cdots a'_{\rho_i} | a_{\rho_i+1} | \cdots | a_n] \\ \otimes a''_1 \cdots a''_{\rho_0} [a''_{\rho_0+1} | \cdots | a''_{\rho_1} | a''_{\rho_1+1} \cdots a''_{\rho_2} | \cdots | a''_{\rho_{i-1}+1} | \cdots | a''_{\rho_i}], \end{aligned}$$

for even i ,

$$\begin{aligned} h_n^t([a_1 | \cdots | a_n]) = \sum_{0 \leq \rho_0 < \rho_1 < \cdots < \rho_i \leq n} [a'_1 | \cdots | a'_{\rho_0} | a'_{\rho_0+1} \cdots a'_{\rho_1} | \cdots | a'_{\rho_{i-1}+1} | \cdots | a'_{\rho_i}] \\ \otimes a''_1 \cdots a''_{\rho_0} [a''_{\rho_0+1} | \cdots | a''_{\rho_1} | a''_{\rho_1+1} \cdots a''_{\rho_2} | \cdots | a''_{\rho_{i-1}+1} | \cdots | a''_{\rho_i}], \end{aligned}$$

Let us sketch the method of computation for completeness sake. Let S be the contracting homotopy for $B(A)$, then $t = S \otimes 1 + \varepsilon \otimes S$ is a contracting homotopy for $B(A) \otimes B(A)$. Define $h_0^0 = \Delta$, then $h_0^0 S_0 = t_0 h_0^0$ determine h_1^0 by $h_1^0(ax) = \Delta(a) h_1^0(x)$. Inductively h^0 is obtained easily. Define $h_0^1 = t_0(h_0^0 + \rho h_1^0)$, then h_1^1 is calculated by $h_1^1 S_0 = t_1(h_1^0 + \rho h_2^0) S_0 + t_1 h_1^0$ and $h_1^1(ax) = \Delta(a) h_1^1(x)$. Repeat this process, we get the above formula.

3. $\mathfrak{S}(\Gamma, \Lambda)$ associated with a pair of Hopf algebras (Γ, Λ)

Let (Γ, Λ) be a pair of connected locally finite cocommutative Hopf algebras over Z_2 such that the subhopf algebra Λ is central in Γ . Then we have a sequence of Hopf algebras

$$\Lambda \xrightarrow{i} \Gamma \xrightarrow{\pi} \Omega = \Gamma / I(\Lambda) \cdot \Gamma$$

where the inclusion i and the projection π are morphisms of Hopf algebras (see [1]). In this setting we are going to associate with a pair of Hopf algebras (Γ, Λ) a graded differential algebra $\mathfrak{S}(\Gamma, \Lambda) = \{C, \delta, F, \cup\}$ with a decreasing filtration F and with cup- i -products \cup_i so that behaviors of algebraic Steenrod operations can be discussed in the spectral sequence $\{E_r, d_r\}$ associated with $\mathfrak{S}(\Gamma, \Lambda)$.

Recall the filtration in the bar construction $B(\Gamma)$ which Adams introduced in [1]. For each integer p define a subcomplex $F_p B(\Gamma)$ of $B(\Gamma)$ such that $F_p B(\Gamma)_n$ is the Γ -submodule of $B(\Gamma)_n = \Gamma \otimes I(\Gamma)^n$ generated by elements of the form $\gamma[\gamma_1 | \dots | \gamma_n]$ with the property that $\gamma_s \in I(\Lambda)$ for at least $(n-p)$ values of s . Then it is immediate to see that F is the canonical increasing filtration in $B(\Gamma)$.

Define the product filtration \tilde{F} in $B(\Gamma) \otimes B(\Gamma)$ by

$$\tilde{F}_p(B(\Gamma) \otimes B(\Gamma)) = \bigcup_{p \geq s \geq 0} F_{p-s} B(\Gamma) \otimes F_s B(\Gamma).$$

Then $(B(\Gamma) \otimes B(\Gamma), \tilde{F})$ is a resolution of $\Gamma \otimes \Gamma$ -module Z_2 with the increasing filtration \tilde{F} . Let $\Delta: \Gamma \rightarrow \Gamma \otimes \Gamma$ be the cocommutative diagonal and let ρ be the twisting chain map of $B(\Gamma) \otimes B(\Gamma)$. Then we have

Theorem 1. *There exists a sequence of Δ -homomorphisms*

$$h^i: B(\Gamma) \rightarrow B(\Gamma) \otimes B(\Gamma)$$

for $i=0, 1, \dots, n, \dots$ such that 1) h^0 is a Δ -chain map which preserves grading and filtration, 2) h^i is a Δ -chain homotopy connecting h^{i-1} and ρh^{i-1} which preserves grading, raises homological dimension by i , and satisfies the filtration condition

$$h^i(F_p B(\Gamma)) \subset \tilde{F}_{\alpha}(B(\Gamma) \otimes B(\Gamma))$$

for $\alpha = \min\{2p, p+i\}$.

Proof. In virtue of Proposition 6 it remains only to prove that h_n^i shown in §2 satisfies the filtration condition. By denoting

$$\Delta(\gamma) = \sum \gamma' \otimes \gamma''$$

the three formulas $h_n^i([\gamma_1 | \dots | \gamma_n])$ show that for each j with $n \geq j \geq 1$ exactly one of the three elements γ_j , γ'_j , and γ''_j appears solely between bars. For example,

if i is odd, each of the elements $\gamma'_1, \dots, \gamma'_{\rho_0}, \gamma''_{\rho_0+1}, \dots, \gamma''_{\rho_1}, \gamma'_{\rho_1+1} \dots, \gamma'_{\rho_2}, \dots, \gamma''_{\rho_{i-1}+1}, \dots, \gamma''_{\rho_i}, \gamma_{\rho_i+1}, \dots, \gamma_n$ appears solely in $| \cdot |$. It follows that if $[\gamma_1 | \dots | \gamma_n] \in F_p B(\Gamma)_n$, each term of the sum on the right hand sides of the formulas contains at least $(n-p)$ elements in $I(\Lambda)$. By definition of the product filtration \check{F} we obtain

$$h^i(F_p B(\Gamma)) \subset \check{F}_{p+i}(B(\Gamma) \otimes B(\Gamma)).$$

If $p \geq i$, then the proof is complete, because $\text{Min } \{p+i, 2p\} = p+i$. If $i \geq p$, it is seen that among i products

$$\gamma'_{\rho_0+1} \dots \gamma'_{\rho_1}, \gamma''_{\rho_1+1} \dots \gamma''_{\rho_2}, \dots, \gamma'_{\rho_{i-1}+1} \dots \gamma'_{\rho_i}$$

(or $\gamma''_{\rho_{i-1}+1} \dots \gamma''_{\rho_i}$ if i is even) there exist at least $(i-p)$ products contained in $I(\Lambda)$. Otherwise, at least $(p+1)$ products are not contained in $I(\Lambda)$. Then γ_s are not in $I(\Lambda)$ for at least $(p+1)$ values of s . This is a contradiction. It follows that each term of the sum for $h_n^i([\gamma_1 | \dots | \gamma_n])$ has at least

$$(n-p) + (i-p) = n + i - 2p$$

elements in $I(\Lambda)$. Therefore,

$$h^i(F_p B(\Gamma)) \subset \check{F}_{2p}(B(\Gamma) \otimes B(\Gamma))$$

if $i \geq p$, where $\text{Min } \{p+i, 2p\} = 2p$. This completes the proof.

Now let us dualize what we have obtained in this section. Let (C, δ) be the cochain complex $\text{Hom}_\Gamma(B(\Gamma), Z_2)$ over Z_2 . For each integer p define a subcomplex $F^p(C)$ by the image of

$$\text{Hom}_\Gamma(B(\Gamma)/F_{p-1}B(\Gamma), Z_2)$$

under the dual of the projection

$$p: B(\Gamma) \rightarrow B(\Gamma)/F_{p-1}B(\Gamma).$$

Then it is seen that (C, δ, F) is a cochain complex with a decreasing filtration. Let us call it Adams filtered complex associated with (Γ, Λ) .

Theorem 2. *Let (C, δ, F) be Adams filtered complex associated with a pair of Hopf algebras over Z_2 . Then there exist a Z_2 -linear map $\bigcup_i: C \otimes C \rightarrow C$ such that $\mathfrak{S}(\Gamma, \Lambda) = \{C, \delta, F, \bigcup_i\}$ is a graded differential algebra with a decreasing filtration F and with cup- i -products in the sense of Definition 1.*

Proof. Let $h^i: B(\Gamma) \rightarrow B(\Gamma) \otimes B(\Gamma)$ be the Δ -homomorphism in Theorem 1 and define $\bigcup_i: C \otimes C \rightarrow C$ by $h^{i*}\chi$ as was considered in Definition 2. Since \bigcup_i

is the cup- i -product in $C = \text{Hom}_{\Gamma}(B(\Gamma), Z_2)$, it is easy to see that \cup_i satisfies all the necessary conditions except the filtration condition. Consequently, it is sufficient to show that if $f \in F^p C^{m,s}$ and $g \in F^q C^{n,t}$, then $f \cup_i g \in F^{\alpha} C^{m+n-i, s+t}$ for $\alpha = \text{Max}\{p+q-i, p, q\}$. Consider first the case when

$$\alpha = \text{Max}\{p+q-i, p, q\} = p+q-i,$$

then

$$\text{Min}\{(\alpha-1)+i, 2(\alpha-1)\} = (\alpha-1)+i = p+q-1$$

except the case when $p=q=i$. By Theorem 1

$$h^i(F_{\alpha-1} B(\Gamma)) \subset \check{F}_{p+q-1}(B(\Gamma) \otimes B(\Gamma)).$$

If

$$h^i(x) = \sum x' \otimes x''$$

for $x \in F_{\alpha-1} B(\Gamma)_{m+n-i, s+t}$, then $x' \in F_{\xi} B(\Gamma)_{p, \theta}$ and $x'' \in F_{\eta} B(\Gamma)_{\sigma, \nu}$ with the property that $\xi + \eta = p+q-1$, $\rho + \sigma = m+n$, and $\theta + \nu = s+t$. Then

$$(f \cup_i g)(x) = \sum_i f(x') \cdot g(x'') = 0,$$

because $\xi < p$ or $\eta < q$. Therefore, $f \cup_i g \in F^{p+q-i} C^{m+n-i, s+t}$. If $\alpha = p$, then $p \geq q$ and $i \geq q$. In this case also,

$$(f \cup_i g)(F_{p-1} B(\Gamma)) = 0$$

can be shown because

$$h^i(F_{p-1} B(\Gamma)) \subset \check{F}_{p-1+i}(B(\Gamma) \otimes B(\Gamma)) \cap \check{F}_{2p-2}(B(\Gamma) \otimes B(\Gamma)).$$

Hence, the proof is completed.

From Theorem 2 and Proposition 1 we obtain

Theorem 3. *Let (Γ, Λ) be a pair of connected locally finite cocommutative Hopf algebras over Z_2 such that Λ is central in Γ , and let $\{E_{\gamma}, d_{\gamma}\}$ be Adams spectral sequence associated with the system $\mathfrak{S}(\Gamma, \Lambda)$. Then there exist algebraic Steenrod operations ${}_B St_i: E_{\gamma}^{p, q} \rightarrow E_{2\gamma-2}^{2p-i, 2q}$ for $\infty \geq \gamma \geq 2$ and ${}_F St_i: E_{\gamma}^{p, 2q+p-i} \rightarrow E_{\gamma}^{p, 2q+p-i}$ for $\infty \geq \gamma \geq 1$.*

4. Some properties of algebraic Steenrod operations

Theorem 4. *${}_B St_i$ and ${}_F St_i$ defined in Adams spectral sequence satisfy Propositions 2, 3, and 4.*

Theorem 5. *Let (Γ, Λ) and (Γ', Λ') be pairs of Hopf algebras over Z_2 , both of which satisfy the conditions stated before, and let E_γ and E'_γ be Adams spectral sequences associated with $\mathfrak{S}(\Gamma, \Lambda)$ and $\mathfrak{S}(\Gamma', \Lambda')$ respectively. If $f: (\Gamma, \Lambda) \rightarrow (\Gamma', \Lambda')$ be a morphism of pairs of Hopf algebras, then f induces a sequence of homomorphisms $\phi_\gamma: E'_\gamma \rightarrow E_\gamma$ for $\gamma \geq 1$ such that*

$$\phi_\gamma {}_F St_i = {}_F St_i \phi_\gamma \quad \text{and} \quad \phi_{2\gamma-2} {}_B St_i = {}_B St_i \phi_\gamma$$

for $\gamma \geq 2$.

Proof. It is obvious that f induces a chain map $B(f): B(\Gamma) \rightarrow B(\Gamma')$ preserving filtrations and gradings. If h^i and h'^i are Δ -homomorphisms in Theorem 1, then we have $h'^i B(f) = (B(f) \otimes B(f)) h^i$. Consequently, $B(f)$ induces a morphism $\mathfrak{S}(f): \mathfrak{S}(\Gamma', \Lambda') \rightarrow \mathfrak{S}(\Gamma, \Lambda)$. By a morphism $\mathfrak{S}(f)$ of the system \mathfrak{S} we mean that $\mathfrak{S}(f)$ is a chain map compatible with gradings, filtrations, and cup- i -products. Therefore, it is straightforward to verify the theorem.

Theorem 6. *Let $\Lambda \xrightarrow{i} \Gamma \xrightarrow{\pi} \Omega$ be a sequence of Hopf algebras as stated before, and let $\{E_\gamma\}$ be Adams spectral sequence associated with (Γ, Λ) . Then the natural maps $B(\pi): B(\Gamma) \rightarrow B(\Omega)$ and $B(i): B(\Lambda) \rightarrow B(\Gamma)$ induce isomorphisms $B(\pi)^*: H^p(\Omega) \rightarrow E_2^{p,0}$ and $B(i)^*: E_2^{0,q} \rightarrow H^q(\Lambda)$ respectively. If $E_2^{p,0}$ and $E_2^{0,q}$ are identified with $H^p(\Omega)$ and $H^q(\Lambda)$ respectively, then ${}_B St_i: E_2^{p,0} \rightarrow E_2^{2p-i,0}$ coincides with ${}_0 Sq_i: H^p(\Omega) \rightarrow H^{2p-i}(\Omega)$, and ${}_F St_i: E_2^{0,q} \rightarrow E_2^{0,2q-i}$ coincides with ${}_0 Sq_i: H^q(\Lambda) \rightarrow H^{2q-i}(\Lambda)$. Moreover, ${}_B St_i: E_\infty^{p,q} \rightarrow E_\infty^{2p-i,2q}$ for $i \leq p$ and ${}_F St_i: E_\infty^{p,q} \rightarrow E_\infty^{p+2q-i}$ for $i \geq p$ are induced by ${}_0 Sq_i: H^{p+q}(\Gamma) \rightarrow H^{2p+2q-i}(\Gamma)$.*

Proof. Adams has shown in [1] that $B(\pi)^*$ and $B(i)^*$ are isomorphisms. Hence, a morphism of pairs of Hopf algebras $\pi: (\Gamma, \Lambda) \rightarrow (\Omega, Z_2)$ induces the isomorphism $\phi_2: E_2'^{p,0} \rightarrow E_2^{p,0}$ for each p , because

$$E_2'^{p,0} = E_\infty'^{p,0} = H^p(\Omega).$$

Since ${}_B St_i: E_2'^{p,0} \rightarrow E_2^{p,0}$ is exactly ${}_0 Sq_i: H^p(\Omega) \rightarrow H^{2p-i}(\Omega)$, we obtain ${}_B St_i \phi_2 = \phi_2 {}_0 Sq_i$. Similarly, $\phi_2 {}_F St_i = {}_0 Sq_i \phi_2$. From the facts that $H^{p+q}(\Gamma)$ is filtered by $F^p H^{p+q}(\Gamma) = Z_\infty^{p,q} / B_\infty^{p,q}$ with the property that $E_\infty^{p,q} = F^p H^{p+q}(\Omega) / F^{p+1} H^{p+q}(\Omega)$ and that ${}_0 Sq_i$ maps $F^p H^{p+q}(\Gamma)$ into $F^{2p-i} H^{2p+2q-i}(\Gamma) \subset H^{2p+2q-i}(\Gamma)$, it is immediate to see that ${}_B St_i: E_\infty^{p,q} \rightarrow E_\infty^{2p-i,2q}$ is induced by ${}_0 Sq_i$. The rest of the proof is obvious. Hence, the proof is complete.

In a subsequent paper the author wishes to discuss higher cohomology operations involved in the Cartan formula and Massey-Uehara products.

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