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Studies on Applications of Stochastic Controls to Finance and Environmental Economics

(確率制御理論のファイナンスと環境経済学への応用に関する研究)

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Abstract

This thesis investigates an agent's optimization problem under uncertainty in finance and environmental economics. Uncertainty is inherent in most economic problems. There is uncertainty about future events and the response of the system to agents' controls. For example, prices of risky assets evolve over time in finance and the stocks of natural resources evolve over time in environmental economics. The systems are described by stochastic differential equations and called diffusion models. The basic sources of uncertainty in diffusion models are Brownian motion and Poisson jump. Since the systems are stochastic, it is difficult for agents to determine what problems they face under the restricted information available to agents. That is, agents must select an optimal decision among all possible decisions in order to solve their problems. Such optimization problems are called stochastic control problems, and are solved by using stochastic control theory. Stochastic control problems are formulated as (classical) stochastic control problems, optimal stopping problems, singular stochastic control problems, and impulse control problems corresponding to considering cases. This thesis deals with a (classical) stochastic control problem, an optimal stopping problem and impulse control problems in order to solve financial and environmental economics problems.

The thesis consists of an introduction and four topics. First, we examine an optimal natural resources management problem under uncertainty with catastrophic risk, and investigate the optimal rate of use of a natural resource. For this purpose, we use stochastic control theory. We assume that, until a catastrophic event occurs, the stock of the natural resource is governed by a stochastic differential equation. We describe the catastrophic phenomenon as a Poisson process. From this analysis, we show the optimal rate of use of the natural resource in explicit form. Furthermore, we present comparative static results for the optimal rate of use of the natural resource.

Second, we investigate the value of tradable emission permits (TEPs) under uncertainty, caused by the effects of an increase in the global mean surface temperature, and expressed as a geometric Brownian motion with a Poisson jump process. The Poisson jump process reflects the development of new technology to reduce CO2 emissions. To this end, we formulate a policy decision-maker's problem using a real options model. The problem is formulated as a search for the optimal timing of an irreversible investment under uncertainty, i.e. as an optimal stopping problem. From this analysis, under a suitable set of sufficient conditions, we show the value of the TEPs and present some numerical examples and comparative static results for their value. The value of the TEPs increases with uncertainty about damage from atmospheric CO2 concentrations, but decreases with the degree of development of new technology.

Third, we investigate a problem in which an agent implements an environmental im-

provement policy under uncertainty. If an emission level of a pollutant reaches a critical level, the agent has to decrease the emission to a certain level in order to improve the environment. The agent's problem is to minimize the expected total discounted cost, which includes the cost of implementing the EIP and the associated damage from the pollutant, under the assumption that a state process of the pollutant follows a geometric Brownian motion. Then, we find critical emission levels of the pollutant, optimal implementation times, optimal implementation size, and the value of the optimal EIP (OEIP), by using an impulse control approach. Then we present some numerical examples and comparative static results for the OEIP. The main results are as follows. An increase in the growth rate of the pollutant, uncertainty, the proportional cost and the constant cost all raise the value of the OEIP.

Finally, we investigate an optimal dividend policy with fixed and proportional transaction costs under a Brownian cash reserve process. The firm's problem is to maximize expected total discounted dividends. To this end, we formulate it as a stochastic impulse control problem, which is approached via quasi-variational inequalities (QVI). Under a suitable set of sufficient conditions, we show the existence of an optimal dividend policy such that whenever the cash reserve reaches a certain level, the firm pays out a dividend. Consequently, it instantaneously reduces to another level. We present some numerical examples and comparative static results for the optimal dividend policy.

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Contents

Chapter 1

Introduction

Uncertainty is inherent in most economic problems. There is uncertainty about future events and the response of the system to agents' controls. For example, prices of risky assets evolve over time in finance and the stocks of natural resources evolve over time in environmental economics. The systems are described by stochastic differential equations and are called diffusion models. The basic sources of uncertainty in diffusion models are Brownian motion and Poisson jump. Brownian motion represents continuous evolution, while Poisson jump represents discontinuous evolution. Since the systems are stochastic, it is difficult for agents to decide they are facing problems under the restricted information available to agents. That is, agents must select an optimal decision among all possible decisions in order to solve their problems. Such optimization problems are called stochastic control problems and are solved by using stochastic control theory. Stochastic control problems are formulated as classical stochastic control problems, optimal stopping problems, singular stochastic control problems, and impulse control problems corresponding to considering cases. Classical stochastic control problems concern situations in which agents continuously control the system. Optimal stopping problems concern situations in which agents decide when to intervene to control the system. Singular stochastic and impulse control problems induce discontinuous change of the system's state at the control time. This thesis investigates the agents' optimization problems under uncertainty in finance and environmental economics. To solve these problems, this thesis deals with a classical stochastic control problem, an optimal stopping problem, and impulse control problems to solve financial and environmental economics problems: a natural resource management problem, the value of tradable emission permits, an environment improvement problem, and a dividend policy problem. Thus, as a preliminary analysis, we first discuss stochastic dynamic programming. Next, we formulate the agent's problem as an optimal stopping problem and an impulse control problem. For this analysis, we refer mainly to Fleming and Soner (1993) and Yong and Zhou (1999).

Stochastic Dynamic Programming

Dynamic programming, which was developed by Richard Bellman (See Bellman (1957)), is appropriate to both deterministic and stochastic optimal control problems. The key principle of dynamic programming is the principle of optimality. In Bellman (1957) p. 83, Bellman stated as follows:

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

The idea of the principle of optimality goes back to the *brachistochrone problem* proposed by Johan Bernoulli in 1696. The problem poses the question: if a small object moves under the effect of gravity, which path makes the trip from one end to the other end in the shortest time? From this proposition, he is regarded as one of the founders of the *calculus of variation.* The solution of the problem was provided by his brother Jakob Bernoulli[1] in 1697. See Section 1 in Kamien and Schwartz (1991) and Chapter 4.7 in Yong and Zhou (1999).

In dynamic programming, the initial values of state variables are fixed and the maximum/minimum value of the performance criterion is considered as a function of this initial value of the state variables. This function is called the *value function.* Whenever the value function is sufficiently differentiable, it is a solution of a nonlinear first order partial differential equation in the deterministic case, or a second order partial differential equation in the stochastic case. The partial differential equation is called the *Hamilton-Jacobi-Bellman (HJB) equation.* If a smooth solution of the HJB equation is given, then we provide sufficient conditions that allow us to conclude that the solution coincides with the value function. This is called *verification technique.*

The discrete-time stochastic version of dynamic programming was already discussed in Bellman (1957). According to Chapter 4.7 of Yong and Zhou (1999), the continuous-time stochastic version of dynamic programming was first studied by Kushner (1962). See also, for example, Fleming and Rishel (1975), Krylov (1980), and Fleming and Soner (1993) for further discussion.

In this part, we first describe Bellman's principle of optimality, the HJB equation. Next, we derive the HJB equation. Finally, we present the verification technique.

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ denote a filtered probability space satisfying the usual conditions, i.e., $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$. Here $\mathcal{F}_t$ is generated by an m-dimensional standard Brownian motion, W_t, i.e., $\mathcal{F}_t = \sigma(W_s, s \leq t)$. Let $u = (u_t)_{t\geq 0}$ be the control processes that takes values in a control space $\mathbb{U}$. The processes of the system state $X^{x_0} = (X_t)_{t\geq 0}$ are given by the following controlled stochastic differential equation:

$$dX_t = b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t, \quad X_0 = x_0 \in \mathbb{R}^n, \tag{1.0.1}$$

where $b : [0,T] \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^{n\times m}$. Let $\mathcal{U}$ be the set of all progressively measurable $\mathbb{U}$-valued processes u. If $u \in \mathcal{U}$, then u is called the *admissible control processes.* Suppose that the purpose of an agent is to minimize the agent's expected total cost, the sum of running cost and terminal cost. Let $f : [0,T] \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$ be a running cost function and $g : \mathbb{R}^n \to \mathbb{R}$ be a terminal cost function. Thus, the expected total discounted cost function is given by

$$J(t, x; u) = \mathbb{E}\left[\int_t^T e^{-rs} f(s, X_s, u_s)ds + e^{-rT} g(X_T)\right], \tag{1.0.2}$$

where $X_t = x$ and $r(\in \mathbb{R}_{++})$ is a constant discount factor. We assume that the following:

[1]He is also known as Jackque I or James I.

Assumption 1.0.1.

$$|b(t,x,u) - b(t',x',u)| \leq C(|x - x'| + |t - t'|), \quad t, t' \in \mathbb{R}_+, \; x, x' \in \mathbb{R}^n, \; u \in \mathbb{U}; \tag{1.0.3}$$

$$|\sigma(t,x,u) - \sigma(t',x',u)| \leq C(|x - x'| + |t - t'|), \quad t, t' \in \mathbb{R}_+, \; x, x' \in \mathbb{R}^n, \; u \in \mathbb{U}; \tag{1.0.4}$$

$$|b(t,x,u)| + |\sigma(t,x,u)| \leq C(1 + |x|^m + |u|^m), \quad t \in \mathbb{R}_+, \; x, \in \mathbb{R}^n, \; u \in \mathbb{U}; \tag{1.0.5}$$

$$|f(t,x,u)| \leq C(1 + |x|^m + |u|^m), \quad t \in \mathbb{R}_+, \; x, \in \mathbb{R}^n, \; u \in \mathbb{U}; \tag{1.0.6}$$

$$|g(x)| \leq C(1 + |x|^m), \quad x, \in \mathbb{R}^n, \tag{1.0.7}$$

for suitable constants C and m.

Ineqs. (1.0.6) and (1.0.7) ensure that $J(t,x;u)$ is well defined. X_t is called a *unique solution*, if ineqs. (1.0.3) – (1.0.5) and the following conditions are satisfied:

$$X_0 = x_0, \quad a.s.; \tag{1.0.8}$$

$$X_t = x_0 + \int_0^t b(s, X_s, u_s) ds + \int_0^t \sigma(s, X_s, u_s) dW_s, \quad t \geq 0, \; a.s.; \tag{1.0.9}$$

$$X_t = Y_t, \quad 0 \leq t < \infty, \; a.s., \tag{1.0.10}$$

where X_t and Y_t are two solutions to eq. (1.0.1). Thus the agent's problem is to choose $u \in \mathcal{U}$ in order to minimize $J(t,x;u)$:

$$V(t,x) = \inf_{u \in \mathcal{U}} J(t,x;u) = J(t,x;u^*), \tag{1.0.11}$$

where V is the value function of the agent's problem eq. (1.0.11) and u^* is the optimal control for the agent's problem eq. (1.0.11). Assumption 1.0.1 leads to the following result, which is useful in the proof of Theorem 1.0.1.

Proposition 1.0.1. *Suppose that Assumption 1.0.1 holds. Then the value function $V(t,x)$ satisfies the following:*

$$|V(t,x)| \leq C(1 + |x|), \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n, \tag{1.0.12}$$

$$|V(t,x) - V(t',x')| \leq C[|x - x'| + (1 + |x| \vee |x'|)|t - t'|^{1/2}], \quad \forall (t,x), (t',x') \in [0,T] \times \mathbb{R}^n, \tag{1.0.13}$$

for suitable constants C.

Proof. See Yong and Zhou (1999) Proposition 4.3.1. □

We are now in a position to present *Bellman's principle of optimality* in the stochastic environment. Bellman's principle of optimality is also called the *dynamic programming principle.* We refer to Theorem 4.3.3. in Yong and Zhou (1999).

Theorem 1.0.1 (Bellman's principle of optimality). *Suppose that Assumption 1.0.1 holds. Then for any initial condition $(t,x) \in [0,T) \times \mathbb{R}^n$ we have*

$$V(t,x) = \inf_{u \in \mathcal{U}} \mathbb{E}\left[\int_t^{t'} e^{-rs} f(s, X_s^{t,x,u}, u_s) ds + e^{-rt'} V(t', X_{t'}^{t,x,u})\right], \quad \forall 0 \le t \le t' \le T. \tag{1.0.14}$$

Proof. Let $\bar{V}(t,x)$ be the right-hand side of eq. (1.0.14). We first show $V(t,x) \ge \bar{V}(t,x)$. For any $\varepsilon > 0$, there exists an $u \in \mathcal{U}$ such that

$$V(t,x) + \varepsilon > J(t,x,u). \tag{1.0.15}$$

Rewriting $J(t,x,u)$, we obtain that

$$\begin{aligned}
J(t,x,u) &= \mathbb{E}\Bigg[\int_t^{t'} e^{-rs} f(s, X_s^{t,x,u}, u_s) ds \\
&\qquad + e^{-rt'} \mathbb{E}\left[\int_{t'}^T e^{-rs} f(s, X_s^{t,x,u}, u_s) ds + e^{-rT} g(X_T^{t,s,u}) \Big| \mathcal{F}_{t'}\right]\Bigg] \\
&= \mathbb{E}\Bigg[\int_t^{t'} e^{-rs} f(s, X_s^{t,x,u}, u_s) ds \\
&\qquad + e^{-rt'} \mathbb{E}\left[\int_{t'}^T e^{-rs} f(s, X_s^{t',X_{t'},u}, u_s) ds + e^{-rT} g(X_T^{t',X_{t'},u}) \Big| \mathcal{F}_{t'}\right]\Bigg] \\
&= \mathbb{E}\left[\int_t^{t'} e^{-rs} f(s, X_s^{t,x,u}, u_s) ds + e^{-rt'} J(t', X_{t'}^{t,x,u})\right] \\
&= \mathbb{E}\left[\int_t^{t'} e^{-rs} f(s, X_s^{t,x,u}, u_s) ds + e^{-rt'} V(t', X_{t'}^{t,x,u})\right] \\
&\ge \bar{V}(t,x).
\end{aligned} \tag{1.0.16}$$

Since ε is arbitrary, it follows from ineqs. (1.0.15) and (1.0.16) that $V(t,x) \ge \bar{V}(t,x)$.

Next, we show $V(t,x) \le \bar{V}(t,x)$. For any $\varepsilon > 0$, by Proposition 1.0.1 there is a $\delta = \delta(\varepsilon)$ such that whenever $|y - y'| < \delta$,

$$|J(t',x;u) - J(t',x';u)| + |V(t',x) - V(t',x')| \le \varepsilon, \quad \forall u \in \mathcal{U}. \tag{1.0.17}$$

Let $(D_j)_{j \ge 1}$ be a *Borel partition*[2] with diameter $\mathrm{diam}(D_j) < \delta$. Choose $x^j \in D_j$. For each j, there is $u^j \in \mathcal{U}$ such that

$$J(t', x^j; u^j) \le V(t', x^j) + \varepsilon. \tag{1.0.18}$$

[2]A Borel partition means that $D_j \in \mathcal{B}(\mathbb{R}^n)$, $\bigcup_{j \ge 1} D_j = \mathbb{R}^n$, and $D_i \cap D_j = \emptyset$ if $i \ne j$.

Hence for any $x^j \in D_j$, combing ineqs. (1.0.17) and (1.0.18), we obtain

$$J(t', x; u^j) \leq J(t', x^j; u^j) + \varepsilon \leq V(t', x^j) + 2\varepsilon \leq V(t', x) + 3\varepsilon. \tag{1.0.19}$$

From $u^j \in \mathcal{U}$, there is a function $\psi^j \in \mathcal{A}^m(\mathbb{U})$ such that

$$u_s^j(\omega) = \psi_s^j(W_{\cdot\wedge s}^j(\omega)), \quad a.s.\ \omega \in \Omega^j,\ s \in [t', T], \tag{1.0.20}$$

where $\mathcal{A}^m(\mathbb{U})$ is the set of all $(\mathcal{B}_{t+}(\boldsymbol{W}^m[0,T]))_{t\geq 0}$-progressively measurable processes η : $[0,T] \times \boldsymbol{W}^m[0,T] \to \mathbb{U}$. Here we put $\boldsymbol{W}^m[0,T] := C([0,T]; \mathbb{R}^m)$ and $\mathcal{B}_{t+}(\boldsymbol{W}^m[0,T]) = \bigcap_{s>t} \mathcal{B}_t(\boldsymbol{W}^m[0,T])$, for all $t \in [0,T)$. Define a new control $\tilde{u}_s(\omega) \in \mathcal{U}$ such that

$$\tilde{u}_s(\omega) = \begin{cases} u_s(\omega), & s \in [t, t'); \\ \psi_s^j(W_{\cdot\wedge s}^j(\omega)), & s \in [t', T] \text{ and } X_s(\omega) \in D_j. \end{cases} \tag{1.0.21}$$

Thus, we obtain

$$V(t, x) \leq J(t, x; \tilde{u}) \tag{1.0.22}$$

Rewriting $J(t, x; \tilde{u})$ yields

$$\begin{aligned} J(t, x; \tilde{u}) &= \mathbb{E}\left[\int_t^{t'} e^{-rs} f(s, X_s^{t,x,u}, u_s) ds \right. \\ &\qquad \left. + e^{-rt'} \mathbb{E}\left[\int_{t'}^{T} e^{-rs} f(s, X_s^{t,x,u}, \tilde{u}_s) ds + e^{-rT} g(X_T^{t', X_{t'}, \tilde{u}}) \middle| \mathcal{F}_{t'}\right]\right] \\ &= \mathbb{E}\left[\int_t^{t'} e^{-rs} f(s, X_s^{t,x,u}, u_s) ds + e^{-rt'} J(t', X_{t'}^{t,x,u}; \tilde{u})\right] \\ &\leq \mathbb{E}\left[\int_t^{t'} e^{-rs} f(s, X_s^{t,x,u}, u_s) ds + e^{-rt'} V(t', X_{s'}^{t,x,u})\right] + 3\varepsilon, \end{aligned} \tag{1.0.23}$$

where the last inequality is due to ineq. (1.0.19). Since ε is arbitrary, it follows from ineqs. (1.0.22) and (1.0.23) that $V(t, x) \leq \bar{V}(t, x)$. Therefore, the proof is completed. □

Eq. (1.0.14) is also called the *dynamic programming equation.*

Next we derive the *Hamilton-Jacobi-Bellman (HJB) equation* of the agent's problem eq. (1.0.11) via the Bellman's optimality principle under smoothness assumptions on the value function. Let $G : [0,T] \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}_{++} \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ be defined by

$$\begin{aligned} G(t, x, u, q, p, P) &:= \frac{1}{2} \operatorname{tr}[P\sigma(t, x, u)\sigma(t, x, u)^\top] + p \cdot b(t, x, u) - rq + f(t, x, u), \\ &\qquad \forall (t, x, u, p, P) \in [0,T] \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}_{++} \times \mathbb{R}^n \times \mathcal{S}^n, \end{aligned} \tag{1.0.24}$$

where $\top$ denotes transposition and $\mathcal{S}^n$ represents the set of all $(n \times n)$ symmetric matrices. The function G is called the *generalized Hamiltonian.*

Theorem 1.0.2 (HJB equation). *Suppose that Assumption 1.0.1 holds and that the value function $V \in C^{1,2}([0,T] \times \mathbb{R}^n)$. Then V is a solution of the following second-order partial differential equation:*

$$-V_t + \sup_{u \in \mathbb{U}} G(t, x, u, -V, -V_x, -V_{xx}) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^n \tag{1.0.25}$$

with the terminal condition

$$V|_{t=T} = g(x), \quad x \in \mathbb{R}^n. \tag{1.0.26}$$

Proof. We refer to Proposition 4.3.5 of Yong and Zhou (1999). Let $\tilde{V}(t,x)$ be the left-hand side of eq. (1.0.25).

i) We first show that $\tilde{V}(t,x) \leq 0$. Let us fix initial condition $(t,x) \in [0,T) \times \mathbb{R}^n$ and take constant control $\bar{u} \in \mathbb{U}$. It follows from eq. (1.0.14) that

$$V(t,x) \leq \mathbb{E}\left[\int_t^{t'} e^{-rs} f(s, X_s^{t,x,\bar{u}}, \bar{u}) ds + e^{-rt'} V(t', X_{t'}^{t,x,\bar{u}})\right]. \tag{1.0.27}$$

Subtract $V(t,s)$ from both sides, divide by $(t'-t)$ and let $t' \downarrow t$ and use Ito formula:

$$\begin{aligned} 0 &\leq \lim_{t' \downarrow t} \frac{1}{t'-t} \mathbb{E}\left[\int_t^{t'} e^{-rs} f(s, X_s^{t,x,\bar{u}}, u_s) ds + [e^{-rt'} V(t', X_{t'}^{t,x,\bar{u}}) - V(t,x)]\right] \\ &= f(t,x,\bar{u}) + \frac{1}{2}\sigma(t,x,\bar{u})\sigma(t,x,\bar{u})^\top V_{xx}(t,x) + b(t,x,\bar{u}) \cdot V_x(t,x) - rV(t,x) + V_t(t,x). \end{aligned} \tag{1.0.28}$$

Since ineq. (1.0.28) holds for all $\bar{u} \in \mathbb{U}$, we have

$$-V_t(t,x) + \sup_{u \in \mathbb{U}} G(t,x,u,-V(t,x),-V_x(t,x),-V_{xx}(t,x)) \leq 0. \tag{1.0.29}$$

Thus we show $\tilde{V}(t,x) \leq 0$.

ii) Next we show that $\tilde{V}(t,x) \geq 0$. For any $\varepsilon > 0$, $0 \leq t < t' \leq T$ with $t' > t$ small enough, there exists a $\tilde{u} \in \mathcal{U}$ such that

$$V(t,x) + \varepsilon(t'-t) \geq \mathbb{E}\left[\int_t^{t'} e^{-rs} f(s, X_s^{t,x,\tilde{u}}, \tilde{u}_s) ds + e^{-rt'} V(t', X_{t'}^{t,x,\tilde{u}})\right]. \tag{1.0.30}$$

Applying the same argument as i), we obtain

$$\begin{aligned} \varepsilon &\geq \lim_{t' \downarrow t} \frac{1}{t'-t} \mathbb{E}\left[\int_t^{t'} e^{-rs} f(s, X_s^{t,x,\tilde{u}}, \tilde{u}_s) ds + [e^{-rt'} V(t', X_{t'}^{t,x,\tilde{u}}) - V(t,x)]\right] \\ &= f(t,x,\tilde{u}) + \frac{1}{2}\sigma(t,x,\tilde{u})\sigma(t,x,\tilde{u})^\top V_{xx}(t,x) + b(t,x,\tilde{u}) \cdot V_x(t,x) - rV(t,x) + V_t(t,x). \end{aligned} \tag{1.0.31}$$

The limit above is derived from the fact that is implied by the uniform continuity of b, σ, f. from Assumption 1.0.1:

$$\lim_{t' \downarrow t} \sup_{x \in \mathbb{R}^n, u \in \mathbb{U}} |\varphi(t', x, u) - \varphi(t, x, u)| = 0, \tag{1.0.32}$$

where $\varphi = b, \sigma, f$. Since ε is arbitrary, we obtain

$$-V_t(t,x) + \sup_{u \in \mathbb{U}} G(t,x,u,-V(t,x),-V_x(t,x),-V_{xx}(t,x)) \geq 0. \tag{1.0.33}$$

Thus we show $\tilde{V}(t,x) \leq 0$. Combining ineqs. (1.0.29) and (1.0.33), the proof is completed. □

Eq. (1.0.25) is called the HJB equation of the agent problem eq. (1.0.11).

Next, we show that if a smooth solution ϕ of the HJB equation with terminal condition is given, then ϕ coincides with the value function and it also gives us the form of the optimal control. This is well known as the *verification theorem*. To show the verification theorem, we introduce the partial differential operator $\mathcal{L}$ associated with the controlled process $(e^{-rt}X_t)_{t\geq 0}$:

$$\mathcal{L}\phi(t,x) := \frac{1}{2}\operatorname{tr}[\sigma(t,x,u)\sigma(t,x,u)^\top \Phi_{xx}(t,x)] + b(t,x,u)\cdot \Phi_x(t,x) - r\phi(t,x). \tag{1.0.34}$$

The following formula is derived from the Ito formula and is called the *Dynkin formula*:

$$\mathbb{E}[e^{-rt'}\phi(t',X_{t'}^{t,x,u})] - \phi(t,x) = \mathbb{E}\left[\int_t^{t'} e^{-rs}[\phi_t(s,X_s^{t,x,u}) + \mathcal{L}\phi(s,X_s^{t,x,u})]ds\right] \tag{1.0.35}$$

Note that the relationship between the operator $\mathcal{L}$ and the generalized Hamiltonian G is as follows.

$$\mathcal{L}\phi(t,x) = G(t,x,u,\phi,\phi_x,\phi_{xx}) - f(t,x,u). \tag{1.0.36}$$

We refer to Theorem III.8.1 in Fleming and Soner (1993) and Theorem 5.5.1. in Yong and Zhou (1999).

Theorem 1.0.3 (Verification theorem). *Suppose that Assumption 1.0.1 holds. Let $\phi \in C^{1,2}([0,T]\times\mathbb{R}^n)$ be a solution of the HJB equation eq. (1.0.25) with terminal condition eq. (1.0.26).*

(I) Then

$$\phi(t,x) \leq J(t,x;u), \quad \forall u \in \mathcal{U},\ (t,x) \in [0,T)\times\mathbb{R}^n. \tag{1.0.37}$$

(II) If there exists an admissible control $u^* \in \mathcal{U}$ such that

$$u^* = \arg\max\{-\phi_t(t,x) + G(t,x,u,-\phi(t,x),-\phi_x(t,x),\phi_{xx}(t,x))\}, \tag{1.0.38}$$

then $\phi = V$, and u^* is an optimal control for the agent's problem eq. (1.0.11).

Proof. (I) For any $u \in \mathcal{U}$, we have

$$\phi_t(t,x) + \mathcal{L}\phi(t,x) + f(t,x,u) \geq 0. \tag{1.0.39}$$

From the terminal condition (1.0.26) and the Dynkin formula we obtain

$$\phi(t,x) = \mathbb{E}\left[\int_t^T -e^{-rs}[\phi_t(s,X_s^{t,x,u}) + \mathcal{L}\phi(s,X_s^{t,x,u})]ds + e^{-rT}g(X_T)\right] \tag{1.0.40}$$

Combining (1.0.39) and (1.0.40) yields

$$\phi(t,x) \leq \mathbb{E}\left[\int_t^T e^{-rs} f(s, X_s^{t,x,u})ds + e^{-rT} g(X_T)\right] \tag{1.0.41}$$

This completes the proof of (I).

(II) To prove the second assertion of the theorem, repeating the argument of (I) and observing that the control u^* achieves equality in eq. (1.0.39). Therefore, from eq. (1.0.36), we obtain $\phi = V$ and u^* is an optimal control. The proof is completed. □

The verification theorem plays an important role in the following chapters. See Theorem 2.3.1, Theorem 3.3.1, Theorem 4.3.1, and Theorem 5.3.1.

Optimal Stopping Problems

Suppose that an agent faces an optimization problem: minimize the agent's expected total costs, the sum of running cost and terminal cost. Furthermore, the agent decides the timing of control in order to minimize expected total costs. Stopping time is a part of the control. This is called an *optimal stopping problem.*

The mathematical origin of the optimal stopping problem is sequential analysis in Wald (1947). See also Chow, Robbins, and Siegmund (1971) for more details. The first study of the optimal stopping problem was Chernoff (1968). The optimal stopping problem is related to the *free boundary problem.* See, for example, van Moerbeke (1974). Bensoussan and Lions (1973a) applied the *variational inequalities (VI) approach* to the optimal stopping problem. See also Bensoussan and Lions (1982) and Chapter 10 of Øksendal (1998). Brekke and Øksendal (1991) and Chapter 10 of Øksendal (1998) present the connection between VI and the *smooth pasting condition.* The smooth pasting condition is also called the *high contact principle* or the *the smooth fit principle* and was first introduced by Samuelson (1965). The smooth pasting condition is essentially a first order condition for optimal stopping problems. See, for example, Merton (1973). See also the excellent text of smooth pasting Dixit (1993). Many of the studies of the optimal stopping problem use the variational inequalities approach to solve these problems. We will discuss VI in Lemma 3.3.2 and the smooth pasting condition in Chapters 3–5. Optimal stopping problems are applied to the valuation of American-type options in finance. See, for example, Jaillet, Lamberton, and Lapeyre (1990) and Mordecki (1999, 2002). In environmental economics, optimal harvesting time problems are solved using the optimal stopping approach. See, for example, Clarke and Reed (1989) and Reed and Clarke (1990). As other example, Tsujimura (2000) investigates the value of tradable emission permits by using an optimal stopping approach. We will discuss this issue in Chapter 3. In this part, we formulate the agent's problem as an optimal stopping problem.

We assume that the drift parameter $b(t,x,u) = b(x)$, the diffusion parameter $\sigma(t,x,u) = \sigma(x)$, and the initial value $x_0 = x$ in eq. (1.0.1). Thus, the state of the system is governed by

$$dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dW_t, \quad X_0^x = x \in \mathbb{R}^n, \tag{1.0.42}$$

where $b : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfy Assumption 1.0.1:

$$|b(x) - b(x')| \leq C|x - x'|, \quad x, x' \in \mathbb{R}^n; \tag{1.0.43}$$

$$|\sigma(x) - \sigma(x')| \leq C|x - x'|, \quad x, x' \in \mathbb{R}^n; \tag{1.0.44}$$

$$|b(x)| + |\sigma(x)| \leq C(1 + |x|^m), \quad x, \in \mathbb{R}^n \tag{1.0.45}$$

for suitable constants C and m.

In the agent's problem given by eq. (1.0.11), we assumed that the control horizon was fixed, $[0, T]$. In this part, we consider the control horizon is a random case. In this context, the terminal time is a control variable and is called *stopping time*. Thus, the agent's cost function is given by

$$J(x;\tau) = \mathbb{E}\left[\int_0^\tau e^{-rt} f(X_t^x)dt + e^{-r\tau} g(X_\tau)\right], \tag{1.0.46}$$

where τ is an $(\mathcal{F}_t)_{t\geq 0}$-stopping time defined by

$$\tau := \inf\{t \geq 0 : X_t^x \notin \mathcal{O}\}. \tag{1.0.47}$$

Here $\mathcal{O} \subseteq \mathbb{R}^n$ is a given open set. Note that

$$u = \tau. \tag{1.0.48}$$

We also assume that f, g satisfy Assumption 1.0.1. Therefore the agent's problem is to choose stopping time to minimize $J(x;\tau)$:

$$V(x) = \inf_{\tau \in \mathcal{U}} J(x;\tau) = J(x;\tau^*), \tag{1.0.49}$$

where V is the value function of the agent's problem eq. (1.0.49) and u^* is the optimal control for the agent's problem eq. (1.0.49).

Impulse Control Problems

Suppose that an agent incurs implementation costs in order to implement a control. There are two types of implementation costs. One is independent of the magnitude of the control. We will call this the fixed cost in Chapters 4 and 5. The other cost depends on the magnitude of the control. We will call this the proportional cost in Chapters 4 and 5. From these costs, the control induces discontinuous change of state at time t. This is reasonable in the context of finance and environmental economics. For the finance example, when companies pay dividends, it incurs both types of transaction costs. Thus, companies do not continuously pay dividends, but pay them out at particular time intervals. We will investigate this issue in Chapter 5, which is based on Ohnishi and Tsujimura (2002a). Another financial example, in the context of consumption/investment problems, is that investors trade securities at discrete-time intervals due to transaction costs. See, for example, Korn (1997), Cadenillas (2000), and Tsujimura (2003b). For environmental economics, if an agent implements the policy in order to reduce a pollutant, the policy incurs both types of implementation costs. Thus, when the policy is implemented, the state of the pollutant jumps the other level. We will investigate this issue in Chapter 4, which is based on Tsujimura (2001). See also Willassen (1998) for the other applications to environmental economics. To represent discontinuous change of state, singular control and impulse control are useful.

Following Chapter 2.7.3 of Yong and Zhou (1999), we formulate the general singular stochastic control and impulse control problems. First, we define the function space: $\mathcal{D}$ is the space of all functions $\varsigma : [0,\infty) \to \mathbb{R}$ that are right-continuous with left limits (RCLL or càdlàg). Define the total variation of ς on $[0,\infty)$ as

$$\int_0^\infty |d\varsigma_t| := |\varsigma|_{[0,\infty)}, \quad \varsigma \in \mathcal{D}, \tag{1.0.50}$$

where $|\varsigma^i|_{[0,\infty)}$ is the total variation of the ith component of ς on $[0,\infty)$. Let $\Delta\varsigma_t := \varsigma_t - \varsigma_{t-}$ and let $\mathbb{T}_\varsigma := \{t \in [0,\infty); \Delta\varsigma_t \neq 0\}$. Furthermore, we define

$$\tilde{\mathcal{D}} = \{\varsigma \in \mathcal{D}; |\varsigma|_{[0,\infty)} < \infty\}. \tag{1.0.51}$$

Let ς_t^{jp} be the pure jump part of $\varsigma \in \tilde{\mathcal{D}}$ and let it be defined by $\varsigma_t^{jp} := \sum_{0\leq s<t} \Delta\varsigma_s$. Let ς_t^c be the continuous part of $\varsigma \in \tilde{\mathcal{D}}$ and let this be defined by $\varsigma_t^c := \varsigma_t - \varsigma_t^{jp}$. Since ς^c is bounded variation, it is differentiable almost everywhere. Thus, it follows that $\varsigma_t^c = \varsigma_t^{ac} + \varsigma_t^{sc}$ for $t \in [0,\infty)$, where $\varsigma_t^{ac} := \int_0^t \dot{\varsigma}_s^c ds$ is the absolutely continuous part of ς and ς_t^{sc} is the singularly continuous part of ς. Thus it follows that the Lebesgue decomposition for $\varsigma \in \tilde{\mathcal{D}}$ is

$$\varsigma_t = \varsigma_t^{ac} + \varsigma_t^{sc} + \varsigma_t^{jp}, \quad t \in [0,\infty). \tag{1.0.52}$$

See also Billingsley (1995) p. 414.

In this part, we also assume that the drift parameter $b(t,x,u) = b(x)$, the diffusion parameter $\sigma(t,x,u) = \sigma(x)$, and the initial value $x_0 = x$ in eq. (1.0.1). Consider the following stochastic differential equation:

$$dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dW_t + d\varsigma_t, \quad X_0^x = x \in \mathbb{R}, \quad \varsigma \in \tilde{\mathcal{D}}, \tag{1.0.53}$$

where $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfy ineqs. (1.0.43) – (1.0.45). Suppose that the agent's objective is to minimize the expected total cost:

$$\begin{aligned} J(x;\varsigma) = \mathbb{E}\Bigg[&\int_0^T e^{-rt} f(X_t^x)dt + \int_0^T e^{-rt} f^a(t)\|\dot{\varsigma}_t^{ac}\|_1 dt \\ &+ \int_0^T e^{-rt} f^s(t)|d\varsigma_t^{sc}| + \sum_{t\in\mathbb{T}_\varsigma} e^{-rt} K(\Delta\varsigma_t) 1_{\mathbb{T}_\varsigma \ni t<T} + e^{-rT} g(X_T^x) 1_{T<\infty}\Bigg], \end{aligned} \tag{1.0.54}$$

where f, f^a, f^s, and g satisfy ineqs. (1.0.6) and (1.0.7). K is a given function. In this context, $\|\dot{\varsigma}_t^{ac}\|_1 dt$ and $|d\varsigma_t^{sc}|$ are the measures generated by the total variations of ς^{ac} and ς^{sc}, respectively, with $\|\cdot\|_1$ denoting the L^1-norm in $\mathbb{R}$. Thus the agent's problem is to minimize (1.0.54) over $\tilde{\mathcal{D}}$. This problem is known as a *singular stochastic control problem.*

A singular stochastic control was initially studied by Bather and Chernoff (1967). See Fleming and Soner (1993) for more details of singular stochastic control. An *impulse control problem* is a special case of the singular stochastic control problem. If ς takes the form of a pure jump process, i.e., if $\varsigma^{ac} \equiv \varsigma^{sc} \equiv 0$, then the agent's problem reduces to an impulse control problem. Furthermore, if $\varsigma^{sc} \equiv \varsigma^{jp} \equiv 0$, then $\varsigma_t^{ac} \equiv \int_0^t \dot{\varsigma}_s^{ac} ds$ and the agent's problem reduces to a standard stochastic control problem.

In this section, we concentrate on impulse control problems. An impulse control problem was initially studied by Bensoussan and Lions (1973b,c) and Bensoussan, Goursat, and Lions (1973), and is solved by using the *quasi-variational inequalities (QVI)* introduced by the work of Bensoussan and Lions in 1973. See also Harrison et al (1983) and Bensoussan and Lions (1984). As for the QVI, we will define it in Definitions 4.3.1 and 5.3.1. Let ζ_i be the ith impulse in $\mathbb{R}_+$, where $\eta(x,\zeta)$ is the new state value following control of the system. Note that $\zeta = \varsigma^{jp}$. Let τ_i be the ith control time such that $\tau_i \to \infty$ as $i \to \infty$. Henceforth, v is an impulse control defined by the following double sequence:

$$v := \{(\tau_i, \zeta_i)\}_{i \geq 0}. \tag{1.0.55}$$

Note that $u = v$. The stochastic differential equation (1.0.53) becomes

$$\begin{cases} dX_t^{x,v} = b(X_t^{x,v})dt + \sigma(X_t^{x,v})dW_t, & \tau_i \leq t < \tau_{i+1} < T, \quad i \geq 0; \\ X_{\tau_i}^{x,v} = \eta\left(X_{\tau_i-}^{x,v}, \zeta_i\right) = X_{\tau_i-}^{x,v} + \zeta_i. \end{cases} \tag{1.0.56}$$

We define the set of admissible impulse controls as follows:

Definition 1.0.1 (Admissible Impulse Control). *An impulse control v is admissible, if the following conditions are satisfied:*

$$0 \leq \tau_i \leq \tau_{i+1}, \quad a.s. \quad i \geq 0; \tag{1.0.57}$$

$$\tau_i \textit{ is an } (\mathcal{F}_t)_{t\geq 0}\textit{-stopping time}, \quad i \geq 0; \tag{1.0.58}$$

$$\zeta_i \textit{ is } \mathcal{F}_{\tau_i}\textit{-measurable}, \quad i \geq 0; \tag{1.0.59}$$

$$P\left[\lim_{i\to\infty} \tau_i \leq \tilde{T} \wedge T\right] = 0, \quad \forall \tilde{T} \in [0,\infty). \tag{1.0.60}$$

The condition given by (1.0.60) means that impulse controls will only occur finitely before a terminal time, $\tilde{T}$. Let $\mathcal{U}$ denote the set of admissible impulse controls.

Let $K : \mathbb{R}_+ \to \mathbb{R}_+$ represent the cost of control. We assume that, for $\zeta, \zeta' \in \mathbb{R}_+$, K satisfies the following:

$$K(\zeta) \geq k, \quad k > 0; \tag{1.0.61}$$

$$K(\zeta + \zeta') \leq K(\zeta) + K(\zeta'); \tag{1.0.62}$$

$$K(\zeta) \leq K(\zeta'), \quad \text{if } \zeta \leq \zeta'. \tag{1.0.63}$$

Ineq. (1.0.62) represents subadditivity with respect to ζ and implies that reasonable $(\mathcal{F}_t)_{t\geq 0}$-stopping times become strictly increasing sequences; i.e., $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_i < \cdots < T(\leq \infty)$. Furthermore, we assume that

$$\mathbb{E}\left[\sum_{i\geq 1} e^{-r\tau_i} K(\zeta_i) 1_{\tau_i < T}\right] < \infty. \tag{1.0.64}$$

Note that (1.0.64) implies

$$\mathbb{E}\left[\sum_{i\geq 1} e^{-r\tau_i} 1_{\tau_i < T}\right] < \infty. \tag{1.0.65}$$

It follows that eq. (1.0.60) holds. See Cadenillas (2000). Now, we define the expected total cost in the context of eq. (1.0.54):

$$J(x;v) = \mathbb{E}\left[\int_0^T e^{-rt} f(X_t^{x,v})dt + \sum_{i\geq 1} e^{-r\tau_i} K(\zeta_i) 1_{\tau_i < T}\right]. \tag{1.0.66}$$

In this context, we assume that

$$J(X_T^{x,v};v) = \mathbb{E}[e^{-rT} g(X_T^{x,v}) 1_{T<\infty}] = 0. \tag{1.0.67}$$

Therefore, the agent's problem is to minimize eq. (1.0.66) over $v \in \mathcal{U}$:

$$V(x) = \inf_{v\in\mathcal{U}} J(x;v) = J(x;v^*), \tag{1.0.68}$$

where $V(x)$ is the value function of the problem, and v^* is an optimal impulse control.

An outline of this thesis is as follows. In Chapter 2, we examine an optimal natural resources management problem under uncertainty with catastrophic risk and investigate the optimal rate of use of a natural resource. For this purpose, we use classical stochastic control theory. We assume that, until a catastrophic event occurs, the stock of the natural resource is governed by a stochastic differential equation. We describe the catastrophic phenomenon as a Poisson process. From this analysis, we show the optimal rate of use of the natural resource in explicit form. Furthermore, we present comparative static results for the optimal rate of use of the natural resource.

In Chapter 3, we investigate the value of tradable emission permits (TEPs) under uncertainty, caused by the effects of an increase in the global mean surface temperature, and expressed as a geometric Brownian motion with a Poisson jump process. The Poisson jump process reflects the development of new technology to reduce CO2 emissions. To this end, we formulate a policy decision-maker's problem using a real options model. The problem is formulated as a search for the optimal timing of an irreversible investment under uncertainty, i.e. as an optimal stopping problem. From this analysis, under a suitable set of sufficient conditions, we show the value of the TEPs and present some numerical examples and comparative static results for their value. The value of the TEPs increases with uncertainty about damage from atmospheric CO2 concentrations, but decreases with the degree of development of new technology.

In Chapter 4, we investigate a problem in which an agent implements an environmental improvement policy under uncertainty. If an emission level of a pollutant reaches a critical level, the agent has to decrease the emission to a certain level in order to improve the environment. The agent's problem is to minimize the expected total discounted cost, which includes a cost to implement the EIP and an associated damage from the pollutant under the assumption that a state process of the pollutant follows a geometric Brownian motion. Then we find critical emission levels of the pollutant, optimal implementation times, optimal intensity of implementation, and the value of the optimal EIP (OEIP) by using an impulse control approach. Furthermore, we show some numerical examples and comparative static results for the OEIP. The main results are as follows. An increase in the growth rate of the pollutant, uncertainty, the proportional cost and the constant cost raises the value of the OEIP.

In Chapter 5, we investigate an optimal dividend policy with fixed and proportional transaction costs under a Brownian cash reserve process. The firm's problem is to maximize expected total discounted dividends. To this end, we formulate it as a stochastic impulse control problem, which is approached via quasi-variational inequalities (QVI). Under a suitable set of sufficient conditions, we show the existence of an optimal dividend policy such that whenever the cash reserve reaches a certain level, the firm pays out a dividend. Consequently, it instantaneously falls to another level. We present some numerical examples and comparative static results for the optimal dividend policy.

Chapter 2

Optimal Natural Resources Management under Uncertainty with Catastrophic Risk

2.1 Introduction

It is important to consider uncertainty in natural resource and environmental economics. This is because the dynamics of the value of the stock of natural resources and pollutants are not deterministic in general. A number of papers have studied the effect of uncertainty in natural resource and environmental economics. See, for example, Johansson (1987), Ueta et al. (1991), and Kolstad (1999). In the real world, we consider not only uncertainty, but also catastrophic risk. This is because natural resources such as forests, fisheries, and groundwater are facing catastrophic risk. For example, forest fires dramatically reduce the stock of trees. Without loss of generality, we study the case of one natural resource. A manager receives benefit flows from using the natural resource. If the manager uses quantities of the natural resource that are excessive given its natural growth rate, the natural resource will become exhausted. Furthermore, the natural resource faces a situation of uncertainty and catastrophic risk from change in the natural environment. Hence, the purpose of this chapter is to examine an optimal natural resource management problem under uncertainty with catastrophic risk and to investigate the optimal rate of use of a natural resource. For this purpose, we use stochastic control theory. To be specific, we use stochastic dynamic programming for an expected discounted problem over an infinite horizon. For an explanation of stochastic control theory, see Fleming and Rishel (1975) and Fleming and Soner (1993).

We assume that, until a catastrophic event occurs, the value of the stock of the natural resource is governed by geometric Brownian motion as in Olsen and Shortle (1996) and Section 6 of Willassen (1998) among other types of dynamics. Clarke and Reed (1989) and Reed and Clarke (1990) discuss other dynamics of natural resource in detail. See also Section 1.3 in Clark (1990). If the finiteness of real carrying capacities is considered, the dynamics of natural resource is assumed to be governed by a stochastic logistic differential equation. In this chapter we assume the dynamics of natural resource is governed by geometric Brownian motion for simplicity. When a catastrophic event occurs, the stock of the natural resource decreases drastically. We describe this phenomenon as a Poisson process as in Reed (1984, 1986)[1]. We also assume that if the value of the stock of the natural resource reaches 0, it is exhausted. Under these assumptions, the manager maximizes expected discounted utility by controlling the rate at which the natural resource is used over an infinite horizon. Then, the manager's problem is to find the optimal value function and the optimal rate of use of the natural resource. We solve the manager's problem by adopting stochastic dynamic programming for the expected discounted problem over the infinite horizon. We then conjecture that the candidate of the value function solves the Hamilton-Jacobi-Bellman (HJB) equation of the problem. This yields the optimal rate of use of the natural resource in explicit form.

To verify that the solution of the HJB equation is the value function, we use a property of the discounted process of the natural resource stock and the Dynkin formula, which, for example, is explained in Fleming and Soner (1993). The property is a supermartingale. We present comparative static results for the optimal rate of use of the natural resource, which can be summarized as follows. An increase in uncertainty or a higher catastrophic risk raises the optimal rate of use of the natural resource. These results imply that the risk-averse manager minimizes future uncertainty and catastrophic risk. An increase in transaction costs reduces the optimal rate of use of the natural resource. If use of the natural resource causes pollution, an externality is generated. A tax on using the natural resource, such as an environmental tax, internalizes the externality. An increase in the rate of this tax reduces use of the natural resource.

Related works are Chapter 12 of Ueta et al. (1991), Lungu and Øksendal (1997), and Yin and Newman (1996) among others. Chapter 12 of Ueta et al. (1991) is similar to this chapter. It investigates the optimal sustainable rate of economic development under uncertainty by using stochastic dynamic programming. As we do, Ueta et al. (1991) present comparative static results for several parameters, and find an uncertainty effect that corresponds to the one obtained in this chapter, but do not consider catastrophic risk. Yin and Newman (1996) study the effect of catastrophic risk on forest investment decisions by using an optimal stopping approach. As in Reed (1984) and this chapter, these authors incorporate a Poisson process to reflect the occurrence of catastrophic events. They show that an optimal threshold governs the implementation of managerial strategies. Similarly, this chapter shows the optimal rate at which the manager uses the natural resource. Lungu and Øksendal (1997) adopt a singular stochastic control approach to study an optimal harvesting strategy, which maximizes the expected total discounted harvested volume. They show that an optimal harvesting strategy exists. They assume that population growth follows a stochastic logistic differential equation. By contrast, in this chapter, the value of the stock value of the natural resource follows geometric Brownian motion with the Poisson process. Given these differences, this chapter contributes to the analysis of natural resource management problems under uncertainty with catastrophic risk.

The contents of this chapter are as follows. Section 2 describes the model. Section 3 studies optimal natural resource management by using the stochastic control approach. In Section 4, we undertake comparative static analysis. Section 5 concludes our study. The derivation of an equation is presented in the Appendix.

[1] Reed (1984) studies the impact of catastrophic events on the rotation problem. Reed (1986) surveys forest management problems.

2.2 The Model

Consider an optimal natural resource management problem under uncertainty with catastrophic risk. As noted in Section 1, natural resources include, for example, fish stocks and forests. These natural resources are facing catastrophic risk, including the extinction of some species. This is why we consider catastrophic risk in our model.

Without loss of generality, we study the case of one natural resource. A manager receives benefit flows from using the natural resource. If the manager uses quantities of the natural resource that are excessive given its natural growth rate, the natural resource will become exhausted. Furthermore, the natural resource faces a situation of uncertainty and catastrophic risk from change in the natural environment. Thus, the manager must control the rate of use of the natural resource under uncertainty with catastrophic risk.

We suppose that the value of the stock of the natural resource, $X = (X_t)_{t\in\mathbb{R}_+}$, is governed by the following stochastic differential equation for $t \in [\mathcal{T}_i, \mathcal{T}_{i+1})$ $(i \in \mathbb{Z}_+)$:

$$dX_t = (\mu X_t - \zeta_t)dt + \sigma X_t dW_t, \quad X_0 = x(\in \mathbb{R}_{++}), \tag{2.2.1}$$

where $\mu(\in \mathbb{R})$ and $\sigma(\in \mathbb{R})$ are constants that are the drift coefficient and the diffusion coefficient, respectively. μ represents the expected growth rate of the stock of the natural resource. σ represents the magnitude of uncertainty with respect to the growth rate of the stock of the natural resource. W_t is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. $\zeta_t(\in \mathcal{U})$ is the rate of use of the natural resource at time t. $\mathcal{U}$ denotes the set of admissible controls and is assumed to be defined on $[0, X_t]$. Then the control process is given by

$$u = \zeta = (\zeta_t)_{t\geq 0}. \tag{2.2.2}$$

$(\mathcal{T}_i)_{i\in\mathbb{Z}_+}$ is the sequence of times for the arrival of the catastrophic event. We represent the event as a Poisson process, $N = (N_t)_{t\in\mathbb{R}_+}$, with intensity $\lambda(\in \mathbb{R}_+)$. Let $\nu := (\nu_i)_{i\in\mathbb{Z}_{++}}$ be the sequence of independent and identically distributed $(-1, 0)$–valued random variables with distribution F. Note that this has a finite mean:

$$\mathbb{E}[\nu] = \int_{-1}^{0} y dF(y) < \infty. \tag{2.2.3}$$

Furthermore, we assume that these three processes are mutually independent. At $t = \mathcal{T}_i(i \in \mathbb{Z}_{++})$, X takes a random-sized jump, the proportion of which, i.e., the relative amplitude of the state just before the jump, is given by ν_i:

$$X_{\mathcal{T}_i} = X_{\mathcal{T}_i-}(1 + \nu_i). \tag{2.2.4}$$

Therefore, for any $t \in \mathbb{R}_+$, X is described by

$$X_t = x \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}\left[\prod_{i=1}^{N_t}(1 + \nu_i)\right] - \int_0^t \zeta_s ds. \tag{2.2.5}$$

If the stock of the natural resource reaches 0, it is exhausted. The time at which exhaustion occurs is represented by T:

$$T := \inf\{t > 0; X_t \notin \mathcal{O}\}, \tag{2.2.6}$$

where we assume that $\mathcal{O} = \mathbb{R}_{++}$. We assume that the manager's utility function, U, is strictly increasing, and strictly concave. It is defined by

$$U(\zeta_t) = \frac{[(1-c)\zeta_t]^\gamma}{\gamma}, \tag{2.2.7}$$

where $c \in (0,1)$ is a transaction cost and $(1-\gamma) \in (0,1)$ is the coefficient of relative risk aversion. In the context of (2.2.7), as $\lim_{\gamma \to 0}$ the above utility function becomes the logarithmic utility function. Let us define the performance criterion function by

$$J(x;\zeta) := \mathbb{E}\left[\int_0^\theta e^{-rt} U(\zeta_t) dt\right], \tag{2.2.8}$$

where $r (\in \mathbb{R}_{++})$ is a discount factor and θ is either $+\infty$ or T. Therefore, the manager's problem is to find the optimal value function, $V(x)$ and the optimal rate of use of the natural resource, $= \zeta^* \in \mathcal{U}$

$$V(x) = \sup_{\zeta \in \mathcal{U}} J(x;\zeta) = J(x;\zeta^*). \tag{2.2.9}$$

2.3 Analysis

In this section, we analyze the model that was introduced in Section 2.2. In addition, we describe the optimal rate of use of the natural resource and the optimal value function. To this end, we introduce the integrodifferential operator, $\mathcal{L}$, given by

$$\begin{aligned} \mathcal{L}L\phi(x) :=& \frac{1}{2}\sigma^2 x^2 \phi''(x) + (\mu x - \zeta_t)\phi'(x) - r\phi(x) \\ &+ \lambda\left[\int_{-1}^0 \phi((1+y)x) dF(y) - \phi(x)\right], \end{aligned} \tag{2.3.1}$$

where ϕ is a continuous function and a candidate of the value function. This operator is provided that

$$\int_{-1}^0 \phi((1+y)x) dF(y) \; (= \mathbb{E}[\phi((1+\nu)x)]) \tag{2.3.2}$$

is well defined.

We conjecture that the candidate of the value function, ϕ, solves the following HJB equation:

$$\sup_{\zeta \in \mathcal{U}} [\mathcal{L}\phi(x) + U(\zeta)] = 0, \tag{2.3.3}$$

with $\phi(0) = 0$. The first-order condition for the HJB equation is

$$-\phi'(x) + U'(\zeta^*) = 0. \tag{2.3.4}$$

This yields

$$\zeta^* = (U')^{-1}\phi(x). \tag{2.3.5}$$

We suppose that a candidate for ϕ is the form given by

$$\phi(x) = Ax^{\gamma}, \tag{2.3.6}$$

where the parameter A is to be determined subsequently. From (2.2.7), (2.3.5) and (2.3.6), the optimal rate of use of the natural resource can be written in explicit form as

$$\zeta^* = (1-c)^{-\frac{\gamma}{\gamma-1}}(\gamma A)^{\frac{1}{\gamma-1}}x. \tag{2.3.7}$$

Substituting (2.3.7) into (2.3.3) gives

$$A^{\frac{1}{\gamma-1}} = \left[\gamma^{\frac{1}{\gamma-1}}c^{-\frac{1}{\gamma-1}}(1-c)^{-1}[\gamma-(1-c)^{-\gamma}]\right]^{-1} f(\gamma,\sigma,\mu,r,\lambda,\nu), \tag{2.3.8}$$

where

$$\Gamma(\gamma,\sigma,\mu,r,\lambda,\nu) := \frac{1}{2}\sigma^2\gamma^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\gamma - r + \lambda[E[(1+\nu)^{\gamma}] - 1], \tag{2.3.9}$$

Note also that $\gamma - (1-c)^{-\gamma}$ is negative given that $c \in (0,1)$ and $\gamma \in (0,1)$. Since we must have $\phi(x) \geq 0$ in our model, it follows that $A^{\frac{1}{\gamma-1}} \geq 0$. Hence,

$$\Gamma(\gamma,\sigma,\mu,r,\lambda,\nu) \leq 0. \tag{2.3.10}$$

Thus, from (2.3.7) and (2.3.8), the optimal rate of use of the natural resource over the infinite horizon becomes

$$\zeta^* = \frac{x}{\gamma-(1-c)^{-\gamma}}\Gamma(\gamma,\sigma,\mu,r,\lambda,\nu). \tag{2.3.11}$$

Some preliminaries are required to verify the optimal value function. First, we obtain the following.

$$\begin{aligned}
\mathbb{E}[(X_t^x)^{\gamma}] &= x^{\gamma}\mathbb{E}\left[\exp\left\{\left(\mu-\frac{1}{2}\sigma^2\right)\gamma t + \sigma\gamma W_t\right\}\right]\mathbb{E}\left[\prod_{i=1}^{N_t}(1+\nu_i)^{\gamma}\right] \\
&= x^{\gamma}\exp\left\{\left[\frac{1}{2}\sigma^2\gamma^2 + \left(\mu-\frac{1}{2}\sigma^2\right)\gamma + \lambda[\mathbb{E}[(1+\nu)^{\gamma}]-1]\right]t\right\} \\
&= x^{\gamma}\exp\{[\Gamma(\gamma,\sigma,\mu,r,\lambda,\nu)+r]t\}.
\end{aligned} \tag{2.3.12}$$

The derivation of eq. (2.3.12) is in the Appendix. (2.3.12) yields the following lemmas.

Lemma 2.3.1. *It follows from (2.2.7) and (2.3.12) that U and ϕ satisfy the following.*

$$\mathbb{E}\left[\int_0^{\infty} e^{-rt}U(\zeta_t)dt\right] < \infty. \tag{2.3.13}$$

$$\mathbb{E}\left[\int_0^{\infty} e^{-rt}|\mathcal{L}\phi(X_t^x)|dt\right] < \infty. \tag{2.3.14}$$

Since the sign of f is non-positive, $(e^{-rt}X_t^x)_{t\geq 0}$ becomes a supermartingale. This property of $(e^{-rt}X_t^x)_{t\geq 0}$ is useful in Theorem 2.3.1, which follows subsequently.

Lemma 2.3.2. *Suppose that the following assumption holds:*
(AS.2.3.1)

$$r - \left(\mu - \frac{1}{2}\sigma^2 - \lambda\mathbb{E}[\nu]\right) > 0.$$

Since $(e^{-rt}X_t^x)_{t\geq 0}$ is a supermartingale and the martingale convergence theorem (see, for example, Theorem 1.10 in Protter (1990)), it follows that

$$\lim_{t\to\infty} \mathbb{E}[e^{-rt}\phi(X_t^x)] = 0. \tag{2.3.15}$$

Proof. (AS.2.3.1) yields

$$\mathbb{E}\left[\int_0^\infty e^{-rt}X_t^x dt\right] < \infty. \tag{2.3.16}$$

Thus, since $(e^{-rt}X_t^x)_{t\geq 0}$ is a supermartingale and (2.3.16), by using martingale convergence theorem, we obtain

$$\lim_{t\to\infty} \mathbb{E}[e^{-rt}X_t^x] = 0. \tag{2.3.17}$$

Therefore, it follows from (2.3.12) and (2.3.17) that we obtain (2.3.15). □

As a final preliminary step, by applying the Dynkin formula to $e^{-rt}\phi(X_t)$, we have

$$e^{-rt}E[\phi(X_t)] - \phi(x) = \mathbb{E}\left[\int_0^t e^{-rs}L\phi(X_s)ds\right]. \tag{2.3.18}$$

For an explanation of the Dynkin formula, see, for example, Section III.2 of Fleming and Soner (1993).

We can now verify the optimal value function of the manager's problem (2.2.9).

Theorem 2.3.1. *Suppose that (AS.2.3.1) holds. Let ϕ be a solution to (2.3.3). From Lemma 2.3.2, we have the followings.*

(i) For all $X_t \in \mathbb{R}_{++}$

$$\phi(x) \geq J(x;\zeta). \tag{2.3.19}$$

(ii) If the optimal natural resource use, ζ^*, is given by

$$\zeta^* = \arg\max[\mathcal{L}\phi(x) + U(\zeta)], \tag{2.3.20}$$

then we obtain:

$$\phi(x) = V(x). \tag{2.3.21}$$

That is the solution for (2.3.3) is equivalent to the optimal value function.

Proof. (i) From the maximization problem, we have

$$L\phi(x) + U(\zeta) \leq 0. \tag{2.3.22}$$

From (2.3.18) and (2.3.22), we obtain

$$e^{-rt}\mathbb{E}[\phi(X_t)] - \phi(x) \leq -\mathbb{E}\left[\int_0^t e^{-rs}U(\zeta_s)ds\right]. \tag{2.3.23}$$

Taking $\lim_{t\to\infty}$ in (2.3.23), from Lemma 2.3.2, the first term on the left-hand side of (2.3.23) becomes zero. Thus, (2.3.22) becomes

$$-\phi(x) \leq -\mathbb{E}\left[\int_0^\infty e^{-rs}U(\zeta_s)ds\right]. \tag{2.3.24}$$

The right-hand side of (2.3.24) is equals to $-J(x;\zeta)$ by (2.2.8). Thus,

$$\phi(x) \geq J(x;\zeta). \tag{2.3.25}$$

(ii) Substituting ζ into ζ^* in the proof of (i), the inequality becomes an equality in (2.3.23). Then we have

$$e^{-rt}\mathbb{E}[\phi(X_t)] - \phi(x) = -\mathbb{E}\left[\int_0^t e^{-rs}U(\zeta_s^*)ds\right]. \tag{2.3.26}$$

Letting $\lim_{s\to\infty}$ in (2.3.26), we have

$$\phi(x) = J(x;\zeta^*) = V(x;\zeta). \tag{2.3.27}$$

This completes the proof. □

2.4 Comparative Static Analysis

In this section, we study the effect of several parameters on the optimal rate of use of the natural resource, ζ^*. Furthermore, we assume that the magnitude of the catastrophic event, $v \in (-1,0)$, is constant. Let us redefine the optimal rate of use of the natural resource, ζ^*, as:

$$\zeta^* = h(x,c,\gamma,\hat{\Gamma}(\gamma,\sigma,\mu,r,\lambda,\nu)) := \frac{x}{\gamma-(1-c)^{-\gamma}}\hat{\Gamma}(\gamma,\sigma,\mu,r,\lambda,\nu), \tag{2.4.1}$$

where $\hat{\Gamma}$ is defined by

$$\hat{\Gamma}(\gamma,\sigma,\mu,r,\lambda,\nu) := \frac{1}{2}\sigma^2\gamma^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\gamma - r + \lambda[(1+\nu)^\gamma - 1], \tag{2.4.2}$$

Proposition 2.4.1. *For all $x \in \mathbb{R}_{++}$ $c \in (0,1)$, $\gamma \in (0,1)$ and $v \in (-1,0)$, the optimal rate of use of the natural resource, ζ^* has the following properties: (i) $\partial\zeta^*/\partial x > 0$; (ii) $\partial\zeta^*/\partial c < 0$; (iii) $\partial\zeta^*/\partial\gamma > 0$; (iv) $\partial\zeta^*/\partial\sigma > 0$; (v) $\partial\zeta^*/\partial\mu < 0$; (vi) $\partial\zeta^*/\partial r > 0$; (vii) $\partial\zeta^*/\partial\lambda > 0$; (viii) $\partial\zeta^*/\partial\nu < 0$.*

Proof. (i)

$$\frac{\partial \zeta^*}{\partial x} = \frac{\partial h}{\partial x} = \frac{1}{\gamma - (1-c)^{-\gamma}} \hat{\Gamma}(\gamma, \sigma, \mu, r, \lambda, \nu) \tag{2.4.3}$$
$$> 0.$$

Since the sign of $\gamma - (1-c)^{-\gamma}$ is negative and given (2.3.10), the inequality in (2.4.3) holds.

(ii)

$$\frac{\partial \zeta^*}{\partial c} = \frac{\partial h}{\partial c} = \frac{\gamma(1-c)^{-\gamma-1} x}{[\gamma - (1-c)^{-\gamma}]^2} \hat{\Gamma}(\gamma, \sigma, \mu, r, \lambda, \nu) \tag{2.4.4}$$
$$< 0.$$

It is obvious that the above inequality holds.

(iii)

$$\begin{aligned} \frac{\partial \zeta^*}{\partial \gamma} &= \frac{\partial h}{\partial \gamma} + \frac{\partial h}{\partial \hat{\Gamma}} \frac{\partial \hat{\Gamma}}{\partial \gamma} \\ &= -\frac{[1 + (\log(1-c))(1-c)^{-\gamma}] x}{[\gamma - (1-c)^{-\gamma}]^2} \hat{\Gamma}(\gamma, \sigma, \mu, r, \lambda, \nu) \\ &\quad + \frac{x}{\gamma - (1-c)^{-\gamma}} \left[\sigma^2 \gamma + \left(\mu - \frac{1}{2} \sigma^2 \right) + \lambda (\log(1+\nu))(1+\nu)^{\gamma} \right] \\ &> 0. \end{aligned} \tag{2.4.5}$$

It is clear that the inequality in (2.4.5) holds.

(iv)

$$\frac{\partial \zeta^*}{\partial \sigma} = \frac{\partial h}{\partial \hat{\Gamma}} \frac{\partial \hat{\Gamma}}{\partial \sigma} = \frac{x}{\gamma - (1-c)^{-\gamma}} \gamma(\gamma - 1)\sigma \tag{2.4.6}$$
$$> 0.$$

Satisfaction of the inequality in (2.4.6) is straightforward.

(v)

$$\frac{\partial \zeta^*}{\partial \mu} = \frac{\partial h}{\partial \hat{\Gamma}} \frac{\partial \hat{\Gamma}}{\partial \mu} = \frac{x}{\gamma - (1-c)^{-\gamma}} \gamma \tag{2.4.7}$$
$$< 0.$$

This inequality holds given the conditions referred to in Proposition 2.4.1.

(vi)

$$\frac{\partial \zeta^*}{\partial r} = \frac{\partial h}{\partial \hat{\Gamma}} \frac{\partial \hat{\Gamma}}{\partial r} = -\frac{x}{\gamma - (1-c)^{-\gamma}} \tag{2.4.8}$$
$$> 0.$$

It is obvious that the above inequality holds.

(vii)

$$\frac{\partial \zeta^*}{\partial \lambda} = \frac{\partial h}{\partial \hat{\Gamma}}\frac{\partial \hat{\Gamma}}{\partial \lambda} = \frac{x}{\gamma - (1-c)^{-\gamma}}\left((1+\nu)^{\gamma} - 1\right) \tag{2.4.9}$$
$$> 0.$$

Since $\nu \in (-1, 0)$, then $(1+\nu)^{\gamma} \in (0,1)$. It therefore follows that the inequality in (2.4.9) holds.

(viii)

$$\frac{\partial \zeta^*}{\partial \nu} = \frac{\partial h}{\partial \hat{\Gamma}}\frac{\partial \hat{\Gamma}}{\partial \nu} = \frac{x}{\gamma - (1-c)^{-\gamma}}\lambda\gamma(1+\nu)^{\gamma-1} \tag{2.4.10}$$
$$< 0.$$

It is easy to confirm (2.4.10).

□

Proposition 2.4.1 affirms the comparative static properties of our model. According to (i), the initial amount of the natural resource, x, increases the optimal rate of use of the natural resource. From (ii), an increase in the transaction cost, c, reduces the optimal rate of use of the natural resource. The transaction cost could be interpreted as a tax rate associated with use of the natural resource. If use of the natural resource causes pollution, an externality is generated. A tax on using the natural resource, such as an environmental tax, internalizes the externality. An increase in the rate of this tax reduces use of the natural resource. If a policy maker raises the tax rate, use of the natural resource is constrained. In the context of (iii), an increase in γ reduces the coefficient of relative risk aversion, $(1 - \gamma)$. Hence, (iii) implies that a decrease in the coefficient of relative risk aversion, raises the optimal rate of use of the natural resource. In (iv), an increase in uncertainty, σ, increases the optimal rate of use of the natural resource. This result implies that the risk–averse manager minimizes future uncertainty. Thus, uncertainty promotes use of the natural resource. From (v), an increase in the expected growth rate of the stock of the natural resource lowers the optimal rate of use of the natural resource. According to (vi), an increase in the discount factor raises the optimal rate of use of the natural resource. In (vii), an increase in catastrophic risk raises the optimal rate of use of the natural resource. This result implies that the risk–averse manager minimizes future catastrophic risk as well as uncertainty. Thus, catastrophic risk promotes use of the natural resource. The meaning of (viii) is that an increase in v reduces the optimal rate of use of the natural resource.

2.5 Conclusion

In this chapter, we examined the optimal natural resource management problem under uncertainty with catastrophic risk and investigated the optimal rate of use of a natural resource. To do so, we used stochastic control theory. To be specific, we used stochastic dynamic programming for an expected discounted problem over an infinite horizon. Having verified that the solution of the HJB equation is the value function, the optimal rate of use of the natural resource was shown in explicit form. We derived comparative static effects on the optimal rate of use of the natural resource, which can be summarized as follows. An increase in uncertainty or a higher catastrophic risk raises the optimal rate of use of the natural resource. These results imply that the risk–averse manager minimizes future uncertainty and catastrophic risk. An increase in transaction costs reduces the optimal rate of use of the natural resource. If use of the natural resource causes pollution, an externality is generated. A tax on using the natural resource, such as an environmental tax, internalizes the externality. An increase in the rate of this tax reduces use of the natural resource.

In this chapter, we assumed that the stock of the natural resource follows geometric Brownian motion with the Poisson process. Other formulations for the dynamics of the natural resource could have been assumed. For example, the following stochastic logistic differential equation (in, for example, Alvarz and Shepp (1998), Section 7 of Willassen (1998), and Alvarz (2000, 2001)[2]) could have been used.

$$dX_t = \mu X_t(1-\delta X_t)dt + \sigma X_t dW_t, \quad X_0 = x, \tag{2.5.1}$$

where δ^{-1} is the carrying capacity of the environment. Incorporation of these dynamics is left to future research. Other interesting extensions are possible. For example, one could investigate the optimal natural resource management problem by using a continuous control approach. (See, for example, Harrison (1985).) This approach yields upper and/or lower thresholds for the use of a natural resource. In addition, the polluting effects of using natural resources could be investigated. Olsen and Shortle (1996) study the renewable resource harvesting case. They assume that the stock of a pollutant follows a stochastic process and solve for the optimal control of pollutant emissions, thus finding a solution to the harvesting problem. They identify the conditions for optimal management of pollutant emissions and harvests.

2.6 Appendix

Recall that W_t is standard Brownian motion. Thus, by the moment generating function of W_t, we obtain:

$$\begin{aligned} E\left[\exp\left\{\left(\mu-\frac{1}{2}\sigma^2\right)\gamma t+\sigma\gamma W_t\right\}\right] &= \exp\left\{\left(\mu-\frac{1}{2}\sigma^2\right)\gamma t\right\} E[\exp\{\sigma\gamma W_t\}] \\ &= \exp\left\{\left[\frac{1}{2}\sigma^2\gamma^2+\left(\mu-\frac{1}{2}\sigma^2\right)\gamma\right]t\right\}. \end{aligned} \tag{2.6.1}$$

[2] Alvarz and Shepp (1998) is extended in Alvarz (2000, 2001) for a more general setting.

Since N is the Poisson process, the remainder of the first equality in (2.3.12) can be straightforwardly rewritten as:

$$\begin{aligned} E\left[\prod_{i=1}^{N_t}(1+\nu_i)^\gamma\right] &= \sum_{n=0}^{\infty} E\left[\prod_{i=1}^{n}(1+\nu_i)^\gamma\right] P(N_t = n) \\ &= \sum_{n=0}^{\infty} E[(1+\nu)^\gamma]^n \frac{(\lambda t)^n \exp\{-\lambda t\}}{n!} \\ &= \exp\{\lambda[E[(1+\nu)^\gamma] - 1]t\}. \end{aligned} \tag{2.6.2}$$

Chapter 3

The Value of Tradable Emission Permits of CO2 Using a Real Options Model

3.1 Introduction

The Intergovernmental Panel on Climate Change (IPCC) provides projections about future climate change by using climate models and emission scenarios in IPCC (1996). Unfortunately, because the climate system is too complicated, it is impossible to make projections about the effects of future climate change. Decision-makers who enforce policies to stabilize global warming must deal with uncertainty about these effects. This means policies aimed at stabilizing global warming should include flexible mechanisms. IPCC (1996) states that we have to reduce greenhouse gas (GHG) emissions to stabilize global warming. Since CO2 is the main GHG emitted by human activities, we have to immediately reduce CO2 emissions by 50–70% in order to prevent further increases in CO2 concentrations. It is, however, almost impossible to achieve this goal. Therefore, we have to discuss the available economic instruments. These instruments, which include subsidies, taxes, and tradable emission permits for CO2, are discussed by Bertram, Stephens, and Wallace (1990) and Jenkins and Lamech (1992). Mullins and Baron Mullins and Baron (1997) and the IEA Workshop Report IEA (1998) provide detailed discussions of emission trading systems. Emission permits trading systems are also the focus of this chapter.

The principle underlying an emission permits trading system is as follows. Agents with high marginal costs of mitigating CO2 emissions can acquire emission reductions from other agents with lower marginal costs. This assists both agents, the buyers and the sellers, to reduce their emissions at least cost. Then, the agents with the high marginal costs promote the development of technology to reduce CO2 emissions, and transfer technology to the other agents with lower marginal costs.

In this chapter, we investigate the value of tradable emission permits (TEPs) under uncertainty, which is caused by the effects of an increase in the global mean surface temperature and is expressed as a geometric Brownian motion with a Poisson jump process. The Poisson jump process reflects the development of new technology to reduce CO2 emissions. To this end, we formulate the policy decision-maker's problem using a real options model[1]. To be specific, the problem is formulated as a search for the optimal timing of an irreversible investment under uncertainty, that is, as an optimal stopping problem. From this analysis, under a suitable set of sufficient conditions, we show the value of the TEPs. Furthermore, we present some numerical examples and comparative static results for the value of the TEPs. The main results are as follows: the value of the TEPs increases with uncertainty about damage from atmospheric CO2 concentrations, whereas it decreases with the degree of development of new technology.

Related studies are as follows. Chao and Wilson (1993) examine the price of emission allowances for SO2 when investments in scrubbers are irreversible and demand for SO2 follows a Winner process. They show that investments in scrubbers are reduced if there is greater uncertainty about future market conditions. Section 12.3 of Dixit and Pindyck (1994) examines the timing of environmental policies. Neither of these previous studies deals explicitly with technological innovation. By contrast, we explicitly discuss technological innovation and express it as a Poisson jump process. Thus, this chapter contributes to the study of the value of TEPs under uncertainty. The Poisson jump process is briefly discussed by McDonald and Siegel (1986), and Farzin, Huisman, and Kort (1998) studies the optimal timing of technology adoption. Unlike Farzin, Huisman, and Kort (1998), we simultaneously deal not only with technology but also with other forms of uncertainty.

The remainder of this chapter is organized as follows. Section 3.2 describes the model. Section 3.3 analyses the model presented in Section 3.2. Section 3.4 presents the numerical and comparative static results for the value of the TEPs. Section 3.5 concludes this study. The derivation of a sensitivity analysis with respect to the degree of development of new technology is provided in the appendix.

3.2 The Model

We consider the problem faced by a policy decision-maker aiming to stabilize global climate change. We assume that the policymaker has to implement the policy, which is to install equipment to reduce CO2 emissions. In addition, we assume that the equipment cannot be utilized for other purposes, that is, the equipment has the property of irreversibility. The policy decision-maker can decide to implement the irreversible investment in the present period, but he or she can also delay implementation. When the policy decision-maker undertakes the irreversible investment expenditure, he or she gives up the possibility of waiting for new information on the effects of global warming. As a result, an opportunity cost occurs. If investment to reduce CO2 emission is delayed, the flow level of CO2 emissions does not change. That is, we can interpret the right to delay an investment as a permit to discharge CO2, that is, it is a TEP. To solve the problem, we formulate it as an optimal stopping problem.

Let Q_t be the benefit from economic activities, and $X_t M_t$ be damage from atmospheric CO2 concentrations, M_t. Let $B(Q_t, X_t, M_t) : \mathbb{R}_+ \times \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}$ denote the flow of net benefit associated with the benefit Q_t and damage $X_t M_t$:

$$B(Q_t, X_t, M_t) = Q_t - X_t M_t, \tag{3.2.1}$$

[1]For examples of the real options approach, see Pindyck (1991), Dixit and Pindyck (1994), and Tsujimura (2003a). Tsujimura (2003a) investigates the value of options in the context of finance and environmental economics, respectively.

where X_t is a variable that stochastically shifts over time to reflect damage due to atmospheric CO2 concentrations. The variable will be discussed later in more detail. Let M_t be given by

$$dM_t^m = (D_t - \kappa M_t^m)dt, \quad M_0^m = m, \tag{3.2.2}$$

where D_t is the flow of CO2 emissions, and $\kappa(\in (0,1))$ denotes a constant parameter which shows natural depreciation of atmospheric CO2 concentrations.

We discuss a variable shift parameter X_t in detail. When we consider damage caused by droughts, heavy rain, floods, and so on, we find that the scale of such damage is not the same in every period, but grows in proportion to the increase in atmospheric CO2 concentrations. Furthermore, even if atmospheric CO2 concentration does not change, the damage increases if the growth rate of population is positive. If we introduce an emission permits trading system to stabilize global climate change, trading the TEPs creates incentives to develop and install technology for controlling CO2 discharges, as we stated in Section 3.1. In this chapter, we explicitly examine this effect on the stochastic process of a shift parameter, X_t. In general, technology does not appear constantly, but in sudden bursts after the progress of R&D. Therefore, we express this phenomenon as a Poisson process $N := (N_t)_{t\geq 0}$ with intensity parameter $\lambda(>0)$, where λ represents the degree of development of new technology. Thus, the shift parameter X_t jumps at the random time $\mathcal{T}_1, \cdots, \mathcal{T}_n, \cdots$ and the relative change in its value at a jump time is given as a constant $\nu(\in (0,1))$. We assume that, until technology appears, the shift parameter follows a geometric Brownian motion. Assume that a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t\geq 0})$, satisfying the usual conditions, is given. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is generated by a Brownian motion process $W := (W_t)_{t\geq 0}$ and the Poisson process N. We assume that W and N are mutually independent. This description[2] can be formalized by allowing the following, on the interval $[\mathcal{T}_n, \mathcal{T}_{n+1})$:

$$dX_t^x = \mu X_t^x dt + \sigma X_t^x dW_t, \quad X_0^x = x, \tag{3.2.3}$$

where $\mu(\in \mathbb{R})$ and $\sigma(\in \mathbb{R})$ are constants. At $t = \mathcal{T}_n$, the jump is given by $X_{\mathcal{T}_n}^x - X_{\mathcal{T}_n -}^x = -\nu X_{\mathcal{T}_n -}^x$, so that:

$$X_{\mathcal{T}_n}^x = X_{\mathcal{T}_n -}^x (1-\nu). \tag{3.2.4}$$

At the generic time t, X_t can be given by the following:

$$X_t^x = x \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t \right\} \left[\prod_{i=1}^{N_t} (1-\nu_i) \right], \tag{3.2.5}$$

where $u_i = u$. Eq. (3.2.5) is easily seen to be the solution of

$$dX_t^x = \mu X_t^x dt + \sigma X_t^x dW_t - \nu X_{t-}^x dN_t. \tag{3.2.6}$$

The expectation of X_t^x is given by

$$\mathbb{E}[X_t^x] = x e^{(\mu - \lambda\nu)t}, \quad x \in \mathbb{R}_{++}. \tag{3.2.7}$$

[2]This description is based on Section 7.2 of Lamberton and Lapeyre (1996) and Subsection 3.1.1 of Runggaldier (2003).

Assume the following:
(AS.3.2.1)

$$\mu < \lambda\nu;$$

(AS.3.2.2)

$$\mathbb{E}\left[\int_0^{\infty} e^{-rt} B(Q_t, X_t^x, M_t^m)\right] dt < \infty.$$

Under the assumption (AS.3.2.1), the discounted shift variable process $(e^{-rt}X_t)_{t\geq 0}$ is a super-martingale, where $r(\in \mathbb{R}_{++})$ is a discount factor.

If the policy decision-maker implements the policy, his or her benefit decreases by the amount of $a(\in \mathbb{R}_{+})$, and CO2 emissions are reduced by the amount of $c(\in \mathbb{R}_{+})$. Then, eqs. (3.2.1) and (3.2.2) become as follows:

$$B(Q_t, X_t^x, M_t^m) = \begin{cases} B_1(X_t^x, M_t^m) = Q - X_t^x M_t^m, & t < \tau, \\ B_2(X_t^x, M_t^m) = (Q-a) - X_t^x M_t^m, & t \geq \tau, \end{cases} \tag{3.2.8}$$

where Q and a are constant and

$$dM_t = \begin{cases} = (D - \kappa M_t)dt, & t < \tau, \\ = ((D-c) - \kappa M_t)dt, & t \geq \tau, \end{cases} \tag{3.2.9}$$

where D and c are constant, and τ denotes the policy implementation time. If, therefore, the policy decision-maker does not reduce the emissions of CO2, he or she suffers heavier damage. Then, the policy decision-maker needs to implement the policy. Hence, the policy decision-maker's net expected total discounted benefit is

$$J(x, m; \tau) = \mathbb{E}\left[\int_0^{\infty} e^{-rt} B(Q_t, X_t^x, M_t^m) dt - e^{-r\tau} K\right], \quad Q, m \in \mathbb{R}_{+},\ x \in \mathbb{R}_{++}, \tag{3.2.10}$$

where K is the cost of implementing the policy, i.e. the investment cost to install the equipment, and is assumed to be a constant. Therefore, the policy decision-maker's problem is to choose the policy implementation time τ in order to maximize eq. (3.2.10):

$$V(x, m) = \max_{\tau} J(x, m; \tau) = J(x, m; \tau^*), \quad x \in \mathbb{R}_{++}, m \in \mathbb{R}_{+}, \tag{3.2.11}$$

where V is the value function of the policy decision-maker's problem and τ^* is the optimal policy implementation time. We assume that V is $C^{2,1}$. Consequently, we refer to the policy decision-maker's problem as an optimal stopping problem.

3.3 Analysis

In this section, we analyze the model that was introduced in Section 3.2. We show the value function, the optimal policy implementation time, and the value of the TEPs. In addition, we prove that the policy decision-maker does not implement the policy when the state of X_t does not arrive at x^*. On the other hand, the policy decision-maker implements the policy when the state of X_t is greater than or equal to x^*. To this end, we first introduce two operators, $\mathcal{G}$ and $\mathcal{H}$:

$$\mathcal{G}\phi(x,m) := \frac{1}{2}\sigma^2 x^2 \phi_{xx}(x,m) + \mu x \phi_x(x,m) + (D - \kappa m)\phi_m(x,m), \tag{3.3.1}$$

$$\mathcal{H}\phi(x,m) := \frac{1}{2}\sigma^2 x^2 \phi_{xx}(x,m) + \mu x \phi_x(x,m) + ((D-c) - \kappa m)\phi_m(x,m), \tag{3.3.2}$$

where $\phi : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}$ is a candidate function of the value function of the policy decision-maker's problem, eq. (3.2.11), and is equal to $C^{2,1}$. Let $\varphi(x,m)$ be a candidate function of the value function in the region where the policy has not been implemented, and let $\psi(x,m)$ be a candidate function of the value function in the region where the policy has been implemented. Thus, the policy implementation time is defined by

$$\tau = \inf\{t > 0; (x,m) \in \mathcal{S}\}, \tag{3.3.3}$$

where $\mathcal{S}$ is the policy implementation region defined by

$$\mathcal{S} := \{(x,m); \phi(x,m) \leq \psi(x,m) - K\}. \tag{3.3.4}$$

$\varphi(x,m)$ satisfies the following partial differential equation:

$$\begin{aligned}[\mathcal{G}\varphi](x,m) - \lambda[\varphi(x,m) - \varphi((1-\nu)x,m)] - r\varphi(x,m) + B_1(Q,x,m) = 0,\\ Q, m \in \mathbb{R}_+,\ x \in \mathbb{R}_{++}.\end{aligned} \tag{3.3.5}$$

On the other hand, $\psi(x,m)$ satisfies the partial differential equation:

$$\begin{aligned}[\mathcal{H}\psi](x,m) - \lambda[\psi(x,m) - \psi((1-\nu)x,m)] - r\psi(x,m) + B_2(Q,x,m) = 0,\\ Q, m \in \mathbb{R}_+,\ x \in \mathbb{R}_{++}.\end{aligned} \tag{3.3.6}$$

We calculate the solution of eq. (3.3.5). First, we fix two real numbers A and γ and try a function φ of the form:

$$\varphi(x) = Ax^\gamma, \quad x \in \mathbb{R}_{++}. \tag{3.3.7}$$

Substituting this equation into the corresponding homogeneous equation, we obtain

$$Ax^\gamma \Gamma(\gamma) = 0, \quad x \in \mathbb{R}_{++}, \tag{3.3.8}$$

where the function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$\Gamma(\gamma) = \frac{1}{2}\sigma^2\gamma(\gamma - 1) - \mu\gamma - (r + \lambda) + \lambda(1-\nu)^\gamma. \tag{3.3.9}$$

Lemma 3.3.1. *Assume that (AS.3.2.1) holds. Then, the nonlinear equation* $\Gamma(\gamma) = 0$ *has two real roots, the larger one,* γ_1*, which satisfies*

$$1 \leq \gamma_1, \tag{3.3.10}$$

and the smaller one, γ_2*, which satisfies*

$$\gamma_2 < 0. \tag{3.3.11}$$

Proof. It is straightforward that $\Gamma(\gamma)$ is a strictly convex function. We have

$$\Gamma(0) = -r < 0; \quad \Gamma(1) = \mu - r - \lambda\nu \leq 0. \tag{3.3.12}$$

Therefore, the nonlinear equation $\Gamma(\gamma) = 0$ has two distinct real roots, γ_2 and γ_1, such that $\gamma_2 < 0$ and $1 < \gamma_1$, respectively. □

Next, we find a particular solution of $\varphi(x, m)$ by using the method of undetermined coefficients. A trial solution is assumed to be given by:

$$\varphi(x, m) = b_1 Q + b_2 x + b_3 m + b_4 x m. \tag{3.3.13}$$

Substituting eq. (3.3.13) into eq. (3.3.5), and rearranging the result, gives

$$b_1 = \frac{1}{r},\ b_2 = -\frac{D}{\delta\rho},\ b_3 = 0,\ b_4 = -\frac{1}{\rho}, \tag{3.3.14}$$

where $\delta := r - \mu + \lambda\nu$, and $\rho := r - \mu + \kappa + \lambda\nu$. Substituting, the particular solution of eq. (3.3.5) is given by

$$\varphi(x, m) = \frac{Q}{r} - \frac{Dx}{\delta\rho} - \frac{xm}{\rho}. \tag{3.3.15}$$

Remark 3.3.1. *If the policy decision-maker never implements the policy, the particular solution, (3.3.15), is also derived as the expected total discounted benefit of the policy decision-maker. From eq. (3.2.2)* M_t^m *is*

$$M_t^m = \frac{D}{\kappa} + e^{-\kappa t}\left(m - \frac{D}{\kappa}\right). \tag{3.3.16}$$

From eqs. (3.3.16) and (3.2.7), we have the expected net benefit as follows:

$$\begin{aligned}\varphi(x, m) &= \mathbb{E}\left[\int_0^\infty e^{-rt}(Q - X_t^x M_t^m)dt\right]\\ &= \int_0^\infty \left[e^{-rt}Q - e^{(\mu-\lambda\nu)t}X_t^x\left[\frac{D}{\kappa} + e^{-\kappa t}\left(m - \frac{D}{\kappa}\right)\right]\right]dt\\ &= \frac{Q}{r} - \frac{Dx}{(r-\mu+\lambda\nu)(r-\mu+\kappa+\lambda\nu)} - \frac{xm}{(r-\mu+\kappa+\lambda\nu)}.\end{aligned} \tag{3.3.17}$$

Thus, we can write a solution of eq. (3.3.5) as:

$$\varphi(x, m) = A_1 x^{\gamma_1} + A_2 x^{\gamma_2} + \frac{Q}{r} - \frac{Dx}{\delta\rho} - \frac{xm}{\rho}, \tag{3.3.18}$$

where A_1 and A_2 are constants to be determined. From Lemma 3.3.1, γ_2 is negative and the power of x goes to infinity as x goes to zero. To prevent the value from diverging, we set $A_2 = 0$. Then, we have:

$$\varphi(x, m) = A_1 x^{\gamma_1} + \frac{Q}{r} - \frac{Dx}{\delta\rho} - \frac{xm}{\rho}, \tag{3.3.19}$$

In a similar way, a solution of $\psi(x,m)$ is derived as:

$$\psi(x,m) = \frac{Q-a}{r} - \frac{(D-c)x}{\delta\rho} - \frac{xm}{\rho}. \tag{3.3.20}$$

Note that the solution is equivalent to the particular solution with $Q = Q-a$ and $D = D-c$ in eq. (3.3.20), since the TEPs have already been exercised.

Remark 3.3.2. *The first term on the right-hand side of eq. (3.3.19) is the value of waiting to invest in the equipment that reduces CO2 emissions. In other words, it is the value of the tradable CO2 emission permits. The rest of the terms represent the present value of the policy decision-maker's benefit. On the other hand, $\psi(x,m)$ is the value function in the region where the TEPs have been exercised, namely, where the CO2 emission control policy has been implemented. The policy reduces the decision-maker's benefit, Q, by the amount of a, and the CO2 emissions, D, by the amount of c. Consequently, the right-hand side of eq. (3.3.20) is the present value of the policy decision-maker's net benefit.*

Let us define a function $\phi(x,m) : \mathbb{R}_{++} \times \mathbb{R}_{+} \to \mathbb{R}$ by

$$\phi(x,m) = \begin{cases} \varphi(x,m), & x \in (0, x^*), \\ \psi(x,m) - K, & x \in [x^*, \infty), \end{cases} \tag{3.3.21}$$

where A_1 and x^* are uniquely determined by the following simultaneous equations. These equations are well-known value matching and smooth pasting conditions:

$$\varphi(x,m) = \psi(x,m) - K, \quad x = x^*; \tag{3.3.22}$$

$$\varphi_x(x,m) = \psi_x(x,m), \quad x = x^*, \tag{3.3.23}$$

where x^* is the critical value of x. In other words, if the value of the shift parameter, x, arrives at x^*, then the policy decision-maker implements the policy. From eqs. (3.3.22) and (3.3.23) we obtain:

$$x^* = \frac{\gamma_1\delta\rho}{(\gamma_1 - 1)c}\left(\frac{a}{r} + K\right); \tag{3.3.24}$$

$$A_1 = \frac{c}{\gamma_1\delta\rho}\left[\left(\frac{a}{r} + K\right)\frac{\gamma_1\delta\rho}{(\gamma_1 - 1)c}\right]^{1-\gamma_1}. \tag{3.3.25}$$

The following lemma is useful for verifying the value function.

Lemma 3.3.2. *Assume that (AS.3.2.1) and (AS.3.2.2) hold. Then, the function $\phi(x,m)$: $\mathbb{R}_{++} \times \mathbb{R}_{+} \to \mathbb{R}$ satisfies the following variational inequalities:*

(I) For any $x \in \mathbb{R}_{++}$ $\quad (x \neq x^*)$,

$$\phi(x,m) \geq \psi(x,m) - K. \tag{3.3.26}$$

(II) For any $x \in \mathbb{R}_{++}$,

$$[\mathcal{G}\phi](x,m) - \lambda[\phi(x,m) - \phi((1-\nu)x,m)] - r\phi(x,m) + B_1(Q,x,m) \leq 0. \quad (3.3.27)$$

(III) For any $x \in \mathbb{R}_{++}$,

$$\begin{aligned} \{\phi(x,m) - \psi(x,m) - K\} \{[\mathcal{G}\phi](x,m) - \lambda[\phi(x,m) - \phi((1-\nu)x,m)] \\ -r\phi(x,m) + B_1(Q,x,m)\} = 0 \end{aligned} \quad (3.3.28)$$

Proof. (I) Define a function $h(x) : \mathbb{R}_{++} \to \mathbb{R}$ by the form:

$$\begin{aligned} h(x) &:= \varphi(x,m) - \psi(x,m) - K \\ &= A_1 x^{\gamma_1} + \frac{Q}{r} - \frac{Dx}{\delta\rho} - \frac{xm}{\rho} - \left(\frac{Q-a}{r} - \frac{(D-c)x}{\delta\rho} - \frac{xm}{\rho} - K\right) \\ &= A_1 x^{\gamma_1} + \frac{a}{r} - \frac{cx}{\delta\rho} + K. \end{aligned} \quad (3.3.29)$$

Differentiating $h(x)$ with respect to x, we obtain

$$h'(x) = \gamma_1 A_1 x^{\gamma_1 - 1} - \frac{c}{\delta\rho}. \quad (3.3.30)$$

From (3.3.10), the right-hand side of eq. (3.3.30) strictly increases from $-c/(\delta\rho)$ to $+\infty$ as x moves from 0 to $+\infty$. Its unique zero point is $x = x^*$. The sign of the derivative of $h(x)$ is negative for $x \in (0, x^*)$, whereas it is positive for $x \in (x^*, \infty)$. Thus, first $h(x)$ strictly decreases from $a/r + K$ to 0 as x moves 0 to x^* and then it strictly increases from 0 to $+\infty$ as x goes from x^* to $+\infty$. Therefore, ineq. (3.3.26) holds for $x \in \mathbb{R}_{++}$.

(II) To show that ineq. (3.3.27) holds for $x \in \mathbb{R}_{++}$, we divide the region of x into $(0, x^*)$ and (x^*, ∞).

First, we consider the region $x \in (0, x^*)$. We have $\phi(x,m) = \varphi(x,m)$ for $x \in (0, x^*)$. Then, we obtain

$$\begin{aligned} &[\mathcal{G}\phi](x,m) - \lambda[\phi(x,m) - \phi((1-\nu)x,m)] - r\phi(x,m) + B_1(Q,x,m) \\ &= [\mathcal{G}\varphi](x,m) - \lambda[\varphi(x,m) - \varphi((1-\nu)x,m)] - r\varphi(x,m) + B_1(Q,x,m) \\ &= 0. \end{aligned} \quad (3.3.31)$$

Next, we consider the region $x \in (x^*, \infty)$. We have to further divide the region into $(1-\nu)x > x^*$ and $(1-\nu)x < x^*$. First, we consider the region $(1-\nu)x > x^*$. From eq. (3.3.21) and (I), ineq. (3.3.27) holds. On the other hand, for $(1-\nu)x < x^*$ we have

$$[\mathcal{G}\phi](x,m) - \lambda[\phi(x,m) - \varphi((1-\nu)x,m)] - r\phi(x,m) + B_1(Q,x,m) \leq 0. \quad (3.3.32)$$

Then, we have to show the following:

$$\begin{aligned} [\mathcal{H}\psi](x,m) - \lambda[(\psi(x,m) - K) - \varphi((1-\nu)x,m)] \\ - r(\psi(x,m) - K) + B_1(Q,x,m) \leq 0. \end{aligned} \quad (3.3.33)$$

Note that we use the partial differential operator, $\mathcal{H}$, in the region where the policy has been implemented in ineq. (3.3.33). To show ineq. (3.3.33), substituting $\psi_x = -(D-c)/\delta\rho - m/\rho$, $\phi^2_{xx} = 0$, $\phi^A_m = -x/\rho$ into ineq.(3.3.33), we obtain

$$(r+\lambda)\left(\frac{a}{r}+K\right)\frac{-(r-\mu+\lambda)c}{\delta\rho}x + \lambda A_1[(1-\nu)x]^{\gamma_1} \leq 0. \tag{3.3.34}$$

Replacing x by x^* in ineq. (3.3.34) and substituting (3.3.24) in ineq. (3.3.34) yields

$$\left(\frac{a}{r}+K\right)\left(\frac{\mu\gamma_1-(r+\lambda\nu)}{\gamma_1-1}\right) \leq 0. \tag{3.3.35}$$

From $\gamma_1 > 1$ and eq. (3.3.12), it is clear that $\mu\gamma_1 - (r+\lambda\nu) < 0$. Therefore, for $x \in (x^*, \infty)$, ineq. (3.3.27) holds.

(III) Obviously, we have this equality from the proof of (I) and (II).

□

We are now in a position to show that the candidate function, ϕ, is the value function of the problem eq. (3.2.11) and the optimal policy implementation time.

Theorem 3.3.1. *Assume that (AS.3.2.1) and (AS.3.2.2) hold. Suppose that $\phi(x) : \mathbb{R}_{++} \rightarrow \mathbb{R}$ is defined by eq. (3.3.21). For any $x \in \mathbb{R}_{++}$ and $m \in \mathbb{R}_+$ we have*

$$J(x,m) \leq \phi(x,m). \tag{3.3.36}$$

From Lemma 3.3.2, for $x \notin \mathcal{S}$ we have:

$$[\mathcal{G}\phi](x,m) - \lambda[\phi(x,m) - \phi((1-\nu)x,m)] - r\phi(x,m) + B_1(Q,x,m) = 0, \quad (x,m) \notin \mathcal{S}, \tag{3.3.37}$$

Thus, the function ϕ is the value function of the problem eq. (3.2.11):

$$\phi(x,m) = V(x,m). \tag{3.3.38}$$

Furthermore, the optimal policy implementation region, $\mathcal{S}^*$, and the optimal policy implementation time, τ^*, are given by the following:

$$\mathcal{S}^* := \{(x,m) \in (\mathbb{R}_{++} \times \mathbb{R}_+);\ \phi(x,m) < \psi(x,m) - K\}; \tag{3.3.39}$$

$$\tau^* := \inf\{t \geq 0; (x,m) \in \mathcal{S}^*\}. \tag{3.3.40}$$

Proof. Let us define a new stochastic process $\Upsilon := (\Upsilon_t)_{t\geq 0}$ by

$$\begin{aligned}\Upsilon_t &:= e^{-rt}\phi(X^x_t, M^m_t) - \phi(x,m) \\ &- \int_0^t e^{-rt}[[\mathcal{G}\phi](X^x_s, M^m_s) - \lambda[\phi(X^x_s, M^m_s) - \phi((1-\nu)X^x_s, M^m_s)] - r\phi(X^x_s, M^m_s)]ds.\end{aligned} \tag{3.3.41}$$

Then, the process Υ is a 0-mean martingale (see, for example, Davis (1993)). Thus, applying the optional sampling theorem for martingale for any policy implementation time τ. Thus, we have the Dynkin formula:

$$\begin{aligned}\mathbb{E}[e^{-r(\tau\wedge t)}\phi(X^x_{\tau\wedge t}, M^m_{\tau\wedge t})] = \phi(x,m) + \mathbb{E}\bigg[\int_0^{\tau\wedge t} & e^{-r(\tau\wedge t)}\{[\mathcal{G}\phi](X^x_{\tau\wedge t}, M^m_{\tau\wedge t}) \\ & - \lambda[\phi(X^x_{\tau\wedge t}, M^m_{\tau\wedge t}) - \phi((1-\nu)X^x_{\tau\wedge t}, M^m_{\tau\wedge t})] - r\phi(X^x_{\tau\wedge t}, M^m_{\tau\wedge t})\}ds\bigg].\end{aligned} \tag{3.3.42}$$

From (II) of Lemma 3.3.2 we have

$$\mathbb{E}\left[\int_0^{\tau\wedge t} e^{-r(\tau\wedge t)}B(Q_{\tau\wedge t}, X^x_{\tau\wedge t}, M^m_{\tau\wedge t}) + e^{-r(\tau\wedge t)}\phi(X^x_{\tau\wedge t}, M^m_{\tau\wedge t})\right] \le \phi(x,m). \tag{3.3.43}$$

From (I) of Lemma 3.3.2 we obtain

$$\mathbb{E}\left[\int_0^{\tau\wedge t} e^{-r(\tau\wedge t)}B(Q_{\tau\wedge t}, X^x_{\tau\wedge t}, M^m_{\tau\wedge t}) + e^{-r(\tau\wedge t)}[\psi(X^x_{\tau\wedge t}, M^m_{\tau\wedge t}) - K]\right] \le \phi(x,m). \tag{3.3.44}$$

Taking $\liminf_{t\to+\infty}$ from both sides of ineq. (3.3.44) and using Fatou's lemma, we have

$$\mathbb{E}\left[\int_0^{\tau} e^{-r\tau}B(Q_{\tau}, X^x_{\tau}, M^m_{\tau}) + e^{-r\tau}[\psi(X^x_{\tau}, M^m_{\tau}) - K]\right] \le \phi(x,m). \tag{3.3.45}$$

From Bellman's principle of optimality, for any x, m we have

$$J(x,m) = \mathbb{E}\left[\int_0^{\tau} e^{-r\tau}B(Q_{\tau}, X^x_{\tau}, M^m_{\tau})dt + e^{-r\tau}[J(X^x_{\tau}, M^m_{\tau}) - K]\right]. \tag{3.3.46}$$

Therefore, we obtain

$$J(x,m) \le \phi(x,m). \tag{3.3.47}$$

On the other hand, from (3.3.37) we see that all ineqs. (3.3.43) – (3.3.45) become equalities. Thus, we obtain

$$\mathbb{E}\left[\int_0^{\tau} e^{-r\tau}B(Q_{\tau}, X^x_{\tau}, M^m_{\tau}) + e^{-r\tau}[\psi(X^x_{\tau}, M^m_{\tau}) - K]\right] = \phi(x,m). \tag{3.3.48}$$

Therefore, we verify that eq. (3.3.38). Furthermore, the optimal policy implementation region S^* and the optimal policy implementation time τ^* are defined by (3.3.39) and (3.3.40), respectively. The proof is completed. □

3.4 Numerical Examples

In this section, we numerically evaluate the value of the TEPs. Table 3.1 shows the value of the parameters and the concept and/or sources of parameters. Table 3.2 illustrates the numerical results. In addition, we study the sensitivity of γ_1, x^*, A_1, and the value of the TEPs for a 20% change of various parameters, σ, λ, K, and c. In each case, one of the parameters is allowed to vary, while the others are fixed at the base levels. The results of the numerical examples are as follows. Considering uncertainty about damage from atmospheric CO2 concentrations, σ, the value of the TEPs increases in σ. If σ is increased by 20%, then x^* rises from 2.33972 to 2.89503, or about 23.7%. Since this means that the incentive to wait for new information becomes strong, the policy decision-maker delays policy implementation. Similarly γ_1 and A_1 are lowered by about 13.0% and 8.4%, respectively. Then, the value of the TEPs rises by about 19.3%.

In regard to λ, the value of the TEPs decreases in keeping with the degree of development of new technology that can reduce the emissions of CO2. If λ increases by 20%, then we find that x^* is 22.8% above the base case. Since this shows that the incentive to wait for new information becomes strong, as for σ, the policy implementation is deferred. Similarly, γ_1 is increased by about 3.4%, and A_1 is lowered by about 45.2%. Therefore, the value of the TEPs falls by about 5.9% as a result of synthesizing these change, contrary to the case where σ was allowed to vary.

When K is allowed to vary, the value of the TEPs increases with the sunk cost of reducing CO2 emissions. If K goes up by 20%, then x^* also increases by about 2.3%. Since the inducement to wait for new information becomes strong, policy implementation is delayed. Likewise, A_1 decreases by about 2.7%. γ_1, however, does not change, since K is not included in eq. (3.3.9). Then, the value of the TEPs rises from 54.84637 to 56.08967 due to the effects of these changes.

With regard to c, if c is increased by 20%, x^* is decreased by about 16.7%. Since this means that the policy decision-maker is not willing to wait for new information, policy implementation is promoted. By contrast, A_1 is increased by about 50.3%. γ_1, however, does not change, since c is not included in eq. (3.3.9). The result of these changes is that the value of the TEPs does not change with changes in the amount of CO2 discharges being curtailed.

3.5 Conclusion

Increasingly, markets for TEPs will become more important economic instruments for stabilizing global warming. The Kyoto protocol emphasizes the importance of the TEPs. Therefore, we have studied the value of the TEPs by using a real options model, and obtained the value of the TEPs as the opportunity cost of investment in an irreversible project, i.e. the value of waiting to invest. The main results that we obtained were as follows: an increase in uncertainty raises the value of the TEPs; by contrast, an increase in jump probability lowers the value of the TEPs.

In this chapter, we assume that the dynamics for Q and D are constants until the policy is implemented. However, they vary according to economic conditions. In addition, we assume that the jump size u is constant. An interesting extension of this chapter would be to allow the parameter to follow stochastic processes. According to IPCC (1996), to stabilize global climate change we need to immediately reduce CO2 emissions by 50–70% and further tighten the reductions in coming decades. This announcement led us to discuss our model with an upper limit of atmospheric CO2 concentrations. In this chapter, we assume that the amount of discharges of CO2 being curtailed, c is exogenously given. As an interesting extension, the parameter c could be derived from the policy decision-maker's optimization problem. We leave these problems for future research.

Table 3.1: The basic value of parameters

parameters	value	concept or/and sources
Q_t	10330.3	US \$ billion (1990) Source: OECD (1994) the aggregate of GDP (1990) of members of OECD.
a	3	US \$ billion (1990). Source: Nordhaus (1991)
D_t	3.565	ppmv. annual change of atmospheric CO2 concentrations at 1990 (=353*0.05+1.8)
c	0.1728	ppmv. the case of a 5% reduction of CO2 emissions (=3.565*0.05)
M_t	353	ppmv. atmospheric CO2 concentration in 1990
K	7.67	US \$ billion(1990). Source: Nordhaus (1991) per billion ton C (=59*0.05*2.6 : reduction cost \$ 2.6 per t C).
r	0.05	IPCC(1996) states 3–6%
μ	0.01	(3.565/353=0.01009)
σ	0.2	Assume that uncertainty is 20%.
λ	0.05	Assume that technology which reduces CO2 emissions by 30% appears every twenty years.
ν	0.3	Assume that the technology reduces CO2 emissions by 30%.
κ	0.005	Source: Nordhaus (1991).

ppmv: parts per million by volume

Table 3.2: Numerical results

	standard	$\sigma = 0.16$	$\lambda = 0.04$	$K = 6.136$	$c = 0.12136$
		$\sigma = 0.24$	$\lambda = 0.06$	$K = 9.204$	$c = 0.18204$
γ_1	2.23381	2.61707	2.15678	–	–
		2.03449	2.31076	–	–
X^*	2.33972	2.38238	2.46515	2.28668	2.92465
		2.89503	2.87350	2.39276	1.94977
A_1	8.21306	4.31522	8.35650	8.44873	4.98915
		7.52385	4.50397	7.98902	12.34187
$A_1 X^{*\gamma_1}$	54.84637	41.84729	58.49859	53.60307	54.84637
		65.41388	51.62654	56.08967	54.84637

3.6 Appendix

In appendix, we present the sensitivity analysis of γ_1, x^*, A_1, and the value of the TEPs with respect to λ. First, we define the functions of γ_1, x^*, and A_1, and the value of the TEPs,respectively:

$$\Gamma(\gamma_1(\lambda), \lambda) := \frac{1}{2}\sigma^2\gamma_1(\gamma_1 - 1) + \mu\gamma_1 - (r + \lambda) + \lambda \exp[\gamma_1 \log(1 - \nu)]; \tag{3.6.1}$$

$$x^*(\gamma_1(\lambda), \lambda) := \frac{\gamma_1 \delta \rho}{(\gamma_1 - 1)c}\left(\frac{a}{r} + K\right); \tag{3.6.2}$$

$$A_1(x^*(\lambda), \gamma_1(\lambda), \lambda) := \frac{c}{\gamma_1 \delta \rho} \exp[(1 - \gamma_1) \log x^*]; \tag{3.6.3}$$

$$TEP(A_1(\lambda), x^*(\lambda), \gamma_1(\lambda)) := A_1 \exp[\gamma_1 \log x^*]. \tag{3.6.4}$$

Taking a partial derivative eq. (3.6.1) with respect to $\lambda = 0.05$, we have

$$\left.\frac{d\gamma_1}{d\lambda}\right|_{\lambda=0.05} = -\frac{\Gamma_\lambda}{\Gamma_{\gamma_1}}. \tag{3.6.5}$$

Since $\Gamma_\lambda < 0$ and $\Gamma_{\gamma_1} > 0$, the sign of eq. (3.6.5) is positive. Then, the sign of the change of γ_1 is the same as that of λ. By differentiating eq. (3.6.2) with respect to $\lambda = 0.05$, we obtain

$$\left.\frac{dx^*}{d\lambda}\right|_{\lambda=0.05} = \frac{\partial x^*}{\partial \gamma_1}\frac{d\gamma_1}{d\lambda} + \frac{\partial x^*}{\partial \lambda}. \tag{3.6.6}$$

Since $\partial x^*/\partial\gamma_1 < 0$, $d\gamma_1/d\lambda > 0$, $\partial x^*/\partial\lambda > 0$, and $(\partial x^*/\partial\gamma_1)(d\gamma_1/d\lambda) < \partial x^*/\partial\lambda$, the sign of eq. (3.6.6) is positive. Hence, the sign of the change of x^* is the same as that of λ. Taking a partial derivative eq. (3.6.3) with respect to $\lambda = 0.05$, we obtain

$$\left.\frac{dA_1}{d\lambda}\right|_{\lambda=0.05} = \frac{\partial A_1}{\partial x^*}\frac{dx^*}{d\lambda} + \frac{\partial A_1}{\partial \gamma_1}\frac{d\gamma_1}{d\lambda} + \frac{\partial A_1}{\partial \lambda}. \tag{3.6.7}$$

Since $\partial A_1/\partial x^* < 0$, $dx^*/d\lambda > 0$, $\partial A_1/\partial\gamma_1 < 0$, $d\gamma_1/d\lambda > 0$, and $\partial A_1/\partial\lambda < 0$, the sign of eq. (3.6.7) is negative. Then, the sign of the change of A_1 is the opposite to that of λ. By differentiating eq. (3.6.4) with respect to $\lambda = 0.05$, we obtain:

$$\left.\frac{dTEP}{d\lambda}\right|_{\lambda=0.05} = \frac{\partial TEP}{\partial A_1}\frac{dA_1}{d\lambda} + \frac{\partial TEP}{\partial x^*}\frac{dx^*}{d\lambda} + \frac{\partial TEP}{\partial\gamma_1}\frac{d\gamma_1}{d\lambda}. \tag{3.6.8}$$

Since $\partial TEP/\partial A_1 > 0$, $dA_1/d\lambda < 0$, $\partial TEP/\partial x^* > 0$, $dx^*/d\lambda > 0$, $\partial TEP/\partial\gamma_1 > 0$, $d\gamma_1/d\lambda > 0$, and $[(\partial TEP/\partial A_1)(dA_1/d\lambda)+(\partial TEP/\partial\gamma_1)(d\gamma_1/d\lambda)] > (\partial TEP/\partial x^*)(dx^*/d\lambda)$, the sign of eq. (3.6.8) is negative. Then, the sign of the change of the price of the TEPs is the opposite to that of λ.

Chapter 4

Optimal Implementation of an Environmental Improvement Policy with Implementation Costs

4.1 Introduction

The problem of environmental pollution results from human activities. Human activities discharge harmful matter and pollutants; waste, greenhouse gases, and so on. For example, increasing atmospheric greenhouse gases are contributing to climate change. According to IPCC (1996), the scientific assessment of climate change estimated that the global mean surface atmospheric temperature will increase by 1 to 3.5 degrees centigrade by the year 2100. It leads to a number of potentially serious consequences. These include an increase in the incidence of heat waves, floods, and droughts as the global climate changes. These events would significantly affect human welfare as well as natural ecosystems. They are subjects of increasing concern to the world community. To prevent damage from pollutants, we have to employ EIPs; environmental taxes, marketable permits, and subsidies. See, for example, Bertram, Stephens, and Wallace (1990) and Jenkins and Lamech (1992). They examine these market-based policies.

In economic theory, one usually assumes free disposal condition to resolve some problems. See Mas-Colell, Whinston, and Green (1995); Chapter 7 and Part 4. However, by recognizing environmental problems, one must consider disposal cost. Hence, costs are incurred to reduce damage from a worsening environment. In this chapter, we consider the following EIP: If an emission level of a pollutant reaches a critical level, an agent has to reduce the emission to a certain level in order to improve the environment. If not, he or she incurs a higher level of damage. Thus, the agent has to decide when to implement the EIP (or the levels of a pollutant) and the magnitude of implementation of the EIP. In order to solve the EIP, we formulate the agent's problem as an impulse control problem, which is approached by using QVI. Then, we examine optimal implementation times, optimal pollutant emission levels, and optimal magnitude of the EIP. We also evaluate the OEIP. To achieve these goals, we take three steps. We first prove that an EIP induced by QVI is an OEIP. Next, we prove that there is a unique solution to a system of simultaneous equations, which are the well-known value matching and smooth pasting conditions.

Third, we verify that a corresponding candidate function satisfies the QVI for the agent's problem under an additional condition expressed by the given parameters. That is, the guessed candidate function is the optimal value function for the agent's problem, so that the EIP induced by the candidate function is indeed optimal.

Related works are as follows: Neuman and Costanza (1990) studies the management of renewable resources by using an impulse control method. However, their study assumes that the state of the system is deterministic. On the contrary, our analysis assumes the dynamics of the pollutant is stochastic. Willassen (1998) studies the optimal harvesting strategy for an ongoing forest by using an impulse control method. Cortazar, Schwartz, and Salinas (1998), Zepapadeas (1998) and Tsujimura (2000) also study environmental problems by using optimal stopping methods.

This chapter is structured as follows. The next section describes the agent's problem. Section 3 analyzes the agent's problem. Section 4 presents the numerical and comparative static results. Section 5 concludes this chapter.

4.2 The Model

Suppose that an agent has to implement an EIP in order to reduce damage from a pollutant. We assume that implementation of the EIP incurs both fixed and proportional implementation costs.

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ denote a filtered probability space satisfying the usual conditions. In this context, $\mathcal{F}_t$ is generated by a Brownian motion, W_t, in $\mathbb{R}$, i.e., $\mathcal{F}_t = \sigma(W_s, s \leq t)$. Let $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_i \leq \cdots$ be $(\mathcal{F}_t)_{t\geq 0}$-stopping times such that $\tau_i \to +\infty$ as $i \to +\infty$ a.s.. For each i, τ_i assigns an impulse $\zeta_i \in \mathbb{R}_+$, where ζ_i is $\mathcal{F}_{\tau_i}$-measurable. Suppose that the implementing the EIP immediately moves the state of the pollutant from x to a new state $\eta(x, \zeta)$, where $\eta : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}_{++}$ is a given function. In this chapter $\eta(x, \zeta)$ is assumed to be given by $x - \zeta$.

Let X_t be the state of the pollutant defined by the following stochastic differential equations:

$$\begin{cases} dX_t = \mu X_t dt + \sigma X_t dW_t, & \tau_i \leq t < \tau_{i+1}, \ \forall i \geq 0; \\ X_{\tau_i} = \eta(X_{\tau_i^-}, \zeta_i) = X_{\tau_i^-} - \zeta_i; \\ X_{0-} = x \in \mathbb{R}_{++}, \end{cases} \tag{4.2.1}$$

where $\mu(\in \mathbb{R})$ and $\sigma(\in \mathbb{R})$ are constants and $\tau_0 := 0$ and $\tau_{i+1}^- = \tau_i$ if $\tau_{i+1} = \tau_i$. An EIP is defined as a double sequence:

$$v := \{(\tau_i, \zeta_i)\}_{i\geq 0}. \tag{4.2.2}$$

We interpret τ_i as the times when the agent decides to implement the EIP and ζ_i as the magnitude of the EIP.

Definition 4.2.1 (Admissible EIP). *An EIP, v, is admissible, if the following are satisfied:*

$$0 \leq \tau_i \leq \tau_{i+1}, \quad a.s. \quad \forall i \geq 0; \tag{4.2.3}$$

$$\tau_i \textit{ is a } (\mathcal{F}_t)_{t\geq 0}\textit{-stopping time}, \quad \forall i \geq 0; \tag{4.2.4}$$

$$\zeta_i \textit{ is } \mathcal{F}_{\tau_i}\textit{-measurable}, \quad \forall i \geq 0; \tag{4.2.5}$$

$$P\left[\lim_{i\to\infty} \tau_i \leq \tilde{T}\right] = 0, \quad \forall \tilde{T} \in [0,\infty). \tag{4.2.6}$$

The condition given by (4.2.6) means that the EIP will be implemented finitely before a terminal time, $\tilde{T}$. Let $\mathcal{U}$ denote the set of admissible EIPs. If the EIP v is given by (4.2.2), then the state of the pollutant, $X^{x,v} := (X_t^{x,v})_{t\geq 0}$ is given by

$$\begin{cases} dX_t^{x,v} = \mu X_t^{x,v} ds + \sigma X_t^{x,v} dW_t; & \tau_{i-1} \leq t < \tau_i < \infty; \\ X_{\tau_i}^{x,v} = \eta(X_{\tau_i^-}^{x,v}, \zeta_i) = X_{\tau_i^-}^{x,v} - \zeta_i; & i = 1, 2, \cdots; \\ X_{\tau_0}^{x,v} = x. \end{cases} \tag{4.2.7}$$

Let $D : \mathbb{R}_{++} \to \mathbb{R}$ be a continuous function satisfying

$$\mathbb{E}\left[\int_0^\infty e^{-rt} D(X_t^{x,v}) dt\right] < \infty, \quad x \in \mathbb{R}_{++}, v \in \mathcal{U}, \tag{4.2.8}$$

$r(\in \mathbb{R}_{++})$ is a constant discount rate. We interpret $D(x)$ as the damage function associated with the state of the pollutant and given by

$$D(x) := ax^2, \quad x \in \mathbb{R}_{++}, \tag{4.2.9}$$

where $a(\in \mathbb{R}_{++})$ is the proportional damage parameter. Let v_0 represent the EIP which the agent does not implement forever. In this case, the expected present value of the flow of $D(x)$ is written as

$$\mathbb{E}\left[\int_0^\infty e^{-rt} D(X_t^{x,v_0}) dt\right] = \frac{ax^2}{r - 2\mu - \sigma^2}, \quad x \in \mathbb{R}_{++}, \tag{4.2.10}$$

By (4.2.8), we assume the following inequality holds:
(AS.4.2.1)

$$r - 2\mu - \sigma^2 > 0.$$

Let $K : \mathbb{R}_+ \to \mathbb{R}_{++}$ represent the cost to implement the EIP and is given by

$$K(\zeta) = k_0 + k_1\zeta, \quad \zeta \in \mathbb{R}_+, \tag{4.2.11}$$

where $k_0(\in \mathbb{R}_{++})$ and $k_1(\in \mathbb{R}_{++})$ are the fixed and proportional cost of implementing of the EIP, respectively. Note that $K(\zeta)$ satisfies subadditivity with respect to ζ:

$$K(\zeta + \zeta') \leq K(\zeta) + K(\zeta'), \quad \zeta, \zeta' \in \mathbb{R}_+. \tag{4.2.12}$$

This implies that $(\mathcal{F}_t)_{t\geq 0}$-stopping times hold strictly increasing sequences, i.e., $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_i < \cdots$.

The damage function, $D(x)$, does not contain the pollutant's fixed effect, because, in general, the damage from pollutants occurs when the pollutants are discharged. Therefore, we define the damage function as eq. (4.2.9). On the other hand, the implementation cost, $K(\zeta)$, involves the fixed implementation cost. This includes the cost of deciding whether to implement the EIP or not. The decision-making requires research on the magnitude of damage, forecasts of future environmental conditions and so on. In this chapter, the decision-making cost is assumed to be constant. Øksendal (1999) studies the effect of the fixed cost in impulse control problems and shows it has significant effects.

The agent's expected total discounted cost function associated with the EIP v is defined by

$$J(x;v) = \mathbb{E}\left[\int_0^\infty e^{-rt}D(X_t^{x,v})dt + \sum_{i=1}^{\infty} e^{-r\tau_i}K(X_{\tau_i^-}^{x,v}, X_{\tau_i}^{x,v})1_{\{\tau_i<\infty\}}\right]. \tag{4.2.13}$$

Therefore, the agent's problem is to choose $v \in \mathcal{U}$ in order to minimize $J(x;v)$:

$$V(x) = \inf_{v\in\mathcal{U}} J(x;v) = J(x;v^*), \tag{4.2.14}$$

where V is the value function of the agent problem given by eq. (4.2.14) and v^* is an OEIP.

4.3 Analysis

In the previous section, we formulated the agent's problem as a stochastic impulse control problem. From that formulation, we naturally guess that, under an OEIP, the agent implements the EIP whenever the state of the pollutant reaches a threshold. In order to verify this conjecture, we take three steps in this section. We first prove that a QVI policy, which is introduced later, is an OEIP for the agent's problem given by eq. (4.2.14). Next we prove that there exists a unique solution to a system of simultaneous equations which are the well-known value matching and smooth pasting conditions. Then we verify that a corresponding candidate function satisfies the QVI for the agent's problem under an additional condition. That is, the guessed candidate function is the optimal value function for the agent's problem given by eq. (4.2.14), so that the EIP induced by the candidate function is indeed optimal. To take these steps, we first introduce some notation.

Suppose that $\phi : \mathbb{R}_{++} \to \mathbb{R}$ is a continuous function. Let $\mathcal{M}$ denote the implementation operator on the space of functions ϕ defined by

$$\mathcal{M}\phi(x) = \inf_{\zeta\in[0,x)} \{\phi(\eta(x,\zeta)) + K(\zeta)\}. \tag{4.3.1}$$

We assume that ϕ is a twice continuously differentiable function on $\mathbb{R}$ except on the boundary of the considered region. Let us define an operator $\mathcal{L}$ of the $X^{x,v}$ as follows:

$$\mathcal{L}\phi(x) = \frac{1}{2}\sigma^2x^2\phi''(x) + \mu x\phi'(x) - r\phi(x). \tag{4.3.2}$$

Since ϕ is not necessarily C^2 in the whole region, we must apply the generalized Dynkin formula for ϕ. The formula is available if ϕ is *stochastically* C^2. That is, we use the generalized Dynkin formula on the set which has a *Green measure* of $X^{x,v}$ zero. The Green measure of $X^{x,v}$ is the expected total occupation measure $\mathbb{G}(\Xi; x, v)$ defined by

$$\mathbb{G}(\Xi; x, v) = \mathbb{E}\left[\int_0^\infty X_t^{x,v} 1_\Xi dt\right], \tag{4.3.3}$$

where 1_Ξ is the indicator of a Borel set $\Xi(\subset \mathbb{R}_{++})$. A continuous function $\phi : \mathbb{R}_{++} \to \mathbb{R}$ is called stochastically C^2 with respect to $X^{x,v}$ if $\mathcal{L}\phi(x)$ is well defined point wise for almost all x with respect to the Green measure $\mathbb{G}(\cdot; x, v)$. Henceforth, we assume that ϕ is stochastically C^2 with respect to $X^{x,v}$. The following equality, which is called the *generalized Dynkin formula*, will be used in the proof of Theorem 4.3.1.

$$\mathbb{E}[e^{-r\theta-}\phi(X_{\theta-}^{x,v})] = \mathbb{E}[e^{-r\tau_i}\phi(X_{\tau_i}^{x,v})] + \mathbb{E}\left[\int_{\tau_i}^{\theta-} e^{-rt}\mathcal{L}\phi(X_t^{x,v})dt\right], \tag{4.3.4}$$

for all i, all bounded stopping times θ such that $\tau_i \le \theta \le \tau_{i+1}$. See, for example, Brekke and Øksendal (1991) for more details.

Definition 4.3.1 (QVI). *The following relations are called the quasi-variational inequalities (QVI) for the problem given by (4.2.14):*

$$\mathcal{L}\phi(x) + D(x) \ge 0, \quad \textit{for a.a. } x \textit{ w.r.t. } \mathbb{G}(\cdot; x, v), \quad \forall v \in \mathcal{U}; \tag{4.3.5}$$

$$\phi(x) \le \mathcal{M}\phi(x); \tag{4.3.6}$$

$$[\mathcal{L}\phi(x) + D(x)][\phi(x) - \mathcal{M}\phi(x)] = 0, \quad \textit{for a.a. } x \textit{ w.r.t. } \mathbb{G}(\cdot; x, v), \quad x \in \mathbb{R}_{++}. \tag{4.3.7}$$

Definition 4.3.2 (QVI policy). *Let ϕ be a solution of the QVI. Then the following EIP $\hat{v} = \{(\hat{\tau}_i, \hat{\zeta}_i)\}_{i\ge 0}$ is called a QVI policy:*

$$(\hat{\tau}_0, \hat{\zeta}_0) = (0, 0); \tag{4.3.8}$$

$$\hat{\tau}_i = \inf\{t \ge \hat{\tau_{i-1}}; X_t^{x,\hat{v}} \notin H\}; \tag{4.3.9}$$

$$\hat{\zeta}_i = \arg\inf\left\{\phi\left(\eta\left(X_{\hat{\tau}_{i-}}^{x,\hat{v}}, \zeta\right)\right) + K(\zeta); \zeta\right\}. \tag{4.3.10}$$

In this context, H is the continuation region defined by

$$H := \{x; \phi(x) < \mathcal{M}\phi(x)\}, \tag{4.3.11}$$

and $X_t^{x,\hat{v}}$ is the result of applying the EIP $\hat{v}$.

We can now prove that a QVI policy is an OEIP. The following Theorem 4.3.1 is a minor modification of Theorem 3.1. in Brekke and Øksendal (1998).

Theorem 4.3.1. *(I) Let a continuous function $\phi : \mathbb{R}_{++} \to \mathbb{R}$ be a solution of the QVI and satisfy the following:*

$$\phi \text{ is stochastically } C^2 \text{ w.r.t. } X^{x,v}; \tag{4.3.12}$$

$$\lim_{t\to\infty} e^{-rt}\phi(X_t^{x,v}) = 0, \quad a.s., \quad \forall v \in \mathcal{U}; \tag{4.3.13}$$

$$\text{the family } \{\phi(X_\tau^{x,v})\}_{\tau<\infty} \text{ is uniformly integrable w.r.t. } P, \quad \forall v \in \mathcal{U}. \tag{4.3.14}$$

Then we obtain

$$\phi(x) \leq J(x;v) \quad \forall v \in \mathcal{U}. \tag{4.3.15}$$

(II) Suppose that, in addition to (4.3.5)–(4.3.7) and (4.3.12)–(4.3.14), we have

$$\mathcal{L}\phi(x) + D(x) = 0, \quad x \in H. \tag{4.3.16}$$

Furthermore, suppose $\hat{v} \in \mathcal{U}$, i.e., the EIP is admissible. Then, we obtain

$$\phi(x) = J(x;\hat{v}). \tag{4.3.17}$$

Hence, we have

$$\phi(x) = V(x) = J(x;\hat{v}). \tag{4.3.18}$$

Therefore $\hat{v}$ is optimal.

Proof. (I) Assume that ϕ satisfies (4.3.12) – (4.3.14). Choose $v \in \mathcal{U}$. Let $\theta_{i+1} := \tau_i \vee (\tau_{i+1} \wedge s)$ for any $s \in \mathbb{R}_+$. Then, by the generalized Dynkin formula, (4.3.4), we obtain

$$\mathbb{E}[e^{-r\theta_{i+1}-}\phi(X_{\theta_{i+1}-}^{x,v})] = \mathbb{E}[e^{-r\tau_i}\phi(X_{\tau_i}^{x,v})] + \mathbb{E}\left[\int_{\tau_i}^{\theta_{i+1}} e^{-rt}[\mathcal{L}\phi](X_t^{x,v})dt\right]. \tag{4.3.19}$$

Hence, from (4.3.5) we obtain

$$\mathbb{E}[e^{-r\theta_{i+1}-}\phi(X_{\theta_{i+1}-}^{x,v})] \geq \mathbb{E}[e^{-r\tau_i}\phi(X_{\tau_i}^{x,v})] - \mathbb{E}\left[\int_{\tau_i}^{\theta_{i+1}} e^{-rt}D(X_t^{x,v})dt\right]. \tag{4.3.20}$$

Taking $\lim_{s\to\infty}$ and using the dominated convergence theorem, we have

$$\mathbb{E}[e^{-r\tau_{i+1}-}\phi(X_{\tau_{i+1}-}^{x,v})] - \mathbb{E}[e^{-r\tau_i}\phi(X_{\tau_i}^{x,v})] \geq -\mathbb{E}\left[\int_{\tau_i}^{\tau_{i+1}} e^{-rt}D(X_t^{x,v})dt\right]. \tag{4.3.21}$$

Summing from $i = 0$ to $i = m$ yields

$$\begin{aligned}\phi(x) &+ \sum_{i=1}^{m}\mathbb{E}[e^{-r\tau_i}\phi(X_{\tau_i}^{x,v}) - e^{-r\tau_i-}\phi(X_{\tau_i-}^{x,v})] \\ &- \mathbb{E}[e^{-r\tau_{m+1}-}\phi(X_{\tau_{m+1}-}^{x,v})] \leq \mathbb{E}\left[\int_0^{\tau_{m+1}} e^{-rt}D(X_t^{x,v})dt\right].\end{aligned} \tag{4.3.22}$$

For all $\tau_i < \infty$, following the EIP, the state of the pollutant jumps immediately from $X^{x,v}_{\tau_i-}$ to a new state $\eta(X^{x,v}_{\tau_i-}, \zeta_i)$. Thus, by eq. (4.3.1) and $\eta(X^{x,v}_{\tau_i-}, \zeta_i) = X^{x,v}_{\tau_i}$, we obtain

$$\phi(\eta(X^{x,v}_{\tau_i-}, \zeta_i)) \geq \mathcal{M}\phi(X^{x,v}_{\tau_i-}) - K(\zeta_i), \quad \tau_i < \infty. \tag{4.3.23}$$

Therefore, we have

$$\begin{aligned}\phi(x) &+ \sum_{i=1}^{m} \mathbb{E}\left[[e^{-r\tau_i}\mathcal{M}\phi(X^{x,v}_{\tau_i-}) - e^{-r\tau_i-}\phi(X^{x,v}_{\tau_i-})]\, 1_{\{\tau_i<\infty\}} \right] \\ &\leq \mathbb{E}\left[\int_0^{\tau_{m+1}} e^{-rt} D(X^{x,v}_t)dt + e^{-r\tau_{m+1}-}\phi(X^{x,v}_{\tau_{m+1}-}) + \sum_{i=1}^{m} e^{-r\tau_i} K(\zeta_i) 1_{\{\tau_i<\infty\}} \right].\end{aligned} \tag{4.3.24}$$

It follows from eq. (4.3.6) that

$$\mathcal{M}\phi(X^{x,v}_{\tau_i-}) - \phi(X^{x,v}_{\tau_i-}) \geq 0. \tag{4.3.25}$$

Hence, we obtain

$$\phi(x) \leq \mathbb{E}\left[\int_0^{\tau_{m+1}} e^{-rt} D(X^{x,v}_t)dt + e^{-r\tau_{m+1}-}\phi(X^{x,v}_{\tau_{m+1}-}) + \sum_{i=1}^{m} e^{-r\tau_i} K(\zeta_i) 1_{\{\tau_i<\infty\}} \right]. \tag{4.3.26}$$

Taking $\lim_{m\to\infty}$, by using (4.3.13), (4.3.14) and the dominated convergence theorem, we obtain

$$\phi(x) \leq \mathbb{E}\left[\int_0^{\infty} e^{-rt} D(X^{x,v}_t)dt + \sum_{i=1}^{\infty} e^{-r\tau_i} K(\zeta_i) 1_{\{\tau_i<\infty\}} \right]. \tag{4.3.27}$$

Therefore, eq. (4.3.15) is proven.

(II) Assume that eq. (4.3.16) holds and $\hat{v}$ is the QVI policy. Then repeat the argument in part (i) for $v = \hat{v}$. Then, ineqs. (4.3.20)-(4.3.27) become equalities. Thus, we obtain

$$\phi(x) = \mathbb{E}\left[\int_0^{\infty} e^{-rt} D(X^{x,\hat{v}}_t)dt + \sum_{i=1}^{\infty} e^{-r\tau_i} K(X^{x,\hat{v}}_{\tau_i-}, X^{x,\hat{v}}_{\tau_i}) 1_{\{\tau_i<\infty\}} \right]. \tag{4.3.28}$$

Hence, we obtain eq. (4.3.17). Combining eq. (4.3.17) with ineq. (4.3.15), we obtain

$$\phi(x) \leq \inf_{v\in\mathcal{U}} J(x; v) \leq J(x; \hat{v}) = \phi(x). \tag{4.3.29}$$

Therefore, $\phi(x) = V(x)$ and $v^* = \hat{v}$ is optimal, i.e., the solution of the QVI is the value function and the QVI policy is optimal. The proof is completed.

□

Next, from the agent's problem given by eq. (4.2.14), we conjecture that, under a suitable set of sufficient conditions on the parameters, an OEIP is specified in the following form by two thresholds: whenever the state of the pollutant reaches a level $\overline{x}$, the agent implements the EIP, so that it instantaneously falls to another level β. To show the validity of this conjecture, we first prove there exist unique parameters $\overline{x}$ and β (and another unknown parameter A_1), by examining a system of simultaneous equations, which are the well-known value matching and smooth pasting conditions.

Let an OEIP be denoted by $v^*(\tau^*, \zeta^*)$ characterized by parameters $(\beta, \overline{x})$ with $0 < \beta < \overline{x} < \infty$ such that

$$\tau_i^* := \inf\{t > \tau_{i-1}^*; X_{t-}^{x,v^*} \notin (0, \overline{x})\}, \tag{4.3.30}$$

$$\zeta_i^* := X_{\tau_i^*-}^{x,v^*} - X_{\tau_i^*}^{x,v^*} = \overline{x} - \beta. \tag{4.3.31}$$

Thus, eq. (4.2.7) becomes

$$\begin{cases} dX_t^{x,v^*} = \mu X_t^{x,v^*} dt + \sigma X_t^{xv^*} dW_t; \quad \tau_{i-1}^* \le t < \tau_i^* < \infty; \\ X_{\tau_i^*}^{x,v^*} = \eta(X_{\tau_i^*-}^{x,v^*}, \zeta_i^*); \quad i = 1, 2, \cdots; \\ X_{\tau_0}^{x,v^*} = x. \end{cases} \tag{4.3.32}$$

Therefore, when the agent implements the OEIP, the value function seems to satisfy

$$\begin{aligned} V(x) &= V(\eta(x, \zeta^*)) + K(\zeta^*) \\ &= V(\beta) + k_0 + k_1(x - \beta), \quad x \in [\overline{x}, \infty). \end{aligned} \tag{4.3.33}$$

Assume that V is stochastically C^2 with respect to X^{x,v^*}. Under the assumed OEIP, if the pollutant level is within the continuation region $(0, \overline{x})$, it remains in that region thereafter. However if the initial level of the pollutant x is $x = \overline{x} + \varepsilon$, where $\varepsilon > 0$, then the OEIP is $\zeta = (\overline{x} + \varepsilon) - \beta$. Thus, we have

$$V(\overline{x} + \varepsilon) = V(\beta) + k_0 + k_1(\overline{x} + \varepsilon - \beta). \tag{4.3.34}$$

Substituting x into $\overline{x}$ in eq. (4.3.33) and subtracting from eq. (4.3.34), we obtain

$$V(\overline{x} + \varepsilon) - V(\overline{x}) = k_1 \varepsilon. \tag{4.3.35}$$

Dividing eq. (4.3.35) by ε and taking $\lim_{\varepsilon \to 0}$ in eq. (4.3.35), we get

$$V'(\overline{x}) = k_1. \tag{4.3.36}$$

By (4.3.30) and (4.3.31), the agent's expected total discounted cost function $J(x; v)$ is minimized at $\zeta = \overline{x} - \beta$. Hence, by the first order condition for the minimization $\partial[V(\eta(\overline{x}, \zeta)) + K(\zeta)]/\partial\zeta \,|_{\zeta = \overline{x} - \beta} = 0$, we obtain

$$V'(\beta) = k_1, \tag{4.3.37}$$

because V is stochastically C^2. Dixit (1991) and Constantinides and Richard (1978) discussed equations to similar eqs. (4.3.36) and (4.3.37) for more details. Furthermore, we can conjecture that eq. (4.3.16) holds in the continuation region $(0, \overline{x})$. Following the standard methods of ordinary differential equations, the general solution of eq (4.3.16) is given by

$$\phi(x) = A_1 x^{\gamma_1} + A_2 x^{\gamma_2} + \frac{ax^2}{r - 2\mu - \sigma^2}, \quad x \in (0, \overline{x}), \tag{4.3.38}$$

where A_1 and A_2 are constants to be determined, and γ_1 and γ_2 are the solutions to the characteristic equation:

$$\frac{1}{2}\sigma^2\gamma^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\gamma - r = 0, \quad \gamma \in \mathbb{R}. \tag{4.3.39}$$

Hence, we obtain

$$\gamma_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \left[\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}\right]^{\frac{1}{2}}, \tag{4.3.40}$$

$$\gamma_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \left[\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}\right]^{\frac{1}{2}}. \tag{4.3.41}$$

Since $\gamma_2 < 0$, to prevent the value from diverging, we set the coefficient $A_2 = 0$. Thus, we have

$$\phi(x) = A_1 x^{\gamma_1} + \frac{ax^2}{r - 2\mu - \sigma^2}, \quad x \in (0, \overline{x}). \tag{4.3.42}$$

Remark 4.3.1. *The first term on the right-hand side of eq. (4.3.42) is the value of the option to implement the EIP at some times in the future. In other words, when the state of the pollutant achieves the threshold $\overline{x}$, the agent exercises the option and implements the EIP. Then, we can evaluate the EIP by calculating the first term on the right-hand side of eq. (4.3.42). Since we are solving a cost minimization problem, the sign of A_1 is negative. Thus, we evaluate the EIP by changing the sign of the first term on the right-hand side of eq. (4.3.42):*

$$EIP = -A_1 x^{\gamma_1}. \tag{4.3.43}$$

The second term, $ax^2/(r - 2\mu - \sigma^2)$, is the expected present value of the flow of $D(x)$ when the agent never implements the EIP by eq. (4.2.10).

Let us define $\phi(x)$ by

$$\phi(x; A_1, \overline{x}, \beta) := \begin{cases} A_1 x^{\gamma_1} + \frac{ax^2}{r - 2\mu - \sigma^2}, & x \in (0, \overline{x}), \\ \phi(\beta) + c + b(x - \beta), & x \in [\overline{x}, \infty). \end{cases} \tag{4.3.44}$$

Hereafter we suppress A_1, $\overline{x}$, and β in $\phi(x; A_1, \overline{x}, \beta)$ for tractability unless we need to pay attention to these parameters. These parameters are uniquely determined by following simultaneous equations:

$$\phi(\overline{x}) = \phi(\beta) + k_0 + k_1(\overline{x} - \beta), \tag{4.3.45}$$

$$\phi'(\overline{x}) = k_1, \tag{4.3.46}$$

$$\phi'(\beta) = k_1. \tag{4.3.47}$$

In order to verify that there are unique solutions A_1, $\overline{x}$, and β of eqs. (4.3.45) – (4.3.47), we investigate how $\overline{x}$ and β depend on A_1. We refer to Øksendal (1999). From eq. (4.3.45), let us define $\Phi(x)$ as

$$\Phi(x) := A_1[x^{\gamma_1} - \beta^{\gamma_1}] + \frac{a(x^2-\beta^2)}{r-2\mu-\sigma^2} - k_0 - k_1(x-\beta), \tag{4.3.48}$$

where we fix $A_1(<0)$. Figure 4.1 illustrates $\Phi(x)$. The first, second and third derivatives of eq. (4.3.48) are given by

$$\Phi'(x) = \gamma_1 A_1 x^{\gamma_1 - 1} + \frac{2ax}{r-2\mu-\sigma^2} - k_1, \tag{4.3.49}$$

$$\Phi''(x) = \gamma_1(\gamma_1-1)A_1 x^{\gamma_1-2} + \frac{2a}{r-2\mu-\sigma^2}, \tag{4.3.50}$$

$$\Phi'''(x) = \gamma_1(\gamma_1-1)(\gamma_1-2)A_1 x^{\gamma_1-3}. \tag{4.3.51}$$

See Figure 4.2, 4.3, and 4.4, respectively. Since $\gamma_1 > 2$ and $A_1 < 0$, $\Phi'''(x)$ is negative. Hence, $\Phi'(x)$ has its unique maximum point, $\breve{x}(A_1)$. From eq. (4.3.50) $\Phi''(x) = 0$ if, and only if

$$A_1 = -\frac{1}{\gamma_1(\gamma_1-1)} x^{2-\gamma_1} \frac{2a}{r-2\mu-\sigma^2}. \tag{4.3.52}$$

From eq. (4.3.49), $\Phi'(\breve{x}(A_1)) > 0$ if, and only if

$$\begin{aligned} \breve{x} &> \frac{r-2\mu-\sigma^2}{2a}\frac{\gamma_1-1}{\gamma_1-2}k_1 \\ &=: \tilde{x}. \end{aligned} \tag{4.3.53}$$

Thus $\Phi'(\breve{x}(A_1)) > 0$ if, and only if $\breve{x} > \tilde{x}$. This implies that

$$\gamma_1 A_1 \breve{x}^{\gamma_1-1} > \gamma_1 \tilde{A}_1 \tilde{x}^{\gamma_1-1} \quad \text{or} \quad A_1 > \tilde{A}_1, \tag{4.3.54}$$

where, from (4.3.52) and (4.3.53), $\tilde{A}_1$ is given by

$$\tilde{A}_1 = -\frac{1}{\gamma_1(\gamma_1-1)}\left(\frac{r-2\mu-\sigma^2}{2a}\right)^{1-\gamma_1}\left(\frac{\gamma_1-1}{\gamma_1-2}k_1\right)^{2-\gamma_1}. \tag{4.3.55}$$

Note that $\Phi'(0) = -k_1 < 0$. Therefore, for any $\tilde{A}_1 < A_1 < 0$, eqs. (4.3.46) and (4.3.47) have two solutions $\overline{x}(A_1)$ and $\beta(A_1)$ such that $0 < \beta(A_1) < \breve{x}(A_1) < \overline{x}(A_1)$. Hereafter we assume that $\tilde{A}_1 < A_1 < 0$.

From the above preliminary analysis, we examine how $\overline{x}$ and β depend on A_1. To this end, we first differentiate eq. (4.3.46) with respect to A_1 and obtain

$$\overline{x}'(A_1) = -\left[\gamma_1(\gamma_1 - 1)A_1\overline{x}^{\gamma_1-2} + \frac{2a}{r - 2\mu - \sigma^2}\right]^{-1} \gamma_1\overline{x}(A_1)^{\gamma_1-1} > 0. \tag{4.3.56}$$

Since $\gamma_1(\gamma_1 - 1)A_1\overline{x}^{\gamma_1-2} + 2a/(r - 2\mu - \sigma^2) = \Phi''(\overline{x}(A_1)) < 0$, ineq. (4.3.56) holds. Ineq. (4.3.56) means $\overline{x}(A_1)$ increases in A_1. Thus, by eq. (4.3.46) we have

$$\lim_{A_1\uparrow 0} \gamma_1 A_1\overline{x}(A_1)^{\gamma_1-1} + \frac{2a\overline{x}(A_1)}{r - 2\mu - \sigma^2} - k_1 = \lim_{A_1\uparrow 0} \frac{2a\overline{x}(A_1)}{r - 2\mu - \sigma^2} - k_1. \tag{4.3.57}$$

It follows that

$$\lim_{A_1\uparrow 0} \overline{x}(A_1) = +\infty. \tag{4.3.58}$$

Note that $\Phi'(\check{x}(A_1)) > 0$ if, and only if $\check{x}(A_1) > \tilde{x}$ or $A_1 > \tilde{A}_1$ and $\overline{x}(A_1) > \check{x}(A_1)$. If A_1 decreases to $\tilde{A}_1$, $\overline{x}(A_1)$ decreases to $\tilde{x}$, i.e.,

$$\lim_{A_1\downarrow \tilde{A}_1} \overline{x}(A_1) = \tilde{x}. \tag{4.3.59}$$

Similarly, we differentiate eq. (4.3.47) with respect to A_1 and by $\Phi''(\beta(A_1)) > 0$ to obtain

$$\beta'(A_1) < 0. \tag{4.3.60}$$

Ineq. (4.3.60) implies $\beta(A_1)$ decreases in A_1. Thus, by eq. (4.3.47) we have

$$\lim_{A_1\uparrow 0} \gamma_1 A_1\beta(A_1)^{\gamma_1-1} + \frac{2a\beta(A_1)}{r - 2\mu - \sigma^2} - k_1 = \lim_{A_1\uparrow 0} \frac{2a\beta(A_1)}{r - 2\mu - \sigma^2} - k_1. \tag{4.3.61}$$

From eq. (4.3.61) we obtain

$$\lim_{A_1\uparrow 0} \beta(A_1) = \frac{r - 2\mu - \sigma^2}{2a} k_1. \tag{4.3.62}$$

Furthermore, we have

$$\lim_{A_1\downarrow \tilde{A}_1} \beta(A_1) = \tilde{x}. \tag{4.3.63}$$

We are now ready to show the existence of unique solutions of simultaneous equations. In this context, we assume the following:
(AS.4.3.1)

$$\frac{a}{r - 2\mu - \sigma^2} > k_1,$$

where $a/(r - 2\mu - \sigma^2)$ is the present value of damage caused by the state of the pollutant. If the above inequality does not hold, it will never be optimal to implement the EIP as far as $k_0 > 0$.

Theorem 4.3.2. *Assume that (AS.4.2.1) and (AS.4.3.1) hold. Then, there exists a uniquely solution to the simultaneous equations (4.3.45) – (4.3.47): A_1^*, $\overline{x}(A_1^*)$, and $\beta(A_1^*)$ with $0 < \beta(A_1^*) < \overline{x}(A_1^*) < +\infty$.*

Proof. By eq. (4.3.45) we obtain

$$g(A_1) = k_0, \tag{4.3.64}$$

where $g(A_1) := A_1[\overline{x}(A_1)^{\gamma_1} - \beta(A_1)^{\gamma_1}] + a[\overline{x}(A_1)^2 - \beta(A_1)^2]/(r - 2\mu - \sigma^2) - k_1[\overline{x}(A_1) - \beta(A_1)]$. The derivative of g with respect to A_1 is

$$\begin{aligned}\frac{\partial g}{\partial A_1} &= [\overline{x}(A_1)^{\gamma_1} - \beta(A_1)^{\gamma_1}] + A_1[\gamma_1 \overline{x}(A_1)^{\gamma_1 - 1}\overline{x}'(A_1) - \gamma_1 \beta(A_1)^{\gamma_1 - 1}\beta'(A_1)] \\ &\quad + \frac{2a}{r - 2\mu - \sigma^2}[\overline{x}(A_1)\overline{x}'(A_1) - \beta(A_1)\beta'(A_1)] - k_1[\overline{x}'(A_1) - \beta'(A_1)].\end{aligned} \tag{4.3.65}$$

From (4.3.56) and (4.3.60), the first and third terms on the right hand-side of eq. (4.3.65) are positive, while the second and fourth terms are negative. The sign of (4.3.65) depends on the relation of these terms. To investigate the relation, first, we suppose that

$$\overline{x}(A_1)^{\gamma_1} - \beta(A_1)^{\gamma_1} - b[\overline{x}'(A_1) - \beta'(A_1)] > 0. \tag{4.3.66}$$

Eqs. (4.3.58) and (4.3.62) reveal that the left-hand side of (4.3.66) is positive. From eqs. (4.3.59) and (4.3.63), it follows that (4.3.66) is zero. Hence ineq. (4.3.66) holds. Secondly, suppose that

$$A_1[\gamma_1 \overline{x}(A_1)^{\gamma_1 - 1}\overline{x}'(A_1) - \gamma_1 \beta(A_1)^{\gamma_1 - 1}\beta'(A_1)] + \frac{2a}{r - 2\mu - \sigma^2}[\overline{x}(A_1)\overline{x}'(A_1) - \beta(A_1)\beta'(A_1)] > 0. \tag{4.3.67}$$

To show ineq. (4.3.67) we require that

$$\frac{2a}{r - 2\mu - \sigma^2}\overline{x}(A_1)^{2-\gamma_1} > -A_1\gamma_1, \tag{4.3.68}$$

$$\frac{2a}{r - 2\mu - \sigma^2}\beta(A_1)^{2-\gamma_1} > A_1\gamma_1. \tag{4.3.69}$$

It follows from eqs. (4.3.58) and (4.3.59) that we have

$$1 > \frac{1}{\gamma_1(\gamma_1 - 1)}. \tag{4.3.70}$$

Since it is obvious that ineq. (4.3.70) holds, we obtain ineq. (4.3.68). On the other hand, eqs. (4.3.62) and (4.3.63) imply that the minimum of $\beta(A_1)$ is attained when A_1 goes to 0. Thus, we obtain

$$\left(\frac{r - 2\mu - \sigma^2}{2a}\right)^{1-\gamma_1} k_1^{2-\gamma_1} > 0. \tag{4.3.71}$$

Since it is clear that ineq. (4.3.71) holds, we have ineq. (4.3.69). Thus, ineq. (4.3.67) follows. Both ineqs. (4.3.66) and (4.3.67) imply that ineq. (4.3.65) is positive. Therefore, it follows that

$$\lim_{A_1 \uparrow 0} g(A_1) = +\infty, \tag{4.3.72}$$

$$\lim_{A_1 \downarrow \bar{A}_1} g(A_1) = 0. \tag{4.3.73}$$

From eqs. (4.3.72) and (4.3.73) there exists A_1^* such that $g(A_1^*) = k_0$, by using the mean value theorem. Therefore, there exist unique solutions to the simultaneous equations (4.3.45) – (4.3.47): A_1^*, $\overline{x}(A_1^*)$, and $\beta(A_1^*)$ with $0 < \beta(A_1^*) < \overline{x}(A_1^*) < +\infty$. The proof is completed. □

As the third step, we show the candidate function of the value function of the agent's problem given by eq. (4.2.14) satisfies the QVI under an additional condition. Thus, the candidate function is the value function of the agent's problem. Therefore, the QVI policy induced by the candidate function is indeed optimal. To this end, we first examine the following. For $x \in (0, \overline{x})$, by (4.3.44) the first and second derivative of $\phi(x)$ are, respectively

$$\phi'(x) = \gamma_1 A_1 \overline{x}^{\gamma_1 - 1} + \frac{2a\overline{x}}{r - 2\mu - \sigma^2}, \tag{4.3.74}$$

$$\phi''(x) = \gamma_1(\gamma_1 - 1) A_1 \overline{x}^{\gamma_1 - 2} + \frac{2a}{r - 2\mu - \sigma^2}. \tag{4.3.75}$$

Note that eq. (4.3.75) equals to eq. (4.3.50). It is obvious that there exist A_1 such that $\Phi'(\overline{x}(A_1)) = \Phi'(\beta(A_1)) = 0$ from Theorem 4.3.2. It follows that $\phi'(\overline{x}) = \phi'(\beta) = b$. Furthermore, since $\Phi'(x(A_1))$ has a unique maximum point,

$$\Phi'(x) \begin{cases} < 0, & x \in (0, \beta) \text{ or } (\overline{x}, \infty), \\ = 0, & x = \beta \text{ or } \overline{x}, \\ > 0, & x \in (\beta, \overline{x}). \end{cases} \tag{4.3.76}$$

Therefore it follows that

$$\phi'(x) \begin{cases} < k_1, & x \in (0, \beta) \text{ or } (\overline{x}, \infty), \\ = k_1, & x = \beta \text{ or } \overline{x}, \\ > k_1, & x \in (\beta, \overline{x}). \end{cases} \tag{4.3.77}$$

Theorem 4.3.3. *Assume that (A.1) and (A.2) hold. Let A_1^*, $\overline{x}^* = \overline{x}(A_1^*)$, and $\beta^* = \beta(A_1^*)$ with $0 < \beta^* < \overline{x}^* < \infty$ be the solutions of the simultaneous equations (4.3.45) – (4.3.47), whose existence is assured by Theorem 4.3.2. Furthermore, we assume that the*

implementation cost satisfies the following:
(AS.4.3.2)

$$(r-2\mu-\sigma^2)\left(\frac{\gamma_1-1}{\gamma_1-2}\right)k_1$$
$$> (r-\mu)k_1+\left[(r-\mu)^2k_1^2+4ar\left[\frac{r-2\mu-\sigma^2}{2a}\left(-\frac{(\gamma_1-1)^2}{2\gamma_1(\gamma_1-2)}\right)k_1^2+k_0\right]\right]^{\frac{1}{2}}.$$

Then, $\phi(x)=\phi(x;A_1^*,\overline{x}^*,\beta^*)$ satisfies the QVI. According to Theorem 4.3.1, ϕ is the value function of the agent's problem given by eq. (4.2.14) and the QVI policy induced by ϕ is optimal. That is, the EIP given by (4.3.30) and (4.3.31) is an OEIP.

Proof. First we show ϕ satisfies the QVI.

(I) Consider ineq. (4.3.5) for two distinct cases, $x\in(0,\overline{x}^*)$ or $x\in[\overline{x}^*,\infty)$.

(i) If $x\in(0,\overline{x})$, it is clear from (4.3.44) and the derivation of eq. (4.3.42) that

$$\mathcal{L}\phi(x)+D(x)=0. \tag{4.3.78}$$

(ii) If $x\in[\overline{x}^*,\infty)$, by (4.3.44) we have

$$\mathcal{L}\phi(x)+D(x)=\mu x k_1-r[\phi(\beta)+k_0+k_1(x-\beta)]+ax^2. \tag{4.3.79}$$

If (4.3.79) is positive, we have

$$\overline{x}(A_1)>\frac{1}{2a}\left\{(r-\mu)k_1+\left[(r-\mu)^2k_1^2+4ar\left[\phi(\beta(A_1))+k_0-k_1\beta(A_1)\right]\right]^{\frac{1}{2}}\right\} \tag{4.3.80}$$

Cadenillas and Zapatero (1999) assumes that inequalities similar to ineq. (4.3.80) hold, to prove Theorem 4.1 in Cadenillas and Zapatero (1999). We express ineq. (4.3.80) with given parameters. Note that $\Phi'(\check{x}(A_1))>0$ if and only if $\check{x}>\tilde{x}$ implies $A_1>\tilde{A}_1$. Taking $\lim_{A_1\to\tilde{A}_1}$ both sides of ineq. (4.3.80), by (4.3.53), (4.3.55), (4.3.59), (4.3.72) and (A.3) we obtain (4.3.79) is positive.

Therefore, ϕ satisfies ineq. (4.3.5).

(II) Next we examine ineq. (4.3.6). We refer to Theorem 1 of Constantinides and Richard (1978)[1]. We divide the region into $(0,\beta^*)$, $[\beta^*,\overline{x}^*)$ and $[\overline{x}^*,\infty)$.

(i) For $x\in(0,\beta^*)$, by (4.3.77) $\zeta=0$ is optimal. Then, we have

$$\begin{aligned}\mathcal{M}\phi(x)&=\inf_{\zeta\in(0,\beta^*)}\{\phi(\eta(x,\zeta))+K(\zeta)\}\\&=[\phi(\eta(x,\zeta))+K(\zeta)]_{\zeta=0}\\&=\phi(x)+k_0\\&>\phi(x).\end{aligned} \tag{4.3.81}$$

[1]Constantinides and Richard (1978) is extended by Mizuta (2000).

(ii) For $x \in [\beta, \overline{x})$, since equality in (4.3.77) holds at $x = \beta^*$, $\zeta = x - \beta^*$ is optimal. Thus, we obtain

$$\begin{aligned}\mathcal{M}\phi(x) &= \inf_{\zeta\in(0,x-\beta^*]}\{\phi(\eta(x,\zeta)) + K(\zeta)\} \\ &= [\phi(\eta(x,\zeta)) + K(\zeta)]_{\zeta=x-\beta^*} \\ &= \phi(\beta^*) + k_0 + k_1(x-\beta^*) \\ &> \phi(x).\end{aligned} \tag{4.3.82}$$

Inequality holds by eq. (4.3.45) and (4.3.76).

(iii) For $x \in [\overline{x}^*, \infty)$, since equality in (4.3.77) holds at $x = \overline{x}^*$ or β^*. Hence either $\zeta = x - \overline{x}^*$ or $\zeta = x - \beta^*$ is optimal. Thus, we have

$$\begin{aligned}\mathcal{M}\phi(x) &= \min\left[\inf_{\zeta\in(0,x-\overline{x}^*]}\{\phi(\eta(x,\zeta)) + K(\zeta)\},\ \inf_{\zeta\in(x-\overline{x}^*,x)}\{\phi(\eta(x,\zeta)) + K(\zeta)\}\right] \\ &= \min\left[[\phi(\eta(x,\zeta)) + K(\zeta)]_{\zeta=x-\overline{x}^*},\ [\phi(\eta(x,\zeta)) + K(\zeta)]_{\zeta=x-\beta^*}\right] \\ &= \phi(\beta^*) + k_0 + k_1(x-\beta^*) \\ &= \phi(x).\end{aligned} \tag{4.3.83}$$

Here, third equality holds by eq. (4.3.45).

Therefore, ϕ satisfies ineq. (4.3.6).

(III) It follows immediately from foregoing consideration that ϕ also satisfies eq.(4.3.7).

Therefore, ϕ satisfies the QVI. That is, the candidate function is a solution of the QVI. ϕ is also the value function of the agent's problem given by eq. (4.2.14) from Theorem 4.3.1. Furthermore, the QVI policy induced by ϕ is optimal. The proof is completed. □

4.4 Numerical Examples

In this section we calculate A_1^*, $\overline{x}^*$ and β^* by using a numerical method and evaluate the size of optimal implementation, ζ^* and the OEIP. Furthermore, we present a comparative static analysis of ζ^*, β^* and the OEIP by changing parameters, because such evaluation gives us economic intuitions. When the agent decides to implement the EIP, these results are useful.

The base case parameters used are listed in Table 4.1. The results of the numerical examples are presented in Table 4.2. Furthermore, we vary parameters by ±10%. First, we find the results from comparative statics to ζ^*: *The optimal implementation size ζ^* is increasing in the discount rate, r, the diffusion parameter, σ, the proportional cost parameter, k_1 and the fixed cost parameter, k_0. On the other hand, ζ^* is decreasing in the drift parameter, μ and the proportional damage parameter, a.* The meaning is as follows. When the discount rate is high, future damage from the pollutant is more serious. Hence, optimal implementation size increases in the discount rate. Since the diffusion parameter means uncertainty about damage, an increase in uncertainty raises optimal implementation size. Optimal implementation size also increases in the cost of implementing the OEIP.

On the other hand, since the drift parameter is the pollutant's growth rate of the, higher pollutant growth rate decreases optimal implementation size. Similarly, the proportional damage parameter decreases optimal implementation size.

Second, comparative static analysis of the OEIP gives us the following results: *An increase in the growth rate of the pollutant, uncertainty, the proportional cost and the fixed cost raises the value of the OEIP, while an increase in the discount rate and the proportional damage parameter decrease the value of the OEIP.* These results have the following implications. Since γ_1, A_1^* and $\overline{x}^*$ affect the OEIP, the comparative static analysis of the OEIP is complicated. Although γ_1, A_1^* and $\overline{x}^*$ do not have the same effect of a change in the parameters, we have plausible results. The OEIP increases in the growth rate of the pollutant, uncertainty, proportional implementation cost and fixed implementation cost, while the OEIP decreases in the discount rate and the proportional damage parameter. Since the OEIP leads to flexibility for the agent's decision, from Remark 4.3.1, higher uncertainty raises the value of the OEIP. The effects of proportional and fixed implementation costs have the same direction.

4.5 Conclusion

In this chapter, we study general environmental improvement policy by using an impulse control method. We present some numerical examples and comparative static results for the OEIP. They give us some economic implications. The main results are as follows. An increase in the growth rate of the pollutant, uncertainty, proportional cost and constant cost raises the value of the OEIP.

The paper considers one pollutant for simplicity. However there are many pollutants in the real world. Thus, it is important to extend our study to consider a model with many pollutants. The other interesting extensions are to consider technological progress that improves the environment, and to generalize the damage function, for example $D(x) := ax^\rho$, $\rho > 0$, and nonlinear implementation cost functions. As a first step, Ohnishi and Tsujimura (2002b) investigate quadratic type implementation cost functions.

Table 4.1: Base Case Parameters

r	μ	σ	a	b	c
0.06	0.01	0.15	0.25	5	5

Table 4.2: The results of numerical examples.

	γ_1	$-A_1$	$\overline{x}$	β	ζ^*	OEIP
base case	2.3656	8.3385	2.1946	0.3217	1.8729	53.5330
$r: +10\%$	2.4783	5.1543	2.3255	0.3769	1.9486	41.7354
$: -10\%$	2.2472	15.2801	2.0599	0.2675	1.7924	77.5109
$\mu: +10\%$	2.3205	10.0666	2.1727	0.3077	1.8650	60.9423
$: -10\%$	2.4116	6.9882	2.2167	0.3360	1.8808	47.6500
$\sigma: +10\%$	2.2363	13.6144	2.2634	0.3122	1.9513	84.6020
$: -10\%$	2.5178	5.5399	2.1256	0.3314	1.7942	36.9834
$a: +10\%$	–	9.4114	2.0611	0.2893	1.7717	52.0798
$: -10\%$	–	7.2921	2.3538	0.3618	1.9920	55.2498
$b: +10\%$	–	8.1951	2.2657	0.3617	1.9039	56.7300
$: -10\%$	–	8.4902	2.1238	0.2827	1.8411	50.4375
$c: +10\%$	–	8.2628	2.2672	0.3183	1.9489	57.2877
$: -10\%$	–	8.4206	2.1184	0.3256	1.7928	49.7248

Notes: γ_1 is independent of parameters a, b, and c.
OEIP is calculated by $-A_1\overline{x}^{\gamma_1}$.

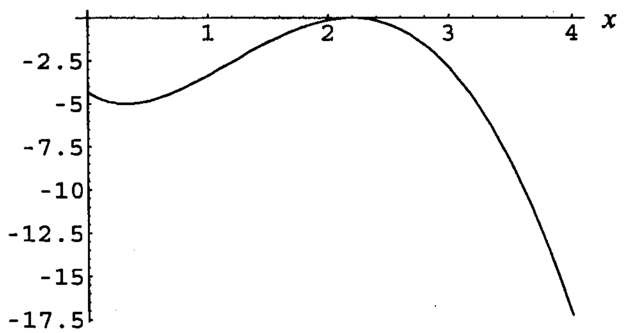

Figure 4.1: $\Phi(x)$

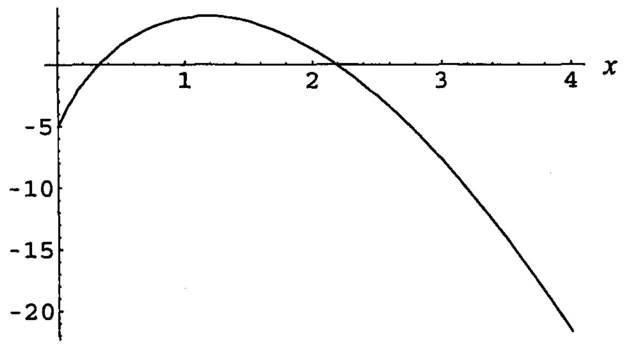

Figure 4.2: $\Phi'(x)$

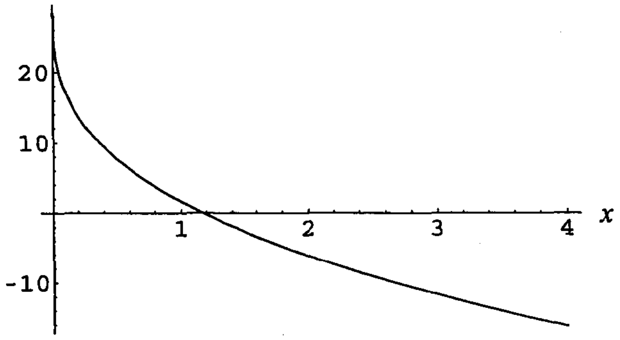

Figure 4.3: $\Phi''(x)$

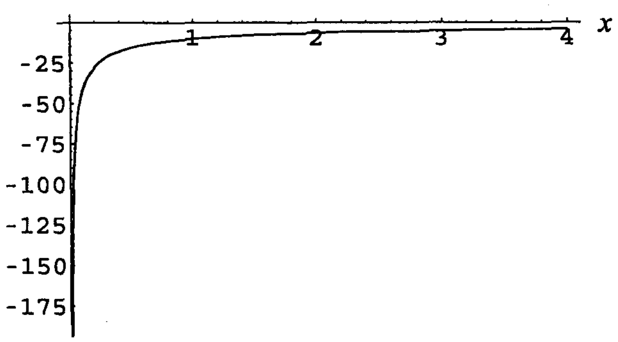

Figure 4.4: $\Phi'''(x)$

Chapter 5

Optimal Dividend Policy with Transaction Costs under a Brownian Cash Reserve

5.1 Introduction

In recent times many financial decision making and optimization problems with transaction costs have been studied in the fields of mathematical finance, optimal portfolio management and contingent claim analysis. We observe two types of transaction costs in actual financial markets. One set of costs is proportional to the amount or volume of transactions, and the other is independent of them. When an agent faces some financial decision making and optimization problem with these transaction costs under uncertainty, he/she could formulate and solve the problem through various methods in stochastic control theory. If the agent faces proportional transaction costs only, he/she could solve the problem by using a singular stochastic control method; see, for example, Chapter 8 of Fleming and Soner (1993) concerning the theory of singular stochastic control. On the other hand, if the agent faces fixed transaction costs or both fixed and proportional transaction costs, the problem could be approached via a stochastic impulse control method; see, for example, Bensoussan and Lions (1984) concerning the theory of stochastic impulse control. Until now, various financial problems in the presence of these two types of transaction costs have been studied such as consumption/investment problems. The study of consumption/investment problems using continuous stochastic processes was initiated by Merton (1969, 1971). Although Merton's work does not deal with transaction costs, a number of studies have extended Merton's work by considering transaction costs. An excellent survey of stochastic models of consumption/investment problems with transaction costs can be found in Cadenillas (2000). Stochastic singular control can be used to solve consumption/investment problems with proportional transaction costs. See, for example, Davis and Norman (1990). For stochastic singular control, see, for example, Fleming and Soner (1993). However, impulse control is needed to solve consumption/investment problems with fixed and proportional transaction costs.

The first application of impulse control theory to a consumption/investment problem was studied by Eastham and Hastings (1988). Hastings (1992) extends Eastham and Hastings (1988) by incorporating a risky asset that follows a jump-diffusion process. These two studies investigated the problem in the context of a finite time horizon, while Chapter 6.3 of Korn (1997), Korn (1998), and Section 5 of Cadenillas (2000) study the problem in the context of an infinite time horizon. Section 5 of Korn (1998) and Section 3.iii of Korn (1999) study the problem of maximizing the expected utility of terminal wealth. Øksendal and Sulem (2002) allow consumption to take place at any time, i.e., they combined stochastic and impulse control. These studies investigated the behavior of an investor who trades in one risky asset (or stock) and one risk-free asset (or bond).

This chapter examines a dividend policy problem with fixed and proportional transaction costs. Stochastic models of the optimal dividend policy have been in use since the middle of the 1990s. They assume that a firm has a stochastic cash flow and optimizes the timing of dividend payouts and their amounts. See, for example, Jeanblanc-Picqué and Shiryaev (1995), Radner and Shepp (1996), Asmussen and Taksar (1997), Højgaard and Taksar (1999, 2001), and Asmussen, Højgaard, and Taksar (2000). An excellent survey of stochastic models of the optimal dividend policy can be found in Taksar (2000).

We assume that the firm's cash reserve is governed by a Brownian motion with a drift, and that, when the firm pays out dividends, it incurs both fixed and proportional transaction costs. For example, the fixed transaction costs could be the costs needed for the firm's decision making, while the proportional transaction costs may be taxes. Further, we suppose that the firm goes bankrupt when the cash reserve falls to zero. The firm's problem is to maximize the expected total discounted dividends paid out to stockholders. To this end, we formulate it as a stochastic impulse control problem, which is approached via quasi-variational inequalities (QVI). We naturally guess that, under an optimal dividend policy, the firm pays out a fixed dividend whenever the cash reserve reaches a predetermined threshold. More precisely, we conjecture that, under a suitable set of sufficient conditions on the given problem parameters, an optimal dividend policy is in the following form specified by two critical cash reserve levels: whenever the cash reserve reaches a predetermined level, $\overline{x}$, the firm pays out a fixed dividend, so that it instantaneously reduces to another predetermined level, β. Namely, the firm always pays out a fixed dividend, $\overline{x} - \beta$, at each (random) dividend time. In order to show the validity of this conjecture, we take three steps in this chapter. We first prove that a QVI policy, introduced later, is an optimal dividend policy for the firm's problem. Next we prove that there exist uniquely the parameters, $\overline{x}$ and β (and another unknown parameter A) by examining a system of simultaneous equations, which are the well-known value matching and smooth pasting conditions; see, for example, Dixit and Pindyck (1994). Then we verify a corresponding candidate function that satisfies the QVI for the firm's problem under an additional condition, that bounds the threshold level, $\overline{x}$ from below. That is, the guessed candidate function is the optimal value function for the firm's problem, so that the above dividend policy induced by it is indeed optimal. Finally, we present some numerical examples and comparative static results for the amount of the optimal dividend and the expected duration between the successive optimal dividend times, which are summarized as follows. Increases in uncertainty and of both the fixed and proportional transaction costs raise the amount of the optimal dividend, and hasten the expected optimal dividend time.

Related papers are as follows. Radner and Shepp (1996)[1], Asmussen and Taksar (1997), and Jeanblanc-Picqué and Shiryaev (1995) examine optimal dividend problems under a Brownian cash reserve, and obtain optimal dividend policy: whenever the cash reserve reaches a threshold, the firm pays out dividends. Radner and Shepp (1996) and Asmussen and Taksar (1997) consider optimal dividend problems without transaction costs. On the other hand, Jeanblanc-Picqué and Shiryaev (1995) examine optimal dividend policies under three types of dividend process: (A) the dynamics of the dividend process, $Z = \{Z_t\}$, follows $dZ_t = u(X_t)dt$, $Z_0 = Z_0(X_0)$, where u and Z_0 are arbitrary measurable functions with upper bounds, and X_t represents the cash reserve at t; (B) the dividend process is accumulated by each dividend at each dividend time according to eq. (5.2.1) in this chapter; (C) the process is an arbitrary, non-negative, non-decreasing, non-anticipating, and right continuous process. Because Jeanblanc-Picqué and Shiryaev (1995) also considers a fixed transaction cost in case (B), case (B) is the one related to this chapter. In order to deal with this case, Jeanblanc-Picqué and Shiryaev (1995) also use a stochastic impulse control approach. Then, the optimal value function and the corresponding optimal control are found. They also solve a system of simultaneous equations to find three unknown parameters which correspond to A, $\overline{x}$, and β in this chapter, and show that the parameters can be found uniquely by solving the simultaneous equations. The differences between Jeanblanc-Picqué and Shiryaev (1995) and this chapter are as follows. Recall that when the cash reserve reaches the threshold, the firm pays out dividends. After paying out dividends the firm's new problem starts from another initial cash reserve level, β. Then, the candidate function changes into the sum of the candidate function of the new cash reserve level and the net dividends. By using this, Jeanblanc-Picqué and Shiryaev (1995) show that when the cash reserve exceeds the threshold, the candidate function satisfies the inequality corresponding to ineq. (5.3.5), which is one of the QVI. On the other hand, we verify that the candidate function requires an additional condition to satisfy ineq. (5.3.5). The condition gives us a lower bound for the threshold, $\overline{x}$, and is expressed only by the given parameters of the problem. See the assumption (AS.5.3.1) of Theorem 5.3.3 and Remark 5.3.1 in Section 5.3. Jeanblanc-Picqué and Shiryaev (1995) also show that the candidate function satisfies the inequality corresponding to ineq. (5.3.6), which is another one of the QVI, by using the value matching and smooth pasting conditions. On the other hand, we shows that the candidate function satisfies the ineq. (5.3.6) by using a different method. We divide the cash reserve into regions, and verify that the candidate function satisfies ineq. (5.3.6) for each region. Furthermore, we present numerical examples and analyze the comparative statics for the amount of the optimal dividend and the optimal expected dividend time.

This chapter is organized as follows. The next section describes the firm's problem. Section 3 is devoted to its analysis. Section 4 concludes this chapter. An Appendix is given in the last section.

[1] Radner and Shepp (1996) carefully discusses the reason why the cash reserve is governed by a Brownian motion in Section 1. Furthermore, it is shown that if the accumulated net revenue follows a geometric Brownian motion, the model defined in this chapter leads to an empty problem, and all the profits of the firm are drawn at once.

5.2 The Model

We assume that a firm's accumulated net profits are governed by a Brownian motion with drift as in Radner and Shepp (1996). Let $(W_t)_{t\geq 0}$ be a Brownian motion process given on a filtered probability space, $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$, satisfying the usual conditions. Here $\mathcal{F}_t$ is generated by W_t in $\mathbb{R}$, i.e., $\mathcal{F}_t = \sigma(W_s, s \leq t)$. The firm pays out dividends by $\zeta (\in \mathbb{R}_+)$ at each dividend time. Let Z_t denote the total dividends up to time t and be defined by

$$Z_t = \sum_{i\geq 0} \zeta_i 1_{\{\tau_i \leq t\}}, \tag{5.2.1}$$

where τ_i is ith dividend time such that $\tau_i \to +\infty$ as $i \to +\infty$ a.s. We put $\tau_0 - := 0$ and $\tau_{i+1}- = \tau_i$ if $\tau_{i+1} = \tau_i$. ζ_i represents the ith dividend and is non-negative for all i. These τ_i and ζ_i correspond, respectively, to the stopping times and impulses in impulse control theory. A dividend policy is defined as the following double sequence:

$$v := \{(\tau_i, \zeta_i)\}_{i\geq 0}. \tag{5.2.2}$$

The remainder is accumulated in a cash reserve. If the dividend policy v is given by (5.2.2), then the cash reserve of the firm, $X^{x,v} := (X_t^{x,v})_{t\geq 0}$ is given by

$$\begin{cases} dX_t^{x,v} = \mu dt + \sigma dW_t - dZ_t, & 0 \leq t \leq T; \\ X_{0-}^{x,v} = x, \end{cases} \tag{5.2.3}$$

where $\mu (\in \mathbb{R})$ and $\sigma (\in \mathbb{R})$ are constants, and T represents a bankruptcy time and is defined by

$$T = \inf\{t > 0; X^{x,v} \leq a\}, \tag{5.2.4}$$

where $a (\in \mathbb{R})$ is a critical level at which the firm becomes bankrupt. The critical level, a, acts as an absorbing barrier for X_t. Once the cash reserve decreases to the critical level, a, the firm cannot receive more investment and becomes bankrupt. In this chapter we assume that $a = 0$ without loss of generality. Furthermore, we assume that when the firm becomes bankrupt, the investors do not need to pay a penalty and the illiquid assets of the firm have no salvage value. Following Definition 1.0.1, we define the set of admissible dividend policy as follows:

Definition 5.2.1 (Admissible Dividend Policy). *A dividend policy, v, is admissible, if the following are satisfied:*

$$0 \leq \tau_i \leq \tau_{i+1}, \quad a.s. \quad i \geq 0; \tag{5.2.5}$$

$$\tau_i \textit{ is an } (\mathcal{F}_t)_{t\geq 0}\textit{-stopping time}, \quad i \geq 0; \tag{5.2.6}$$

$$\zeta_i \textit{ is } \mathcal{F}_{\tau_i}\textit{-measurable}, \quad i \geq 0; \tag{5.2.7}$$

$$P\left[\lim_{i\to\infty} \tau_i \leq \tilde{T}\right] = 0, \quad \forall \tilde{T} \in [0, \infty). \tag{5.2.8}$$

The condition given by eq. (5.2.8) means that dividend policies will only occur finitely before a terminal time, $\tilde{T}$. Let $\mathcal{U}$ denote the set of admissible dividend policies.

Let $K : \mathbb{R}_+ \to \mathbb{R}_{++}$ represent net dividends defined by

$$K(\zeta) := k_1\zeta - k_0, \tag{5.2.9}$$

where $(1-k_1)(\in (0,1))$ is proportional to the transaction costs and $k_0(\in \mathbb{R}_{++})$ is a fixed transaction cost. Note that $K(\zeta)$ satisfies superadditivity with respect to ζ:

$$K(\zeta+\zeta') \geq K(\zeta) + K(\zeta'), \quad \zeta, \zeta' \in \mathbb{R}_+. \tag{5.2.10}$$

This implies that reasonable $(\mathcal{F}_t)_{t\geq 0}$-stopping times become strictly increasing sequences, i.e., $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_i < \cdots$. The expected total discounted dividends function associated with the dividend policy $v \in \mathcal{U}$ is defined by

$$J(x;v) = \mathbb{E}\left[\sum_{i=1}^{\infty} e^{-r\tau_i} K(\zeta_i) 1_{\{\tau_i < T\}}\right], \tag{5.2.11}$$

where $r(\in \mathbb{R}_{++})$ is a discount factor. We assume the following:
(AS.5.2.1)

$$r > \mu.$$

Therefore the firm's problem is to choose $v \in \mathcal{U}$ in order to maximize $J(x;v)$:

$$V(x) = \sup_{v\in\mathcal{U}} J(x;v) = J(x;v^*), \tag{5.2.12}$$

where V is the value function of the firm's problem eq. (5.2.12) and v^* is an optimal dividend policy.

5.3 Analysis

In the previous section, we formulated the firm's problem as a stochastic impulse control problem. From that formulation, we naturally guess that, under an optimal dividend policy, the firm pays out a fixed dividend whenever the cash reserve reaches a threshold. In order to verify this conjecture, we take three steps in this section. We first prove that a QVI policy, which is introduced later, is an optimal dividend policy for the firm's problem (5.2.12). Next we prove that there uniquely exist the solutions of a system of simultaneous equations, which are the well-known value matching and smooth pasting conditions. Then we verify that a corresponding candidate function satisfies the QVI for the firm's problem under an additional condition, which bounds the threshold level from below. That is, the guessed candidate function is the optimal value function for the firm's problem (5.2.12), so that the dividend policy induced by the candidate function is indeed optimal. To take these three steps, we first introduce some notations.

Suppose that $\phi : \mathbb{R}_{++} \to \mathbb{R}$ is a continuous function. Let $\mathcal{M}$ denote the dividend operator on the space of functions ϕ defined by

$$\mathcal{M}\phi(x) = \sup_{\zeta\in[0,x)} \{\phi(\eta(x,\zeta)) + K(\zeta)\}, \tag{5.3.1}$$

where $\eta(x, \zeta)$ is a new cash reserve value after paying out dividends. We assume that ϕ is a twice continuously differentiable function on $\mathbb{R}$, C^2, except on the boundary of the considered region. Let us define an operator $\mathcal{L}$ of the X^v by

$$\mathcal{L}\phi(x) = \frac{1}{2}\sigma^2\phi''(x) + \mu\phi'(x) - r\phi(x). \tag{5.3.2}$$

Since ϕ is not necessarily C^2 in the whole region, we must apply the generalized Dynkin formula for ϕ. The formula is available if ϕ is stochastically C^2. That is, we use the generalized Dynkin formula on the set which has a Green measure of X^v zero. Here the Green measure of X^v is the expected total occupation measure $\mathbb{G}(\cdot, x; v)$ defined by

$$\mathbb{G}(\Xi, x; v) = \mathbb{E}\left[\int_0^\infty X_t^{x,v} 1_\Xi dt\right], \tag{5.3.3}$$

where 1_Ξ is the indicator of a Borel set $\Xi(\subset \mathbb{R}_{++})$. A continuous function $\phi : \mathbb{R}_{++} \to \mathbb{R}$ is called *stochastically* C^2 with respect to $X^{x,v}$ if $\mathcal{L}\phi(x)$ is well defined pointwise for almost all x with respect to the Green measure $\mathbb{G}(\cdot, x; v)$. Hereafter we assume that ϕ is stochastically C^2 with respect to $X^{x,v}$. The following equality, which is called the *generalized Dynkin formula*, will be used in the proof of Theorem 5.3.1.

$$\mathbb{E}[e^{-r\theta-}\phi(X_{\theta-}^{x,v})] = \mathbb{E}[e^{-r\tau_i}\phi(X_{\tau_i}^{x,v})] + \mathbb{E}\left[\int_{\tau_i}^{\theta-} e^{-rt}\mathcal{L}\phi(X_t^{x,v})dt\right], \tag{5.3.4}$$

for all i, all bounded stopping times θ such that $\tau_i \leq \theta \leq \tau_{i+1}$.

Definition 5.3.1 (QVI). *The following relations are called the QVI for the firm's problem (5.2.12):*

$$\mathcal{L}\phi(x) \leq 0, \quad \text{for a.a. } x \text{ w.r.t. } \mathbb{G}(\cdot, x; v),\ \forall v \in \mathcal{U}; \tag{5.3.5}$$

$$\phi(x) \geq \mathcal{M}\phi(x); \tag{5.3.6}$$

$$[\mathcal{L}\phi(x)][\phi(x) - \mathcal{M}\phi(x)] = 0, \quad \text{for a.a. } x \text{ w.r.t. } \mathbb{G}(\cdot, x; v),\ \forall v \in \mathcal{U}. \tag{5.3.7}$$

.

Definition 5.3.2 (QVI policy). *Let ϕ be a solution of the QVI. Then, the following dividend policy $\hat{v} = \{(\hat{\tau}_i, \hat{\zeta}_i)\}_{i\geq 0}$ is called a QVI policy:*

$$(\hat{\tau}_0, \hat{\zeta}_0) = (0, 0); \tag{5.3.8}$$

$$\hat{\tau}_i = \inf\{t \geq \hat{\tau}_{i-1}; X_t^{x,\hat{v}} \notin H\}; \tag{5.3.9}$$

$$\hat{\zeta}_i = \arg\max\{\phi(\eta(X_{\hat{\tau}_i-}^{x,\hat{v}}, \zeta_i)) + K(\zeta_i); \zeta\}. \tag{5.3.10}$$

In this context, H is the continuation region defined by

$$H := \{x; \phi(x) > \mathcal{M}\phi(x)\}, \tag{5.3.11}$$

and $X_t^{x,\hat{v}}$ is the result of applying the dividend policy $\hat{v} = \{(\hat{\tau}_i, \hat{\zeta}_i)\}_{i\geq 0}$.

Now we are in a position to prove that a QVI policy is an optimal dividend policy. The following Theorem 5.3.1 is a minor modification of Theorem 3.1 of Brekke and Øksendal (1998). Because Brekke and Øksendal (1998) discuss combined continuous control and impulse control, we cannot apply their result directly to our problem. So, we present a modified version of theirs:

Theorem 5.3.1. *(I) Let $\phi : \mathbb{R}_{++} \to \mathbb{R}$ be a solution of the QVI and satisfy the following:*

$$\phi \text{ is stochastically } C^2 \text{ w.r.t. } X^{x,v}; \tag{5.3.12}$$

$$\lim_{t\to\infty} e^{-rt}\phi(X_t^{x,v}) = 0, \quad a.s. \quad \forall v \in \mathcal{U}; \tag{5.3.13}$$

$$\text{the family } \{\phi(X_\tau^{x,v})\}_{\tau<\infty} \text{ is uniformly integrable w.r.t. } P \quad \forall v \in \mathcal{U}. \tag{5.3.14}$$

Then, we obtain

$$\phi(x) \geq J(x;v) \quad \forall v \in \mathcal{U}. \tag{5.3.15}$$

(II) From (5.3.5) - (5.3.7) and (5.3.11), we have

$$\mathcal{L}\phi(x) = 0, \quad x \in H. \tag{5.3.16}$$

Suppose $\hat{v} \in \mathcal{U}$, i.e., the dividend policy is admissible. Then, we obtain

$$\phi(x) = J(x;\hat{v}). \tag{5.3.17}$$

Hence we have

$$\phi(x) = V(x) = J(x;\hat{v}), \tag{5.3.18}$$

and therefore $\hat{v}$ is optimal.

Proof. (I) Assume that ϕ satisfies (5.3.12) - (5.3.14). Choose $v \in \mathcal{U}$. Let $\theta_{i+1} := \tau_i \vee (\tau_{i+1} \wedge s)$ for any $s \in \mathbb{R}_+$. Then, by the generalized Dynkin formula, (5.3.4), we obtain

$$\mathbb{E}[e^{-r\theta_{i+1}-}\phi(X^{x,v}_{\theta_{i+1}-})] = \mathbb{E}[e^{-r\tau_i}\phi(X^{x,v}_{\tau_i})] + \mathbb{E}\left[\int_{\tau_i}^{\theta_{i+1}-} e^{-rt}\mathcal{L}\phi(X_t^{x,v})dt\right]. \tag{5.3.19}$$

Hence from (5.3.5) we obtain

$$\mathbb{E}[e^{-r\theta_{i+1}-}\phi(X^{x,v}_{\theta_{i+1}-})] \leq \mathbb{E}[e^{-r\tau_i}\phi(X^{x,v}_{\tau_i})]. \tag{5.3.20}$$

Taking $\lim_{s\to\infty}$ we have by the dominated convergence theorem

$$\mathbb{E}[e^{-r\tau_{i+1}-}\phi(X^{x,v}_{\tau_{i+1}-})] \leq \mathbb{E}[e^{-r\tau_i}\phi(X^{x,v}_{\tau_i})]. \tag{5.3.21}$$

Summing from $i = 0$ to $i = m$ yields

$$\phi(x) + \sum_{i=1}^{m} \mathbb{E}[e^{-r\tau_i}\phi(X_{\tau_i}^{x,v}) - e^{-r\tau_i-}\phi(X_{\tau_i-}^{x,v})] \geq \mathbb{E}[e^{-r\tau_{m+1}-}\phi(X_{\tau_{m+1}-}^{x,v})]. \quad (5.3.22)$$

Since after paying out dividends the cash reserve jumps immediately from $X_{\tau_i-}^{x,v}$ to a new cash reserve level $\eta(X_{\tau_i-}^{x,v}, \zeta_i)$ for all $\tau_i < T$, by eq. (5.3.1) and $\eta(X_{\tau_i-}^{x,v}, \zeta_i) = X_{\tau_i}^{x,v}$ we obtain

$$\phi(\eta(X_{\tau_i-}^{x,v}, \zeta_i)) \leq \mathcal{M}\phi(X_{\tau_i-}^{x,v}) - K(\zeta_i). \quad (5.3.23)$$

And if $\tau_i = T$, then we have

$$\phi(X_{\tau_i}^{x,v}) = 0. \quad (5.3.24)$$

Thus ineq. (5.3.22) becomes

$$\phi(x) + \sum_{i=1}^{m} \mathbb{E}[[e^{-r\tau_i}\mathcal{M}\phi(X_{\tau_i-}^{x,v}) - e^{-r\tau_i-}\phi(X_{\tau_i-}^{x,v})]1_{\{\tau_i<T\}}]$$
$$\geq \mathbb{E}\left[\sum_{i=1}^{m} e^{-r\tau_i}K(\zeta_i)1_{\{\tau_i<T\}} + e^{-r\tau_{m+1}-}\phi(X_{\tau_{m+1}-}^{x,v})\right]. \quad (5.3.25)$$

It follows from eq. (5.3.6) that

$$\mathcal{M}\phi(X_{\tau_i-}^{x,v}) - \phi(X_{\tau_i-}^{x,v}) \leq 0. \quad (5.3.26)$$

Ineqs. (5.3.25) and (5.3.26) yields

$$\phi(x) \geq \mathbb{E}\left[\sum_{i=1}^{m} e^{-r\tau_i}K(\zeta_i)1_{\{\tau_i<T\}} + e^{-r\tau_{m+1}-}\phi(X_{\tau_{m+1}-}^{x,v})\right]. \quad (5.3.27)$$

Taking $\lim_{m\to\infty}$ and using (5.3.13), (5.3.14) and the dominated convergence theorem we obtain

$$\phi(x) \geq \mathbb{E}\left[\sum_{i=1}^{\infty} e^{-r\tau_i}K(\zeta_i)1_{\{\tau_i<T\}}\right]. \quad (5.3.28)$$

Therefore eq. (5.3.15) is proved.

(II) Assume that eq. (5.3.16) holds and $\hat{v} = \{(\hat{\tau}_i, \hat{\zeta}_i)\}_{i\geq 1}$ is the QVI policy. Then, repeat the argument in part (I) for $v = \hat{v}$. Then, all the ineqs. (5.3.20) – (5.3.28) except eq. (5.3.24) become equalities. Thus we obtain

$$\phi(x) = \mathbb{E}\left[\sum_{i=1}^{\infty} e^{-r\hat{\tau}_i}K(\hat{\zeta}_i)1_{\{\hat{\tau}_i<T\}}\right]. \quad (5.3.29)$$

Hence we obtain eq. (5.3.17). Combining eq. (5.3.17) with ineq. (5.3.15), we obtain

$$\phi(x) \geq \sup_{v\in\mathcal{U}} J(x; v) \geq J(x; \hat{v}) = \phi(x). \quad (5.3.30)$$

Therefore $\phi(x) = V(x)$ and $v^* = \hat{v}$ is optimal, i.e., the solution of the QVI is the value function and the QVI policy is optimal.

□

Next, from the firm's problem (5.2.12), we conjecture that, under a suitable set of sufficient conditions on the given problem parameters, an optimal dividend policy is specified in the following form by two critical cash reserve levels: whenever the cash reserve reaches a level, $\overline{x}$, the firm pays out a fixed dividend, so that it instantaneously reduces to another level, β. Hence, the firm always pays out a fixed dividend, $\overline{x}-\beta$, at each (random) dividend time, τ_i. In order to show the validity of this conjecture, we first prove that there uniquely exist the parameters, $\overline{x}$ and β (and another unknown parameter A) by examining a system of simultaneous equations, which are the well-known value matching and smooth pasting conditions.

Let an optimal dividend policy be denoted by $v^*=(\tau^*,\zeta^*)$, characterized by parameters $(\beta,\overline{x})$ with $0<\beta<\overline{x}<\infty$ such that

$$\tau_i^* := \inf\{t>\tau_{i-1}^*; X_{t-}^{x,v^*} \notin (0,\overline{x})\}, \tag{5.3.31}$$

$$\zeta_i^* := X_{\tau_i^*-}^{x,v^*} - X_{\tau_i^*}^{x,v^*} = \overline{x}-\beta. \tag{5.3.32}$$

Eq. (5.2.3) becomes

$$\begin{cases} dX_t^{x,v^*} = \mu dt + \sigma dW_t - dZ_t, & 0 \le t \le T; \\ X_{\tau_0-}^{x,v^*} = x. \end{cases} \tag{5.3.33}$$

Therefore, when the firm pays out dividends, the value function seems to satisfy

$$\begin{aligned} V(x) &= V(\eta(x,\zeta^*)) + K(\zeta^*) \\ &= V(\beta) + k_1(x-\beta) - k_0, \quad x \in [\overline{x},\infty). \end{aligned} \tag{5.3.34}$$

Assume that V is stochastically C^2 with respect to X^{x,v^*}. Under the assumed optimal policy, if the initial level of the cash reserve x is $x=\overline{x}+\varepsilon$, where $\varepsilon \in \mathbb{R}_{++}$, then the optimal amount of dividends is $\zeta^*=(\overline{x}+\varepsilon)-\beta$. Thus we have

$$V(\overline{x}+\varepsilon) = V(\beta) + k_1((\overline{x}+\varepsilon)-\beta) - k_0. \tag{5.3.35}$$

Substituting x into $\overline{x}$ in eq. (5.3.34) and subtracting from eq. (5.3.35), we obtain

$$V(\overline{x}+\varepsilon) - V(\overline{x}) = k_1\varepsilon. \tag{5.3.36}$$

Dividing eq. (5.3.36) by ε and taking $\lim_{\varepsilon\to 0}$ in eq. (5.3.36), we obtain

$$V'(\overline{x}) = k_1. \tag{5.3.37}$$

By (5.3.31) and (5.3.32), the expected total discounted dividends function, J^v, is maximized at $\zeta^*=\overline{x}-\beta$. Hence, by the first order condition for the maximization $\partial[V(\eta(\overline{x},\zeta))+K(\zeta)]/\partial\zeta\mid_{\zeta=\overline{x}-\beta}=0$ we obtain

$$V'(\beta) = k_1, \tag{5.3.38}$$

because V is stochastically C^2. Furthermore, we can conjecture that eq. (5.3.16) holds in the continuation region $(0, \overline{x})$. Following the standard methods of ordinary differential equations, the general solution of eq (5.3.16) is in the form given by

$$\phi(x) = A_1 e^{\gamma_1 x} + A_2 e^{\gamma_2 x}, \quad x \in (0, \overline{x}), \tag{5.3.39}$$

where A_1 and A_2 are constants to be determined, and γ_1 and γ_2 are the solutions to the characteristic equation:

$$\frac{1}{2}\sigma^2\gamma^2 + \mu\gamma - r = 0, \quad \gamma \in \mathbb{R}. \tag{5.3.40}$$

Eq. (5.3.40) has two distinct real roots:

$$\gamma_1 = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2 r}}{\sigma^2}, \tag{5.3.41}$$

$$\gamma_2 = \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2 r}}{\sigma^2}. \tag{5.3.42}$$

Since we have $\phi(0) = 0$, we obtain $A_1 + A_2 = 0$ from eq. (5.3.39). It follows that

$$\phi(x) = A_1 e^{-\alpha x}(e^{\delta x} - e^{-\delta x}), \quad x \in (0, \overline{x}), \tag{5.3.43}$$

where $\alpha := \mu/\sigma^2$ and $\delta := \sqrt{\mu^2 + 2\sigma^2 r}/\sigma^2$. Rearranging eq. (5.3.43), we obtain

$$\phi(x) = A e^{-\alpha x} \sinh(\delta x), \quad x \in (0, \overline{x}), \tag{5.3.44}$$

where $A := 2A_1$.

Let us define $\phi(x; A, b, \beta)$ by

$$\phi(x; A, b, \beta) := \begin{cases} A e^{-\alpha x} \sinh(\delta x), & x \in (0, \overline{x}), \\ \phi(\beta) + k_1(x - \beta) - k_0, & x \in [\overline{x}, \infty). \end{cases} \tag{5.3.45}$$

Hereafter we suppress A, $\overline{x}$, and β in ϕ for tractability unless we need to pay attention to these parameters. We draw ϕ and ϕ' in Figures 5.1 and 5.2, respectively.[2] A, $\overline{x}$, and β are parameters that are uniquely determined by the following simultaneous equations. These equations are the well-known *value matching* and *smooth pasting conditions*:

$$\phi(\overline{x}) = \phi(\beta) + k_1(\overline{x} - \beta) - k_0, \tag{5.3.46}$$

$$\phi'(\overline{x}) = k_1, \tag{5.3.47}$$

$$\phi'(\beta) = k_2. \tag{5.3.48}$$

[2]We have used Mathematica to draw the figures.

In order to verify that there uniquely exist the solutions, A, $\overline{x}$, and β, of the simultaneous equations (5.3.46) – (5.3.48), we investigate how $\overline{x}$ and β depend on A. We will refer to the method of Øksendal (1999). From (5.3.46) let us define $\Phi(x)$ as

$$\Phi(x) := A[e^{-\alpha x}\sinh(\delta x) - e^{-\alpha\beta}\sinh(\delta\beta)] - k_1(x-\beta) + k_0, \tag{5.3.49}$$

where we fix $A \in \mathbb{R}_{++}$. See Figure 5.3. The first, second and third derivatives of $\Phi(x)$ are respectively as follows:

$$\Phi'(x) = Ae^{-\alpha x}[-\alpha\sinh(\delta x) + \delta\cosh(\delta x)] - k_1. \tag{5.3.50}$$

$$\Phi''(x) = Ae^{-\alpha x}[(\alpha^2+\delta^2)\sinh(\delta x) - 2\alpha\delta\cosh(\delta x)]. \tag{5.3.51}$$

$$\Phi'''(x) = Ae^{-\alpha x}[-\alpha(\alpha^2+3\delta^2)\sinh(\delta x) + \delta(3\alpha^2+\delta^2)\cosh(\delta x)]. \tag{5.3.52}$$

See Figures 5.4 – 5.6, respectively. It is obvious that $\Phi'''(x)$ is positive from (AS.5.2.1), $A \in \mathbb{R}_{++}$ and $0 < \alpha < \delta$. Hence, $\Phi'(x)$ has its unique minimum point at $\breve{x}$. $\Phi''(\breve{x}) = 0$ yields

$$\breve{x} = \frac{\operatorname{arctanh}(\frac{2\alpha\delta}{\alpha^2+\delta^2})}{\delta}. \tag{5.3.53}$$

$\Phi'(\breve{x}) < 0$ leads to

$$\begin{aligned} A &< \frac{e^{\alpha\breve{x}}k}{-\alpha\sinh(\delta\breve{x}) + \delta\cosh(\delta\breve{x})} \\ &=: \breve{A}. \end{aligned} \tag{5.3.54}$$

For any $0 < A < \breve{A}$, eqs. (5.3.47) and (5.3.48) have two solutions $\overline{x}(A)$ and $\beta(A)$ such that $0 < \beta(A) < \breve{x} < \overline{x}(A)$. Hereafter we assume that $0 < A < \breve{A}$.

From the above preliminary paragraph we examine how $\overline{x}$ and β depend on A. We first differentiate eq. (5.3.47) with respect to A and obtain

$$\begin{aligned} \overline{x}'(A) &= \frac{e^{-\alpha\overline{x}(A)}[\alpha\sinh(\delta\overline{x}(A)) - \delta\cosh(\delta\overline{x}(A))]}{Ae^{-\alpha\overline{x}(A)}[(\alpha^2+\delta^2)\sinh(\delta\overline{x}(A)) - 2\alpha\delta\cosh(\delta\overline{x}(A))]} \\ &= \frac{-\phi'(\overline{x}(A))}{A\Phi''(\overline{x}(A))}. \end{aligned} \tag{5.3.55}$$

The second equality holds from eqs. (5.3.44) and (5.3.51). It is obvious that the sign of $\overline{x}'(A)$ is negative. This means that $\overline{x}(A)$ decreases in A. Rewriting eq. (5.3.47) gives

$$Ae^{-\alpha\overline{x}(A)}[\delta\cosh(\delta\overline{x}(A)) - \alpha\sinh(\delta\overline{x}(A))] = k_1. \tag{5.3.56}$$

If $\overline{x}(A)$ does not go to infinity as A goes to zero, then eq. (5.3.56) does not hold. Thus, it follows that

$$\lim_{A\downarrow 0}\overline{x}(A) = +\infty. \tag{5.3.57}$$

On the other hand, it follows from eq. (5.3.53) and (5.3.54) that

$$\lim_{A\uparrow\check{A}} \overline{x}(A) = \check{x}. \tag{5.3.58}$$

Similarly we obtain

$$\begin{aligned}\beta'(A) &= \frac{e^{-\alpha\beta(A)}[\alpha\sinh(\delta\beta(A)) - \delta\cosh(\delta\beta(A))]}{Ae^{-\alpha\beta(A)}[(\alpha^2+\delta^2)\sinh(\delta\beta(A)) - 2\alpha\delta\cosh(\delta\beta(A))]} \\ &= \frac{-\phi'(\beta(A))}{A\Phi''(\beta(A))}.\end{aligned} \tag{5.3.59}$$

The second equality holds from eqs. (5.3.44) and (5.3.51). Thus the sign of $\beta'(A)$ is positive. It implies $\beta(A)$ increases in A. Rewriting eq. (5.3.48) yields

$$Ae^{-\alpha\beta(A)}[\delta\cosh(\delta\beta(A)) - \alpha\sinh(\delta\beta(A))] = k_1. \tag{5.3.60}$$

From the same reason as in the case of $\overline{x}(A)$ we have

$$\lim_{A\downarrow 0} \beta(A) = -\infty, \tag{5.3.61}$$

$$\lim_{A\uparrow\check{A}} \beta(A) = \check{x}. \tag{5.3.62}$$

Now we are ready to show that there exist unique solutions of the simultaneous equations, (5.3.46) – (5.3.48).

Theorem 5.3.2. *Suppose that (AS.5.2.1) holds. Then, there uniquely exist the solutions of the simultaneous equations (5.3.46) – (5.3.48):* A^*, $\overline{x}(A^*)$, *and* $\beta(A^*)$ *with* $0 < \beta(A^*) < \overline{x}(A^*) < +\infty$.

Proof. By eq. (5.3.46) we require

$$g(A) = -k_0, \tag{5.3.63}$$

where $g(A) := A[e^{-\alpha\overline{x}(A)}\sinh(\delta\overline{x}(A)) - e^{-\alpha\beta(A)}\sinh(\delta\beta(A))] - k[\overline{x}(A) - \beta(A)]$. The derivative of g with respect to A is

$$\begin{aligned}\frac{dg}{dA} &= [e^{-\alpha\overline{x}(A)}\sinh(\delta\overline{x}(A)) - e^{-\alpha\beta(A)}\sinh(\delta\beta(A))] \\ &\quad + Ae^{-\alpha\overline{x}(A)}[-\alpha\overline{x}'(A)\sinh(\delta\overline{x}(A)) + \delta\overline{x}'(A)\cosh(\delta\overline{x}(A))] \\ &\quad - Ae^{-\alpha\beta(A)}[-\alpha\beta'(A)\sinh(\delta\beta(A)) + \delta\beta'(A)\cosh(\delta\beta(A))] \\ &\quad - k[\overline{x}'(A) - \beta'(A)] \\ &= e^{-\alpha\overline{x}(A)}\sinh(\delta\overline{x}(A)) - e^{-\alpha\beta(A)}\sinh(\delta\beta(A))\end{aligned} \tag{5.3.64}$$

From $d\phi(x)/dx = -\alpha Ae^{-\alpha x}\sinh(\delta x) + \delta Ae^{-\alpha x}\cosh(\delta x)$ and eqs. (5.3.47) – (5.3.48), it follows that summation from the second line to fourth line of eq. (5.3.64) becomes 0. This affirms that the second equality of eq. (5.3.64) holds. Furthermore, since (AS.5.2.1) yields

$0 < \alpha < \delta$, $e^{-\alpha x}\sinh(\delta x)$ increases in x. Thus we find that the sign of dg/dA is positive. Therefore it follows that

$$\lim_{A \downarrow 0} g(A) = -\infty, \tag{5.3.65}$$

$$\lim_{A \uparrow \bar{A}} g(A) = 0. \tag{5.3.66}$$

From eqs. (5.3.65) and (5.3.66) there exists A^* such that $g(A^*) = -k_0$ by using the mean value theorem. Therefore there uniquely exist solutions of the simultaneous equations (5.3.46) – (5.3.48): A^*, $\overline{x}(A^*)$, and $\beta(A^*)$ with $0 < \beta(A^*) < \overline{x}(A^*) < +\infty$. □

Next we show that the candidate function of the value function of the firm's problem eq. (5.2.12), satisfies the QVI under an additional condition, that bounds $\overline{x}$ from below. Thus, the candidate function is the value function of the firm's problem (5.2.12). Therefore, the QVI policy induced by the candidate function is indeed optimal. To this end, we first examine the following. For $x \in (0, \overline{x})$, by (5.3.45) the second derivative of $\phi(x)$ is

$$\phi''(x) = Ae^{-\alpha x}[(\alpha^2 + \delta^2)\sinh(\delta x) - 2\alpha\delta\cosh(\delta x)]. \tag{5.3.67}$$

Note that eq. (5.3.67) equals eq. (5.3.51). It is obvious that there exists A such that $\Phi'(\overline{x}) = \Phi'(\beta) = 0$ from Theorem 5.3.2. It follows that $\phi'(\overline{x}) = k_1$ and $\phi'(\beta) = k_1$. Furthermore, since $\Phi'(x)$ has a unique minimum point,

$$\Phi'(x) \begin{cases} > 0, & x \in (0, \beta) \text{ or } (\overline{x}, \infty), \\ = 0, & x = \beta \text{ or } \overline{x}, \\ < 0, & x \in (\beta, \overline{x}). \end{cases} \tag{5.3.68}$$

Therefore, it follows that

$$\phi'(x) \begin{cases} > k_1, & x \in (0, \beta) \text{ or } (\overline{x}, \infty), \\ = k_1, & x = \beta \text{ or } \overline{x}, \\ < k_1, & x \in (\beta, \overline{x}). \end{cases} \tag{5.3.69}$$

We now verify that the candidate function of the value function satisfies the QVI. That is, the candidate function is a solution of the QVI. Thus, the candidate function is the value function of the firm's problem eq. (5.2.12), using Theorem 5.3.1. Therefore, the QVI policy induced by the candidate function is optimal.

Theorem 5.3.3. *Under the assumption (AS.5.2.1), let A^*, $\overline{x}^* = \overline{x}(A^*)$ and $\beta^* = \beta(A^*)$ be solutions of the simultaneous equations (5.3.46) – (5.3.48), whose existence is assured by Theorem 5.3.2. Furthermore, we assume the following inequality:*
(AS.5.3.1)

$$e^{2\delta\overline{x}^*} \geq \frac{r + \mu(1+\alpha)}{r - \mu(1-\alpha)}.$$

Then, $\phi^*(x) = \phi(x; A^*, \overline{x}^*, \beta^*)$ satisfies the QVI. According to Theorem 5.3.1, ϕ^* is the optimal value function of the firm's problem eq. (5.2.12), and the QVI policy induced by ϕ^* is optimal. That is, the dividend policy given by (5.3.31) and (5.3.32) is an optimal dividend policy.

Proof. First we show that ϕ satisfies the QVI, (5.3.5) – (5.3.7).

(I) Consider ineq. (5.3.5) for two distinct cases, $x \in (0, \overline{x}^*)$ or $x \in [\overline{x}^*, \infty)$.

(i) If $x \in (0, \overline{x}^*)$, it is clear from (5.3.45) and the derivation of eq. (5.3.44) that

$$\mathcal{L}\phi^*(x) = 0. \tag{5.3.70}$$

(ii) If $x \in [\overline{x}^*, \infty)$, by (5.3.45) we have to show

$$\mathcal{L}\phi^*(x) = \mu k_1 - r[\phi^*(\beta^*) + k_1(x - \beta^*) - k_0]. \tag{5.3.71}$$

If the sign of the right-hand side of eq. (5.3.71) is non-positive, we have

$$\phi^*(\beta^*) + k_1(x - \beta^*) - k_0 \geq \frac{\mu k_1}{r}. \tag{5.3.72}$$

Because $\phi^*(x) = \phi^*(\beta^*) + k_1(x - \beta^*) - k_0$ for $x \in [\overline{x}^*, \infty)$, we rewrite eq. (5.3.72) as

$$\phi^*(x) \geq \frac{\mu k_1}{r}. \tag{5.3.73}$$

Note here that the minimum value of x for $x \in [\overline{x}^*, \infty)$ is attained at $x = \overline{x}^*$. Thus we have to show only

$$\phi^*(\overline{x}^*) \geq \frac{\mu k_1}{r}. \tag{5.3.74}$$

From (5.3.45) and eq. (5.3.56), we can rewrite ineq. (5.3.74) as

$$e^{\delta \overline{x}^*}\left(1 - \frac{\mu(1-\alpha)}{r}\right) - e^{-\delta \overline{x}^*}\left(1 + \frac{\mu(1+\alpha)}{r}\right) \geq 0. \tag{5.3.75}$$

It is enough to show ineq. (5.3.75). Because ineq. (5.3.75) is equivalent to the assumption (AS.5.3.2), ineq. (5.3.75) holds. Thus, we obtain that the sign of the right-hand side of eq. (5.3.71) is non-positive.

Therefore, ϕ^* satisfies ineq. (5.3.5).

(II) Next we show ϕ^* satisfies ineq. (5.3.6). We refer to Theorem 1 of Constantinides and Richard (1978), and divide the region into $(0, \beta^*)$, $[\beta^*, \overline{x}^*)$ and $[\overline{x}^*, \infty)$. For each region, we show that ϕ^* satisfies ineq. (5.3.6).

(i) For $x \in (0, \beta^*)$, by (5.3.69) $\zeta = 0$ is optimal. Then, we have

$$\begin{aligned}
\mathcal{M}\phi^*(x) &= \sup_{\zeta \in (0, \beta^*)} \{\phi^*(\eta(x, \zeta)) + K(\zeta)\} \\
&= [\phi^*(\eta(x, \zeta)) + K(\zeta)]_{\zeta=0} \\
&= \phi^*(x) - k_0 \\
&< \phi^*(x).
\end{aligned} \tag{5.3.76}$$

(ii) For $x \in [\beta^*, \overline{x}^*)$, because equality in (5.3.69) holds at $x = \beta^*$, we obtain that $\zeta = x - \beta^*$ is optimal. Thus, it leads to the following:

$$\begin{aligned}\mathcal{M}\phi^*(x) &= \sup_{\zeta \in (0, x-\beta^*]} \{\phi^*(\eta(x,\zeta)) + K(\zeta)\} \\ &= [\phi^*(\eta(x,\zeta)) + K(\zeta)]_{\zeta = x - \beta^*} \\ &= \phi^*(\beta^*) + k_1(x - \beta^*) - k_0 \\ &< \phi^*(x).\end{aligned} \tag{5.3.77}$$

From $\Phi'(x)$ being a convex function with $\Phi'(0) = k_0$ and the derivation of $\tilde{x}$, we find that $\Phi'(x)$ decreases in x for $x \in (0, \tilde{x})$. On the other hand, $\Phi'(x)$ increases in x for $x \in (\tilde{x}, \infty)$. Thus, we obtain that $\phi^*(x) \geq \phi^*(\beta^*) + k_1(x - \beta^*) - k_0$ for all x. It follows that the inequality of (5.3.77) holds.

(iii) For $x \in [\overline{x}^*, \infty)$, since equality in (5.3.69) holds at $x = \overline{x}^*$ or β^*. Hence either $\zeta = x - \overline{x}^*$ or $\zeta = x - \beta^*$ is optimal. Therefore we have

$$\begin{aligned}\mathcal{M}\phi^*(x) &= \max\left[\sup_{\zeta \in (0, x-\overline{x}^*]} \{\phi^*(\eta(x,\zeta)) + K(\zeta)\},\ \sup_{\zeta \in (x-\overline{x}^*, x)} \{\phi^*(\eta(x,\zeta)) + K(\zeta)\}\right] \\ &= \max\left[[\phi^*(\eta(x,\zeta)) + K(\zeta)]_{\zeta = x - \overline{x}^*},\ [\phi^*(\eta(x,\zeta)) + K(\zeta)]_{\zeta = x - \beta^*}\right] \\ &= \phi^*(\beta^*) + k_1(x - \beta^*) - k_0 \\ &= \phi^*(x).\end{aligned} \tag{5.3.78}$$

Rearranging eq. (5.3.46), we have $\phi^*(\overline{x}^*) + k_1(x - \overline{x}^*) = \phi^*(\beta^*) + k_1(x - \beta^*) - k_0$. It follows that $\phi^*(\overline{x}^*) + k_1(x - \overline{x}^*) - k_0 < \phi^*(\beta^*) + k_1(x - \beta^*) - k_0$. Hence, the third equality of (5.3.78) holds.

Thus, ϕ satisfies ineq. (5.3.6).

(III) It follows immediately from the foregoing consideration that ϕ^* also satisfies eq.(5.3.7).

Therefore, the candidate function, ϕ^*, satisfies the QVI. That is, ϕ^* is a solution of the QVI. ϕ^* is also the value function of the firm's problem eq. (5.2.12) by Theorem 5.3.1. Furthermore, the QVI policy induced by ϕ^*, that is, the dividend policy given by (5.3.31) and (5.3.32), is optimal. This completes the proof. □

Remark 5.3.1. *Jeanblanc-Picqué and Shiryaev (1995) show that the candidate function, ϕ^*, satisfies $\mathcal{L}\phi^*(x) \leq 0$ for $x \in [\overline{x}^*, \infty)$ as follows.*

$$\begin{aligned}\mathcal{L}\phi^*(x) &= \mu k_1 - r[\phi^*(\beta^*) + k_1(x - \beta^*) - k_0] \\ &\leq \mu k_1 - r[\phi^*(\beta^*) + k_1(\overline{x}^* - \beta^*) - k_0] \\ &= \mu k_1 - r\phi^*(\overline{x}^*) \\ &= 0.\end{aligned}$$

The first inequality above holds from $x \geq \overline{x}^*$ in this region. The second equality above holds by (5.3.46). On the other hand, we show that ϕ^* requires the condition (AS.5.3.1), to satisfy ineq. (5.3.6). That is, we show that if the equality above is substituted by *inequality, $\phi^*(\overline{x}^*) \geq \mu k_1/r$, holds. The condition (AS.5.3.1) gives us a lower bound of the threshold, $\overline{x}$, and is expressed only by given parameters of the problem. As to the condition (AS.5.3.1), Cadenillas and Zapatero (1999) derive a similar condition. Cadenillas and Zapatero (1999) study the optimal policy of a central bank in foreign exchange markets by using impulse control theory. If $\overline{x}^* > -1/k_1[\phi^*(\beta^*) - k_1\beta^* - k_0] + \mu/r$, which is derived by applying (48) of Theorem 4.1 of Cadenillas and Zapatero (1999) to our model, then the right-hand side of eq. (5.3.71) is non-positive. The parameter to be determined, β^*, is included in the right-hand side of this condition. On the other hand, our condition (AS.5.3.1) is given by external parameters. If the condition (AS.5.3.1) holds, the right-hand side of eq. (5.3.71) is non-positive in this chapter. Nevertheless, the unknown parameter $\overline{x}^*$ is included in the condition (AS.5.3.1). However it is easy to verify that $\overline{x}^*$, which is provided by numerical methods, satisfies the condition (AS.5.3.1). Note that the condition (AS.5.3.1) gives the lower bound for $\overline{x}^*$. Then, we can confirm that the condition (AS.5.3.1) holds by using the relations between $\overline{x}$ and $\check{x}$, $\check{x} < \overline{x}^*$, that is , substituting $\check{x}$ into $\overline{x}^*$ in the condition (AS.5.3.1).*

5.4 Numerical Examples

In this section we calculate A^*, $\overline{x}^*$ and β^* by using a numerical method, the Newton method, and evaluate the amount of the ith optimal dividend, ζ_i^*, and the ith optimal dividend time, τ_i^*. We have used Mathematica to write the program. Furthermore, we present a comparative statics analysis of ζ_i^* and τ_i^* by changing parameters. The comparative statics analysis provides us with an economic implication. When the firm decides to pay out dividends with transaction costs under uncertainty, these results are useful. Here we compute the ith optimal dividend time, τ_i^* by using the expected first passage time. Following standard methods,[3] we obtain

$$\mathbb{E}[\tau_i^*] = \frac{\overline{x}^*(\exp\{-2\alpha\beta^*\} - 1)}{\mu(\exp\{-2\alpha\overline{x}^*\} - 1)}. \tag{5.4.1}$$

We assume that the hypothetical value of the parameters are as follows: $r = 0.05$, $\mu = 0.04$, $\sigma = 0.2$, $k_1 = 0.7$, and $k_0 = 0.05$. The results of the numerical examples are presented in Table 1. Furthermore, we vary parameters by $\pm 10\%$ and present the change of A^*, $\overline{x}^*$, β^*, ζ_i^* and $\mathbb{E}[\tau_i^*]$. This illustrates the comparative statics analysis. First, the comparative statics results for the amount of the ith optimal dividend, ζ_i^*, are as follows. *The amount of the ith optimal dividend is increasing in the drift parameter, the diffusion parameter and both the fixed and proportional transaction costs. On the other hand, ζ_i^* is decreasing in the discount rate.* Next, the comparative statics analysis of the expected ith optimal dividend time, $\mathbb{E}[\tau_i^*]$, gives us the following results: *An increase in the drift parameter delays the expected ith optimal dividend time, whereas increases in the discount rate, the diffusion parameter and both the fixed and proportional transaction costs hasten the expected ith optimal dividend time.* Note that the magnitude of uncertainty of the cash reserve is measured by the diffusion parameter. It remains true that, because $(1 - k_1)$ is the proportional transaction cost, a decrease in k_1 raises the proportional transaction cost.

[3]See, for example, Section 8.4.1 of Ross (1996).

From a shareholder's perspective these results of the comparative statics analysis are interpreted as follows. An increase in the discount rate means that shareholders prefer today's dividends to future ones. Thus, the optimal threshold and the optimal amount of dividends decrease in the discount rate, while the optimal expected dividend times are hastened. The drift parameter represents the trend of the cash reserve's dynamics. Shareholders prefer to continue to invest in the firm rather than receive dividends in the higher trend case. It follows that the optimal threshold, the optimal amount of the dividends, and the optimal expected dividend times all increase in the trend of the cash reserve's dynamics. As shareholders want to avoid future uncertainty and prefer earlier dividends, the optimal expected dividend times are hastened and the optimal amount of dividends increases in uncertainty. Shareholders would like to avoid transaction costs as well. Thus, an increase in the fixed and proportional transaction costs hastens the optimal expected dividend times, and raises the amount of the optimal dividend at any time.

5.5 Conclusion

This chapter examines an optimal dividend policy with transaction costs under uncertainty. We assume that the firm's cash reserve is governed by a Brownian motion, and that, when the firm pays out dividends, it incurs both fixed and proportional transaction costs. Further, we suppose that the firm becomes bankrupt when the cash reserve falls to zero. The firm's problem is to maximize the expected total discounted dividends paid out to stockholders. To this end, we formulate it as a stochastic impulse control problem, which is approached via the QVI. Then, we conjecture that, under a suitable set of sufficient conditions on the given problem parameters, an optimal dividend policy is in the following form, specified by two critical cash reserve levels: whenever the cash reserve reaches a predetermined level, $\overline{x}$, the firm pays out a fixed dividend, so that it instantaneously reduces to another predetermined level, β. Hence, the firm always pays out a fixed dividend, $\overline{x} - \beta$, at each dividend time. Note that the values of $\overline{x}$ and β are internally determined. In order to show the validity of this conjecture, we first prove that there uniquely exist the parameters, A, $\overline{x}$, and β by examining a system of simultaneous equations, which are the well-known value matching and smooth pasting conditions. Then, we verify that the corresponding candidate function satisfies the QVI for the firm's problem under the additional condition (AS.5.3.1), which bounds the threshold level, $\overline{x}$, from below. That is, the guessed candidate function is the optimal value function for the firm's problem, so that the above dividend policy induced by it is indeed optimal. Finally, we present some numerical examples and comparative static results for the amount of the optimal dividend and the expected duration between the successive optimal dividend times, which are summarized as follows. An increase in uncertainty and of both the fixed and proportional transaction costs raises the amount of the optimal dividend, and hastens the expected optimal dividend time.

In this chapter, the firm's future revenues are governed by a Brownian motion. To make the cash reserve dynamics more relevant for the real world, one needs to consider more general cash reserve dynamics than a dynamics generated by the Brownian motion. An interesting extension of this chapter would be to allow the cash reserve to follow a Brownian motion with a Poisson process. The Poisson process represents the default phenomenon. That is, when we deal with the Poisson process, we develop the model with credit risks. This is currently one of the popular issues in the theoretical and practical analyses of finance. Other interesting extensions would be to consider the upper bound of the dividend and the penalty of bankruptcy. These are to be considered in future works.

Table 5.1: The results of numerical examples.

	A^*	$\overline{x}^*$	β^*	ζ_i^*	$\mathbb{E}[\tau_i^*]$
base case	0.6050	1.0499	0.3125	0.7374	13.9015
r : +10%	0.5545	0.9885	0.2736	0.7148	12.0923
: −10%	0.6677	1.1218	0.3585	0.7632	16.0592
μ : +10%	0.6521	1.0915	0.3524	0.7390	14.7177
: −10%	0.5622	1.0032	0.2673	0.7359	12.7370
σ : +10%	0.6287	1.0917	0.3098	0.7819	13.0939
: −10%	0.5849	1.0029	0.3110	0.6918	14.6759
k_1 : +10%	0.6722	1.0344	0.3223	0.7120	14.0652
: −10%	0.5382	1.0679	0.3012	0.7667	13.7018
k_0 : +10%	0.5987	1.0662	0.3023	0.7638	13.7218
: −10%	0.6117	1.0328	0.3233	0.7094	14.0816

Base case parameters value are as follows: $r = 0.05$, $\mu = 0.04$, $\sigma = 0.2$, $k_1 = 0.7$, and $k_0 = 0.05$.

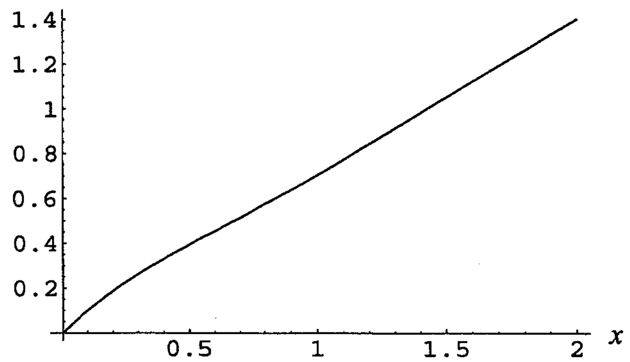

Figure 5.1: $\phi(x)$

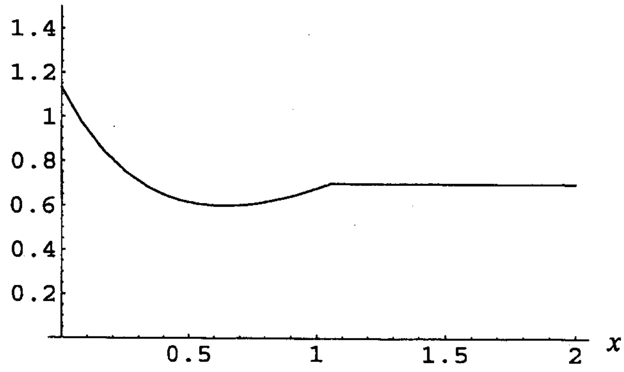

Figure 5.2: $\phi'(x)$

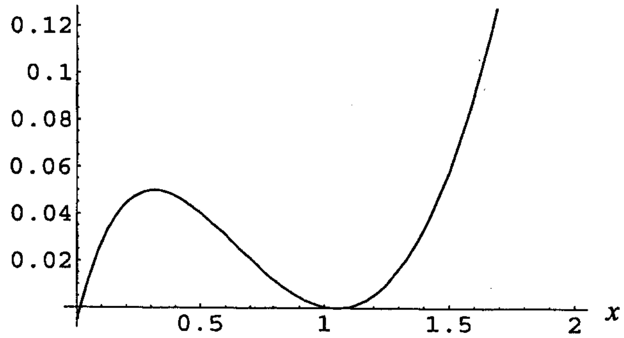

Figure 5.3: $\Phi(x)$

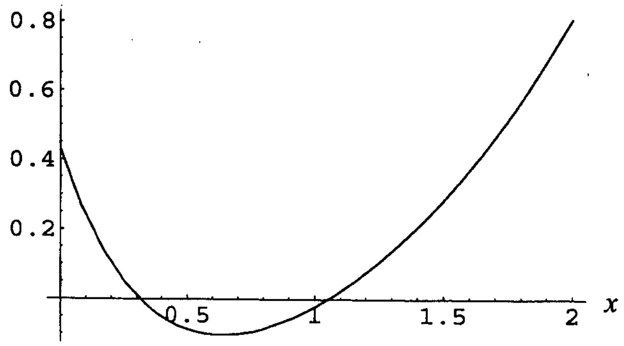

Figure 5.4: $\Phi'(x)$

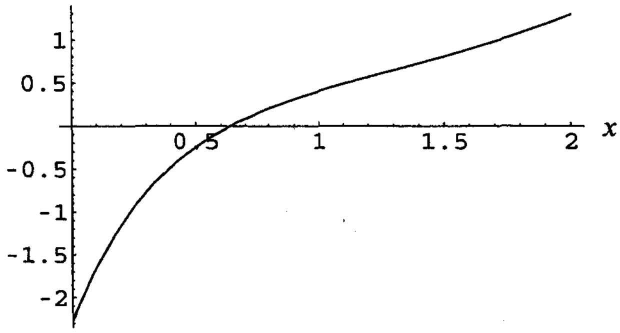

Figure 5.5: $\Phi''(x)$

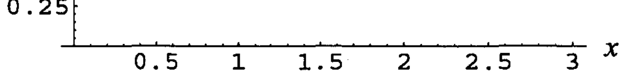

Figure 5.6: $\Phi'''(x)$

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