

Title	A note on regular self-injective rings
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Citation	Osaka Journal of Mathematics. 1984, 21(3), p. 679-682
Version Type	VoR
URL	https://doi.org/10.18910/9832
rights	
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A NOTE ON REGULAR SELF-INJECTIVE RINGS

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(Received March 25, 1983)

(Revised June 17, 1983)

Introduction

Let R be an arbitrary ring and let M be a finitely generated right R -module. Then the dual module $M^* = \text{Hom}_R(M, R)$ of M is a left R -module. If R is a left noetherian ring or finite dimensional algebra, then M^* is also finitely generated. But this property does not hold for a general ring. However, we prove that R has this property if R is a regular right self-injective ring (Proposition).

The purpose of this paper is to prove the following theorems.

Theorem 1. *Let R be a regular ring. Then the following statements are equivalent.*

- 1) R is a right and left self-injective ring.
- 2) For every finitely generated non-singular right (resp. left) R -module, the dual module is a non-zero finitely generated left (resp. right) R -module.

In particular, if R is a commutative regular ring, then we have the following theorem.

Theorem 2. *Let R be a commutative regular ring. Then the following conditions are equivalent.*

- 1) R is a self-injective ring.
- 2) For every finitely generated R -module, the dual module is also finitely generated.

Throughout this paper, we assume that R is a ring with identity element and all modules are unitary. We denote the maximal right quotient ring of R by Q .

Let M be a right R -module. Then we denote the right (resp. left) annihilator ideal by $r(M)$ (resp. $l(M)$), i.e. $r(M) = \{r \in R \mid Mr = 0\}$, (resp. $l(M) = \{r \in R \mid rM = 0\}$).

We denote the category of right R -modules by $\text{Mod-}R$. Let M be a right R -module. Then M is said to be a *cogenerator* in $\text{Mod-}R$ if $\text{Hom}_R(-, M)$ is a faithful functor. In particular, if R is an injective cogenerator in $\text{Mod-}R$, then R is said to be a *right PF-ring*.

1. Proofs of the theorems

Proposition. *Let R be a regular right self-injective ring. Then for any finitely generated right (resp. left) R -module M , the dual module M^* is also finitely generated.*

Proof. Let M be a right R -module and let $\{m_1, m_2, \dots, m_n\}$ be a set of generators of M . Then we prove Proposition by the induction on the number of generators. If $n=1$, then we may assume that $M=R/I$ for some right ideal I of R . In this case, the dual module M^* is isomorphic to $l(I)$. On the other hand, since R is a right self-injective regular ring, R is a Baer ring, so that $l(I)=eR$ for some idempotent element e of R . Therefore we have that $M^*\cong Re$, whence Proposition holds in the case $n=1$. Let $n>1$ and assume Proposition holds in the case where the number of generators is less than n . Set $\bar{M}=m_2R+m_3R+\dots+m_nR$ and take the exact sequence $0\rightarrow\bar{M}\rightarrow M\rightarrow M/\bar{M}\rightarrow 0$. Then we have the exact sequence $0\rightarrow(M/\bar{M})^*\rightarrow M^*\rightarrow\bar{M}^*\rightarrow 0$, since R is self-injective. From the induction hypothesis, $(M/\bar{M})^*$ and \bar{M}^* are finitely generated. Hence M^* is also finitely generated. Next let M be a finitely generated left R -module and let $0\rightarrow K\rightarrow F\rightarrow M\rightarrow 0$ be an exact sequence with a finitely generated and free left R -module F . Then we obtain the exact sequence $0\rightarrow M^*\rightarrow F^*\rightarrow C\rightarrow 0$, where C is the cokernel of the map $M^*\rightarrow F^*$. Since K^* is a torsionless right R -module, C is a finitely generated torsionless right R -module. We claim that C is projective. Since C is a finitely generated right R -module, C^* is also finitely generated. Note that C^* is non-zero, since C is torsionless. Hence we have that C^* is projective since R is a regular ring. Consequently, C^{**} is also projective and C is isomorphic to a finitely generated submodule of C^{**} . Then [1, Theorem 1.11] shows that C is projective. Thus we obtain that M^* is a direct summand of F^* , which implies that M^* is finitely generated.

Proof of Theorem 1.

1) \Rightarrow 2). Since R is a right and left self-injective regular ring, by the above Proposition and [1; Theorem 9.2], 1) implies 2).

2) \Rightarrow 1). Let I be a non-essential right ideal of R . We consider the right R -module $M=R/I$. Then there exists an exact sequence $0\rightarrow(M/Z_r(M))^*\rightarrow M^*$ where $Z_r(M)$ is the singular submodule of M . Since I is a non-essential right ideal of R , $M\neq Z_r(M)$. Therefore by our assumption, $(M/Z_r(M))^*$ is a non-zero and finitely generated left R -module, whence M^* is also non-zero. The same is true for any non-essential left ideal. In this case, Theorem of Utumi [4; Theorem 3.3] shows that the maximal right quotient ring of and the maximal left quotient ring of R coincide. Next let a be any element of the maximal quotient ring Q of R . Then we set $J=\{r\in R|ar\in R\}$ and $K=\{r\in R|ra\in R\}$.

Clearly J is an essential right ideal and K is an essential left ideal of R . We set $N = aR + R \subset Q$, then by the assumption, N^* is non-zero finitely generated. We claim that $N^* \cong K$ as left R -modules. For any element f of N^* , we put $f(a) = x$ and $f(1) = y$. Then we have, for any element z of I , $f(az) = yaz$. Hence we obtain that $x = ya$ since Q is a non-singular right R -module. Now we define an R -homomorphism φ from N^* to K by $\varphi(f) = f(1)$. Evidently, φ is a well-defined R -homomorphism. We shall show that φ is injective. If $\varphi(f) = f(1) = 0$, then from $x = ya$, it follows that $f(a) = 0$, whence φ is injective. Finally, we shall show that φ is surjective. Let k be any element of K . Then we define the element f_k in N^* by the left multiplication by k . Clearly, $\varphi(f_k) = k$. Consequently, we have $N^* \cong K$ as claimed. This implies that K is finitely generated since N^* is finitely generated. On the other hand, since K is an essential left ideal of R , K must be equal to R . Therefore we obtain that a is in R , which shows that R is a right and left self-injective ring.

Proof of Theorem 2.

1) \Rightarrow 2). This is a direct consequence of the above Proposition.
 2) \Rightarrow 1). Let Q be a maximal quotient ring of R and let a be any element of Q . We set $M = aR + R$. By the proof of Theorem 1, it suffices to prove that M^* is non-zero. We set $I = \{r \in R \mid ar \in R\}$. For any element i of I , we can define R -homomorphisms $f_i: M \rightarrow R$ by the multiplication map by i . Hence $f_i \in M^*$, so that M^* is non-zero. Therefore R is a self-injective ring.

REMARK. In the above Theorems, we can not drop the assumption that R is a regular ring. For example, let $Z_{(p)}$ be the *Prufer group* (p a prime), and Z_{p^∞} be the ring of the p -adic integers, considered as the endomorphism ring of Z_{p^∞} . Define a multiplication by $(a, x)(b, y) = (ab, ay + xb)$, for $a, b \in Z_{(p)}$ and $x, y \in Z_{p^\infty}$, on the additive group $R = Z_{(p)} \oplus Z_{p^\infty}$. Then by [3], R is a commutative quasi-local *PF*-ring but not a noetherian ring. Furthermore we shall show that R does not have the property: For any finitely generated R -module M , the dual module M^* is also finitely generated. In order to show this fact, we prove the following result. "Let R be a right *PF*-ring and suppose that R satisfies the above property. Then R is a *QF*-ring". In fact, let M be a finitely generated right R -module. Then since R is a right *PF*-ring, M is torsionless. From the assumption, M^* is also finitely generated. Now since we have an exact sequence $F \rightarrow M^* \rightarrow 0$ with a finitely generated and free left R -module F , we obtain an exact sequence $0 \rightarrow M^{**} \rightarrow F^*$. Since M is torsionless, M can be embedded in a free R -module. In this case, by the Theorem of Faith and Walker [2], R is a *QF*-ring. Therefore in our example, R does not satisfy this property. Hence R is a self-injective ring but not a regular

ring, and does not satisfy the property: For any finitely generated R -module M , the dual module M^* is also finitely generated.

References

- [1] K.R. Goodearl: Von Neumann regular rings, Pitman. London. 1979.
- [2] C. Faith and E.A. Walker: *Direct sum representations of injective modules*, J. Algebra **5** (1967), 203–221.
- [3] B.L. Osofsky: *A generalization of quasi-Frobenius rings*, J. Algebra **4** (1966), 373–387.
- [4] Y. Utumi: *On rings of which any one sided quotient rings are two-sided*, Proc. Amer. Math Soc. **14** (1963), 141–147.

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