Title	A version of Landweber's filtration theorem for v_n-periodic Hopf algebroids
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Citation	Osaka Journal of Mathematics. 1995, 32(3), p. 689-699
Version Type	VoR
URL	https://doi.org/10.18910/9833
rights	
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Baker A.
Osaka J. Math.
32 (1995), 689-699

A VERSION OF LANDWEBER'S FILTRATION THEOREM FOR v_n-PERIODIC HOPF ALGEBROIDS

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(Received November 11, 1993)

0. Introduction

In the theory of comodules over the Hopf algebroids $MU_*(MU)$ and $BP_*(BP)$, a major result is Peter Landweber's Filtration Theorem and its important corollary, the Exact Functor Theorem [7]. A careful examination of the proof of Landweber's Filtration Theorem shows that it depends on the fact that a comodule M_* over $MU_*(MU)$ (or $BP_*(BP)$) which is finitely generated over MU_* (or BP_*) is *connective*, i.e., $M_n=0$ if n is sufficiently small. This condition on M_* allows an inductive proof of the Filtration Theorem based upon the fact that an element in M_* of smallest degree is coaction primitive. Clearly such an approach is not possible for a comodule over a *periodic* Hopf algebroid since there is no analogous notion of smallest degree. It is therefore reasonable to ask if there are counterexamples to the Filtration Theorem in this context.

In this paper we investigate the situation for some periodic Hopf algebroids which play an important rôle in algebraic topology. These are associated to Noetherian completions of the spectra $E(n)$ of Johnson and Wilson [6], well known to be 'lifts' of the Morava K-theories $K(n)$.

In §1, we describe the Hopf algebroids of interest in terms inspired by Morava ; in particular, we show that they are *split*. A similar result was also proved by Hopkins and Ravenel in the unpublished preprint [5]. It is crucial in this and related work that these Hopf algebroids should be viewed as topologized objects.

In §2, we prove an analogue of the Landweber's Filtration Theorem for finitely generated discrete topologized comodules over our Hopf algebroids. The proof is essentially a well known result in p-group theory. It is also the case that any comodule induced from a connective one over a connective Hopf algebroid for which Landweber's Filtration Theorem applies has such a filtration ; in particular, comodules such as $E(n)_*(X)$ over $E(n)_*(E(n))$ have this property as remarked by Doug Ravenel in [9].

In §3, we give counterexamples to the Landweber Filtration Theorem for arbitrary topological comodules, by exhibiting examples with no non-trivial coaction primitives.

Finally, in the Appendix we give an account of some basic facts about *twisted group rings*, probably well known to algebraists, but not as familiar to topologists.

I would like to thank the following for help and advice over many years on the subject matter of this paper and related things: Francis Clarke, Mike Hopkins, John Hunton, Doug Ravenel, Neil Strickland, and Urs Würgler.

The author would like to acknowledge the support of SERC, MSRI, Max-Planck-Institute (Bonn), Osaka Prefecture, and the Universities of Manchester and Glasgow.

1. Some periodic Hopf algebroids

Let p be a prime and $1\leqslant n$. Then there is a (graded) Hopf algebroid $(\Gamma(n)_*, E(n)_*)$, where

$$\begin{aligned} E(n)_* &= \boldsymbol{Z}_{(p)}[v_1, \cdots, v_{n-1}, v_n, v_n^{-1}], \\ \Gamma(n)_* &= E(n)_* \underset{BP_*}{\otimes} BP_*(BP) \underset{BP_*}{\otimes} E(n)_* \\ &= E(n)_*(t_k:\ k\geqslant 1). \end{aligned}$$

In fact, from [6] we know that there is a ring spectrum $E(n)$ for which $\Gamma(n)_*\cong E(n)_*(E(n))$, but we will proceed entirely algebraically. In this description, the elements $t_k\in\Gamma(n)_{2(p^k-1)}$ satisfy algebraic relations of the form

$$v_n t_k^{p^n} - v_n^{p^k} t_k \equiv 0 \qquad \bmod I_n,$$

where $I_n \lhd E(n)_*$ is the maximal graded ideal $(v_0, v_1, \cdots, v_{n-1})\lhd E(n)_*$, where as usual we set $v_0=p$.

Now we can consider the invariant ideal I_n as generating a Hopf ideal in $\Gamma(n)_*$, which we will often denote by $I_n\lhd\Gamma(n)_*$ when no confusion seems likely. Then the I_n-adic completions of the rings $E(n)_*$ and $\Gamma(n)_*$ are by definition the Hausdorff completions

$$\widehat{E(n)}_* = \varprojlim_k E(n)_*/I_n^k,$$

$$\widehat{\Gamma(n)}_* = \varprojlim_k \Gamma(n)_*/I_n^k,$$

which give rise to a topological Hopf algebroid $(\widehat{\Gamma(n)}_*, \widehat{E(n)}_*)$. We remark that from a topological perspective this is an anomalous object, since although there is a spectrum $\widehat{E(n)}$ satisfying $\widehat{E(n)}_*\cong\pi_*(\widehat{E(n)})$, we do not have $\widehat{\Gamma(n)}_*\cong \widehat{E(n)}_*(\widehat{E(n)})$. Instead, we have

$$\widehat{\Gamma(n)}_* \cong \pi_*((\widehat{E(n)}\wedge\widehat{E(n)})_{K(n)}),$$

where $(\ )_{K(n)}$ denotes localization with respect to Morava K-theory $K(n)_*(\ )$ in the sense of Bousfield [4] (see also [9] and [8]). The Hopf algebroid $\widehat{\Gamma(n)}_*$ is essentially that discussed by Hopkins and Ravenel in [5], although we give an independent account.

Now we need to recall the existence of the *n-th Morava stabilizer group* $\boldsymbol{S}_n$. Further details to augment the following brief account may be found in [8].

For p a prime and $n \geqslant 1$, let $\boldsymbol{F}_{p^n}$ be the Galois field of p^n elements. Let $W(\boldsymbol{F}_{p^n})$ be the ring of Witt vectors of $\boldsymbol{F}_{p^n}$, i.e., the unique unramified extension of the ring of p-adic integers $\boldsymbol{Z}_p$. The group $\boldsymbol{S}_n$ is a profinite p-group which is a subgroup of the group of units in a central division algebra $\boldsymbol{D}_n$ over the p-adic numbers $\boldsymbol{Q}_p$ of degree n^2 and which contains the field of fractions of $W(\boldsymbol{F}_{p^n})$ (and in fact any degree n extension of $\boldsymbol{Q}_p$). The elements of this group are expressible uniquely in the form of a convergent series

$$\alpha = 1 + \sum_{k \geqslant 1} \alpha_k S^k,$$

where $\alpha_k \in W(\boldsymbol{F}_{p^n})$ satisfies $\alpha_k^{p^n} = \alpha_k$ for all k. Here S is an element of $\boldsymbol{D}_n$ for which $S^n = p$. Moreover, the formula

$$S \gamma S^{-1} = \gamma^p$$

holds for any $\gamma \in W(\boldsymbol{F}_{p^n})$ which also satisfies $\gamma^{p^n} = \gamma$; thus conjugation by S agrees with the lift of Frobenius to $W(\boldsymbol{F}_{p^n})$, which we will denote by $(\ )^\sigma$.

In order to state our first major result, we need to convert our $\boldsymbol{Z}$-graded objects into $\boldsymbol{Z}/2$-graded objects. We do this in a standard way which is equivalent to introducing a periodicity of degree 2. Thus if M_* is a module over $\widehat{E(n)}_*$ we first construct the object

$$\widehat{E(n)}_*[u_n]/(u_n^{p^n-1} - v_n) \underset{\widehat{E(n)}_*}{\otimes} M_*.$$

If M_* is a comodule over $\widehat{\Gamma(n)}_*$, we can extend the coaction to a coaction

$$\begin{aligned} &\widehat{E(n)}_*[u_n]/(u_n^{p^n-1} - v_n) \underset{\widehat{E(n)}_*}{\otimes} M_* \xrightarrow{\psi} \\ &\quad (\widehat{E(n)}_*[u_n]/(u_n^{p^n-1} - v_n) \underset{\widehat{E(n)}_*}{\otimes} \widehat{\Gamma(n)}_* \underset{\widehat{E(n)}_*}{\otimes} \widehat{E(n)}_*[u_n]/(u_n^{p^n-1} - v_n)) \underset{\widehat{E(n)}_*}{\otimes} M_* \end{aligned}$$

which makes $\widehat{E(n)}_*[u_n]/(u_n^{p^n-1} - v_n) \underset{\widehat{E(n)}_*}{\otimes} M_*$ into a comodule over the Hopf algebroid

$$\widehat{E(n)}_*[u_n]/(u_n^{p^n-1} - v_n) \underset{\widehat{E(n)}_*}{\otimes} \widehat{\Gamma(n)}_*$$

To define this extended coaction, we first extend to $\widehat{E(n)}_*[u_n]/(u_n^{p^n-1} - v_n)$ using the formula

$$\psi(u_n)=u_n(v_n^{-1}\psi v_n)^{1/(p^n-1)}.$$

The latter is meaningful since the formula $\psi v_n \equiv v_n \bmod I_n$ allows us to expand the element $(v_n^{-1}\psi v_n)^{1/(p^n-1)}$ by the binomial expansion which converges in the I_n-adic topology on $\widehat{\Gamma(n)}_*$. The general case now follows using the product formula for the coaction. Having obtained this $\boldsymbol{Z}$-graded extended comodule where everything is 2-periodic, we can now take the 0 and 1 components which form a $\boldsymbol{Z}/2$-graded object whose grading we will denote by $\bullet$ with values $+$ and $-$. Thus we have the $\boldsymbol{Z}/2$-graded ring $\widehat{E(n)}_\bullet$ and the module $M_\bullet$ over it, which is also a comodule over the Hopf algebroid $(\widehat{\Gamma(n)}_\bullet, \widehat{E(n)}_\bullet)$. If the odd grading in $M_\bullet$ is trivial, then we can and frequently will identify $M_\bullet$ with the ungraded object M_+. We may recover the original $\boldsymbol{Z}$-graded object by tensoring $M_\bullet$ with the ring $\boldsymbol{Z}_p[u_n, u_n^{-1}]$ (over $\boldsymbol{Z}_p$). This actually sets up an equivalence of categories but we will not dwell on this.

We will frequently consider the $\boldsymbol{Z}/2$-graded object $\widehat{E(n)}_\bullet$ as a ring with unique maximal ideal $(p, v_1', \cdots, v_{n-1}')$ generated by the elements

$$v_k'=u_n^{-(p^k-1)}v_k\in\widehat{E(n)}_0.$$

We will denote this by I_n since it is unlikely to be confused with the ideal $I_n \lhd \widehat{E(n)}_*$.

We also need to define an action of the infinite cyclic group generated by S, $\langle S\rangle$, on the ring of continuous functions $\boldsymbol{S}_n \rightarrow W(\boldsymbol{F}_{p^n})$, $\mathrm{Cont}(\boldsymbol{S}_n, W(\boldsymbol{F}_{p^n}))$; this action is given by

$$f^S(\alpha)=f(S\alpha S^{-1})^{\sigma^{-1}}.$$

The fixed point set will be denoted by $\mathrm{Cont}(\boldsymbol{S}_n, W(\boldsymbol{F}_{p^n}))^{\langle S\rangle}$. Of course, this can be considered as a $\boldsymbol{Z}/2$-graded object concentrated in degree 0.

Theorem 1.1. *There is an isomorphism of topological $\boldsymbol{Z}/2$-graded algebras over $\boldsymbol{Z}_p$,*

$$\begin{aligned}\widehat{\Gamma(n)}_\bullet &\cong \mathrm{Cont}(\boldsymbol{S}_n, \widehat{E(n)}_\bullet \underset{\boldsymbol{Z}_p}{\otimes} W(\boldsymbol{F}_{p^n}))^{\langle S\rangle}\\ &\cong \widehat{E(n)}_\bullet \underset{\boldsymbol{Z}_p}{\widehat{\otimes}} \mathrm{Cont}(\boldsymbol{S}_n, W(\boldsymbol{F}_{p^n}))^{\langle S\rangle}.\end{aligned}$$

Moreover, there is a $\boldsymbol{Z}_p$-Hopf algebroid $(\widehat{E(n)}_\bullet \widehat{\otimes}_{\boldsymbol{Z}_p} \mathrm{Cont}(\boldsymbol{S}_n, W(\boldsymbol{F}_{p^n}))^{\langle S\rangle}, \widehat{E(n)}_\bullet)$ and the above isomorphism is one of Hopf algebroids. Thus, the Hopf algebroid $(\widehat{\Gamma(n)}_\bullet, \widehat{E(n)}_\bullet)$ is split in the sense of [8].

Proof. If we reduce modulo I_n, it is a by now well established fact that there is an isomorphism

$$K(n)_{\bullet}K(n)\cong\widehat{\Gamma(n)}_{\bullet}/I_n\cong\mathrm{Cont}(\boldsymbol{S}_n,\ \boldsymbol{F}_{p^n})^{\langle S\rangle},$$

where $\boldsymbol{F}_{p^n}$ is discrete and S acts on it as Frobenius. This isomorphism can be lifted to show that for each k satisfying $0\leqslant k<\infty$ we have

$$\begin{aligned}\widehat{\Gamma(n)}_{\bullet}/I_n^k&\cong\mathrm{Cont}(\boldsymbol{S}_n,\ E(n)_{\bullet}/I_n^k\underset{\boldsymbol{Z}_p}{\otimes}W(\boldsymbol{F}_{p^n}))^{\langle S\rangle}\\&\cong\mathrm{Cont}(\boldsymbol{S}_n,\ W(\boldsymbol{F}_{p^n})/(p^k))^{\langle S\rangle}\underset{\boldsymbol{Z}_p}{\otimes}\widehat{E(n)}_{\bullet}/I_n^k\end{aligned}$$

The details make use of results from [1] and also (see [8], [2] for example) the fact that the generators t_j of $K(n)_{\bullet}K(n)$ are identifiable with the locally constant functions

$$t_j:1+\sum_{i\geqslant1}\alpha_iS^i\longmapsto\overline{\alpha_j}\in\boldsymbol{F}_{p^n}$$

which are known to generate $\mathrm{Cont}(\boldsymbol{S}_n,\ \boldsymbol{F}_{p^n})$. The work of [2] then gives liftings of these t_j to continuous function $t_j':\boldsymbol{S}_n\to\widehat{E(n)}_{\bullet}\otimes_{\boldsymbol{Z}_p}W(\boldsymbol{F}_{p^n})$. For each k as above, we can then show the desired result since $\widehat{E(n)}_{\bullet}/I_n^k$ is finitely generated over $\boldsymbol{Z}/(p^k)$.

The result on the Hopf algebroid structure requires the lifting theory of [3] to establish that there is a continuous multiplicative action of $\boldsymbol{S}_n$ upon $\widehat{E(n)}_{\bullet}\otimes_{\boldsymbol{Z}_p}$ $W(\boldsymbol{F}_{p^n})$ which dualises to give a continuous coaction as required. □

In [5], Hopkins and Ravenel give a different proof of this result. We refer the reader to the Appendix for a discussion of twisted group rings which is also relevant to this material.

Now suppose that $M_{\bullet}$ is a finitely generated topological $E(n)_{\bullet}$ module which is also a topological left comodule over $\widehat{\Gamma(n)}_{\bullet}\cong\mathrm{Cont}(\boldsymbol{S}_n,\ \widehat{E(n)}_{\bullet}\otimes_{\boldsymbol{Z}_p}W(\boldsymbol{F}_{p^n}))^{\langle S\rangle}$. Then if

$$\psi(m)=\sum f'\otimes m',$$

we can define a continuous action of $\boldsymbol{S}_n$ on $M_{\bullet}$ by

$$\alpha\cdot m=\sum f'(\alpha)m'\quad\text{for } \alpha\in\boldsymbol{S}_n.$$

In the particular case of *discrete* topological comodule, we see that for each $m\in M_{\bullet}$ we have for some k that $I_n^k\cdot m=0$. We then obtain that the minimal subcomodule $M(m)_{\bullet}\subseteq M_{\bullet}$ containing m also satisfies $I_n^k\cdot M(m)_{\bullet}=0$ and therefore is a comodule over the quotient $\mathrm{Cont}(\boldsymbol{S}_n,\ \widehat{E(n)}_{\bullet}/I_n^k\otimes_{\boldsymbol{Z}_p}W(\boldsymbol{F}_{p^n}))$. In terms of the action of $\boldsymbol{S}_n$ we have

Proposition 1.2. *A discrete topological $\widehat{\Gamma(n)}_{\bullet}$-comodule $M_{\bullet}$ admits the structure of a topological $\widehat{E(n)}_{\bullet}[\boldsymbol{S}_n]$-module with the property that for each m*

$\in M_\bullet$, Stab$(m)\subseteq S_n$ is an open subgroup.

Such S_n-modules are usually called *proper* in the language of Galois cohomology. The converse to Proposition (1.2) is

Proposition 1.3. *A proper topological $\widehat{E(n)}_\bullet[S_n]$-module $M_\bullet$ admits the structure of a discrete topological $\widehat{\Gamma(n)}_\bullet$-comodule.*

In effect, this sets up an equivalence of categories between discrete comodules and proper modules over these algebras.

2. A Filtration Theorem for discrete comodules over $\widehat{\Gamma(n)}_\bullet$

We now investigate the structure of discrete $\boldsymbol{Z}/2$-graded comodules over the Hopf algebroid $\widehat{\Gamma(n)}_\bullet$ which are finitely generated over $\widehat{E(n)}_\bullet$. By our last section these are equivalent to discrete modules over the group ring $\widehat{E(n)}_\bullet[S_n]$. The following result is a variant on a standard fact from group cohomology.

Proposition 2.1. *Let G be a profinite p group and M a non-zero discrete abelian group in which every element has order a power of p. Suppose that G acts properly on M as a group of automorphisms. Then the set of G-invariant elements M^G is non-zero.*

Proof. Let $m\in M$ be a non-zero element. Then the orbit of m under G is finite, of order a power of p. The subgroup $M(m)$ of M generated by all the G-translates of m is similarly finite since these all have finite order. But this subgroup is itself a G-submodule of M. Hence we obtain

$$|M(m)|=\sum_{m'}|\mathrm{Orb}_G(m')|,$$

where the sum ranges over a complete set of representatives m' of the G-orbits in $M(m)$. Since each orbit has to have order a power of p, we find that at least one orbit apart from that containing 0 has order 1. □

An immediate consequence is the following.

Proposition 2.2. *Let $M_\bullet$ be a finite discrete topological comodule over $\widehat{\Gamma(n)}_\bullet$. Then the group of primitive elements with respect to the coaction ψ_M: $M_\bullet\to\widehat{\Gamma(n)}_\bullet\otimes_{\widehat{E(n)}_\bullet}M_\bullet$ is non-zero.*

Proof. The group of primitives is equal to the group of fixed points under the associated action of S_n. $\square$

We can now deduce a version of Landweber's Filtration Theorem for such comodules.

Theorem 2.3. *Let $M_\bullet$ be a finite discrete comodule over $\widehat{\Gamma(n)}_\bullet$. Then there is a sequence of subcomodules $0=M_0\subseteq M_1\subseteq\cdots\subseteq M_l=M$ for which*

$$M_k/M_{k-1}\cong\widehat{E(n)}_\bullet/I_n=\boldsymbol{F}_p.$$

Proof. By induction on the order of $M_\bullet$. We carry out the induction by making use of the fact that there is a non-trivial proper subcomodule constructed by taking a primitive $m\neq 0$ (this is possible by (2.2)) and considering the comodule

$$\widehat{E(n)}_\bullet m=\{\lambda\cdot m : \lambda\in\widehat{E(n)}_\bullet\}.$$

Setting $M_1=\widehat{E(n)}_\bullet m$ we can now appeal to the hypothesis on $M_\bullet/M_1$. $\square$

Corollary 2.4. *Let M_* be a comodule over $\Gamma(n)_*$ which is a finitely generated I_n torsion module over $E(n)_*$, i.e., every element is annihilated by a power of I_n. Then there is a sequence of subcomodules $0=M_0\subseteq M_1\subseteq\cdots\subseteq M_l=M$ for which*

$$M_k/M_{k-1}\cong\widehat{E(n)}_\bullet/I_n=\boldsymbol{F}_p.$$

Proof. Such a comodule becomes a discrete module over $\widehat{E(n)}_*$ which is also a comodule over $\widehat{\Gamma(n)}_\bullet$ and we can apply Theorem (2.4) to the associated $\boldsymbol{Z}/2$ graded module over $\widehat{E(n)}_\bullet$, $\widehat{E(n)}_\bullet\otimes_{E(n)_*}M_*$. $\square$

Of course, there are other topological comodules over $\widehat{\Gamma(n)}_\bullet$, which might reasonably possess an analogue of a Landweber filtration. However, in §3 we will show this to be false in general. This implies that the v_n-periodic situation is radically different from that for $BP_*(BP)$ comodules. However, as remarked upon by D. Ravenel in [9], for any $BP_*(BP)$ comodule N_*, the induced $E(n)_*(E(n))$ comodule $N'_*=E(n)_*\otimes_{BP_*}N_*$ *does* possess a filtration with quotients of the form $E(n)_*/I_k$ where $0\leqslant k\leqslant n$; a similar fact is true for comodules over $\widehat{\Gamma(n)}_\bullet$.

3. Some counterexamples

In this section we show that an arbitrary topological comodule over $\widehat{\Gamma(n)}_\bullet$ need not have any non-zero primitives. For ease of exposition we will confine our attention to the details of one type of example for the case of $n=1$; however, this gives rise to a related example for each $n\geqslant 1$.

The ring $\widehat{E(1)}_0$ is just the ring of p-adic integers $\boldsymbol{Z}_p$ and the stabilizer group $\boldsymbol{S}_1$ is equal to the group of strict p-adic units

$$1+p\boldsymbol{Z}_p=\{\alpha\in\boldsymbol{Z}_p : \alpha=1+\alpha' p,\ \alpha'\in\boldsymbol{Z}_p\}.$$

We can form the (pro-)group ring

$$\begin{aligned}\boldsymbol{Z}_p[\boldsymbol{S}_1]&=\boldsymbol{Z}_p[1+p\boldsymbol{Z}_p]\\&=\varprojlim_k \boldsymbol{Z}_p[(1+p\boldsymbol{Z}_p)/(1+p^{k+1}\boldsymbol{Z}_p)]\\&=\boldsymbol{Z}_p[[\tilde{\gamma}]]\end{aligned}$$

where we set $\tilde{\gamma}=[1+p]-1$, using the convention that the element α of the group $1+p\boldsymbol{Z}_p$ is denoted by $[\alpha]$ when viewed as an element of this group ring (hence we have $[0]=1$.

For any integer $m\geqslant 1$, the polynomial X^m-p is irreducible over $\boldsymbol{Z}_p$, by Eisenstein's test, and hence the quotient ring $\boldsymbol{Z}_p[[X]]/(X^m-p)$ is an integral domain. Now setting $X=\tilde{\gamma}$ and ensuring that $m\geqslant 2$, we see that the quotient ring $\boldsymbol{Z}_p[[\tilde{\gamma}]]/(\tilde{\gamma}^m-p)$ is a finitely generated $\boldsymbol{Z}_p$ module which is a $\boldsymbol{Z}_p[\boldsymbol{S}_1]$ module with no non-trivial fixed points. For example, when $m=p-1$ this quotient ring is isomorphic to the ring of integers in the cyclotomic field extension of $\boldsymbol{Q}_p$ obtained by adjoining p-th roots of 1.

Corresponding to such a module over $\boldsymbol{Z}_p[\boldsymbol{S}_1]$, we have a finitely generated comodule over $\widehat{\Gamma(1)}_{\bullet}$ which has no non-trivial primitives under the coaction. Thus there is no filtration of Landweber type for such a comodule.

This example can be generalised to the case of $n>1$, using the semi-direct product decomposition from [2],

$$\boldsymbol{S}_n\cong\boldsymbol{S}_n^1\rtimes(1+p\boldsymbol{Z}_p).$$

This gives a representation of $\boldsymbol{S}_n$ through $\boldsymbol{S}_1$ on the above module. Hence we may conclude that filtrations of Landweber type need not exist for such comodules over $\widehat{\Gamma(n)}_*$.

Appendix: Twisted group rings and their duals

In this Appendix we give an account of the algebra of *twisted group rings* and their dual function algebras. This is surely well known material but we believe it worthwhile to give an account relevant to this paper.

Let G be a group, initially assumed to be finite, $\boldsymbol{k}$ a commutative ring with 1, and R a commutative $\boldsymbol{k}$ algebra on which G acts on the left via $\boldsymbol{k}$ automorphisms (we denote this action by ${}^{\gamma}r$). The *twisted group algebra of G over R* is the free left R module on the elements of G endowed with the product

$$(r\alpha)\cdot(s\beta)=r{}^{\alpha}s\alpha\beta$$

whenever r, $s \in R$ and α, $\beta \in G$. We can view $R\{G\}$ as a bimodule over R, with right action given by

$$(r\alpha)\cdot s = r\,{}^{\alpha}s\alpha.$$

If the action of G on R is trivial this is the usual group ring over R.

Suppose that M is a left R module on which G acts by left $\boldsymbol{k}$ linear automorphisms. Then M is a left $R\{G\}$ module if and only if for $t \in R$, $m \in M$ and $\gamma \in G$,

$$\gamma\cdot(tm) = {}^{\gamma}tm.$$

Now suppose that M is an $R\{G\}$ module and consider the left R module $\mathrm{Map}(G, M)$ of all functions $\theta : G \to M$, endowed with pointwise product by elements of R. There are two natural ways to let G act on $\mathrm{Map}(G, M)$, and we will distinguish these by using the notations $\mathrm{Map}_{\mathrm{L}}$ and $\mathrm{Map}_{\mathrm{D}}$ for the resulting $\boldsymbol{k}[G]$ modules.

We define the action of G on $\mathrm{Map}_{\mathrm{L}}$ by

$$\begin{aligned}(\alpha\cdot\theta)(\gamma) &= ({}^{\alpha}\theta)(\gamma)\\ &= \theta(\gamma\alpha).\end{aligned}$$

Clearly this is left R linear and in general it does *not* give rise to a left $R\{G\}$ module structure.

We define the action of G on $\mathrm{Map}_{\mathrm{D}}$ by

$$\begin{aligned}(\alpha\cdot\theta)(\gamma) &= ({}_{\alpha}\theta)(\gamma)\\ &= \alpha\theta(\gamma\alpha).\end{aligned}$$

This *does* give rise to a left $R\{G\}$ module structure, as is easily verified.

In fact, the two structures just defined are equivalent in the sense of the following result.

Proposition A1. *There is an isomorphism of* $\boldsymbol{k}[G]$ *modules*

$$\Phi : \mathrm{Map}_{\mathrm{L}}(G, M) \to \mathrm{Map}_{\mathrm{D}}(G, M),$$

where $\Phi(\theta)(\gamma) = \gamma^{-1}\theta(\gamma)$. *Moreover, for* $r \in R$ *and* $\alpha \in G$ *we have*

$$\Phi({}^{\alpha}r\theta) = {}^{\alpha}r_{\alpha}\Phi(\theta).$$

Again let M be an $R\{G\}$ module and consider the action map

$$\begin{aligned}\psi_M : G \times M &\to M\,;\\ (\gamma, m) &\longmapsto \gamma m.\end{aligned}$$

Adjoint to this is a coaction map

$$\begin{aligned}\psi_M : M &\to \mathrm{Map}_{\mathrm{L}}(G, M);\\ m &\longmapsto (\psi(m) : \gamma \mapsto \gamma m).\end{aligned}$$

Now for G finite, we can identify the set $\mathrm{Map}(G, M)$ with $M \otimes_R \mathrm{Map}(G, R)$ by using the correspondence

$$\theta \longleftrightarrow \sum_\gamma \theta(\gamma) \otimes \delta_\gamma$$

where $\delta_\gamma : G \to R$ is the characteristic function

$$\delta_\gamma(\alpha) = \begin{cases} 1 & \text{if } \alpha = \gamma, \\ 0 & \text{otherwise.}\end{cases}$$

Under this isomorphism, the above coaction ψ_M corresponds to a map

$$\begin{aligned}\psi_M : M &\to M \underset{R}{\otimes} \mathrm{Map}(G, R);\\ m &\longmapsto \sum_\gamma \gamma m \otimes \delta_\gamma.\end{aligned}$$

Notice that we have

$$\begin{aligned}\psi_M(\alpha(rm)) &= \sum_\gamma {}^{\gamma\alpha} r \gamma \alpha m \otimes \delta_\gamma\\ &= \sum_\gamma r \alpha m \otimes {}^{\gamma\alpha} r \delta_\gamma\\ &= \sum_\gamma \gamma m \otimes {}^{\gamma} r\, {}^{\alpha}\delta_\gamma\end{aligned}$$

since ${}^{\alpha}\delta_\gamma = \delta_{\gamma\alpha}$. Hence we interpret the right hand factor $\mathrm{Map}(G, R)$ as the $R\{G\}$ module $\mathrm{Map}_{\mathrm{D}}(G, R)$ in order to ensure that ψ_M is a homomorphism of $R\{G\}$ modules where we take the G action on the codomain to be that on the right hand factor only.

For the special case of $R = \boldsymbol{k}$ and $M = R$ with R a $\boldsymbol{k}$ algebra on which G acts as above, we obtain an isomorphism

$$\mathrm{Map}(G, R) \cong R \underset{\boldsymbol{k}}{\otimes} \mathrm{Map}(G, \boldsymbol{k})$$

and we can make this into an isomorphism of $\boldsymbol{k}\{G\}$ algebras in two different ways, namely

$$\mathrm{Map}_{\mathrm{D}}(G, R) \cong (R \underset{\boldsymbol{k}}{\otimes} \mathrm{Map}(G, \boldsymbol{k}))_{\mathrm{D}}$$

and

$$\mathrm{Map}_{\mathrm{L}}(G, R) \cong R \underset{\boldsymbol{k}}{\otimes} \mathrm{Map}_{\mathrm{R}}(G, \boldsymbol{k})_{\mathrm{R}}$$

where in the first we take the diagonal action and in the second the action on the right hand factor alone.

Let us now consider a (left) $R\{G\}$ module M. Then there is a canonical R

module structure on M given by

$$r\cdot m=(re)m$$

where e is the identity element in G. However, for each $\gamma\in G$, there is another module structure given by

$$r\cdot m={}^{\gamma}r\gamma m.$$

We can thus identify such an $R\{G\}$ module M with the $\boldsymbol{k}$ module M together with the collection of R module structures indexed on G and with products of the form

$$\varphi_{\gamma}: r\cdot m \longmapsto {}^{\gamma}r\gamma m$$

for $\gamma\in G$. Of course, such structures fit into commutative diagrams of the form

$$\begin{array}{ccc} R\otimes_{k}M & \xrightarrow{\varphi_e} & M \\ \| & & \downarrow \gamma \\ R\otimes_{k}M & \xrightarrow{\varphi_{\gamma}} & M \end{array}$$

as γ varies over G.

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